Trichromatic Diagrams for Understanding Qubits

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The general context

We build on the stream of work of categorical semantics for quantum information processing, first initiated in the seminal paper of Abramsky and Coecke [1]. In this tradition, Coecke and Duncan developed and made extensive use of a calculus of dichromatic diagrams to express quantum protocols and quantum states [2,3]. This diagrammatic calculus turned out to be universal for quantum computing. The graphical calculus has been used to prove many statements useful to quantum computing, including some about graph states [9], measurement-based quantum computation [10], and a multitude of protocols [3]. Unfortunately, this calculus is known to be incomplete (with respect to stabilizer quantum mechanics) in a sense that is formalized in the main matter.

The research problem

The red-green calculus of [2,3,9,10] talks of two complementary observables in qubits. It is by now well known that it is possible to fit 3 complementary observables in qubits, and no more [21]. We thus set out to find a calculus which speaks of three complementary observables in a nice way—with the hope of developing a more "complete" theory—and contrast it with the existing red-green calculus

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Your contribution

The Bloch sphere is a well-understood [16] way to understand single-qubit unitaries as rotations of a 2-sphere—the elements of the group $SO(3)$. Although the red-green calculus is universal, and thus able to express all these unitaries, it does not express some of the crucial equations between them.

We developed the trichromatic calculus, which expresses the presence of 3 complementary observables in qubits, and also how they relate to each other through rotations of the Bloch sphere. We also showed that this was equivalent to adding an Euler angle decomposition of the Hadamard gate to the red-green calculus. This is in turn equivalent to Van den Nest's theorem in the red-green setting [9, 20].

Arguments supporting its validity

The trichromatic calculus is equivalent to the dichromatic calculus with the addition of the Euler angle decomposition of the hadamard gate. Thus, it precludes all the improvements that are currently known to the dichromatic calculus, and does so in an elegant way. It is not yet known if these calculi are complete or not. It does seem, however, like the trichromatic calculus offers greater promise than the ad-hoc addition of the Euler angle decomposition to the dichromatic calculus.

Summary and future work

We refer to the section at the end of the report for a detailed exposition of future work.

Contents

1 Preliminaries

We assume the reader is comfortable with the basic notions of category theory and quantum information theory, as well as Dirac notation and standard bases in $\mathbb{C}^2$. Here we present some additional definitions and notation which will be useful throughout.

Definition 1.1 (†-SMC)**.** We refer the reader to [17] for the definition of †-symmetric monoidal category.

Definition 1.2 (**FdHilb**)**.** The †-SMC of complex Hilbert spaces and linear maps between them.

Definition 1.3 ($\mathbf{FdHilb}_{wp}$)**.** The †-SMC of complex Hilbert spaces and linear maps modulo the relation $f \equiv g$ if $\exists\theta : f = e^{i\theta} g$. The reason for this relation is that maps which are different but only up to a phase are indistinguishable.

Definition 1.4 ($\mathbf{FdHilb}_Q$)**.** The full subcategory of $\mathbf{FdHilb}_{wp}$ generated by the objects $\left\{ \underbrace{Q \otimes \cdots \otimes Q}_{n} \;\middle|\; n \geq 0 \right\}$, where $Q := \mathbb{C}^2$. This is essentially the category of qubits, and the main "linear" category we will be considering.

We also use the following category which incarnates stabilizer quantum mechanics—as defined in [16]—and define it in a similar fashion as in [6].

Definition 1.5 (**Stab**)**.** The subcategory of $\mathbf{FdHilb}_Q$ generated by the following linear maps:

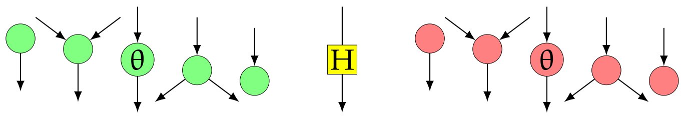
- single-qubit Clifford unitaries : $Q \to Q$
- $$\delta_{\mathbf{Stab}} : Q \to Q \otimes Q = \begin{cases} |0\rangle & \mapsto |00\rangle \\ |1\rangle & \mapsto |11\rangle \end{cases} \quad (1)$$
- $$\varepsilon_{\mathbf{Stab}} : Q \to 1 = \begin{cases} |+\rangle & \mapsto 1 \\ |-\rangle & \mapsto 0 \end{cases} \quad (2)$$

Definition 1.6 (C_4)**.** The rotation group of the square. Of abstract group type $\mathbb{Z}/4\mathbb{Z}$ with elements denoted $\{0, 1, 2, 3\}$ and its operation written additively.

2 Red and Green diagrams

2.1 RG generators

In a similar manner to the work of [9, 10], we formalize the red-green calculus—described in detail in [2,3]—as a category. We define a category **RG** where the objects are n-fold monoidal products of an object $*$, denoted $*^n$. In **RG**, a morphism from $*^m$ to $*^n$ is a dichromatic diagram (Also called Open graphs in [8]) from m wires to n wires, built from the generators below:



(3)

where θ was allowed to be any real number in [3], we restrict θ to take values in C_4, restricting it to 4 values. Except for H, each generator is of one of two colours, hence the name dichromatic. Additionally, the identity morphism on $*$ is represented as the straight wire. The generators θ and θ are called phase gates.

2.2 RG relations

RG diagrams are also subject to the equations depicted in Figure 1[1]. The motivations behind these rules are explained in detail in [2,3].

