The CP^*-construction: A Category of Classical and Quantum Channels

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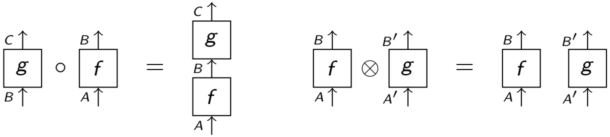
A category for protocols

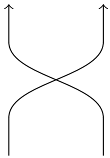
- Fix a category **V**. Think of the objects as state spaces, morphisms as *pure* state evolution.
- Goal: construct a category that is useful for reasoning about quantum protocols.
- To accomplish this, we should generalise in two ways:
  1. pure states $\Longrightarrow$ mixed states
  2. quantum data $\Longrightarrow$ quantum + classical data
- Concretely:
  1. $|\psi\rangle \in H$ $\Longrightarrow$ $\rho \in \mathcal{L}(H)$
  2. operators in $\mathcal{L}(H)$ $\Longrightarrow$ elements in C*-algebra A
- Abstractly:
  1. **V** $\Longrightarrow$ $\mathrm{CPM}[\mathbf{V}]$
  2. $\mathrm{CPM}[\mathbf{V}]$ $\Longrightarrow$ category of "abstract C*-algebras"

Compact closed categories

- Objects are represented as wires, morphisms are boxes
- Horizontal and vertical composition:

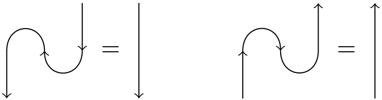


- Crossings (symmetry maps):

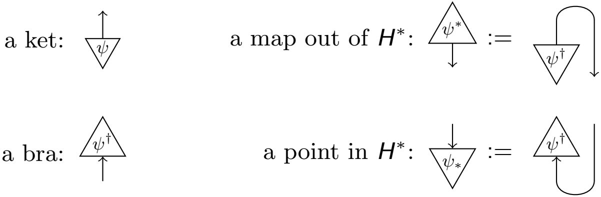


Turning stuff upside-down: duals and daggers

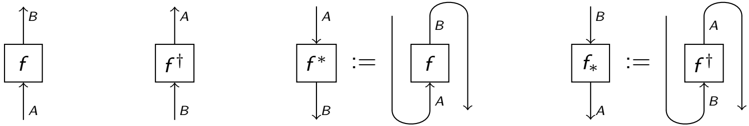
- Compact closure: all objects H have duals H^*, characterised by duality maps. Think: dual space.



- We define a functor $\dagger\colon \mathbf{V}^{\mathrm{op}} \to \mathbf{V}$ that respects all the compact closed structure, and $(f^\dagger)^\dagger = f$. Think: conjugate-transpose.
- This gives us 4 ways to represent (the data of) a ket:



- ...or any other map for that matter:



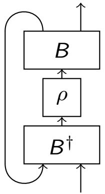
Completely positive maps

- To see how we construct abstract CPMs, consider the concrete case. Any CPM can be represented using Kraus matrices:

$$\Theta(\rho) = \sum_i B_i \rho B_i^\dagger$$

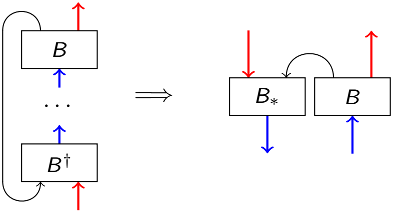
- We can eliminate the sum by purification. Let $B = \sum_i |i\rangle \otimes B_i$, then:

$\Theta(\rho) =$

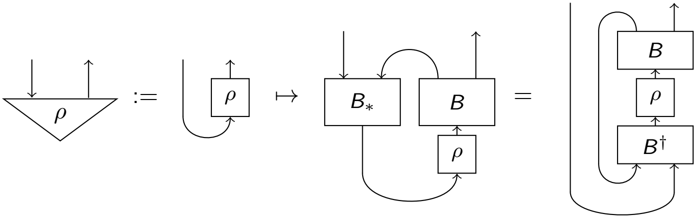


Completely positive maps (cont'd)

- In a compact closed category, maps $\rho : A \to A$ are the same as points $\hat{\rho} : I \to A^* \otimes A$, and operators $\Theta : [A \to A] \to [B \to B]$ are the same as first order maps $\hat{\Theta} : A^* \otimes A \to B^* \otimes B$.

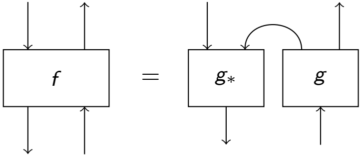


- This is equivalent to the trace-based definition of Θ, up to bending some wires.

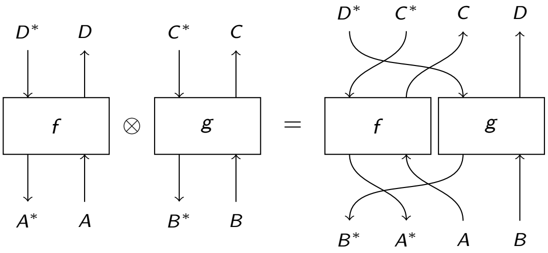


The category $\mathrm{CPM}[\mathbf{V}]$

- The category $\mathrm{CPM}[\mathbf{V}]$ has the same objects as $\mathbf{V}$
- A morphism from A to B is a $\mathbf{V}$-morphism from $A^* \otimes A$ to $B^* \otimes B$, such that there exists same X and some map $g\colon A \to X \otimes B$ where:



