

The ZX calculus is incomplete for non-stabilizer quantum mechanics



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We prove that the ZX -calculus is incomplete for non-stabilizer quantum mechanics. We suggest additional rules to be integrated with the graphical calculus and possible further incompleteness sources. We express a simple quantum error correction circuit in Selinger's CPM construction in an attempt to obtain a graphical construction framework for general stabilizer error codes. We also provide a simple framework for the integration of gate approximation errors into the ZX -calculus, including a simple way of propagating upper approximation bounds through quantum circuits.

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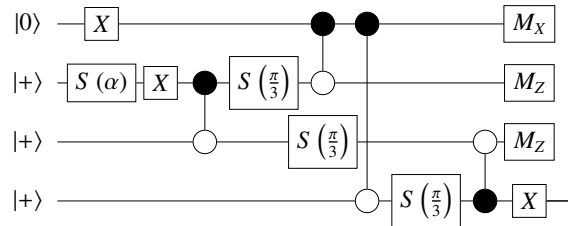
0.1 Making waves: An introduction for physicists

While being the uncontested workhorse in the field for more than eighty years, the Dirac-von Neumann formalism of quantum mechanics is neither terribly elegant, nor universally practical. The use of Hilbert spaces results in *structural overkill*: Every physical system Ψ corresponds to an infinite number of unit vector representations $e^{i\theta} |\Psi\rangle$, $\theta \in \mathbb{R}$. However, the true clumsiness of the Dirac-von Neumann approach lies in its failure to adequately capture the *compositionality* of the theory. The following example is taken from a *quantum secret sharing protocol*[30, p.99]:

$$\begin{aligned} & [\langle 0 + + |_{123} \otimes X_4] \circ [\text{Id}_{12} \otimes \text{CNot}_{43}] \circ \left[\text{Id}_{123} \otimes S_4 \left(\frac{\pi}{3} \right) \right] \\ & \circ [\text{CNot}_{14} \otimes \text{Id}_{23}] \circ [\text{CNot}_{12} \otimes \text{Id}_{34}] \circ \left[S_2 \left(\frac{\pi}{3} \right) \otimes S_3 \left(\frac{\pi}{3} \right) \otimes \text{Id}_{14} \right] \\ & \circ [\text{Id}_{14} \otimes \text{CNot}_{23}] \circ [X_1 \otimes (X_2 \circ S_2(\alpha)) \otimes \text{Id}_{34}] \circ |0 + ++\rangle_{1234} \end{aligned}$$

It needs to be shown that this expression is proportional to $|0\rangle + e^{i\alpha} |1\rangle$. This can be done brute-force, using the matrix representations of the individual gates. The above circuit involves the manipulation of four distinct physical qubits, which are the smallest informational unit in the realm of quantum mechanics. Matrix size scales as 2^n , meaning that 4 qubits result in 16×16 matrices, with roughly $n^3 = 4096$ scalar multiplications performed per matrix multiplication. Performing these calculations on a classical computer won't work beyond roughly a dozen qubits. Many recent important quantum computation models, such as *measurement-based quantum computation*, depend on thousands or millions of highly entangled qubits. But even if tractable, brute-force calculations yield little physical insight. If one wishes to understand the physical operation

of the described system better, one may resort to circuit representation:

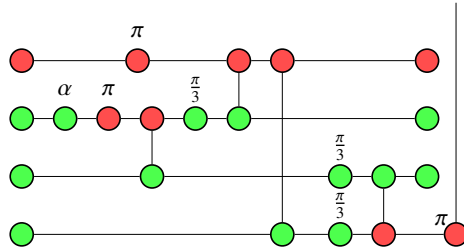


How does one arrive at this circuit diagram? Each qubit corresponds to a distinct line. The tensor product \otimes is represented by *juxtaposition*. *Unitary evolution*, corresponding to \circ , is represented by sequential composition. Two-qubit gates, such as CNot, are depicted by connecting the lines of the respective qubits, which intuitively indicates that there is some *information flow* between control and gate. The power of this pictorial representation lies in abstracting away the well-known identity $(- \circ -) \otimes (- \circ -) = (- \otimes -) \circ (- \otimes -)$, which allows to trace individual qubits through the system. Provided that only standard gates are being employed and the network isn't too dense, this method can be used to avoid the complexity of matrix manipulations. However, questions remain: Firstly, can the map *Dirac-von Neumann term* \mapsto *circuit diagram* be expressed in a mathematically rigorous fashion? Secondly, there are some obvious rewrites that can be directly performed on the circuit diagram, for example, $\text{CNot}_{12} \circ \text{CNot}_{12} = \text{Id}_{12}$ corresponding to:

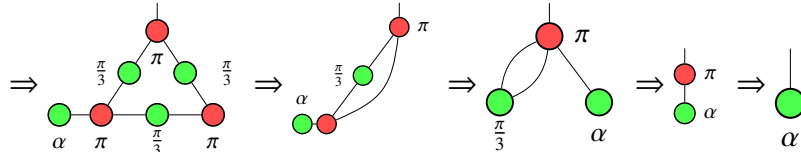
$$\begin{array}{c} \bullet \bullet \\ \circ \circ \end{array} := \begin{array}{c} \text{---} \\ \text{---} \end{array} .$$

Are there further opportunities for graphical manipulation? Apart from being intrinsically suited to human abilities, this could open new avenues for efficient automatized manipulation beyond physically-unaware sparse-matrix methods.

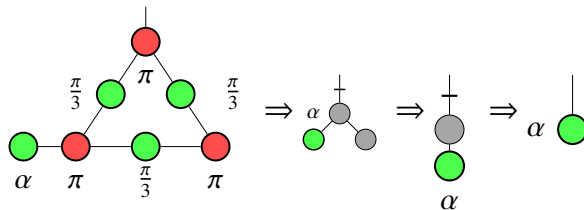
In their seminal paper[3], Abramsky and Coecke developed a rigorous mathematical framework for translating Dirac-von Neumann terms into circuit diagrams based on *category theory*. The study of interactions between complementary observables lead to a simple, yet powerful, set of graphical rewrite rules, dubbed the *ZX-calculus*. Returning to the above example, the *ZX-calculus* allows to convert the circuit diagram into the following form:



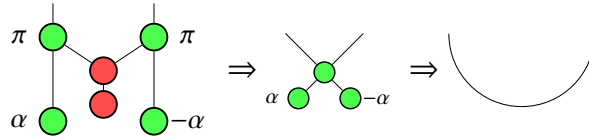
A short series of rewrites transforms this diagram to the expected form:



Categorical quantum mechanics doesn't stop there. The mathematical framework is sufficiently abstract to accommodate a wide range of graphical calculi. Instead of interactions between complementary qubits, one can study the interactions of three-partite entangled states. This results in the graphical *GHZ/W-calculus*. The *ZX-calculus* diagram from above may now be represented as follows:



Selinger[40] discovered a generalisation of the ZX-calculus describing density matrix interactions graphically. For example, suppose one would like to apply a conditional X-gate to the above example state with probability $\frac{1}{2}$, leaving the state intact otherwise. In von Neumann-formalism the resulting density matrix is obtained by treating both scenarios individually. In Selinger's *CPM construction*, this corresponds to the following graphical manipulation:



, where the resultant diagram corresponds to the maximally mixed state $|0\rangle\langle 0| + |1\rangle\langle 1|$. This may not immediately seem like a clear advantage. Measurement correlations of and conditional gates on n -qubit entangled states, however, in the worst case result in 2^n distinct scenarios to be considered. The CPM construction comes with clear and simple *high-level* rules that help to prune unnecessary complexity, whereas such strategies may be hard to come by using the Dirac-von Neumann notation.

Yet again, *Categorical quantum mechanics* offers much more than just a rigorous framework for graphical calculi. It provides an abstraction away from Hilbert spaces based on information-theoretic axioms. Many characteristic features of *quantum information processing*, such as the *no-cloning theorem*[47], *quantum teleportation*[10], *entanglement swapping*[34], *quantum steering*[39] and *gate teleportation*[24], are trivially encompassed[13]. The formalism also seems ideal for the study of *multi-partite entanglement*[15].

One way to better understand the oddness of quantum mechanics is to compare and contrast quantum phenomena within a large space of generalised theories. Categorical quantum mechanics helped to clarify

issues around one particularly popular such theory, namely *Spekken's toy model*[44]. This in turn may have *interpretational* consequences[14].

But why is it so important to thoroughly understand the information-theoretic properties of quantum mechanics? Information is always physical, and conversely, every physical theory gives rise to an *associated information processing theory*. It is clear that a successful theory of *quantum gravity* will need to have both quantum mechanics and general relativity as limiting cases. It follows that a theory of *quantum gravity information processing* needs to respect both a *general relativistic information processing theory* and quantum information processing. This latter issue has received a lot of attention recently: Tentative theories of quantum gravity information processing have been suggested[25] the interaction between motion under spatial curvature and quantum entanglement has been studied experimentally[4].

At the same time, there is an ongoing debate questioning the role of *string theory* in physics[2]. Whether or not string theory will ever result in a useful formulation of physical reality seems as of yet undecidable. However, given thirty years of technical difficulty and chronic unfalsifiability, it seems increasingly like a good idea to not put all the eggs in one basket. At the very least, there could be more efficient ways of arriving at the '*ultimate truth*'. Unfortunately, infinite dimensional Hilbert spaces have so far resisted full categorical treatment[29], but the finite dimensional formalism seems to be well-suited to relativistic extensions[1].

This thesis will investigate both fundamental theoretical as well as practical aspects of a variety of graphical languages arising in categorical quantum mechanics. It will proceed as follows:

- We prove, for the first time, that the current version of the ZX-calculus is *incomplete* outside of stabilizer quantum mechanics.

The proof relies on an equivalence with simple geometric justification. We also recommend a more general set of rules that should be added to the ZX-calculus. We conjecture that completeness will require the addition of a potentially infinite number of further, more complex, rules. We also discuss whether a weaker form of completeness, namely *approximate completeness*, may be achieved by the addition of further rules.

- We explore the suitability of Selinger’s CPM construction in proving the correctness of a popular class of *quantum error correction circuits*. In the process, we explain how certain types of noisy channels may be represented and simulated through many-qubit conditional gates probabilistically driven by entangled quantum states. We discuss how the GHZ/W calculus could be extended in order to enable purely graphical verification of quantum error correction protocols.
- We discuss how graphical reasoning may be used in order to trace *gate approximation errors* through quantum circuits in both ZX-calculus and CPM construction. In particular, we give a simple graphical calculus for the calculation of upper bounds. We suggest to add this capability to *Quantomatic*, an automated graph rewriting tool.

0.2 The perks of being a Category

Category theory is a branch of mathematics that tries to understand structure by investigating the *processes* that preserve it. Expressing fundamental physical theories in the language of category theory corresponds

to a change of paradigm: Instead of looking for ever smaller constituents of physical theories, one focusses on the complex interactions between objects instead. This is not entirely a new idea: physicists use Noether's theorem to understand conservation laws by studying the actions that respect them and they also use Feynman diagrams, which provide exactly the same types of process abstraction[8]. .

A category consists of *objects* and *morphisms* between them. Morphisms may be composed by the associative operation \circ , as long as types are preserved. For example, morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ may be composed to form a morphism $g \circ f : A \rightarrow C$. Furthermore, each object A gives rise to an *identity morphism* 1_A . Slightly more formally:

Definition 1. [17] A **category** C consists of: A family $|C|$ of objects A set of morphisms $C(A, B)$ for each pair of objects $A, B \in |C|$ We define a composition operation $- \circ - : C(A, B) \times C(B, C) \rightarrow C(A, C) :: (f, g) \mapsto g \circ f$. The composition operation has to obey two conditions:

1. It must be associative, i.e. for any $f \in C(A, B), g \in C(B, C)$ and $h \in C(C, D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
2. It must have a unit, i.e. for any object $A \in |C|$, there exists a morphism $1_A \in C(A, A)$ called identity, which is such that for any $f \in C(A, B)$ we have $f = f \circ 1_A = 1_B \circ f$.

Many familiar mathematical structures may be categorified: A single object, serving as target and source of all arrows, together with the monoid binary relation determines the category $\mathbf{C}(M)$ of a *monoid* M .

Definition 2. [17] A **monoid** is a triple $(M, \bullet, 1_\bullet)$ with M a set, \bullet an associative multiplication and $1_\bullet \in M$ is its unit.

If each of these arrows has an inverse, then the category corresponds to a *group*. The category of *finite sets and relations* **FRel** consists of finite sets as objects and relations between sets as arrows. One particularly important category is **FdVect**:

Definition 3. *The category **FdVect** $_{\mathbb{K}}$ has finite dimensional vector spaces over \mathbb{K} as objects and all linear maps between these as morphisms. Arrow composition is simply function composition, so associativity and unit elements are trivially addressed.*

Finally, we give an example of a structure that does not define a category: Envision a category **empire**. Take objects to be countries on a map. Define a morphism $f : A \rightarrow B$ to signify that country A borders country B . Now we have a problem: Unless all countries border on each other, this construct does not define a category, as the attribute 'bordering on' is *intransitive*. This makes arrow composition ill-defined. This shows that the concept of a category is broad, but not arbitrary.

Functors are structure-preserving maps between categories. A category that has categories as objects has functors as morphisms. For example: The arrows of the category that has categories of monoids as objects are functors which are determined by *monoid homomorphisms*.