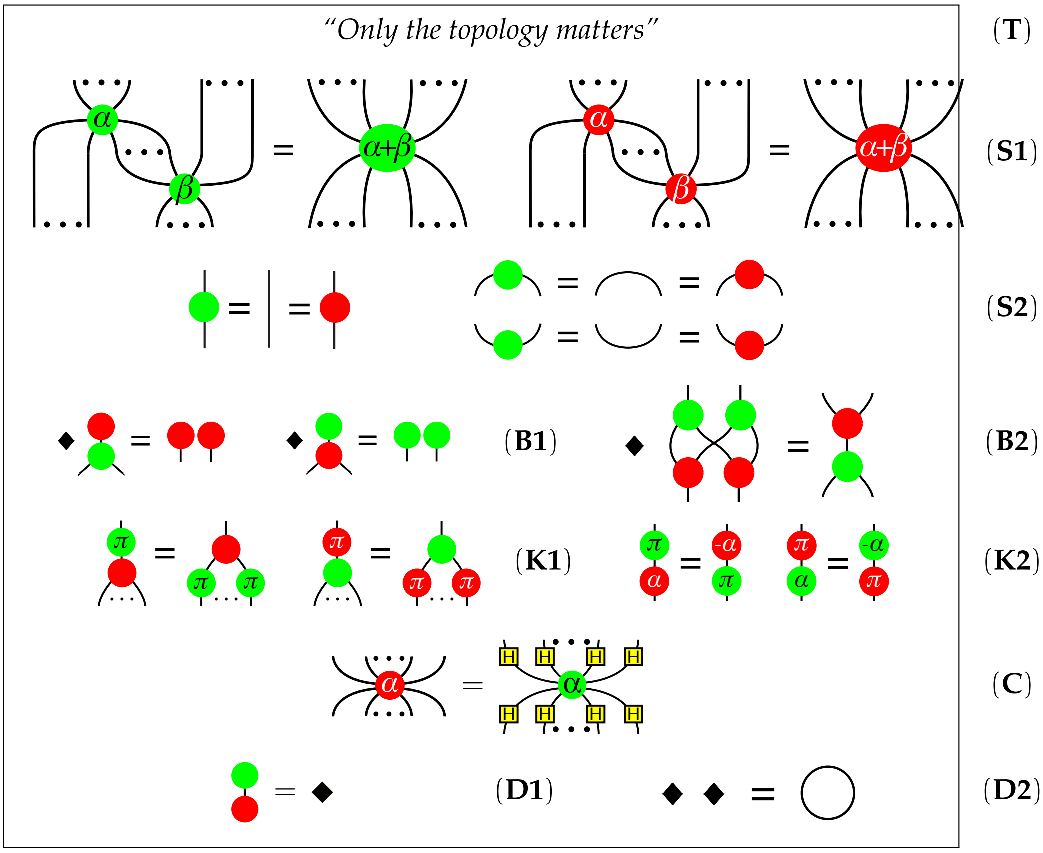
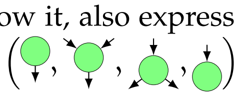
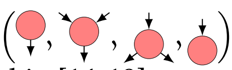


Figure 1: **RG** rules

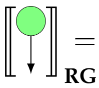
The spider rule gives the subcategory of **RG** generated by the phase gates a group structure. In this case, the group is $C_4 * C_4$, the free product of 2 copies of C_4—one for each colour.

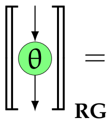
As well, the spider rule, combined with the rule below it, also express the fact that for each colour, the quadruples of generators  and  each form a Frobenius algebra, as defined and exposed in [14, 19].

[1] Figure taken with permission from [3]

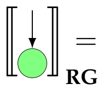
2.3 RG interpretation

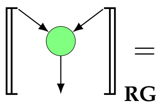
So far we have described a category of diagrams with no quantum content, but in fact these diagrams are meant to represent quantum maps. Here, we provide an interpretation for these diagrams by describing a monoidal functor $[\![\cdot]\!]_{\mathbf{RG}} : \mathbf{RG} \to \mathbf{Stab}$, taking $*$ to $\mathbb{C}^2$ and mapping the morphisms like so (as expressed in Dirac notation):

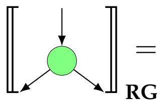
 $= |+\rangle$

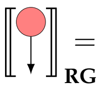


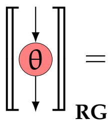
$= |0\rangle\langle 0| + e^{i\frac{\pi}{4}\theta}|1\rangle\langle 1|$

 $= \langle +|$

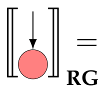
 $= |0\rangle\langle 00| + |1\rangle\langle 11|$

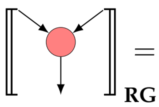
 $= |00\rangle\langle 0| + |11\rangle\langle 1|$

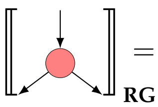
 $= |0\rangle$

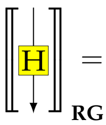


$= |+\rangle\langle +| + e^{i\frac{\pi}{4}\theta}|-\rangle\langle -|$

 $= \langle 0|$

 $= |+\rangle\langle ++| + |-\rangle\langle --|$

 $= |++\rangle\langle +| + |--\rangle\langle -|$



$= |+\rangle\langle 0| + |-\rangle\langle 1|$

The calculus of **RG** turns out to be universal, as exposed in [3].

