The $\mathbb{CP}^*$-construction: A Category of Classical and Quantum Channels

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A category for protocols

- Fix a category $\mathbf{V}$. Think of the objects as state spaces, morphisms as pure state evolution.
- Goal: construct a category that is useful for reasoning about quantum protocols.
- To accomplish this, we should generalise in two ways:
  1. pure states $\implies$ mixed states
  2. quantum data $\implies$ quantum + classical data
- Concretely:
  1. $|\psi\rangle \in H \implies \rho \in \mathcal{L}(H)$
  2. operators in $\mathcal{L}(H) \implies$ elements in C*-algebra $A$
- Abstractly:
  1. $\mathbf{V} \implies \text{CPM}[\mathbf{V}]$
  2. $\text{CPM}[\mathbf{V}] \implies$ category of “abstract C*-algebras”
Compact closed categories

- Objects are represented as wires, morphisms are boxes
- Horizontal and vertical composition:

\[
\begin{array}{ccc}
\text{g} & \circ & \text{f} \\
\downarrow_{B} & & \downarrow_{B} \\
\uparrow_{C} & & \uparrow_{C} \\
\end{array}
\quad = \quad
\begin{array}{ccc}
\text{g} & & \text{f} \\
\downarrow_{B} & & \downarrow_{A} \\
\uparrow_{C} & & \uparrow_{A} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{f} & \otimes & \text{g} \\
\downarrow_{B} & \otimes & \downarrow_{B'} \\
\uparrow_{B} & \otimes & \uparrow_{B'} \\
\end{array}
\quad = \quad
\begin{array}{ccc}
\text{f} & & \text{g} \\
\downarrow_{B} & & \downarrow_{A} \\
\uparrow_{B} & & \uparrow_{A} \\
\end{array}
\]

- Crossings (symmetry maps):

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}
\quad \begin{array}{c}
\downarrow \\
\uparrow \\
\end{array}
\end{array}
\]
Turning stuff upside-down: duals and daggers

- Compact closure: all objects $H$ have duals $H^*$, characterised by duality maps. Think: dual space.

  \[ \begin{align*}
  \psi^\dagger & = \psi \\
  \psi_\dagger & = \psi^*
  \end{align*} \]

- We define a functor $\dagger: V^\text{op} \to V$ that respects all the compact closed structure, and $(f^\dagger)^\dagger = f$. Think: conjugate-transpose.

- This gives us 4 ways to represent (the data of) a ket:

  a ket:  
  \[ \psi \]
  a map out of $H^*$:  
  \[ \psi^* :\]

  a bra:  
  \[ \psi^\dagger \]
  a point in $H^*$:  
  \[ \psi_\dagger \]

- ...or any other map for that matter:
To see how we construct abstract CPMs, consider the concrete case. Any CPM can be represented using Kraus matrices:

$$\Theta(\rho) = \sum_i B_i \rho B_i^\dagger$$

We can eliminate the sum by purification. Let $B = \sum_i |i\rangle \otimes B_i$, then:

$$\Theta(\rho) = \begin{array}{c}
\begin{array}{c}
B \\
\rangle \\
\rho \\
\langle \\
B^\dagger
\end{array}
\end{array}$$
Completely positive maps (cont’d)

- In a compact closed category, maps $\rho : A \to A$ are the same as points $\hat{\rho} : I \to A^* \otimes A$, and operators $\Theta : [A \to A] \to [B \to B]$ are the same as first order maps $\hat{\Theta} : A^* \otimes A \to B^* \otimes B$.

This is equivalent to the trace-based definition of $\Theta$, up to bending some wires.

\[
\begin{align*}
B & \quad \Rightarrow \quad B_* \quad \text{and} \quad B \\
B^\dagger & \\
\downarrow & \\
\rho & \\
\end{align*}
\]
The category $\mathbf{CPM}[\mathbf{V}]$

- The category $\mathbf{CPM}[\mathbf{V}]$ has the same objects as $\mathbf{V}$
- A morphism from $A$ to $B$ is a $\mathbf{V}$-morphism from $A^* \otimes A$ to $B^* \otimes B$, such that there exists same $X$ and some map $g : A \to X \otimes B$ where:

\[
\begin{align*}
&\quad f \quad = \quad g^* \otimes g
\end{align*}
\]

- If $X = A \otimes B$, then $X^* = B^* \otimes A^*$. To maintain this “mirror image”, the monoidal product involves a reshuffling of wires:
In CPM[\textbf{FHilb}], the (normalised) points of an object $A$ are density matrices and maps are CPMs, as required.