Definition 4. *For categories \mathcal{C} and \mathcal{D} , a **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of functions (F_0, F_1) : $F_0 : \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$ maps each object in \mathcal{C} to an object in \mathcal{D} , and $F_1 : \text{arr}(\mathcal{C}) \rightarrow \text{arr}(\mathcal{D})$ maps each arrow $f : A \rightarrow B$ in \mathcal{C} to an arrow $F_1(f) : F_0(A) \rightarrow F_0(B)$ in \mathcal{D} .*

For any category \mathcal{C} , if one forgets that the arrows in \mathcal{C} may be composed and which ones are identity arrows, then one ends up with a category just defining a *graph*. Such graphs may be used in order to draw

commutative diagrams, for example for $f : A \rightarrow B$, $g : B \rightarrow C$, $k : A \rightarrow C$ and $l : C \rightarrow D$ and $g \circ f = l \circ k$, one may draw:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 k \downarrow & & \downarrow g \\
 C & \xrightarrow{l} & D
 \end{array}$$

Let's fix two categories \mathcal{C} , \mathcal{D} and look at the family of functors \mathcal{F} between them. One may form a new category with those functors as objects, and the arrows between these objects being so-called *natural transformations*.

Definition 5. For $F, G \in \mathcal{F}$, a *natural transformation* $\eta : F \rightarrow G$ associates to each object X in \mathcal{C} a morphism $\eta_X : F(X) \rightarrow G(X)$ between objects of \mathcal{D} , such that for every morphism $f : X \rightarrow Y$:

$$\begin{array}{ccc}
 F(X) & \xrightarrow{F(f)} & F(Y) \\
 \eta_X \downarrow & & \downarrow \eta_Y \\
 G(X) & \xrightarrow{G(f)} & G(Y)
 \end{array}$$

It is of course possible to continue by defining morphisms between natural transformations and so on. This leads to interesting insights: Fundamental algebraic operations, like subtraction and addition, may be seen as having resulted from an *infinite categorification* of set-theoretical operations like unions and disjoints[8]. However, the knowledge of functors and natural transformations suffices for many practical applications of category theory.

How can a physical theory be cast into the language of category theory? There are many possibilities. An intuitive one is to construct a category **Phys**, whose objects are physical states and morphisms are processes between such states. This is an example of an *abstract category* which can be concretized in order to reflect a specific physical theory. For example, quantum mechanics implies objects to be Hilbert spaces and morphisms to be linear maps. Classical mechanics suggests a category **ClassMech** with Poisson manifolds as objects and Poisson maps as morphisms[7]. However, there's more to a physical theory.

0.3 Symmetric Monoidal Categories: The mother of all physical theories

Compositionality is a principle in physics that is both heavily debated and ubiquitous: Loosely speaking, it refers to the observation that larger systems may often be understood as a result of smaller subsystems linked together by some kind of binary operation[26]. The properties of these compositions vary across different physical theories: Maxwell's equations of classical electrodynamics, for example, respect the *superposition principle* for charges and currents, which is captured by scalar and vector addition which are a special case of the direct sum \oplus . Similarly, classical mechanics obeys a principle of superposition of forces, again based on the direct sum. In contrast, quantum mechanics uses the direct sum as an internal composition for the superposition of quantum states within a given system. The external composition operation used to tie different quantum mechanical systems together is the tensor product \otimes .

One might thus assume that the major difference between classical mechanics and quantum mechanics is the nature of compositionality be-

tween the two. In fact it turns out that **ClassMech** gives rise to a notion of compositionality that is more tensor-like than direct sum-like, but this issue has so far been poorly understood[7]. In any case, we conclude that compositionality seems a fundamental characteristic of any physical theory[26]. We will see that *symmetric monoidal categories* offer just the right structure to capture compositionality at an abstract level.

Definition 6. A *monoidal category* $(\mathbf{C}, \otimes, \mathbf{I})$ is a category \mathbf{C} equipped with a bifunctor $-\otimes- : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, a distinguished unit object \mathbf{I} , natural unit isomorphisms $\lambda_A : A \simeq \mathbf{I} \otimes A$ and $\rho_A : A \simeq A \otimes \mathbf{I}$, and a natural associativity isomorphism $\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$, which are subject to certain coherence equations, which we omit.

Definition 7. [17] A monoidal category is called *strict* when λ, ρ, α are all identities, i.e. the objects made isomorphic are in fact equal. Every monoidal category is equivalent to a strict monoidal category.

What is still lacking from this definition is though that it shouldn't matter whether we regard system A composed with system B , or system B composed with system A . Defining $A \otimes B = B \otimes A$, however, would violate types: Imagine we transform from system $A \otimes B$ to system $D \otimes E$ by using process f to get from A to D and process g to get from B to E . We can write this as a composite process

$$A \otimes B \xrightarrow{f \otimes g} D \otimes E$$

. If now $A \otimes B = B \otimes A$, then we cannot distinguish anymore which process was applied to which system:

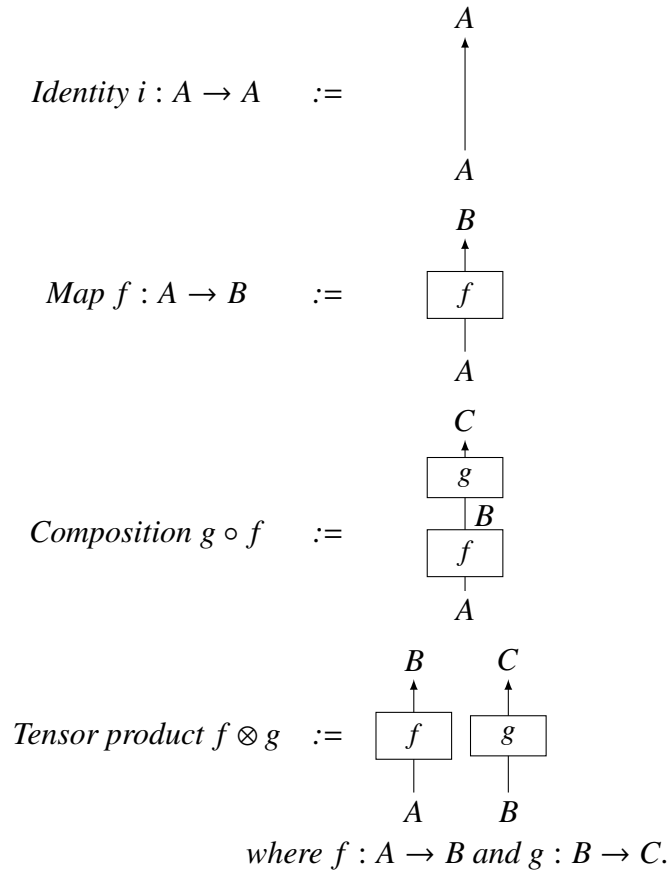
$$A \otimes B = B \otimes A \xrightarrow{f \otimes g} D \otimes E = E \otimes D$$

. This problem can be fixed through the following definition:

Definition 8. *A symmetric monoidal category (SMC) is a monoidal category equipped with a natural symmetry isomorphism $\sigma_{A,B} : A \otimes B \simeq B \otimes A$ such that $\sigma_{A,B}^{-1} = \sigma_{B,A}$, and again subject to some coherence conditions which we omit.*

It has been conjectured that SMCs are the natural categorical description of any physical theory, i.e. the smallest common denominator[26]. A very neat feature of SMCs is that they admit a **graphical calculus**, which can be used to reason in a way that is as powerful as formal reasoning:

Theorem 1. *[17] A formal equational statement holds for SMCs iff it holds up to isomorphism in the graphical calculus. The graphical calculus is given by:*



0.4 An introduction to Categorical Quantum Mechanics

We now apply the concept of a *symmetric monoidal category (SMC)* to quantum mechanics *in finite dimensions*.

Definition 9. [17] *The category **FdHilb** consists of a symmetric monoidal category (SMC) with finite-dimensional complex Hilbert spaces as objects and linear transformations as arrows. Arrow composition is pro-*

vided by matrix multiplication. The monoidal structure is provided by the tensor product \otimes .

Avoiding the ambiguity of *global phases*:

Definition 10. [21] The category \mathbf{FdHilb}_{wp} has the same objects and arrows as \mathbf{FdHilb} , however linear maps are subject to the equivalence condition $f \equiv g$ iff there exists $\theta \in \mathbb{R}$ such that $f = e^{i\theta}g$.

Noticing that the type of a qubit is given by $\mathbf{Qubit} : \mathbb{C}^2$, then what we can do now is compose qubits using the tensor product \otimes and use linear transformations to transform between vector spaces of individual or composed qubits. If we wish to do anything more exciting, we need to reflect more Hilbert space structure in our categorical backbone.

Hilbert spaces come with scalar products. In a vector space V over complex numbers, scalar products are defined as maps $\langle -, - \rangle : V^* \times V \rightarrow \text{scalar}$, where V^* is the dual space of V .

Definition 11. [40] A *compact closed category* is symmetric monoidal category (SMC) where each object A is assigned a dual object A^* , together with a unit map $\eta_A : I \rightarrow A^* \otimes A$ and a counit map $\epsilon_A : A \otimes A^* \rightarrow I$, such that $\lambda_A^{-1} \circ (\epsilon_A \otimes A) \circ \alpha_{A,A^*}^{-1} \circ (A \otimes \eta_A) \circ \rho_A = id_A$ and $\rho_A^{-1} \circ (A^* \otimes \epsilon_A) \circ \alpha_{A^*,A,A^*} \circ (\eta_A \otimes A^*) \circ \lambda_A = id_{A^*}$.

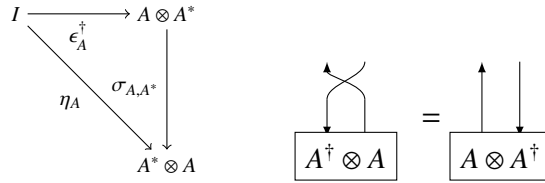
Next, we wish to encode that the morphisms in our category are *unitary*:

Definition 12. [13] A *\dagger -symmetric monoidal category* (\dagger -SMC) is a symmetric monoidal category equipped with an identity-on-objects contravariant endofunctor $(-)^{\dagger} : \mathbf{C}^{op} \rightarrow \mathbf{C}$, which assigns to each morphism $f : A \rightarrow B$ an adjoint morphism $f^{\dagger} : B \rightarrow A$, which coherently preserves

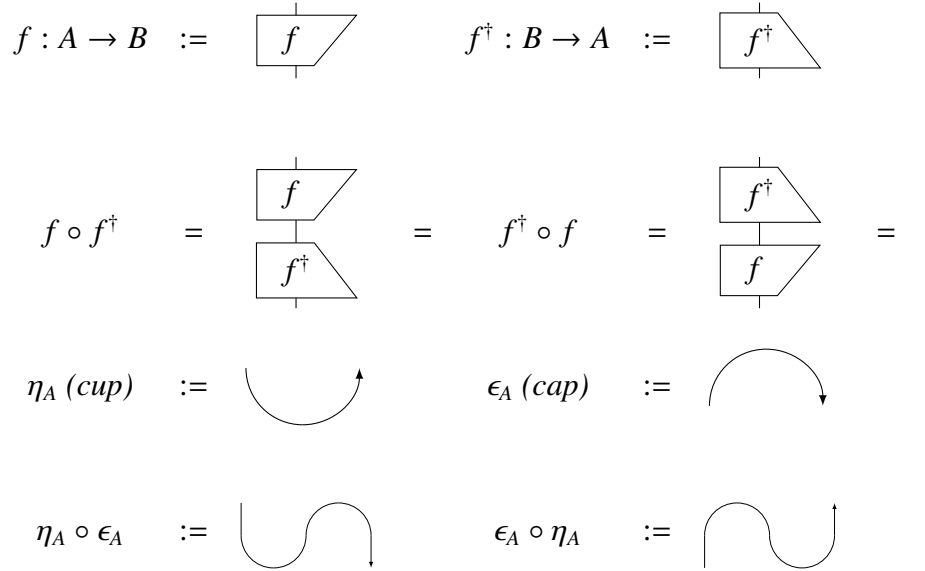
the monoidal structure, i.e.: $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, $1_A^\dagger = 1_A$ and $f^{\dagger\dagger} = f$. Further, for the natural isomorphisms λ , ρ , α and σ of the symmetric monoidal structure, the adjoint and the inverse coincide.

Combining unitarity and inner product structure,

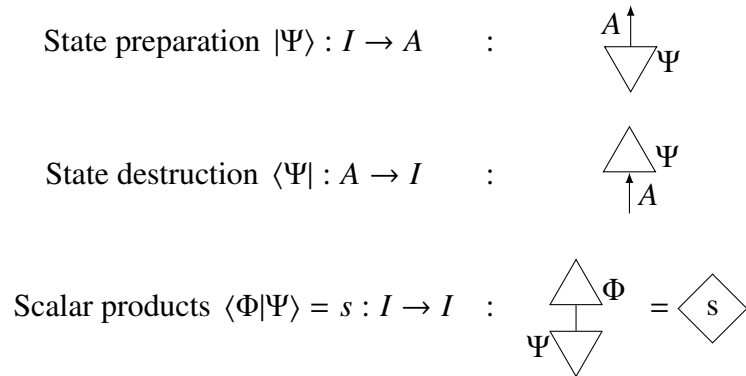
Definition 13. [40] A \dagger -compact closed category is a dagger symmetric monoidal category that is also compact closed, and such that the following diagram commutes:



\dagger -compact closed categories equally admit a diagrammatic calculus:



Regarding qubits, the infamous Dirac notation has a straight-forward analogue in the graphical calculus:



The following important theorem allows custom *yanking* and *bending* of wires:

Theorem 2. [13] *An equation expressed in the symbolic language of a dagger compact category follows from the axioms of dagger compact*

categories iff it holds up to isotopy in the graphical language.