Proposition 2.1. $[\![\cdot]\!]_{\mathbf{RG}}$ *is indeed a functor.*

Proof. This involves checking for each rule $f = g$ in **RG** that $[\![f]\!]_{\mathbf{RG}} = [\![g]\!]_{\mathbf{RG}}$ as linear maps modulo phase. □

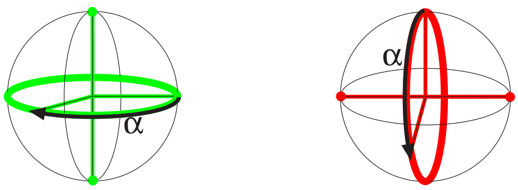
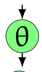
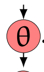
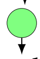
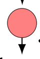


Figure 2: **RG** Bloch spheres

Figure 2 depicts the interpretation of the rotation gates  and . The arrow-tail represents the locations of the deleting points  and . Although the rotations in the axis which is perpendicular to both of these can be expressed in terms of red and green rotations, there is no primitive support for them. This is what we will attempt to remedy in the next section.

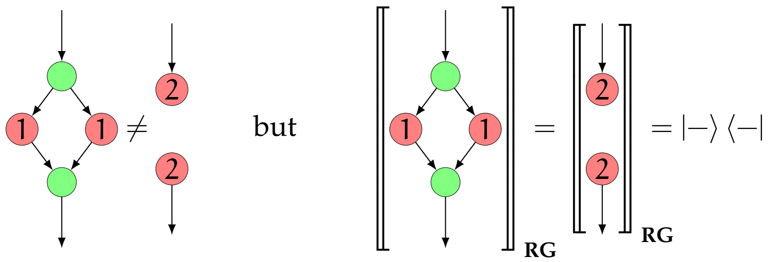
3 Red, Green and Blue diagrams

Here we introduce a theory of trichromatic diagrams as a category **RGB**. Whereas the diagrams in **RG** spoke of two complementary observable structures, **RGB** speaks of three complementary observable structures, the maximum number that can hold in qubits.

One might ask—why look for another diagrammatic theory if **RG** is already universal? The reason is expressed in the following proposition.

Proposition 3.1. $[\![\cdot]\!]_{\mathbf{RG}}$ *is not faithful*

Proof. A counterexample is provided by the following diagrams which are not equal in **RG**, but whose images are equal in **Stab**.



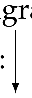
(4)

□

This is a special case of the supplementary rule which was used extensively in [4, 11].

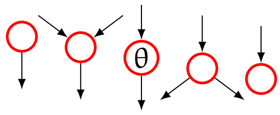
Effectively, this means that **RG** is not complete with respect to **Stab**. That is, there exist diagrams in **RG** which are not equal, but their interpretations are.

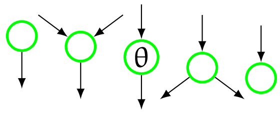
3.1 The category RGB of red-green-blue diagrams

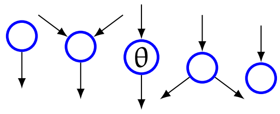
We define a category **RGB** of trichromatic diagrams in a similar manner as we did for the dichromatic category **RG**. **RGB** is a †-SMC with as objects, tensor products of $*$. Morphisms in **RGB** are represented as diagrams and again, the identity morphism on $*$ is represented graphically by: 

the monoidal product is represented by horizontal disjoint union of diagrams and composition is represented by plugging wires.

RGB is generated by (horizontal) disjoint union and (vertical) composition by the following generators:


(5)


(6)



(7)

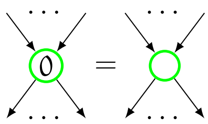
where again θ can be any element of C_4. Of special note is the fact that there is no Hadamard gate in **RGB**. There is instead a pair of gates which take the role of "colour changers" which can be defined in terms of the above generators. They will be defined in the next section.

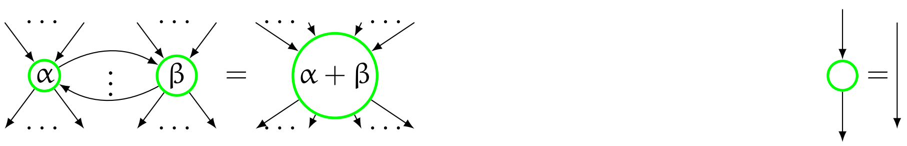
3.2 RGB Rules

These following equations hold in **RGB**.

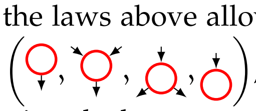
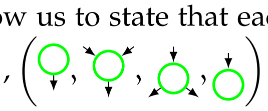
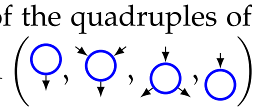
All rules hold under cyclic permutation of colours. (8)

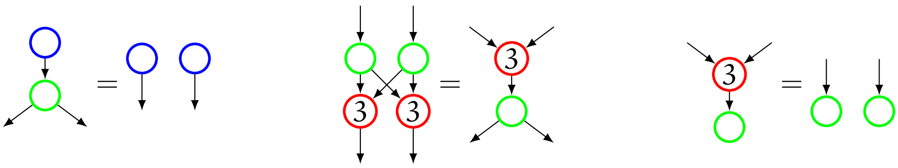
All equations hold under flip of arrows and negation of angles (†). (9)

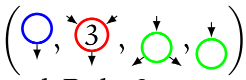
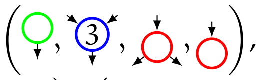
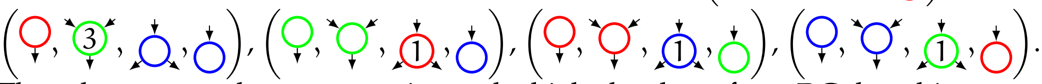

(10)

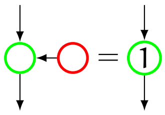


(11)

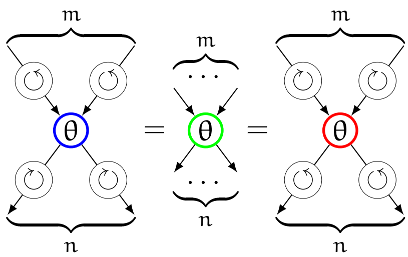
Again, the laws above allow us to state that each of the quadruples of generators ,  and  form Frobenius algebras.

