Algebraic (and Diagrammatic) Structures in Quantum Theory

Aleks Kissinger

Institute for Computing and Information Sciences
Radboud University Nijmegen

Quantum Software in NL 2017
Software = components + composition

- Effective **software** comes from understanding relevant **components** and how they **compose**
- Effective **quantum software** will be exactly the same
- But now the “components” could mean:
  - devices
  - physical processes
  - mathematical processes

\[ \rho \mapsto \sum_i B_i \rho B_i^\dagger \]

- ...each with its own (related) notion of composition
• The moral: take *processes* as primitive
• A process is just a box with inputs and outputs:

![Diagram of processes]

• Composition means forming *diagrams* of processes:

![Diagram of processes]

• A collection of processes that make sense to ‘plug together’ is called a *process theory*
The idea: Describe quantum theory entirely in terms of:

- **processes**
  - \( f \)
  - \( C \rightarrow D \)
  - \( A \rightarrow B \)

- **connectivity**
  - \( g \)
  - \( A \rightarrow C \)
  - \( B \rightarrow D \)

- **interaction**
  - =

Not in terms of:
- Hilbert space
- self-adjoint operators, unitary transformations
- calculations with matrices/complex numbers
- ....

(though some may be emergent notions)
PICTURING QUANTUM PROCESSES
A First Course in Quantum Theory and Diagrammatic Reasoning
BOB COECKE
ALEKS KISSINGER

Cambridge University Press
March 2017 (pre-order)
Why?

- Simpler!

\[
(1 \otimes \sigma \otimes k) \circ (\sigma \otimes 1 \otimes 1 \otimes 1) \circ (f \otimes g \otimes 1 \otimes 1) \circ (h \otimes 1) = \text{vs.} \quad (g \otimes f) \circ (1 \otimes k) \circ (h \otimes 1)
\]

- New perspective = new insights
- Deriving QT from first principles ⇐ ‘diagrammatic backbone’
  + a bit of information theory
  
  e.g. **Pavia 2010** and **Hardy 2011**
  **Hardy (2010):** “we join the quantum picturalism revolution”

- A ‘theory playground’
  
  e.g. QT vs. real/boolean-valued/modal QT,
  local QT (indefinite causal structure),
  Spekken’s toy theory, ...

- New calculational tools, applications in quantum info/computation
Q: What kinds of behaviour can we study using just diagrams, and nothing else?

A: (Non-)separability
Separability for states

- States are processes with no input:

\[ \psi \]

Interpret as: ‘preparing a system (or some systems)’

- Separable states:

\[ \psi = \psi_1 \psi_2 \]

- ...are boring!

- Non-separable := ‘no such \( \psi_1, \psi_2 \) exist’, but this isn’t very helpful
‘Non-separable’ isn’t very helpful, but ‘really non-separable’ is:

**Definition**

A process $\psi$ is called *cup-state* if there exists a process $\phi$, called a *cap-effect*, such that:

$\phi \psi = \psi \phi$.

$\psi$ looks like a state, but it *acts* like a wire.
Cup-states

• By introducing some clever notation:

\[ \bigcirc \psi \bigcirc := \triangle \psi \triangle \quad \bigcirc \phi \bigcirc := \triangle \phi \triangle \]

• Then these equations:

\[
\begin{align*}
\triangle \phi \triangle \bigcirc \psi \bigcirc &= \bigcirc \psi \bigcirc \phi \bigcirc & = & \bigcirc \psi \bigcirc \phi \bigcirc \\
\end{align*}
\]

• ...look like this:

\[
\begin{align*}
\bigcirc \bigcirc &= \bigcirc \bigcirc \\
\end{align*}
\]
Yank the wire!
Quantum teleportation

Quantum state $\psi$
Quantum teleportation

Bob

Aleks

quantum state

quantum state

Diagrammatic Structures in Quantum Theory
Quantum teleportation

Diagram:

- **Aleks**
- **Bob**
- Quantum state $\psi$

Quantum state is being transmitted from **Aleks** to **Bob**.
Quantum teleportation

Quantum state $\psi$ is sent to Aleks and remains at Bob. The entangled state is then created and sent to Aleks, allowing the quantum state to be teleported.
Quantum teleportation

Quantum measurement → $U_i$ → quantum state $\psi$ → entangled state

$\psi$: entangled state
$U_i$: quantum measurement

Aleks
Bob
Quantum teleportation

Bob's fix

quantum measurement

quantum state

U_i

ψ

U_i

entangled state

Diagrammatic Structures in Quantum Theory
...and it works
...and it works
...and it works
...and it works

Aleks \rightarrow Bob

\[ \psi \]
If we change the process theory...