The symbolic language of dagger compact categories allows insightful representations of quantum phenomena, such as *quantum teleportation* and *entanglement swapping*[3]. What our categorical depiction of finite-dimensional quantum mechanics still lacks is a notion of the direct sum \oplus . We omit the formal definition of a biproduct \oplus and its projectors p_i, q_i .

Definition 14. [40] *A biproduct dagger compact closed category is a dagger compact closed category with biproducts, such that $p_i^\dagger = q_i : A_i \rightarrow A_1 \oplus A_2$, for all objects A_1, A_2 and $i = 1, 2$.*

Definition 15. [40] *The category **FdHilb** of finite dimensional Hilbert spaces is biproduct dagger compact closed. The morphisms of this category are linear maps. The biproducts are given by the tensor product \otimes , as well as the direct sum \oplus . The (linear algebra) adjoint of a linear map $f : A \rightarrow B$ is given by the unique map $f^\dagger : B \rightarrow A$ satisfying $\langle f v | w \rangle = \langle v | f^\dagger w \rangle$ for all $v \in A, w \in B$.*

It can be shown that **FdHilb** enriched with biproducts features both *matrix addition* $+$ and *matrix multiplication* \circ . It also naturally embeds *spectral decomposition*, and thus measurement, as [3]

$$U : A \rightarrow \bigoplus_{i=1}^{i=n} A_i, \text{ together with projectors } P_j$$

for A, A_i being objects of **FdHilb**.

However, it is not clear what a graphical language for biproduct dagger compact closed categories should look like. This is largely because only one of the products \otimes and \oplus can be depicted by juxtaposition[40] at a time. Protruding into the third dimension would most likely not result in a practical graphical calculus.

0.5 Drawing sums: A farewell to biproducts

So how can one arrive at a graphical language for *biproduct dagger compact closed categories*? The key insight here is that a full characterisation of the direct sum \oplus might not be required in order to represent the key ingredients of finite-dimensional quantum mechanics. In this manner, Coecke et al. showed that actually spectral decomposition, and thus *quantum measurement*, may be formulated using the tensor product \otimes only[18].

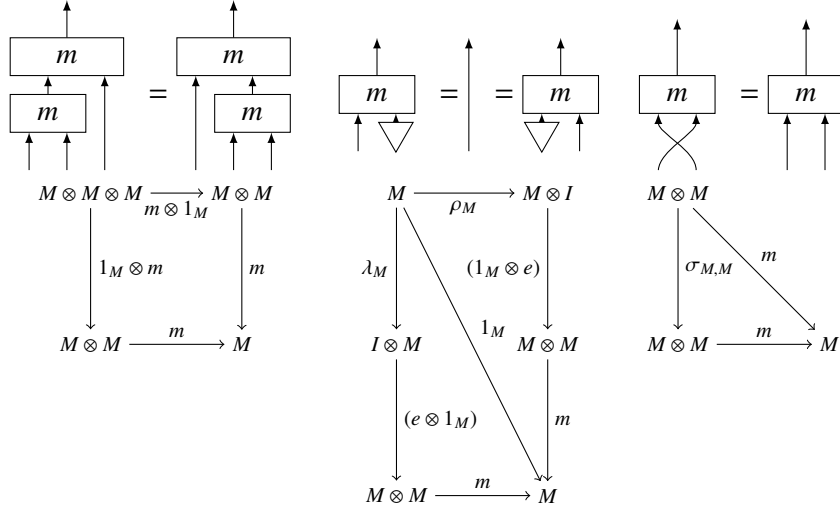
Projectively measuring a quantum state in a given basis produces both a (potentially differing) quantum state and some classical information, which provides knowledge as to which eigenstate of the measurement operator the quantum system has collapsed. A projective measurement is thus of type $\mathcal{M} : A \mapsto X \otimes A$, where A is of type quantum system and X is of type classical information. It is a well-established fact that quantum states may not be cloned[47]. Therefore, for objects representing classical information, there exists a copy map $\delta_{(X)} : X \rightarrow X \otimes X$ and a deletion map $\epsilon_{(X)} : X \rightarrow I$ - but such maps do not exist for general quantum states: A quantum state can only be cloned if it is equal to one of the orthonormal basis states of the measurement operator.

Definition 16. [18] *One defines a **classical object** to be the triple (X, δ, ϵ) of a structure X that can be copied and deleted and the corresponding maps δ and ϵ .*

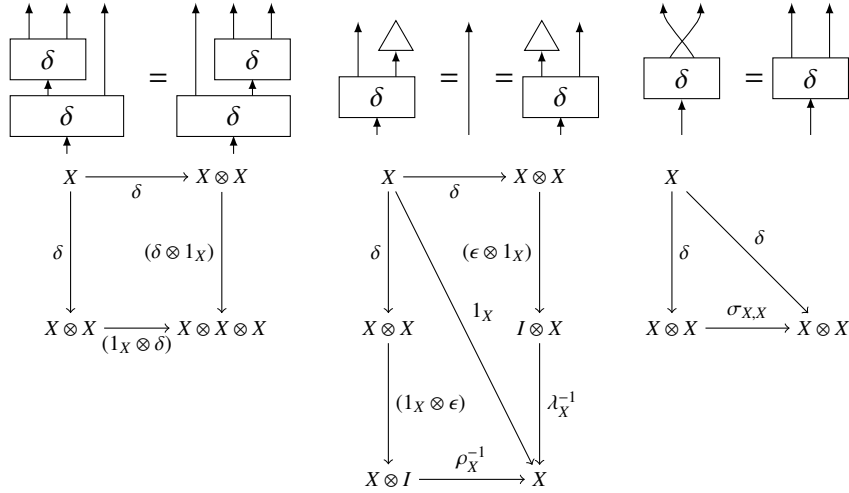
It has been shown that such classical objects encode a commutative \dagger -special Frobenius comonoid in **FdHilb**.

Definition 17. [13] *For the definition of a monoid, see definition 2. An **internal commutative monoid** in an SMC is a triple (M, m, e) , consisting*

of an object M , equipped with a multiplication $m : M \otimes M \rightarrow M$, and a unit $e : I \rightarrow M$, satisfying



Definition 18. [13] An *internal co-commutative monoid* in an SMC is a triple (X, δ, ϵ) , consisting of an object X , equipped with a comultiplication $\delta : X \rightarrow X \otimes X$, and a co-unit $\epsilon : X \rightarrow I$, satisfying



Definition 19. [13] A *Frobenius algebra* in an SMC \mathcal{C} is a quintuple of morphisms $(X, d : X \otimes X \rightarrow X, e : I \rightarrow X, \delta : X \rightarrow X \otimes X, \epsilon : X \rightarrow I)$

which together satisfy the Frobenius law

$$\begin{array}{ccc}
 X \otimes X & \xrightarrow{\delta^\dagger} & X & \xrightarrow{\delta} & X \otimes X \\
 & \searrow (1_A \otimes \delta) & & & \nearrow (\delta^\dagger \otimes 1_A) \\
 & & X \otimes X \otimes X & &
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} \uparrow \uparrow \\ \delta \\ \uparrow \uparrow \\ \delta^\dagger \\ \uparrow \uparrow \end{array}
 =
 \begin{array}{c} \uparrow \\ \delta^\dagger \\ \uparrow \uparrow \\ \delta \\ \uparrow \end{array}
 \end{array}$$

and where (X, m, e) forms an internal commutative monoid and where (X, δ, ϵ) forms an internal co-commutative monoid.

Definition 20. [15] A Frobenius algebra $(A, \mu, \eta, \delta, \epsilon)$ is called a \dagger -Frobenius algebra if $\mu = \delta^\dagger$ and $\eta = \epsilon^\dagger$

Definition 21. [15] A special commutative Frobenius algebra (SCFA) is a commutative Frobenius algebra $\mathcal{A} = (A, \mu, \eta, \delta, \epsilon)$ such that $\mu \circ \delta = 1_A$, i.e.:

Definition 22. [5] A \dagger -special commutative Frobenius algebra (SCFA), or \dagger -SCFA, $(A, \delta^\dagger, \epsilon^\dagger, \delta, \epsilon)$, is a \dagger -Frobenius algebra such that $\delta^\dagger \delta = 1_A$ [13, eq. 38].

$$\begin{array}{c} \uparrow \\ \delta^\dagger \\ \uparrow \uparrow \\ \delta \\ \uparrow \end{array}
 =
 \begin{array}{c} \uparrow \\ \delta^\dagger \\ \uparrow \uparrow \\ \delta \\ \uparrow \end{array}$$

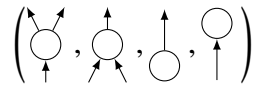
Having introduced SCFAs, it is now time to detail how this structure can be used to express some internal structure of Hilbert spaces: instead of talking about different sets of orthonormal bases, one can talk about SCFAs:

Definition 23. [13] Any orthonormal basis $\{|\Psi_i\rangle\}$ for a Hilbert space \mathcal{H} induces an **observable structure** by considering the linear maps that

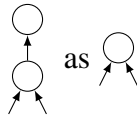
respectively 'copy' and 'uniformly erase' the basis vectors: $\delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} :: |\Psi_i\rangle \mapsto |\Psi_i\rangle \otimes |\Psi_i\rangle$, $\epsilon : \mathcal{H} \rightarrow \mathbb{C} :: |\Psi_i\rangle \mapsto 1$. In fact, all observable structures in **FdHilb** arise from orthonormal bases.

Additionally, one can recover the basis by solving the equation $\delta(|\Psi\rangle) = |\Psi\rangle \otimes |\Psi\rangle$. ϵ is not required to recover the basis.

Now that a tensor-product description of copying and deleting operations has been introduced, it is time to introduce a new graphical notation for SCFAs using dots: Instead of (d, δ, e, ϵ) , one writes



. For simplicity, one can write

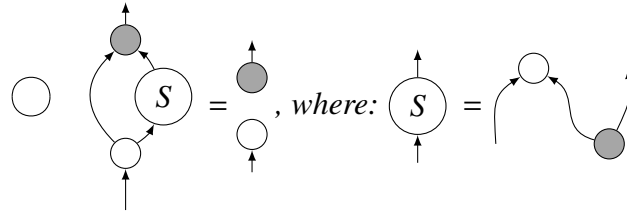


Classical structures with respect to differing bases are denoted by dots with different colors.

0.6 Interacting Complementary Observables

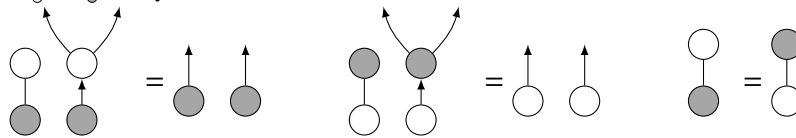
Two observables are complementary iff the measurement outcome of one does not reveal anything about the measurement outcome of the other. Formally:

Definition 24. [31, Def. 3.4] In **FdHilb**, a pair (O_\circ, O_\bullet) of observables is *complementary* iff:



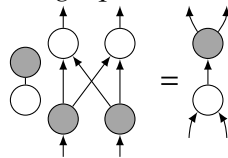
It can be shown that a Hilbert space of dimension 2 only admits 2 + 1 pairwise complementary observables [31]. These are usually denoted X , Y and Z observables. Developing a graphical calculus for qubit interactions, it might seem as if it was most intuitive to thus study interactions of three complementary bases. This has indeed been done[33]. However, any of the three observables may always be expressed through combinations of the other two. This the idea behind ZX-calculus.

Definition 25. [31] It turns out that a pair of complementary observables also O_{\circ}, O_{\bullet} obeys so-called **coherence** conditions:



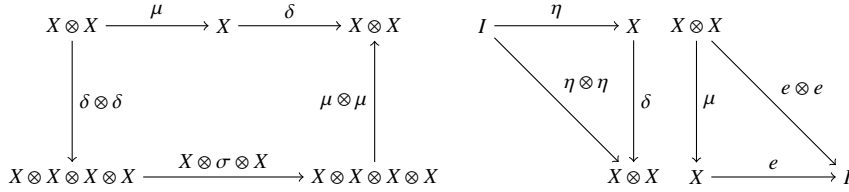
Taking Z and X as complementary bases of interest, it can be shown that these are the only strongly-complementary observables that a Hilbert space of dimension 2 admits:

Definition 26. [31] Two observables O_{\circ}, O_{\bullet} are **strongly complementary** iff they satisfy the following equation:



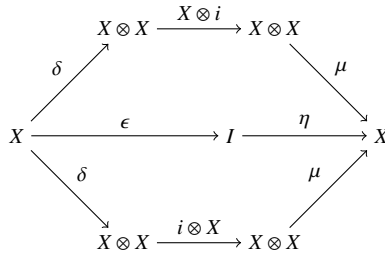
Both coherence and strong complementarity can be summarized in a familiar mathematical structure:

Definition 27. [17] In a monoidal category \mathcal{V} , a **bialgebra** is a tuple $(X, \mu, \eta, \delta, e)$ where (X, μ, η) is a monoid, (X, δ, e) is a comonoid and the following diagrams commute:



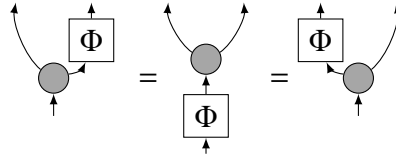
It can be shown that two observables forming a bialgebra always form a Hopf algebra as well:

Definition 28. [17] A **Hopf algebra** is a tuple $(X, \mu, \eta, \delta, \epsilon, i)$ such that $(X, \mu, \eta, \delta, \epsilon)$ forms a bialgebra and the following diagram commutes:



The last essential structure introduced here is the one of *phase*, which is intimately related to quantum phase gates and, together with a Frobenius algebra, leads to the concept of *decorated spiders* in ZX calculus:

Definition 29. [5, Def. 3.2.12] For a Frobenius algebra $\mathcal{A} = (A, \mu, \eta, \delta, \epsilon)$, a map $\Phi : A \rightarrow A$ is called a **phase** for \mathcal{A} if:



The structures introduced in this chapter explain the laws of a particular instance of graphical calculus, namely the ZX-calculus. The addi-

tional structures needed in order derive the GHZ/W calculus and Selinger's CPM construction will be discussed in-situ.