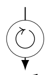
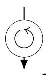
 (12)

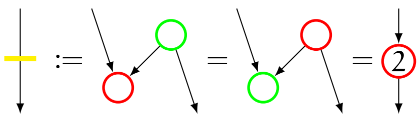
These laws state that state that the quadruple  forms a bialgebra, as defined in [13]. Using Rule 8 and Rule 9, we can also show that the following quadruples form bialgebras: ,

These laws are perhaps not as nice as the bialgabra laws from **RG**, but this is the price to pay for a more expressive, more symmetric theory.

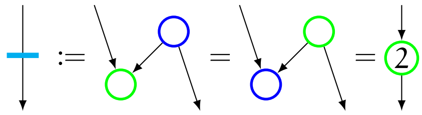
 (13)

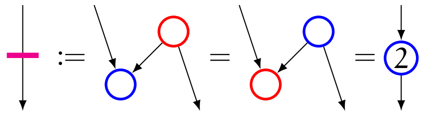
The above equation allows us to express a rotation in term of a multiplication or comultiplication.

 (14)

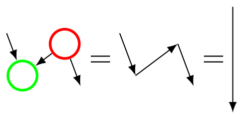
The above rule demonstrates the use of the  and  gates as colour rotation gates. They can be thought of as the analogues of the Hadamard gate from **RG**. They also allow any node of any one colour to be expressed in terms of the other two colours.

 (15)

 (16)

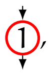
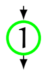
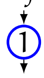
 (17)

We define these convenient "ticks", which are used to flip the direction of arrows. Their notation is inspired by the tick in the GHZ/W calculus of [7]. In the GHZ/W calculus, the tick is defined in as a GHZ "cup" followed by a W "cap", but whereas GHZ/W has only two "colours" and hence has only one kind of tick; **RGB** has three and so has three kinds of ticks—one for each pair of colours. There are no ticks in **RG** simply because the natural definition of a tick reduces to the identity:

 (18)

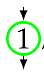
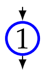
One can notice that the rules which held for the red and green nodes in **RG** don't translate directly to the red and green dots of **RGB**. This is made clearer by the interpretation in **Stab** given below. A translation which does preserve this interpretation is also given further down.

The rules of **RGB** talk a lot about the rotation group of the Bloch sphere, specifically an octahedral subgroup of it. In fact, it is in a sense "complete" for the octahedral group, as demonstrated by the following result

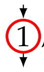
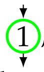
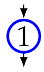
Proposition 3.2. *The octahedral group* O *embeds faithfully into* $\hom_{\mathbf{RGB}}(*, *)$. *It is isomorphic to the group generated by* ,  *and* 

Proof. The group O is of abstract group type S_4 and can be given the following standard presentation:

$$\begin{aligned} O \cong S_4 \cong \langle \tau_1, \tau_2, \tau_3 | \tau_1^2 = \tau_2^2 = \tau_3^2 = e, \\ \tau_1\tau_3 = \tau_3\tau_1, \\ \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2, \\ \tau_2\tau_3\tau_2 = \tau_3\tau_2\tau_3 \rangle \end{aligned}$$

If we look at the subcategory generated by , ,  in $\hom_{\mathbf{RGB}}(*, *)$, and we look at the equations involving these generators on both sides, we obtain the following group:

$$G = \langle \sigma_r, \sigma_g, \sigma_b | \sigma_r^4 = \sigma_g^4 = \sigma_b^4 = \sigma_r^2\sigma_g^2\sigma_b^2 = e,$$
$$\sigma_g\sigma_r = \sigma_b\sigma_g = \sigma_r\sigma_b \rangle$$

where σ_r, σ_g, σ_b represent , ,  respectively and the group law is given by vertical composition—where reading from up to down is the same as reading from right to left. Note that it is clear from this presentation that G is a quotient of the group $C_4 * C_4 * C_4$. We denote the quotient map $q : C_4 * C_4 * C_4 \to G$ and will make use of it later.

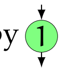
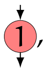
We can then define the following group homomorphisms:

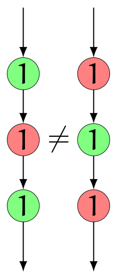
$$f : O \to G = \begin{cases} \tau_1 & \mapsto \sigma_b\sigma_r^2 \\ \tau_2 & \mapsto \sigma_b^2\sigma_g \\ \tau_3 & \mapsto \sigma_g\sigma_r\sigma_g \end{cases} \tag{19}$$

$$g : G \to O = \begin{cases} \sigma_r & \mapsto \tau_1\tau_2\tau_3 \\ \sigma_g & \mapsto \tau_3\tau_1\tau_2 \\ \sigma_b & \mapsto \tau_1\tau_2\tau_3\tau_1\tau_2 \end{cases} \tag{20}$$

after checking that these are indeed group homomorphisms, we can also determine that f and g are inverses, and thus $G \cong O$.

□

This certainly doesn't hold with **RG** and the group generated by  and , where the following are not equal.

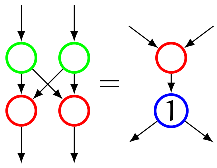


(21)

3.3 Some Derivable equations

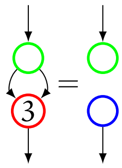
The equations below can be derived from the axioms of **RGB** given above. These equations can often be useful when wanting to demonstrates some more complex equalities in **RGB**

The following equation is perhaps a more convenient form of the bialgebra law.