Diagram:

- **Aleks**
  - Quantum state $\psi$
  - Quantum measurement
  - Operation $U_i$

- **Bob**
  - Operation $U_i^*$
  - Entangled state

Bob’s fix
If we change the process theory...

Diagram:
- **Aleks**
  - $U_i$
  - compare bits
  - secret bit
  - shared random bit

- **Bob**
  - $U_i^\dagger$
  - do nothing OR flip bit

Extracted text:
- Shared random bit
- Compare bits
- Secret bit
If we change the process theory...

...‘classical teleportation’ is one-time-pad crypto
...and it works
...and it works

\[ U_i, \quad U_i^\dagger \]
...and it works

Aleks

Bob

$U_i^\dagger$

$U_i$

$b$
...and it works
On the same page, OTP cryptography and quantum teleportation. OTP cryptography was introduced in 1882, and quantum teleportation was demonstrated 111 years later. Image: Hanson lab@TUDelft.
Quantum algorithms

⇒ simple (diagrammatic) derivations of **Deutsch-Jozsa**, **Bernstein-Vazirani**, **quantum search**, and **hidden subgroup** algorithms.
Quantum circuits + algebraic structure = ZX-calculus

\[ Z(\alpha) := \begin{array}{c}
\alpha
\end{array} \]

\[ X(\alpha) := \begin{array}{c}
\alpha
\end{array} \]
**ZX-calculus** has 4 equations:

\[
\alpha + \beta = \alpha + \beta
\]

...which can prove any equality between Clifford circuits (and a bit more).
Quantum circuit simplification

\[
\begin{align*}
0 \quad X\left(\frac{\pi}{3}\right) \quad S^\dagger \quad X\left(\frac{\pi}{3}\right) \quad \cdots \quad 1 \\
0 \quad X\left(\frac{\pi}{4}\right)^\dagger \\
0 \quad X\left(\frac{\pi}{4}\right)^\dagger \\
0 \quad X
\end{align*}
\]

\[
\begin{align*}
= & \quad \bigoplus \\
= & \quad T^\dagger \\
= & \quad \bigoplus \\
= & \quad \bigoplus \\
= & \quad \bigoplus \\
:= & \quad 0 \quad 1
\end{align*}
\]
Measurement-based quantum computation

\[ \alpha \pi = \ldots = \pi \alpha \]

Aleks Kissinger
Quantum Software in NL 2017

Diagrammatic Structures in Quantum Theory
GHZ/Mermin non-locality

<table>
<thead>
<tr>
<th>quantum theory</th>
<th>any local theory</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
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</table>
Causal structures and indeterminism

\[ \Phi_A \leftrightarrow \Phi_B \]

\[ \Phi_A \quad ? \quad \Phi_B \]

\[ \Phi_A \quad \Rightarrow \quad \Phi_B \]
Quantum error correction

Stabiliser code

\[ Z_{d_1} Z_{d_2} Z_{b_1} Z_{p_2} Z_{p_4} \]
\[ Z_{d_1} Z_{d_3} Z_{b_3} Z_{p_1} Z_{p_4} \]
\[ Z_{d_2} Z_{d_3} Z_{b_2} Z_{p_3} Z_{p_4} \]
\[ Z_{b_4} Z_{p_1} Z_{p_2} Z_{p_3} Z_{p_4} \]
\[ X_{d_1} X_{d_2} X_{b_2} X_{b_4} X_{p_1} \]
\[ X_{d_1} X_{d_3} X_{b_1} X_{b_4} X_{p_3} \]
\[ X_{d_2} X_{d_3} X_{b_3} X_{b_4} X_{p_2} \]
\[ X_{b_1} X_{b_2} X_{b_3} X_{b_4} X_{p_4} \]
Automation

Quantomatic:
Thanks!

- Picturing Quantum Processes. CUP (March 2017)
- Coherent Parity Check Construction for Quantum Error Correction. Chancellor, Kissinger, Zohren, Horsman. (arXiv on Monday!)

http://quantomatic.github.io