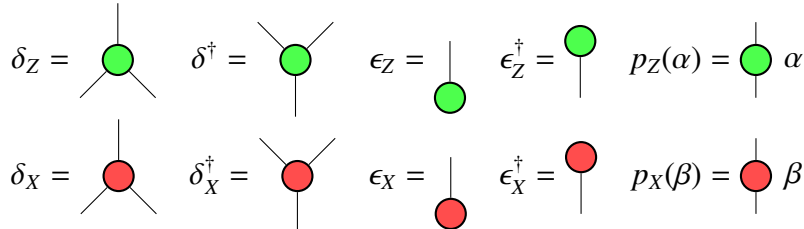
0.7 The ZX-calculus

Definition 30. [21] An *open graph* is a *finite undirected graph* of which some degree-one vertices are labelled as *inputs and outputs*. Open graphs form a *self-dual compact category*, i.e. $A = A^*$, $A \in \text{Obj}(\mathcal{C})$, where the composition of graphs works by identifying respective input and output vertices and deleting the ones that have been joined. The tensor product \otimes corresponds to juxtaposition. Units and counits are given by certain unique empty graphs.

From this, it immediately follows that *open graphs* are subject to the usual graphical reasoning of compact categories, i.e. two open graphs are identical iff they can be deformed into one another.

Definition 31. [9] The *free category* $F(G)$ generated by a graph G has objects of G as nodes and paths of G as arrows. Composition between tuples of arrows is defined by $(f_1, f_2, \dots, f_k) \circ (f_{k+1}, \dots, f_n) = (f_1, f_2, \dots, f_n)$.

Definition 32. [21] The *category* \mathcal{D} is the free category generated by the graphs

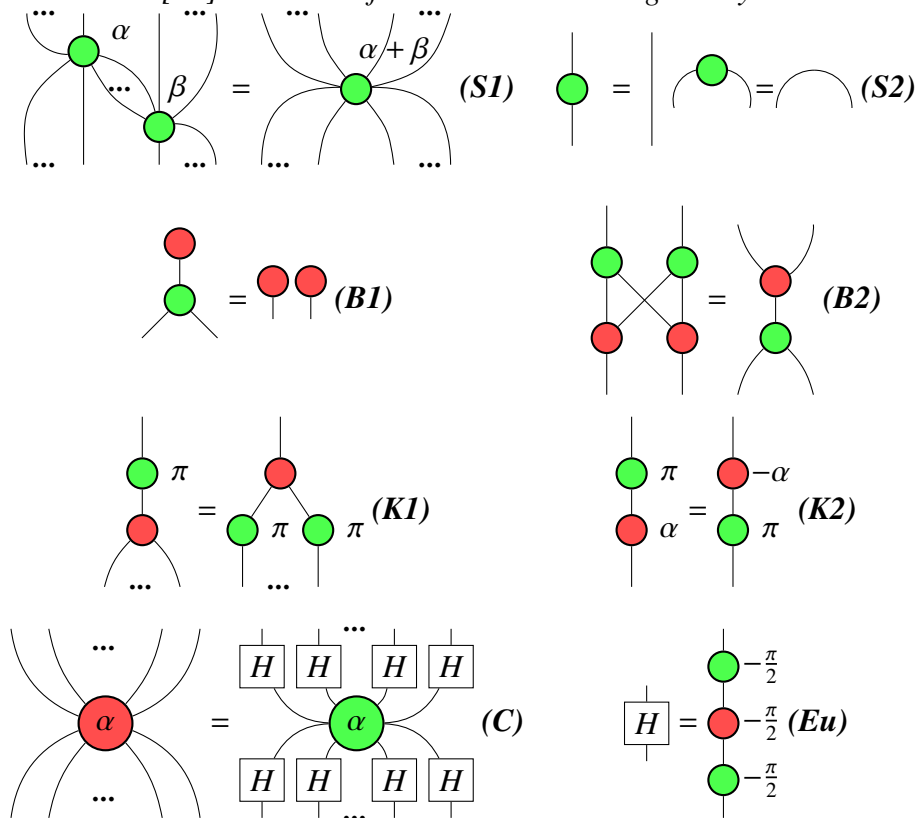


where n and m are the number of inputs and outputs respectively, and

phases $\alpha, \beta \in]0, 2\pi)$. If a phase is equal to zero, then it is omitted. The arrows of \mathcal{D} are called **diagrams**.

Now we define a set of rules for what we call the ZX-calculus:

Theorem 3. [17] *The rules of the ZX-calculus are given by:*



Rule **S1**, **S2** derive from Frobenius algebra enriched with phases, rules **B1** and **B2** stem from bialgebra structure and **K1**, **K2**, **C** and **Euler** may be verified by direct calculation. As all the rules therefore coincide with interaction rules of complementary classical structures^{0.4}, the following theorem applies:

Theorem 4. [21] The *interpretation functor* $\llbracket \cdot \rrbracket : \mathcal{D} \rightarrow \mathbf{FdHilb}_{wp}$ given

by

$$\llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \\ | \end{array} \rrbracket = |0\rangle\langle 00| + |1\rangle\langle 11| \quad \llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \end{array} \rrbracket = |+\rangle \quad \llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \\ | \\ \alpha \end{array} \rrbracket = |0\rangle\langle 0| + e^{i\alpha} |1\rangle\langle 1|$$

$$\llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \\ | \end{array} \rrbracket = |+\rangle\langle ++| + |-\rangle\langle --| \quad \llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \end{array} \rrbracket = |0\rangle \quad \llbracket \begin{array}{c} | \\ | \\ \bullet \\ | \\ | \\ \beta \end{array} \rrbracket = |+\rangle\langle +| + e^{i\beta} |-\rangle\langle -|$$

extends to a **monoidal functor**, i.e. it preserves monoid structure.

Note that green dots are defined with respect to Z-basis and red dots are defined with respect to X-basis. Strictly speaking, the category we are interested in is not \mathcal{D} but rather the category \mathbb{D} , in which all diagrams which are equal under the ZX-calculus rules are indeed **congruent** to each other:

Theorem 5. [21] The category \mathbb{D} is the quotient category of \mathcal{D} by the graphical rules of the ZX-calculus. There is always a canonical functor $\llbracket \cdot \rrbracket_{\sim} : \mathbb{D} \rightarrow \mathbf{FdHilb}_{wp}$ such that

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad} & \mathbb{D} \\ & \searrow \llbracket \cdot \rrbracket & \downarrow \llbracket \cdot \rrbracket_{\sim} \\ & & \mathbf{FdHilb}_{wp} \end{array}$$

commutes.

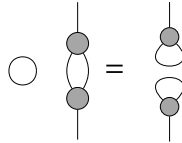
0.8 The GHZ/W calculus

Definition 33. Two states $|\Psi\rangle, |\Phi\rangle$ are **SLOCC-equivalent** if there exist local invertible maps L_i such that $|\Psi\rangle = L_1 \otimes L_2 \otimes \dots \otimes L_n$. Note that L_i

need not be unitary, but can arise through general local operations including classical communication: SLOCC equivalence merely demands that two states can be inter-converted with non-zero probability under such operations. [15]

It can be shown that any three-qubit state is SLOCC-equivalent to either $|GHZ\rangle = |000\rangle + |111\rangle$ or $|W\rangle = |100\rangle + |010\rangle + |001\rangle$. Furthermore, treating both $|GHZ\rangle$ and $|W\rangle$ as graphical dots, the following mathematical structures arise: GHZ-states are *special commutative Frobenius algebras (SCFAs)* and W-states are *anti-special commutative Frobenius algebras (ACFA)*, where ACFA's are defined as follows[15]:

Definition 34. An anti-special commutative Frobenius algebra (ACFA)[5] is a commutative Frobenius algebra such that:



In addition to having a unit and a counit, anti-special Frobenius algebras have canonical disconnecting points which we shall refer to as the anti-unit and anti-counit.

Definition 35. The following dictionary (up to scalars) can be used in order to convert between GHZ/W-, ZX-calculus and matrix formalism[28]:

$$\Leftrightarrow |1\rangle\langle 11| + |0\rangle\langle 01| + |0\rangle\langle 10| \quad (1)$$

$$\Leftrightarrow |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| \quad (2)$$

$$\Leftrightarrow |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| \quad (3)$$

$$\Leftrightarrow |1\rangle\langle 0| + |0\rangle\langle 1| \quad (4)$$

$$\Leftrightarrow |0\rangle\langle 00| + |1\rangle\langle 11| \quad (5)$$

$$\Leftrightarrow |0\rangle \quad (6)$$

$$\Leftrightarrow |00\rangle\langle 0| + |11\rangle\langle 1| \quad (7)$$

$$\Leftrightarrow \langle 0| \quad (8)$$

$$\Leftrightarrow |+\rangle\langle ++| + |-\rangle\langle --| \quad (9)$$

$$\Leftrightarrow |+\rangle \quad (10)$$

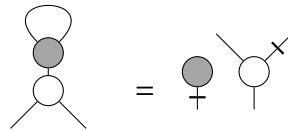
$$\Leftrightarrow |++\rangle\langle ++| + |--\rangle\langle --| \quad (11)$$

$$\Leftrightarrow \langle +| \quad (12)$$

Definition 37. Copy rules



$$(18)$$



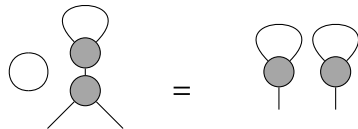
$$(19)$$



$$(20)$$

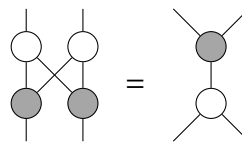


$$(21)$$

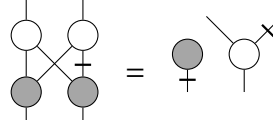


$$(22)$$

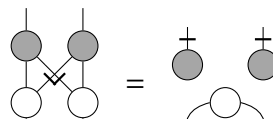
Definition 38. Bialgebra-like rules



$$(23)$$



$$(24)$$



$$(25)$$



$$(26)$$

Definition 39. Hopf-like rules

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \quad (27)$$

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \text{---} \quad (28)$$

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} \quad (29)$$

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} \quad (30)$$

$$\quad (31)$$

There also exist some non-trivial rules for cases where there are more than 2 arms[38].

Definition 40. Scalar rules

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} \quad (32)$$

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} \quad (33)$$

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \text{---} \end{array} = 0 \quad (\text{zero map}) \quad (34)$$

Definition 41. Device rules

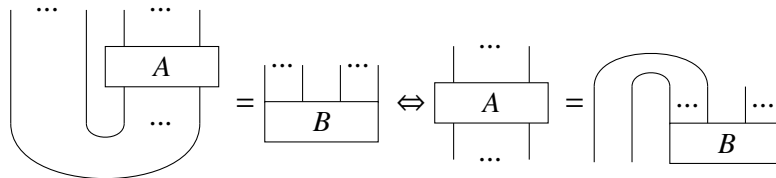
$$\begin{array}{c} \boxed{QMux} \end{array} = \begin{array}{c} \text{Diagram: } QMux \text{ as a network of nodes and wires} \end{array} \quad (35)$$

$$\begin{array}{c} \boxed{QAND} \end{array} = \begin{array}{c} \text{Diagram: } QAND \text{ as a network of nodes and wires} \end{array} \quad (36)$$

0.9 CPM construction

An important relationship in quantum mechanics is the *Choi-Jamiołkowski isomorphism*:

Definition 42. *The Choi-Jamiołkowski isomorphism states that in $Fd\text{-Hilb}$, there is a bijection between points of $A \otimes B$ and maps of the form $A \mapsto B$ [16, Thm 1.2.18]. In graphical notation, this means that [6, Thm 14]:*



with A being an operator from n to m qubits, and B an state of dimension $n + m$.

Quantum easurements are described by a set of projectors $\{P_i\}$. The outcome of a measurement is completely described by assigning probabilities p_i with which a measurement result corresponding to a particular

projector P_i is being measured. Equivalently, a set of tuples $\{(P_i, p_i)\}$ may be formed and by Choi-Jamiołkowski isomorphism??, this set is completely described by forming a vector \vec{B} with p_i as components and the basis states of the P_i as basis states. Vectors of the form of \vec{B} are called *Born vectors* and, unlike vectors describing quantum states, they satisfy $\sum_i p_i = 1$. As will be seen, the concept of a Born vector describing measurement outcomes plays an important role in the CPM-construction, an important extension of ZX-calculus. The traditional formalism, however, is based on matrix manipulation which is often more comfortable when dealing with basis transformations.