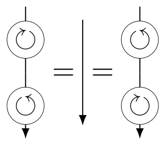
(22)

The following equation is a version of the Hopf law. It shows that in fact, all the bialgebras in **RG** that we had defined previously are also Hopf algebras where the antipode is the identity [13].



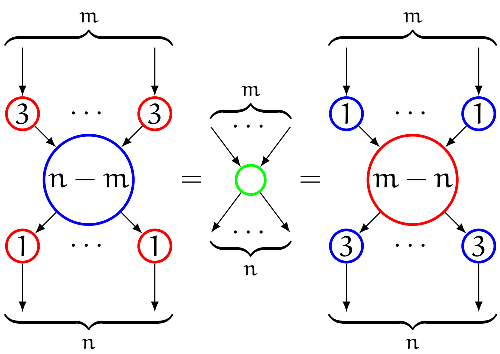
(23)

The following equation shows that colour rotation in one direction is the inverse of colour rotation in the other direction.



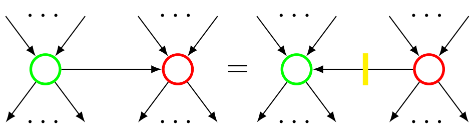
(24)

The following equation demonstrates a convenient way to write a node of a colour in terms of nodes of the two other colours.



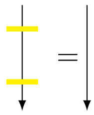
(25)

The following equation shows how ticks can be used to invert arrows in diagrams.



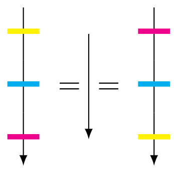
(26)

The following equation shows that two ticks of the same colour annihilate.



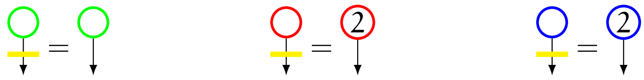
(27)

The following equation shows that three heterochromatic ticks annihilate.



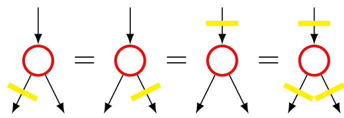
(28)

The remaining equations are useful equations about ticks:



(29)

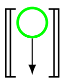
(30)

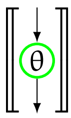


(31)

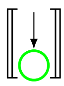
3.4 RGB interpretation

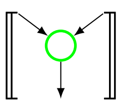
In a similar manner to what we did for **RG**, we provide an interpretation $[\![\cdot]\!]_{\mathbf{RGB}} : \mathbf{RGB} \to \mathbf{Stab}$ of the diagrams in **RGB**

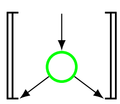
 $_{\mathbf{RGB}} = |+\rangle$

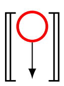


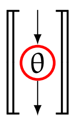
$_{\mathbf{RGB}} = |0\rangle\langle 0| + e^{i\theta\frac{\pi}{4}}|1\rangle\langle 1|$

 $_{\mathbf{RGB}} = \langle +|$

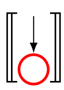
 $_{\mathbf{RGB}} = |0\rangle\langle 00| + |1\rangle\langle 11|$

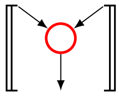
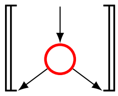
 $_{\mathbf{RGB}} = |00\rangle\langle 0| + |11\rangle\langle 1|$

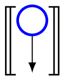
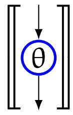
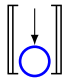
 $_{\mathbf{RGB}} = |0\rangle$

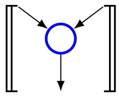
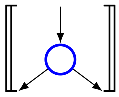


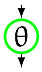
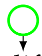
$_{\mathbf{RGB}} = |+\rangle\langle +| + e^{i\theta\frac{\pi}{4}}|-\rangle\langle -|$

 $_{\mathbf{RGB}} = \langle 0|$

 $_{\mathbf{RGB}} = |+\rangle \langle ++| + i\,|-\rangle \langle --|$  $_{\mathbf{RGB}} = |++\rangle \langle +| - i\,|--\rangle \langle -|$

 $_{\mathbf{RGB}} = |i\rangle$  $_{\mathbf{RGB}} = |i\rangle \langle i| + e^{i\theta\frac{\pi}{4}} |-i\rangle \langle -i|$  $_{\mathbf{RGB}} = \langle i|$

 $_{\mathbf{RGB}} = |i\rangle \langle ii| + |-i\rangle \langle -i-i|$  $_{\mathbf{RGB}} = |ii\rangle \langle i| + |-i-i\rangle \langle -i|$

Informally, the difference between **RG** and **RGB** can be seen in Figure 3. The rotational symmetry of the three colours can clearly be seen, and contrasted with Figure 2. For each observable, the straight segment represents the axis of rotation concretized by the  morphism—it is also the line of values which are fixed by that morphism. The arrow shows the direction of positive rotation, and the arrow-tail is where the deleting point  lies.

One can notice that the red deleting point in Figure 3 is at a different location than the red deleting point in Figure 2. It is for this reason that—although the green generators of **RG** and **RGB** are interpreted as the same values in **Stab**—the red generators are mapped to different values. Essentially, this design choice was made so that Rule 8, and the rules involving the colour changers can hold.