In the density matrix formalism, measurement can be expressed by projecting an arbitrary density matrix $\rho$ onto the diagonal w.r.t. some basis:

$$m_Z(\rho) = \text{Diag}(\text{prob}_Z(\rho, 1), \text{prob}_Z(\rho, 2), \text{prob}_Z(\rho, 3), \ldots)$$

...but $\rho$ is an arbitrary state, whereas the RHS is a classical probability distribution. It lives in a tiny corner of $\mathcal{L}(H)$.

We would like objects not just for the whole quantum state space, but for classical or semi-classical subspaces.
There are two ways, due to Selinger, to extend CPM[\textbf{V}] such that CPM[\textbf{FHilb}] will have all of these classical objects:

1. **Freely add biproducts.** All classical objects can be expressed as direct sums of 1D matrix algebras $\mathcal{L}(\mathbb{C})$.
2. **Freely split idempotents.** This effectively adds all subspaces of $\mathcal{L}(H)$ whose associated projection maps $P: \mathcal{L}(H) \to \mathcal{L}(H)$ are CPMs. Subalgebras are a special case.

However, one may be “too small” and one may be “too big”. Some evidence:

1. The objects of CPM[\textbf{Rel}] are fairly degenerate (indiscreet groupoids), so CPM[\textbf{Rel}] $\oplus$ are just sums of degenerate things.
2. Split\dagger (CPM[\textbf{FHilb}]) may have objects which are not physically relevant. (*open problem*)
Another approach: defining “abstract” C*-algebras

- The objects in $\text{CPM}[\mathcal{V}]$ can be thought of as the abstract analogue of matrix algebras. When $\mathcal{V} = \text{FHilb}$, $\mathcal{L}(\mathbb{C}^n) \cong M_n(\mathbb{C})$.

- Rather than starting at $\text{CPM}[\mathcal{V}]$ and trying to extend, start with a notion of abstract C*-algebra, internal to $\mathcal{V}$.

- **Vicary 2008**: dagger-Frobenius algebras in $\text{FHilb}$ are in 1-to-1 correspondence with finite-dimensional C*-algebras

- A dagger-FA on an object $A$ is a tuple $(A, \uparrow, \odot, \odot, \odot)$ such that
  $$(\uparrow)\dagger = \odot \text{ and } (\odot)\dagger = \odot \text{ and}:$$

- \[\begin{align*}
  & \uparrow \downarrow = \uparrow \downarrow \\
  & \uparrow \downarrow = \bigcirc \bigcirc \\
  & \uparrow \downarrow = \bigcirc \bigcirc \\
  & \uparrow \downarrow = \bigcirc \bigcirc \\
  & \uparrow \downarrow = \bigcirc \bigcirc
\end{align*}\]
The category $\mathbb{CP}^*[\mathbf{V}]$
$\mathbf{CP}^*[\mathbf{V}]$ is dagger-compact closed
\textbf{CP}^*\textbf{[FHilb]} and \textbf{CP}^*\textbf{[Rel]}

- \textbf{CP}^*\textbf{[FHilb]} is equivalent to the category of finite-dimensional C*-algebras and completely positive maps
- In \textbf{Rel}, dagger-normalisable Frobenius algebras must be special (loop = identity).
The "pants" algebra
$\text{CPM}[\mathcal{V}] \subseteq \text{CP}^* [\mathcal{V}]$
\text{Stoch}[\mathbf{V}] \subseteq \mathbb{C}P^*[\mathbf{V}]
$\text{CPM}[\mathcal{V}]^\oplus \subseteq \text{CP}^*[\mathcal{V}] \subseteq \text{Split}^\dagger(\text{CPM}[\mathcal{V}])$
Future work

- Generalisation to infinite dimensions
- How many notions from the C*-algebra approach to quantum info can be imported into $\text{CP}^*[\mathcal{V}]$? Already, many can be used verbatim, e.g. commutative subalgebras, POVMs, broadcasting maps, ...
- CBH characterised QM in information-theoretic terms. Often criticised for being too concrete. We have reproduced some parts of their theorem, as well as shown counter-examples (e.g. commutativity is strictly stronger than broadcasting) for $\text{CP}^*[\mathcal{V}]$.
- For $\mathcal{V} \neq \text{FHilb}$, can we make sense of the objects of $\text{CP}^*[\mathcal{V}]$ as state spaces and the morphisms as evolutions? For instance, the category $\text{Stab}$ of stabiliser states and (post-selected) Clifford circuits faithfully embeds into $\text{CP}^*[\text{Rel}]$.
- Can we characterise categories of the form $\text{CP}^*[\mathcal{V}]$ axiomatically, as with $\text{CPM}[\mathcal{V}]$?