A density matrix ρ , of the form $\sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, corresponding to a pure or mixed quantum state has a trace equal to 1 and is a positive operator, i.e. $\langle\phi|\rho|\phi\rangle \geq 0$ for $|\phi\rangle \in \mathcal{H}$. The probability of obtaining a result m when measuring an operator M is given by $\text{tr}(M_m^\dagger M_m \rho)$ and the state of the system after the measurement is given by $|\Psi_i^m\rangle = \frac{M_m |\Psi_i\rangle}{\sqrt{\langle\Psi_i|M_m^\dagger M_m|\Psi_i\rangle}}$, which corresponds to a density matrix of $\frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}$.

An abstract, and fairly broad, description of open quantum processes is achieved by *quantum operations*. In the *operator-sum representation*, the transformation of a quantum system, described as density operator ρ , into another system $\mathcal{E}(\rho)$ is given by:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

, where there is a *completeness relation* for the *operation elements* E_k given by $\sum_k E_k^\dagger E_k \leq I$. The general form of $\mathcal{E}(\rho)$ in the operation sum representation may be motivated by depicting the openness of the principal system ρ as an interaction with an environment ρ_{env} in the form of a product state $\rho \otimes \rho_{\text{env}}$: For a general projective measurement operator

P_m with projectors P_m , we have (after normalisation)

$$\mathcal{E}(\rho) = \frac{P_m E(\rho \otimes \rho_{\text{env}}) U^\dagger P_m}{\text{tr}(P_m U(\rho \otimes \sigma) U^\dagger P_m)}$$

. From this, we can trace out the environment to only receive the principal system ρ after the transformation. This expression may be brought into the form $\frac{\mathcal{E}_m(\rho)}{\text{tr}_E(\mathcal{E}_m(\rho))}$, which can be physically interpreted as the transformation randomly replacing the initial state ρ by the state $\mathcal{E}_m(\rho)$ with probability $\text{tr}_E(\mathcal{E}_m(\rho))$. Choosing an orthonormal basis $|e_k\rangle$ for the principal system and a suitable ensemble decomposition $\sum_j q_j |j\rangle \langle j|$ for the environment, one can express

$$\mathcal{E}(\rho) = \sum_{jk} E_{jk} \rho E_{jk}^\dagger$$

, where $E_{jk} \equiv \sqrt{q_j} \langle e_k | P_m U | j \rangle$. One thus has a description of a generalised projective measurement process on an open system ρ . Purely unitary transformations are included in this treatment as a measurement operator with a single identity projector.

While considering product-state interactions with an environment is physically intuitive, there is an axiomatic approach to quantum operations which sometimes allows system and environment to start out in an entangled state. One defines \mathcal{E} to be a map $\mathcal{E} : \rho \mapsto \sigma$ such that:

1. $0 \leq \text{tr}(\mathcal{E}) \leq 1$. This is chosen out of mathematical convenience such that $\text{tr}(\mathcal{E}(\rho))$ is equal to the probability of the measurement outcome described by \mathcal{E} occurring.
2. \mathcal{E} is a *convex-linear map* on the set of density matrices, i.e. for probabilities $\{p_i\}$, $\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$. This requirement ultimately stems from Bayes' theorem applied to choosing the input state to be randomly selected from an ensemble $\{p_i, \rho_i\}$.

3. \mathcal{E} is a *completely positive map*. This means that $\mathcal{E}(A)$ must be positive if A is positive, and $(I \otimes \mathcal{E})(A)$, where I is the identity map, must be positive if A is positive.

Definition 43. [40] A morphism $f : A \rightarrow B$ in a dagger category is called *positive* if there exists an object B and a morphism $g : A \rightarrow B$ such that $f = g^\dagger \circ g$. In **FdHilb**, positive morphisms correspond exactly to positive operators. The positive matrix of a positive morphism $f : A \rightarrow A$ is defined as $[f] : I \rightarrow A^* \otimes A$. - TODO graphical representation of a positive matrix CPM p. 154

Definition 44. [40] Let A, B be objects in a dagger compact closed category. We say that a morphism $f : A^* \otimes A \rightarrow B^* \otimes B$ is **completely positive** if for all objects C and all positive matrices, $g : I \rightarrow C^* \otimes A^* \otimes A \otimes C$, the morphism

$$(C^* \otimes f \otimes C) \circ g =$$

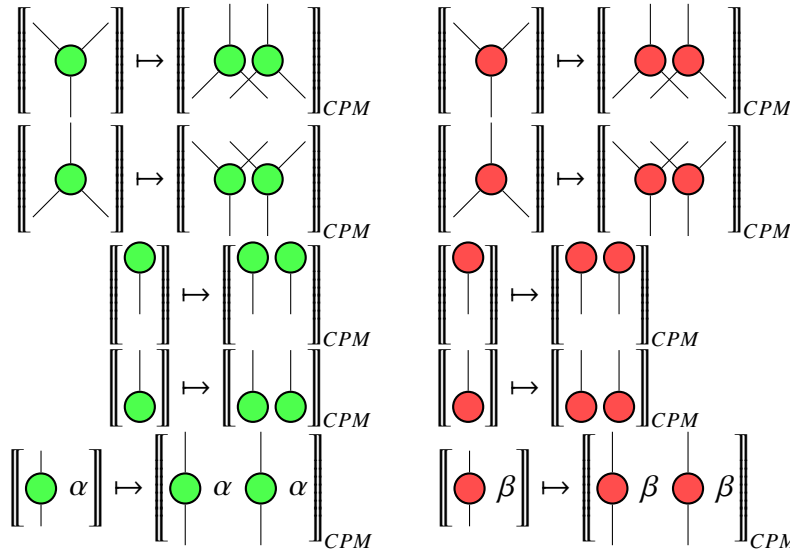
is a positive matrix.

Definition 45. [40] Given a dagger compact closed category C , one defines a new category $CPM(C)$ by specifying a functor $F : C \rightarrow CPM(C)$: F preserves the objects of C , but maps each morphism f in C to $f_* \otimes f$. For $A, B \in CPM(C)$, a morphism $f : A \rightarrow B$ is a completely positive map $f : A^* \otimes A \rightarrow B^* \otimes B$. Even more strikingly, $CPM(C)$ is also dagger compact closed: objects still admit the tensor product from C , morphisms $f : A^* \otimes A \rightarrow B^* \otimes B$, $g : B^* \otimes B \rightarrow C^* \otimes C$ admit a tensor product $C^* \otimes A^* \otimes A \otimes C \xrightarrow{\simeq} A^* \otimes A \otimes C^* \otimes C \xrightarrow{f \otimes g} B^* \otimes B \otimes D^* \otimes D \xrightarrow{\simeq} D^* \otimes B^* \otimes B \otimes D$ which is again a completely positive map. The structural maps $\alpha_{A,B,C}$,

$\lambda_A, \sigma_{A,B}, \eta_A$ and ϵ_A are all given by the images of the respective maps of C under F . IF $f : A^* \otimes A \rightarrow B^* \otimes B$, the f^\dagger in $CPM(C)$ is given by $f^\dagger : B^* \otimes B \rightarrow A^* \otimes A$ in C . The functor $F : C \rightarrow CPM(C)$ preserves the dagger compact closed structure.

Example 1. [40] For $CPM(\mathbf{FdHilb})$, $F(f) = F(g)$ iff $f = \phi g$, where g is a morphism in C and ϕ a unit scalar, i.e. F is faithful up to global phases - which is expected as two quantum states are physically equivalent if they only differ by a unit scalar factor.

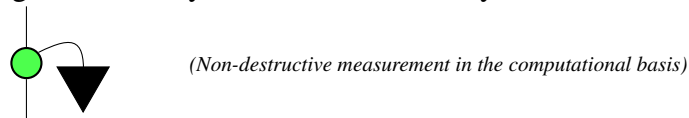
Definition 46. The functor $\llbracket \cdot \rrbracket_{CPM} : C \mapsto CPM(C)$



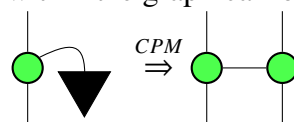
As CPM-construction captures positive density matrices, classical and quantum information flow can be modelled more concisely than in the ZX-calculus alone. In the ZX-calculus, classical information arises only through non-destructive measurement:

Definition 47. A non-destructive measurement is described by [16]

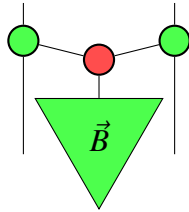
As a consequence, non-destructive measurements are performed simply by applying a copy map corresponding to the orthonormal basis of P to the quantum state to be measured. Now, if the resultant classical information (consisting in the eigenstate that is 'copied out of the diagram' is destroyed before interacting with the environment (i.e. the measurement result is not revealed), the act of measurement does not affect the quantum state. However, if the resultant classical information '*leaves the diagram*', i.e. it interacts with the environment, then clearly the measured state is affected, i.e. it collapses to one of the eigenbases of the projector P . However, this effect constitutes **decoherence**, whose description requires density matrix formalism not captured by ZX-construction. In order to indicate in the ZX-calculus that a measurement has taken place, therefore, one introduces a *ground symbol*, which prevents the copy-map from being inadvertently resolved to an identity.



However, in CPM-construction, it can be shown[16] that by simply joining corresponding grounds by a line, the effect of measurements can be completely described within the graphical formalism, i.e.



Furthermore, post-selection of measurement results can be used to implement conditional phase gates, with the following being a conditional phase-gate:



Importantly, the Born vector \vec{B} may represent basis states of many-qubit measurements and therefore can drive conditional many-qubit measurements.

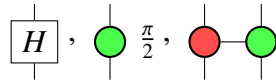
1 | The ZX-calculus is incomplete for non-stabilizer quantum mechanics

1.1 Introduction to the completeness question

The original rules of the ZX-calculus, as put forward by Coecke et al.[13], did not contain the Euler decomposition of the Hadamard gate, **EU**. Subsequently, Duncan and Perdrix proved[22] that the rule **EU** was not derivable from within the original ZX-calculus. The original ZX-calculus was therefore *incomplete*. Informally, incompleteness signifies that there are equations that can be proven to hold in Dirac-von Neumann notation that cannot be proven in the ZX-calculus. This would reduce the power of the graphical calculus and possibly limit its applications in automated reasoning.

Backens finally proved[6] that the current ZX-calculus, which is simply the original ZX calculus extended by **EU**, is indeed complete for an important fragment of quantum mechanics, namely *stabilizer quantum mechanics (SQM)* (see section 2.2). Backens' proof relies on the fact that each SQM state is, under local Clifford operations[23], equivalent to a special entangled state, namely a *graph state*[27]. This allows to abstract

away from matrix representation and instead decide equivalence between different SQM states by performing *local complementations*, which are a class of graph manipulations, between graph states. In this way, Backens showed that SQM states may be represented as so-called rGS-LC diagrams, which are only equivalent iff they are graphically identical. In this way, equivalence can be decided in the ZX-calculus. Stabilizer quantum mechanics is generated only by gates



, so a natural question is how Backens' proof accommodates a further gate α , where α is not divisible by $\frac{\pi}{2}$. Surprisingly, this does not seem trivial[35]. We will prove later that any such attempt must be futile¹⁰.

Ideally, one would wish the ZX-calculus to be as physically expressive as the Dirac-von Neumann formalism. The existence of a functor $[[\cdot]]_{\sim} : \mathbb{D} \rightarrow \mathbf{FdHilb}_{\text{wp}}$, as introduced in 5 implies that the ZX-calculus is *sound*, i.e.

Theorem 6. *If two diagrams \mathcal{D}_1 and \mathcal{D}_2 both $\in \text{Arr}(\mathbb{D})$ are equal, then $[[\mathcal{D}_1]]_{\sim} = e^{i\theta} [[\mathcal{D}_2]]_{\sim}$, $\theta \in \mathbb{R}$.*

This implies that no false equalities may be derived in the ZX-calculus. Secondly, the ZX-calculus is *universal*, meaning that it can express any quantum state and gate. This is easily proven by showing that the ZX-calculus can express any of the set of universal quantum gates[13]. Finally, *completeness* is the inverse of soundness, namely the question of whether

Theorem 7. If for two terms $[[\mathfrak{D}_1]]_{\sim}$ and $[[\mathfrak{D}_2]]_{\sim}$, $[[\mathfrak{D}_1]]_{\sim} = e^{i\theta} [[\mathfrak{D}_2]]_{\sim}$, $\theta \in \mathbb{R}$ then $\mathfrak{D}_1 = \mathfrak{D}_2$, where $\mathfrak{D}_1, \mathfrak{D}_2 \in \text{Arr}(\mathbb{D})$.

i.e. whether $[[\cdot]]_{\sim}$ is faithful.