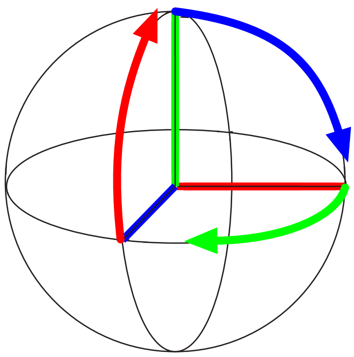
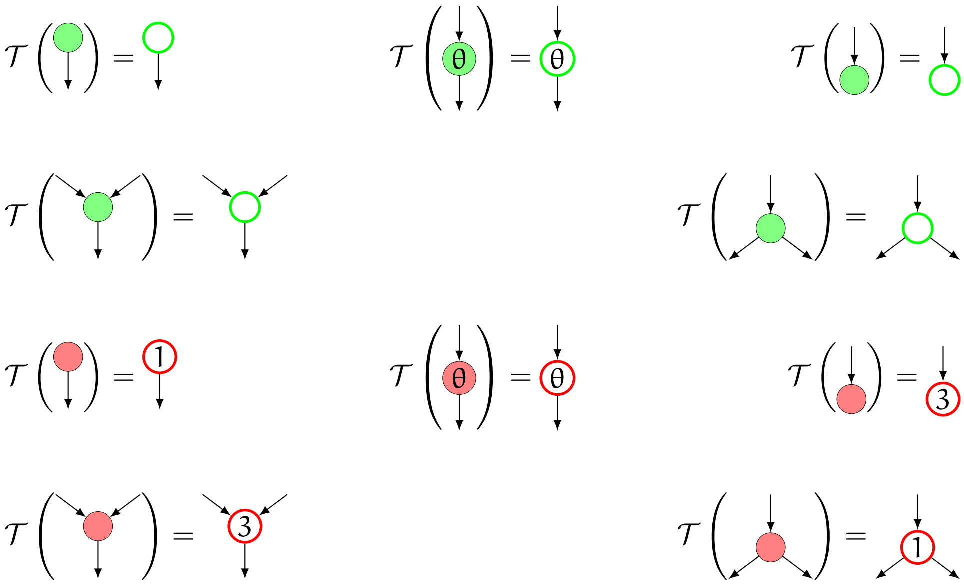


Figure 3: **RGB** Bloch sphere

4 RG to RGB translation

We define a translation from red-green diagrams to red-green-blue diagrams as a functor $\mathcal{T} : \mathbf{RG} \to \mathbf{RGB}$ by first defining $\mathcal{T}$ by its value on the generators of **RG** and then checking that equal diagrams of **RG** are equal under translation.



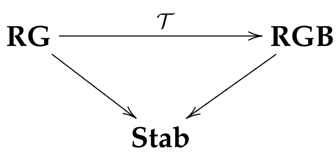
Proposition 4.1. *$\mathcal{T}$ is a functor.*

Proof. It suffices to do a routine check that for all rules $f = g$ in **RG**, we can prove $\mathcal{T}f = \mathcal{T}g$ in **RGB**. □

4.1 Translation preserves interpretation

Finally, the crucial property of this translation is the following

Proposition 4.2. *The following diagram commutes*

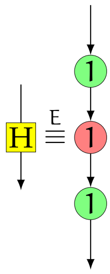


(32)

Proof. This follows from a routine check that for each generator $\varphi : *^m \to *^n$ in **RG**, $\mathcal{T}[\![\varphi]\!]_{\mathbf{RGB}} = [\![\varphi]\!]_{\mathbf{RG}}$ □

5 Euler Decomposition

Let E be the relation on morphisms of **RG** defined by the following:



(33)

This in known as an Euler decomposition of the Hadamard gate. Duncan and Perdrix [9] have shown that this equality is not provable in **RG**.

From this relation, we can produce the quotient category $\mathbf{RG}^+ := \mathbf{RG}/\mathsf{E}$ with quotient functor $\mathcal{Q} : \mathbf{RG} \to \mathbf{RG}^+$. Duncan and Perdrix [9] showed that Van den Nest's theorem [20] is equivalent to the addition of E. This means that Van den Nest's theorem is not provable in **RG** but is provable in $\mathbf{RG}^+$.

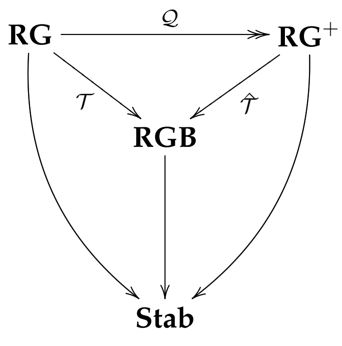
Definition 5.1 (chromatic diagram category)**.** We will use the term chromatic diagram category—usually denoted by **D**—to mean any category of **RG**, **RGB** or $\mathbf{RG}^+$

Lemma 5.1. *For every pair* $\mathsf{f} \overset{\mathsf{E}}{\equiv} \mathsf{g}$*, we have* $\mathcal{T}\mathsf{f} = \mathcal{T}\mathsf{g}$ *and* $[\![\mathsf{f}]\!]_{\mathbf{RG}} = [\![\mathsf{g}]\!]_{\mathbf{RG}}$.