- If $X = A \otimes B$, then $X^* = B^* \otimes A^*$. To maintain this "mirror image", the monoidal product involves a reshuffling of wires:



Classical data

- In $\mathrm{CPM}[\mathbf{FHilb}]$, the (normalised) points of an object A are density matrices and maps are CPMs, as required.
- In the density matrix formalism, meaurement can be expressed by projecting an arbitrary density matrix ρ onto the diagonal w.r.t. some basis:

$$m_Z(\rho) = \mathrm{Diag}(\mathrm{prob}_Z(\rho, 1),\ \mathrm{prob}_Z(\rho, 2),\ \mathrm{prob}_Z(\rho, 3),\ \ldots)$$

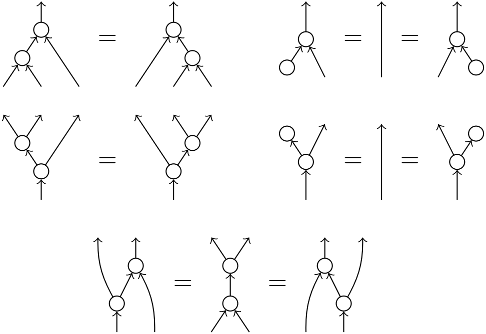
- ...but ρ is an arbitrary state, whereas the RHS is a classical probability distribution. It lives in a tiny corner of $\mathcal{L}(H)$.
- We would like objects not just for the whole quantum state space, but for classical or semi-classical subspaces.

Adding classical objects to $\mathrm{CPM}[\mathbf{V}]$

- There are two ways, due to Selinger, to extend $\mathrm{CPM}[\mathbf{V}]$ such that $\mathrm{CPM}[\mathbf{FHilb}]$ will have all of these classical objects:
  1. **Freely add biproducts.** All classical objects can be expressed as direct sums of 1D matrix algebras $\mathcal{L}(\mathbb{C})$.
  2. **Freely split idempotents.** This effectively adds all subspaces of $\mathcal{L}(H)$ whose associated projection maps $P\colon \mathcal{L}(H) \to \mathcal{L}(H)$ are CPMs. Sub*algebras* are a special case.
- However, one may be "too small" and one may be "too big". Some evidence:
  1. The objects of $\mathrm{CPM}[\mathbf{Rel}]$ are fairly degenerate (indiscreet groupoids), so $\mathrm{CPM}[\mathbf{Rel}]^{\oplus}$ are just sums of degenerate things.
  2. $\mathrm{Split}^{\dagger}(\mathrm{CPM}[\mathbf{FHilb}])$ may have objects which are not physically relevant. *(open problem)*

Another approach: defining "abstract" C*-algebras

- The objects in $\mathrm{CPM}[\mathbf{V}]$ can be thought of as the abstract analogue of matrix algebras. When $\mathbf{V} = \mathbf{FHilb}$, $\mathcal{L}(\mathbb{C}^n) \cong M_n(\mathbb{C})$.
- Rather than starting at $\mathrm{CPM}[\mathbf{V}]$ and trying to extend, start with a notion of abstract C*-algebra, internal to $\mathbf{V}$.
- **Vicary 2008:** dagger-Frobenius algebras in **FHilb** are in 1-to-1 correspondence with finite-dimensional C*-algebras
- A dagger-FA on an object A is a tuple (A,  , ,  , ) such that ()† =  and ()† =  and:



The category $\mathrm{CP}^*[\mathbf{V}]$

- ...

$\mathrm{CP}^*[\mathbf{V}]$ is dagger-compact closed

- ...

$\mathrm{CP}^*[\textbf{FHilb}]$ and $\mathrm{CP}^*[\textbf{Rel}]$

- $\mathrm{CP}^*[\textbf{FHilb}]$ is equivalent to the category of finite-dimensional C*-algebras and completely positive maps
- In **Rel**, dagger-normalisable Frobenius algebras must be special (loop $=$ identity).

The "pants" algebra

- ...

$\mathrm{CPM}[\mathbf{V}] \subseteq \mathrm{CP}^*[\mathbf{V}]$

- ...

Stoch[V] $\subseteq \mathrm{CP}^*$[V]

- ...

$\mathrm{CPM}[\mathbf{V}]^{\oplus} \subseteq \mathrm{CP}^*[\mathbf{V}] \subseteq \mathrm{Split}^{\dagger}(\mathrm{CPM}[\mathbf{V}])$

- ...

Future work

- Generalisation to infinite dimensions
- How many notions from the C*-algebra approach to quantum info can be imported into $\mathrm{CP}^*[\mathbf{V}]$? Already, many can be used verbitim, e.g. commutative subalgebras, POVMs, broadcasting maps, ...
- CBH characterised QM in information-theoretic terms. Often criticised for being too concrete. We have reproduced some parts of their theorem, as well as shown counter-examples (e.g. commutativity is strictly stronger than broadcasting) for $\mathrm{CP}^*[\mathbf{V}]$.
- For $\mathbf{V} \neq \mathbf{FHilb}$, can we make sense of the objects of $\mathrm{CP}^*[\mathbf{V}]$ as state spaces and the morphisms as evolutions? For instance, the category **Stab** of stabiliser states and (post-selected) Clifford circuits faithfully embeds into $\mathrm{CP}^*[\mathbf{Rel}]$.
- Can we characterise categories of the form $\mathrm{CP}^*[\mathbf{V}]$ axiomatically, as with $\mathrm{CPM}[\mathbf{V}]$?