1.2 The ZX-calculus is incomplete: The proof

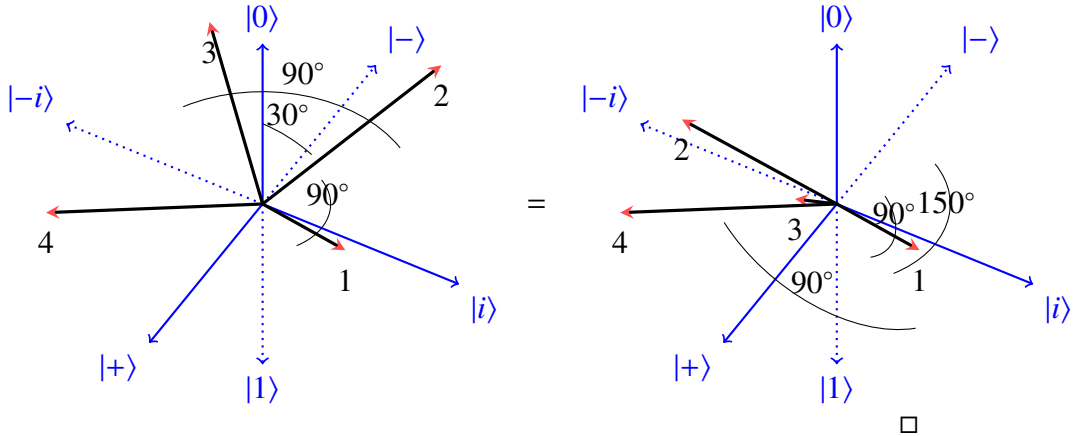
Theorem 8. The following equivalence $(\mathbf{new})_{\alpha}$ between terms may be shown to hold in the Dirac-von Neumann formalism:

$$\left[\begin{array}{c} \text{green circle} \\ \text{red circle} \\ \text{green circle} \end{array} \right]_{\sim} \begin{array}{c} \frac{\pi}{2} \\ \alpha \\ \frac{\pi}{2} \end{array} = \left[\begin{array}{c} \text{red circle} \\ \text{green circle} \\ \text{red circle} \end{array} \right]_{\sim} \begin{array}{c} \frac{\pi}{2} \\ \pi - \alpha \\ \frac{\pi}{2} \end{array}$$

Proof. By direct calculation:

$$p_Z\left(\frac{\pi}{2}\right) p_X(\alpha) p_Z\left(\frac{\pi}{2}\right) \stackrel{?}{=} e^{i\theta} p_X\left(\frac{\pi}{2}\right) p_Z(\pi - \alpha) p_X\left(\frac{\pi}{2}\right), \theta \in \mathbb{R}.$$

However, there is also a neat graphical verification of this equivalence:



Definition 48. [22] The interpretation functor $\llbracket \cdot \rrbracket_n$ is given by

$$\begin{aligned} \llbracket \begin{array}{c} | \\ \textcircled{\alpha} \\ | \end{array} \rrbracket_n &= \llbracket \begin{array}{c} | \\ \textcircled{n\alpha} \\ | \end{array} \rrbracket_{\sim} \\ \llbracket \begin{array}{c} | \\ \textcircled{\alpha} \\ | \end{array} \rrbracket_n &= \llbracket \begin{array}{c} | \\ \textcircled{n\alpha} \\ | \end{array} \rrbracket_{\sim} \end{aligned}$$

and apart from that $\llbracket \cdot \rrbracket_n = \llbracket \cdot \rrbracket_{\sim}$ where clearly, the equivalences defined hold up to complex phases.

Theorem 9. The ZX-calculus is incomplete.

Proof. This is equivalent to saying that $\llbracket \cdot \rrbracket_{\sim}$ is not faithful. We define a category \mathbb{D}_* , which is equal to the category \mathcal{D} quotiented by the rules of the ZX-calculus augmented by the rule **(new) $_{\alpha}$** given by:

$$\begin{array}{ccc} \begin{array}{c} | \\ \textcircled{\alpha} \\ | \end{array} \frac{\pi}{2} & & \begin{array}{c} | \\ \textcircled{\pi - \alpha} \\ | \end{array} \frac{\pi}{2} \\ \begin{array}{c} | \\ \textcircled{\alpha} \\ | \end{array} & = & \begin{array}{c} | \\ \textcircled{\pi - \alpha} \\ | \end{array} \\ \begin{array}{c} | \\ \textcircled{\alpha} \\ | \end{array} \frac{\pi}{2} & & \begin{array}{c} | \\ \textcircled{\pi - \alpha} \\ | \end{array} \frac{\pi}{2} \end{array}$$

In summary:

$$\begin{array}{ccccc} \mathbb{D} & \longleftarrow & \mathcal{D} & \longrightarrow & \mathbb{D}_* \\ & \searrow & \downarrow \llbracket \cdot \rrbracket & \swarrow & \\ & \llbracket \cdot \rrbracket_{\sim} & & \llbracket \cdot \rrbracket_* & \\ & & \mathbf{FdHilb}_{\text{wp}} & & \end{array}$$

Assume that $\mathbb{D} \cong \mathbb{D}_*$. We note that clearly, $\llbracket \cdot \rrbracket_{\sim} = \llbracket \cdot \rrbracket_* = \llbracket \cdot \rrbracket^1$. Replace $\llbracket \cdot \rrbracket$ by $\llbracket \cdot \rrbracket_{-3}$. Clearly, the left triangle of the above diagram still commutes. However, the right diagram does not commute as $\llbracket \cdot \rrbracket_2$ is

¹I thank Aleks Kissinger for clarifying some details with this categorical formulation.

not a functor $\mathbb{D}_* \rightarrow \mathbf{FdHilb}_{\text{wp}}$ as $(\mathbf{new})_{\frac{\pi}{5}}$ does not hold. Therefore, $\mathbb{D} \not\cong \mathbb{D}_*$ and therefore, $(\mathbf{new})_{\frac{\pi}{5}}$ cannot be derived from the rules of the ZX-calculus. \square

Theorem 10. *The ZX-calculus is incomplete for non-stabilizer quantum mechanics.*

Proof. All rules of the ZX-calculus except for **(EU)** are trivially seen to hold for all $n \in \mathbb{Z}$. We note that, given $\text{SU}(2)$ is a double cover of $\text{SO}(3)$,

$$\textcircled{\alpha} = \textcircled{\alpha + 2\pi n}, \quad n \in \mathbb{Z} .$$

Thus, **(EU)** only holds if

$$-\frac{n\pi}{2} = -\frac{\pi}{2} + 2\pi p, \quad p, n \in \mathbb{Z} \Leftrightarrow p = \frac{1}{4}(1 - n)$$

. For $p \in \mathbb{Z}$, we require that $(1 - n) \pmod{4} = 0$ which is fulfilled iff

$$n = 4k + 1, \quad k \in \mathbb{Z} = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

. For which n does $(\mathbf{new})_{\alpha}$ hold? We require

$$n(\pi - \alpha) = n\alpha + 2\pi p, \quad n, p \in \mathbb{Z}$$

So for $p \in \mathbb{Z}$, require $\alpha = \pi k$, $k \in \mathbb{Z}$ by transcendence of π . It follows that

$$p = \frac{1}{2}n(1 - 2k)$$

So we require that $n(1 - 2k) \pmod{2} = 0$, i.e. n even or $k = \frac{1}{2} - \epsilon$, $\epsilon \in \mathbb{Z} = \{\dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\}$.

Therefore, in order for $(\mathbf{new})_\alpha$ to be derivable from (\mathbf{EU}) , we require that $k = \frac{1}{2} - \epsilon$, $\epsilon \in \mathbb{Z}$. In that case, $(\mathbf{new})_\alpha$ lies entirely in stabilizer quantum mechanics, for which the ZX-calculus is complete. We thus conclude that the ZX-calculus is incomplete for non-stabilizer quantum mechanics. \square

Definition 49. *Approximate completeness means completeness with respect to a set of quantum gates allowing for arbitrarily close approximation of arbitrary unitary gates.*

Theorem 11. *The ZX-calculus is approximately incomplete.*

Proof. This follows directly from theorem 10, which proves that the ZX-calculus is only complete for stabilizer quantum mechanics and incomplete upon the addition of any non-Clifford gate. As gate approximation relies on adding at least one non-Clifford gate to the set of Clifford gates[12], the ZX-calculus is approximately incomplete. \square

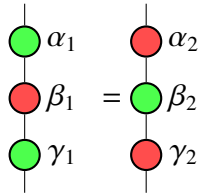
Theorem 12. *The CPM-construction is incomplete.*

Proof. TODO! \square

1.3 New rules for the ZX-calculus

Definition 50. *Euler decomposition.*

Definition 51. *ZXZ-XZX rules.* The rule $(\mathbf{ZXZ} - \mathbf{XZX}_{\alpha_1, \beta_1, \gamma_1})$ is of the form



where at most two non-adjacent gates have phases divisible by $\frac{\pi}{2}$ and $\alpha_1 \geq \gamma_1$ to avoid duplicates.

Note that $(\mathbf{ZXZ} - \mathbf{XZX}_{\alpha_1, \beta_1, \gamma_1})$ includes $(\mathbf{new})_\alpha$.

Theorem 13. *The ZX-calculus augmented by $\mathbf{ZXZ} - \mathbf{XZX}_{\alpha_1, \beta_1, \gamma_1}$ is complete for one-dimensional rotation chains.*

Proof. TODO! □

Theorem 14. *For ZXZ-XZX rules, there exists a **closed form** solution for both*

$$\alpha_1(\alpha_2, \beta_2, \gamma_2), \beta_1(\alpha_2, \beta_2, \gamma_2), \gamma_1(\alpha_2, \beta_2, \gamma_2)$$

and its inverse.

Proof. Using Maple©, we find a closed form solution in matrix formalism (see appendix.1). TODO!

The inverse can be found by simply swapping the sets of variables $(\alpha_1, \beta_1, \gamma_1) \leftrightarrow (\alpha_2, \beta_2, \gamma_2)$ as:

$$\begin{aligned} p_Z(\alpha_1)p_X(\beta_1)p_Z(\gamma_1) &= p_X(\alpha_2)p_Z(\beta_2)p_X(\gamma_2) \\ \Leftrightarrow H p_Z(\alpha_1)p_X(\beta_1)p_Z(\gamma_1)H &= H p_X(\alpha_2)p_Z(\beta_2)p_X(\gamma_1)H \\ \Leftrightarrow p_X(\alpha_1)p_Z(\beta_1)p_X(\gamma_1) &= p_Z(\alpha_2)p_X(\beta_2)p_Z(\gamma_2) \end{aligned}$$

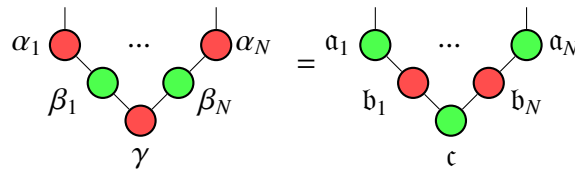
□

Proposition 1. *The ZX-calculus augmented with $(\mathbf{new})_\alpha$ is not complete for stabilizer quantum mechanics augmented with α phase gates.*

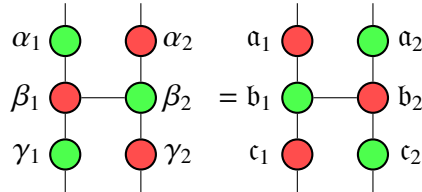
Proof. TODO: Go for alpha-pi/2-alpha or so! And then find a nice example! (maybe one can prove this even!) \square

Proposition 2. *The ZX-calculus augmented by $\mathbf{ZXZ} - \mathbf{XZX}_{\alpha_1, \beta_1, \gamma_1}$ is incomplete.*

Justification. *\mathbf{ZXZ} or \mathbf{XZX} chains may be used as building blocks of complex graph fragments between which there may hold equivalences for non-trivial parameters. The only property that definitely has to be preserved in such an equivalence is type. Two such examples are provided here:*



Here, \mathbf{XZX} -chains are joined at their heads. Below, alternating $\mathbf{ZXZ}/\mathbf{XZX}$ chains are joined at their middle compartments:



Of course, hybrid forms of the above are well imaginable. Neither of these suggested equivalences seems in principle derivable using the ZX calculus augmented with \mathbf{ZXZ} - \mathbf{XZX} rules, thus nurturing the suspicion that the inability of the ZX-calculus to capture component-wise matrix manipulation may not be compensated for with a finite set of additional rules.

2 | Quantum Error Correction

2.1 Introduction

The discovery of quantum error correction independently by Shor and Steane in 1995[45][43] was considered a huge break-through at that time because of the extraordinary experimental difficulties associated with the control of isolated quantum systems. Classically, error correction is well understood[11]. However, the concepts from classical information theory cannot be applied to quantum information theory in a straightforward fashion: Many classical codes are based on repetition of information (to enable recovery in case some of the information is being corrupted by a noisy quantum channel), however, *quantum systems cannot be cloned*. Secondly, in classical systems, a single bit can only be corrupted by flipping it. A qubit, however, may be corrupted by applying an infinitesimal set of different unitary gates, thus stirring the fear that error recognition and correction may use infinite resources. Thirdly, one can simply measure classical bits to find an error syndrome, conclude what error has occurred and then just apply the proper corrections. Qubits, however, are corrupted through measurement and thus more subtle approaches are needed in order to determine error syndromes.

As a result, there exists a large class of quantum error correction

codes that exploit quantum features without classical equivalents. Historically, the most important such class of quantum error correcting codes were so-called *stabiliser codes*. The very first quantum codes discovered belonged to a subset of these, *CSS codes*, which do strongly resemble classical error codes. In order to understand stabilizer codes, one needs to be familiar with the syntax of *stabilizer quantum mechanics* (SQM).

The classification of non-stabilizer codes (*non-additive codes*) is an ongoing field of research[37][48]. Interestingly, some *non-additive* codes have been shown to be more efficient than comparable stabilizer codes[48].

This paper will focus first on stabilizer quantum codes: The simple three-qubit flip code will be used in order to introduce methods related to graphical expression. The 5-qubit stabilizer code subsequently serves as an example of a fully-fledged stabilizer code. A short diversion towards *entanglement-assisted stabilizer formalism* will be made. Finally, a popular non-additive code will be studied. Contrary to many popular treatments on error correction [46], no references to classical coding theory and particularly, despite their historical importance, CSS codes, will be made.

2.2 Stabilizer quantum mechanics

Stabilizer quantum mechanics is a subset of finite-dimensional quantum mechanics based on the n -qubit Pauli group G_n .