Proof. This is proved by a routine check on all the rules of **RG**. □

With the previous lemma, we can now prove the following proposition

Proposition 5.2. *We can lift* $\mathcal{T}$ *and* $[\![\cdot]\!]_{\mathbf{RG}}$ *uniquely to the functors* $\hat{\mathcal{T}} : \mathbf{RG}^+ \to \mathbf{RGB}$ *and* $[\![\cdot]\!]_{\mathbf{RG}^+}$ *such that the following diagram commutes:*

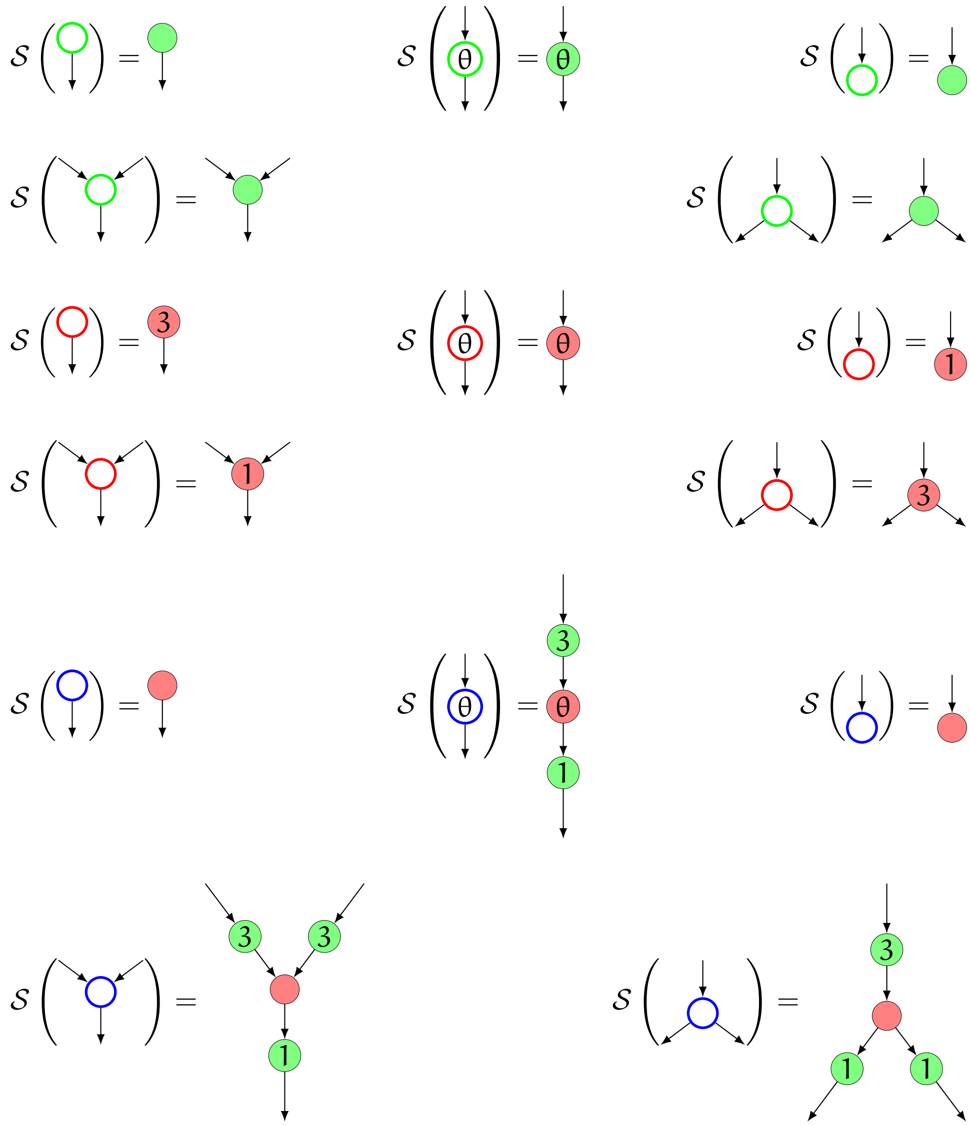


(34)

Proof. The lifting of $\mathcal{T}$ and $[\![\cdot]\!]_{\mathbf{RG}}$ is immediate from the fact that $\mathbf{RG}^+$ is a quotient category and Lemma 5.1 [15].

The lower right 2-cell is proved by using the fact that $\mathcal{Q}$ is an epi. □

We can define a functor from $\mathsf{S} : \mathbf{RG}^+ \to \mathbf{RGB}$ to act the following way on generators:



Theorem 5.3. *$\mathcal{S}$ is the inverse functor of $\hat{\mathcal{T}}$. In other words,* **RGB** *and* $\mathbf{RG}^+$ *are isomorphic categories.*

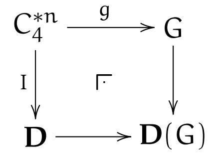
Proof. This requires showing that for each generator φ of **RGB**, one can prove $\hat{\mathcal{T}}(\mathcal{S}\varphi) = \varphi$ from the rules of **RGB** and that for each generator ψ of $\mathbf{RG}^+$, one can prove $\mathcal{S}\left(\hat{\mathcal{T}}\psi\right) = \psi$ from the rules of $\mathbf{RG}^+$. □

In a sense, the extra equations that appear in **RGB** and are not in **RG** are the equations about the rotations of the octahedral subgroup of the rotation group of the Bloch sphere. This isomorphism of categories tells us that these rotation equations are necessary and sufficient to prove Van den Nest's theorem.

6 Extensions

In the description of the red-green calculus given in [2,3,9,10], phases were permitted to be arbitrary real numbers representing arbitrary angles—not just integer multiples of $\frac{\pi}{2}$. We offer a method of extending the present work to make that possible.