Definition 52. *The Pauli group G_n consists of all n -fold tensor products of Pauli matrices σ_k with multiplicative factors $\pm 1, \pm i$ under the operation of matrix multiplication. Importantly, G_1 is generated by $\langle X, Y, Z \rangle$.*

A vector state $|\Psi\rangle$ is a *stabilized* by a quantum operator S iff $S|\Psi\rangle = |\Psi\rangle$. In general, if S is a subgroup of G_n and V_S is a set of n -qubit states

fixed by every element of S , then S is the *stabilizer* of V_S . It can be shown[36, p. 455] that any subgroup of G_n defines a stabilizer for a non-trivial vector space (i.e. the zero vector space) under the conditions that the elements of S commute and $-I \notin S$. In fact, a more compact way of describing a group is by the use of *generators*, i.e. a list of elements g_1, \dots, g_k generates a group G iff any element of G can be decomposed into a product of g_i [36, p. 455]. A useful way of presenting generators of G_n is by using a *check matrix*, which is a $l \times 2n$ matrix with rows corresponding to generators g_1 through g_k . In fact, each generator is encoded by a binary code indicating the positions of Z gates to the left, and the positions of X gates to the right. Two set bits at the same position indicate thus a Y gate. For example, if a generator is given by $I \otimes Z \otimes Z \otimes X \otimes I \otimes Y \otimes Z$, this may be written as $Z_2 Z_3 X_4 Y_6 Z_7$ and thus the corresponding row in the check matrix looks like:

$$0\ 1\ 1\ 0\ 0\ 1\ 1\ | 0\ 0\ 0\ 1\ 0\ 1\ 0$$

An important theorem relates generators of the Pauli group to their stabilized vector spaces:

Theorem 15. *Let $S = \langle g_1, \dots, g_{n-k} \rangle$ be generated by $n - k$ independent and commuting elements from G_n , and such that $-I \notin S$. Then V_S is a 2^k dimensional vector space.*

Studying the dynamics of a vector space $|\Psi\rangle$ stabilized by S under a unitary transformation U directly implies that $U|\Psi\rangle$ is stabilized by USU^\dagger . Importantly, the dynamics of stabilizer quantum mechanics may be described using only a finite set of quantum gates:

Theorem 16. *The set of unitary gates U such that $UG_nU^\dagger = G_n$ is called the normalizer of G_n and it can be generated by the Hadamard, phase and C-NOT gates.*

Similarly, measurements within the stabilizer formalism have simple representations:

Theorem 17. *Let a system be in state $|\Psi\rangle$ with stabilizer $\langle g_1, \dots, g_n \rangle$. A stabilizer measurement is represented by $g \in G_n$, as g is Hermitian. g has eigenvalues ± 1 . If g commutes with all g_i , then the stabilizer of the state after the measurement is unchanged. If g anti-commutes with g_j , then if ± 1 is measured, the stabilizer of the new system $g|\Psi\rangle$ is given by $\langle g_1, \dots, g_{j-1}, \pm g, g_{j+1}, \dots, g_k \rangle$.*

2.3 Stabilizer codes

The fundamental scheme of stabilizer QECC is as follows[36, p. 435]: An arbitrary quantum state ρ is encoded into a quantum error-correcting code, which is itself a subspace C of some larger Hilbert space \mathcal{H} . This code is then transmitted through the noisy channel, whose action is being modelled by a *quantum operation* \mathcal{E} . After transmission, an *error syndrome* is detected through measurement and the original state is subsequently *recovered* through unitary transformations \mathcal{R} . In short, for error correction to succeed, one requires that

$$(\mathcal{R} \circ \mathcal{E})(\rho) \propto \rho \tag{2.1}$$

By constructing an explicit form for \mathcal{R} according to the general error correction scheme and by considering a general projection of ρ into its code-space C using the associated projector P , it can be shown that the following *quantum error-correction conditions* hold[36, p. 437]:

Theorem 18. *Suppose \mathcal{E} has operation elements $\{E_i\}$, called errors. An associated error-correction operation \mathcal{R} correcting \mathcal{E} on C (thus turning*

$\{E_i\}$ into a correctable set of errors) exists iff

$$PE_i^\dagger E_j P = \alpha_{ij} P$$

, where α is some complex Hermitian matrix.

This theorem allows one to calculate whether a given error correction code is able to correct a specific noise process. Importantly, it can be shown[36] that if C corrects \mathcal{E} , then it also corrects the *class of noise processes* \mathcal{F} consisting in arbitrary linear combinations of the operation elements E_i of \mathcal{E} . Furthermore, the proof of theorem 2.3, which has been omitted[36, p. 437], hints at important code construction mechanisms.

A *stabilizer code* which uses n qubits to encode k qubits is referred to as a $[n, k]$ -code.

Definition 53. [36, p. 465] A $[n, k]$ -stabilizer code is defined as the subspace V_S stabilized by a subgroup S of G_n such that $-i \notin S$. S has $n - k$ independent and commuting generators $S = \langle g_1, \dots, g_{n-k} \rangle$.

Definition 54. The normalizer $N(S)$ of a stabilizer group S , consists of all elements E of G_n such that $EgE^\dagger \in S, \forall g \in S$.

Definition 55. Let S be the stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^\dagger E_k \in N(S) - S, \forall j, k$. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.

Definition 56. The distance of a stabilizer code $C(S)$ is defined to be the minimum weight of an element of $N(S) - S$. The weight of an error $E \in G_n$ is defined to be the number of non-identity terms in the tensor product.

Theorem 19. A quantum error correction code with distance $\geq 2t + 1$ is able to correct arbitrary errors on any t qubits.

With these basic definitions in mind, it is now possible to give a concise description of the stabilizer error-correction scheme: By theorem 15 and definition 55, $C(S)$ has a code space of dimension 2^k . The state to be transmitted is encoded into that code space before entering the noisy channel. After leaving the noisy channel, the error syndrome is determined by measuring all $n - k$ stabilizer generators g_i in sequence. This results in $n - k$ measurements β_i , from which corresponding errors E_j may be picked via the relation $E_j g_i E_j^\dagger = \beta_i g_i$. The error is then corrected simply by applying E_j^\dagger .

2.4 The 3-qubit flip code

A particularly simple example of a stabilizer code is the the 3-qubit flip code. It uses $n = 3$ qubits to encode $k = 1$ qubit and can correct X errors on at most one of the qubits at a time. In stabilizer formalism, specifications look as follows: The code stabilizer S is generated by $\langle Z_1 Z_2, Z_2 Z_3 \rangle$. Therefore, S can be explicitly expressed as the set $\{I, Z_1 Z_2, Z_2 Z_3, Z_2 Z_1, Z_3 Z_2, Z_1 Z_3, Z_3 Z_1\}$. S stabilizes the vector space V_S , and by 15 (with $n - k = 2$, n corresponding to G_3), we have that V_S spans a 2^1 -dimensional vector space, subspace of \mathcal{H}_3 . A natural choice of a basis of V_S is $\{|000\rangle, |111\rangle\}$. What set of errors $\{E_i\}$ can be corrected by this code? One can employ 55 in order to check that the set $E = \{I, X_1, X_2, X_3\}$ forms a correctable set of errors as any $E_j^\dagger E_k$, $E_j, E_k \in E, \notin S$ anti-commute with at least one generator of S , so are $\notin N(S) - S$. However, extending E in any way, by e.g. $X_1 X_2$ will results in errors not part of a correctible set of errors, as e.g. $X_1 X_2 X_3$ commutes with all generators of the stabilizer. The actual error E_f is detected by measuring both stabiliser generators. The 4 distinct measurement outcomes thus obtained

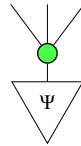
each correspond to errors as follows:

Z_1Z_2	Z_2Z_3	Error E_i
+1	+1	E_0
-1	+1	E_1
-1	-1	E_2
+1	-1	E_3

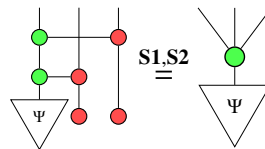
Error correction is of course effected by just applying E_f^\dagger to the system.

In order to understand how the 3-qubit flip code may be expressed in the ZX-calculus, we need the following theorems:

Theorem 20. *The 3-qubit flip code encoding circuit is given by:*

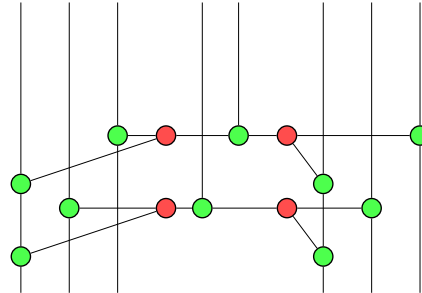


Proof. The projector P projecting the arbitrary qubit $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ onto the code space is given by $P = |000\rangle\langle 000| + |111\rangle\langle 111|$. This projector is easily seen to be implemented by the sequence of gates $CNOT_{12}CNOT_{13}$ acting on initial state $|\Psi 00\rangle$. In ZX-calculus notation, this setup is expressed as:



□

Theorem 21. *The 3-qubit flip code error detection circuit may be expressed as:*



where we use the CPM-construction (the middle three outputs encode the classical error syndrome) and the QAND-gate acts as a logical AND-gate on green (computational) basis states.

Proof. The CPM-construction doubles the 3 qubits. Error detection measurements are performed as required by the stabilizers, and the resultant classical two-dimensional Born vector in computational basis (the middle two outputs) encodes the error syndrome. \square

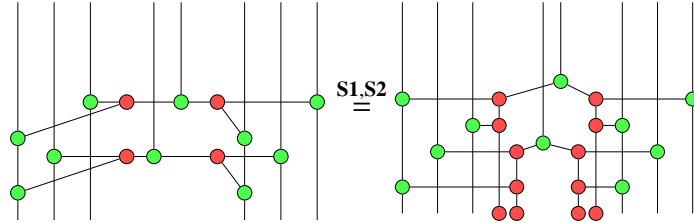
Theorem 22. *The QAND-gate required by 21 may be expressed using the GHZ/W-calculus as:*

$$\boxed{\text{QAND}} = \begin{array}{c} \oplus \quad \oplus \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \oplus \quad \oplus \end{array}$$

Proof. A proof is given in [28]. Also note that in theorem 21, measurements are performed in the appropriate green basis. \square

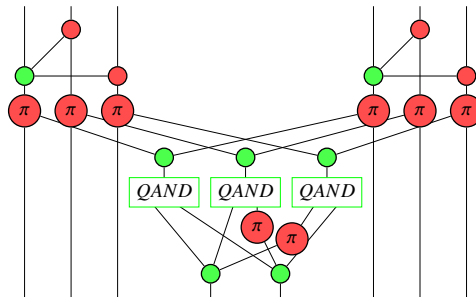
Theorem 23. *Instead of Multipartite measurements, one can use two prepared ancilla qubits and multipartite unitary gates in order to extract the error syndrome.*

Proof. This will serve as a nice example of the power of the CPM construction: While the argument using Dirac/von Neumann notation is relatively involved [36, p. 439], it is straight-forward in ZX-calculus notation:



Here, the LHS denotes multipartite measurements, while the RHS introduces two ancilla qubits prepared in state $|0\rangle$, upon which a series of CNOT-gates is performed. \square

Theorem 24. *The 3-qubit flip code error correction circuit may be expressed in the CPM-construction as:*



where the middle two inputs are connected to the outputs of the error detection circuit.

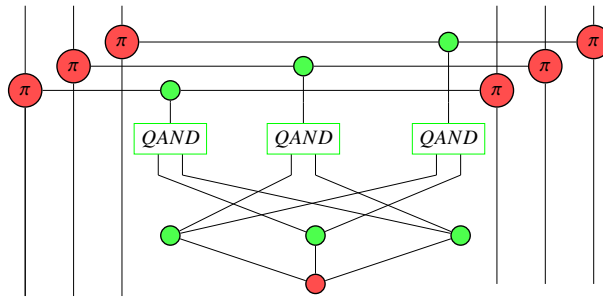
Proof. The E_j^\dagger are represented by conditional X-gates $CX_{1,2,3}$ on the 3 qubits. The Born vectors controlling the X-gates are generated from the error syndrome $\beta_{1,2}$ by the following map:

β_1	β_2	C_1	C_2	C_3
0	0	0	0	0
0	1	0	0	1
1	0	0	1	0
1	1	1	0	0

It is easy to verify that this binary function is given by: $C_1 = \beta_1 \wedge \beta_2$, $C_2 = \beta_1 \wedge (\neg\beta_2)$ and $C_3 = (\neg\beta_1) \wedge \beta_2$. As X-gates are equivalent

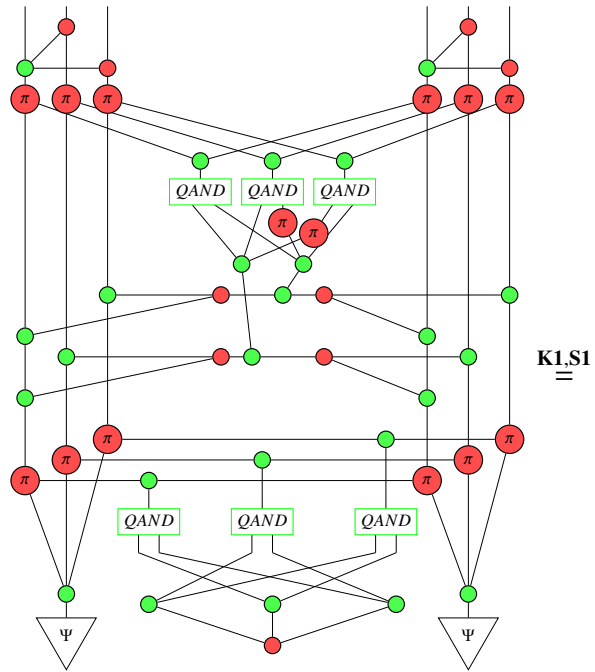
to NOT-gates in the computational basis, the graphical logical network works as advertised. The top of the diagram just consists of two CNOT-gates which, under error-free operation, suffice to decode the transmitted qubits. \square