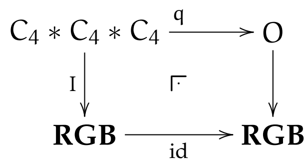
In an n-chromatic diagram, the group of phases for the combined n colours can be seen as a quotient of C_4^{*n}, the free product of n copies of C_4. C_4^{*n} thus maps into $\mathbf{D}$ in a natural way. If $g : C_4^{*n} \to G$ is a group homomorphism, then we can construct the following pushout:



(35)

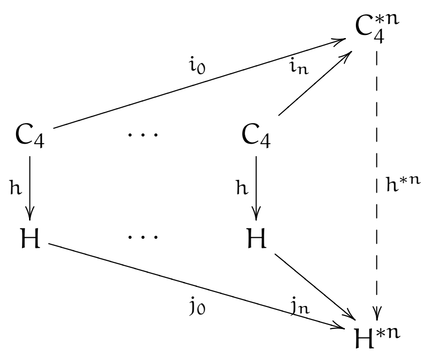
Note that although the homomorphism g is not present in the notation of $\mathbf{D}(G)$, the pushout does depend on it. We omit it so as to avoid clutter.

In the case where $\mathbf{D} = \mathbf{RGB}$, we can internalize some of the relations in **RGB** by using the quotient map we had defined earlier: $q : C_4 * C_4 * C_4 \to O$. The pushout square then becomes:



(36)

One can also suppose that H is an abelian group and $h : C_4 \to H$ is a group homomorphism. Of particular interest to us will be the case where h is monic, in which case, H will be seen as an extension of the phase group. Then we can lift h to a group homomorphism h^{*n} by the unique arrow which makes the following diagram commute:



(37)

h^{*n} is guaranteed to exist and be unique by the universal property of the free product in **Grp**.

If D is a chromatic diagram category with n colours, then there is a functor $I : C_4^{*n} \to D$—where C_4^{*n} is seen as a category with one object—which is defined by the property that $I \circ i_k$ maps C_4 to phase group of the kth colour. From this data, we can construct the following pushout.

$$\begin{array}{ccc} C_4^{*n} & \xrightarrow{h^{*n}} & H^{*n} \\ {\scriptstyle I}\downarrow & \ulcorner & \downarrow \\ \mathbf{D} & \longrightarrow & \mathbf{D}(H^{*n}) \end{array} \tag{38}$$

In the particular case that $H = U(1)$—the group of rotations of the circle—with $h : k \mapsto \frac{\pi}{2}k$, we use the shorthand $\mathbf{D}_{\bigcirc} := \mathbf{D}(U(1))$. The diagrams of $\mathbf{RG}_{\bigcirc}$ were the ones described in [3].

In **RGB**, we end up with the following cube:

$$\begin{array}{ccccccc} C_4 * C_4 * C_4 & & & \xrightarrow{q} & & O & \\ \downarrow & \searrow^{h*h*h} & & & & \downarrow & \searrow \\ & & U(1) * U(1) * U(1) & & \longrightarrow & & S \\ \downarrow & & \downarrow & & & \downarrow & \downarrow \\ \mathbf{RGB} & & & \xrightarrow{\mathrm{id}} & & \mathbf{RGB} & \\ & \searrow & \downarrow & & & & \searrow \downarrow \\ & & \mathbf{RGB}_{\bigcirc} & & \xrightarrow{\mathrm{id}} & & \mathbf{RGB}_{\bigcirc} \end{array} \tag{39}$$

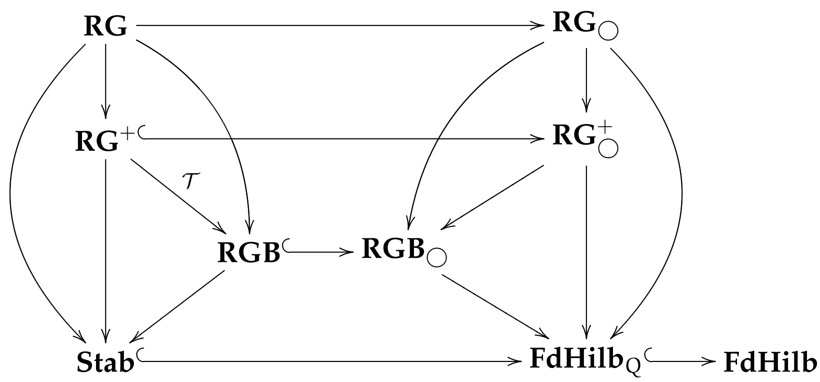
where all the faces are pushouts and S is a group extension of $SO(3)$.

There is an obvious way to additionally define a functor $[\![\cdot]\!]_{\mathbf{D}_{\bigcirc}} : \mathbf{D}_{\bigcirc} \to \mathbf{FdHilb}_Q$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{D} & \longrightarrow & \mathbf{D}_{\bigcirc} \\ \downarrow & & \downarrow \\ \mathbf{Stab} & \longrightarrow & \mathbf{FdHilb}_Q \end{array} \tag{40}$$

7 Future Work

The current formalism as it stands looks like the following



(41)

As mentioned earlier, it is known that **RG** is not complete for **Stab**—that is to say, $[\![\cdot]\!]_{\mathbf{RG}}$ is not a faithful functor. It is not known, however if $[\![\cdot]\!]_{\mathbf{RGB}}$ is faithful or not. In the future, we would hope to either prove that it is (which would be a great result), or show that it's not by providing a counterexample.

Spekkens has developed a toy theory of qubits [18], which was later formalized categorically in [5] as the category **Spek**. **Spek** exposes some features of quantum mechanics, but not all. For example, it does not "have" non-locality [6]. Interpreting the diagrams of **RGB** into **Spek** would hopefully reveal more of the differences between **Spek** and **Stab**.

In our formalization of chromatic diagram categories as †-SMCs, we have spoken very little about the †. Of course, it does play a role, and some work should be done to integrate it more tightly into this formalization. Similarly, we have omitted to talk about any notion of trace. However, a trace can be defined quite easily on these categories, making them traced monoidal categories [12].

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