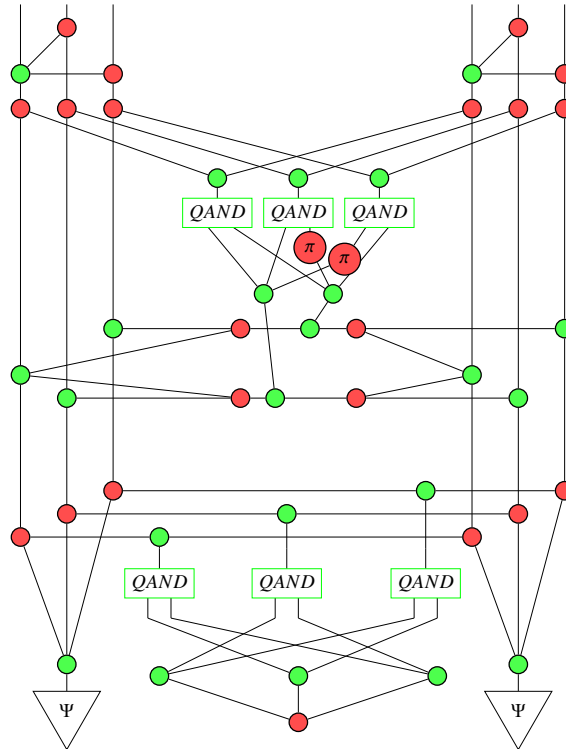
Theorem 25. *The noisy channel may be modelled by the following CPM-construction:*



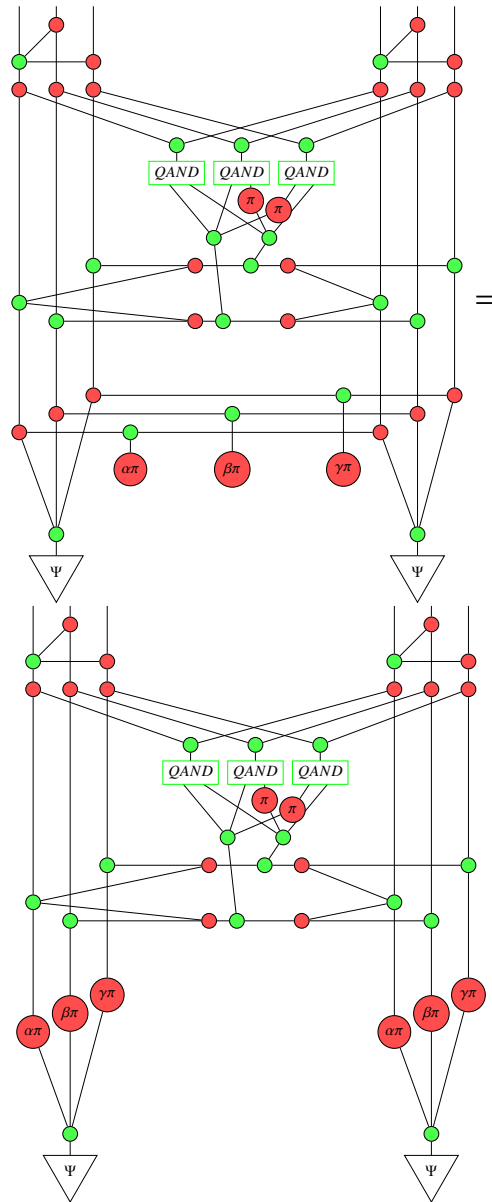
Proof. We only wish to encode the correctable errors. To that end, we model errors as conditional X -gates. From a qualitative perspective, the controlling Born vector $|B\rangle$ will be the (unnormalised) arbitrarily weighted sum of all possible control codes, i.e.: $|B\rangle = |000\rangle + \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle$, $\alpha, \beta, \gamma > 0$. For simplicity (and without loss of generality), we set $\alpha = \beta = \gamma = 1$. The representation of this Born vector in the CPM construction may be found in two steps: First, by preparing a 'resource' state - in this case, a GHZ-state in the X -basis. This state is given by $|+++\rangle + |--\rangle = |000\rangle + |110\rangle + |101\rangle + |011\rangle$, i.e. it has the appropriate number of terms and even the amplitudes demanded. All that is left in step 2 is to feed this into a logical circuit converting each term individually to $|(2 \wedge 3)(1 \wedge 3)(1 \wedge 2)\rangle$, which is precisely what the graphical network does. \square

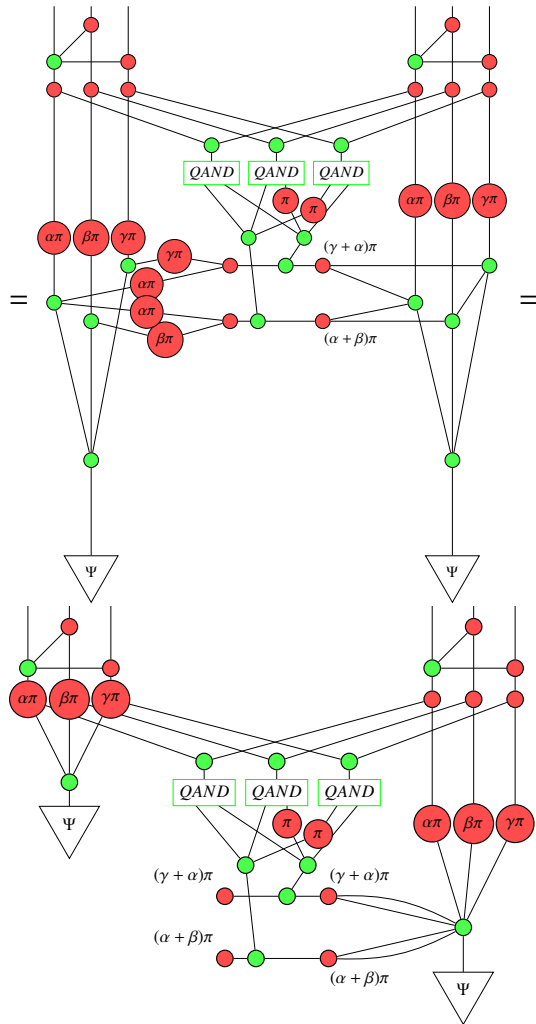
So, therefore the complete circuit is given by:

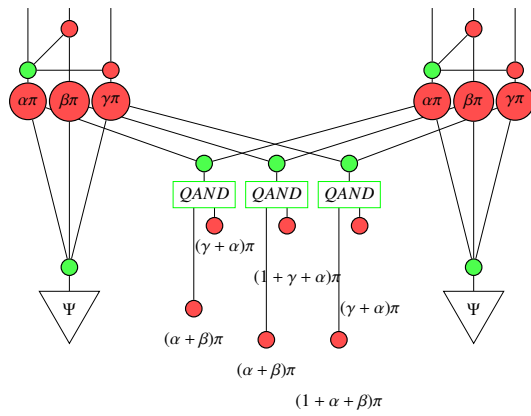
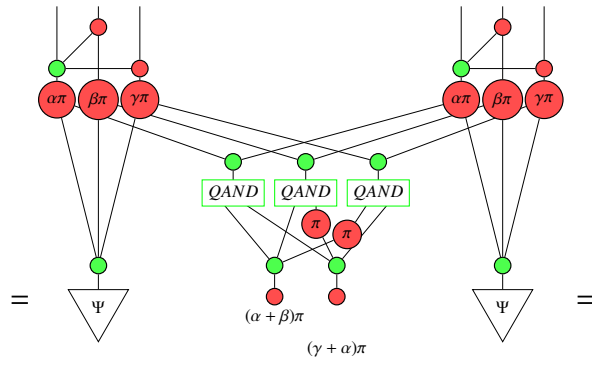


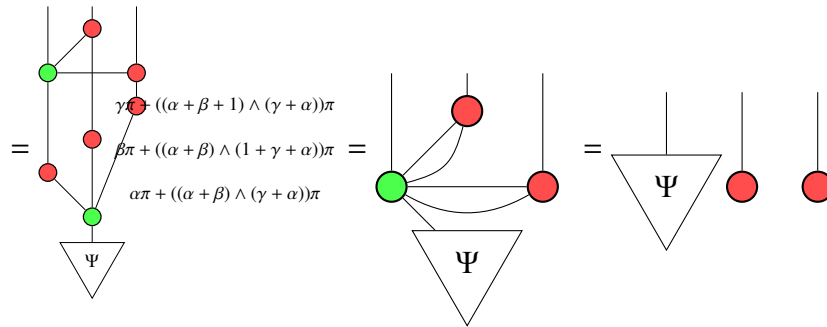


It is not clear how the above circuit can be simplified using only the rules of the ZX and GHZ/W calculi. A recent paper on the graphical verification of Steane's 7 qubit-code works around this problem by using a conditional form of the ZX-calculus[20]. We will employ a hybrid method, where the noisy channel is modelled conditionally, but the correction circuit is not. Replacing the control circuit of the noisy channel for the moment with a Born vector of form $|\alpha\beta\gamma\rangle$, where $\alpha, \beta, \gamma = 0, 1$ and either $\alpha, \beta, \gamma = 0$ or at best one of them is 1, we find that we can reduce this circuit:









2.5 Towards a graphical method of quantum error correction code construction

Above, we have taken first steps toward a purely graphical verification of quantum error correcting codes. In order to design a graphical theory of quantum error correction code construction,

3 | Gate approximations

3.1 Introduction

Some quantum algorithms require a large set of phase gates to be implemented, e.g. Shor's factorising algorithm requires the implementation of $U^{\frac{1}{2^n}}$ gates [42]. However, experimentally, gates of arbitrary phase require elaborate manufacturing and calibration processes. Especially in the era of mass manufacture, it is preferable to be able to implement all quantum circuits using only a (small) finite set of phase gates. Fortunately, it was shown by Solovay and Kitaev that Clifford gates, together with any other non-Clifford gate, suffice in order to approximate arbitrary quantum gates to arbitrary precision[19]. Mosca et al. and Selinger recently discovered a seemingly optimal algorithm for gate approximations using Clifford gates, together with the $\frac{\pi}{4}$ [32][41].

A remarkable characteristic of the ZX-calculus is that the topology of quantum information flow in experimental implementations of quantum circuits can be modelled very graphically similarly. This is illustrated, e.g. by the circuit in section 0.1, where the quantum circuit diagram, which itself would look very similar to a schematic drawing of an actual implementation setup, has a very similar representation in the ZX-calculus. This graphical (and topological) similarity of graphical

language and experimental setup allows to identify device-related *error sources* in the graphical language. A simple way of representing error sources is as follows: Let **abstract**(\cdot) be the map

experimental device \mapsto unapproximated representation in ZX-calculus

. Each experimental device has a set of errors attached to it. These errors may include systematic or statistical measurement imperfections, but also *gate approximation errors*, i.e. errors arising through deliberate gate approximations. In order to represent such errors in the ZX-calculus, one can simply assign a label to the subgraph **abstract**(*a particular experimental device*) which encodes the set of errors of the underlying device.

These labels will in the following be encoded in **black** and correspond to **primary error labels**.

3.1.1 Graphical error propagation

We will, in the following, restrict ourselves to gate approximation errors only. In principle, however, the ideas outlined in this section should be easily generalisable for other kinds of errors as well.

Given a representation in the ZX-calculus augmented with error labels, a question of immediate theoretical and practical concern is: Given two subgraphs \mathcal{G}_A and \mathcal{G}_B labelled with gate approximation errors ϵ and κ respectively, what are the gate approximation error labels on graphical composites of the two? I.e., given a set of *primary error sources*, one would like to track the propagation of these errors through arbitrary subgraphs. The assignment of propagated errors to subgraphs through labelling is referred to as *secondary error labelling*.

TODO: Give graphical depictions of this! TODO: Give rules for tensor and circle composition!

How can secondary errors be calculated graphically? Gate approximation errors are calculated as the trace distance between the perfect unitary ρ and the gate approximated unitary σ [36]:

$$D(\rho, \sigma) \equiv \frac{1}{2} \text{tr}|\rho - \sigma|$$

The difference between density matrices ρ, σ is of course is not easily calculated graphically. However, there is a way to circumvent this. The fidelity between two density matrices τ, λ is given by[36]

$$F(\tau, \sigma) \equiv \text{tr} \sqrt{\sqrt{\tau} \lambda \sqrt{\tau}}$$

If τ, σ are *pure*, then

$$F(|\tau\rangle, |\sigma\rangle) = |\langle \tau | \sigma \rangle|$$

where $|\tau\rangle, |\sigma\rangle$ are obtained by purification[36, p.110]. In this case, it can be shown that

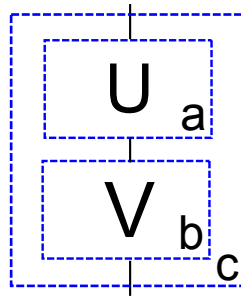
$$D(|\tau\rangle, |\sigma\rangle) = \sqrt{1 - F(|\tau\rangle, |\sigma\rangle)^2}$$

So in summary, the trace distance D can always be calculated easily from the fidelity F given that density matrices have been purified.

It is easy to see that the fidelity between two pure states $|\tau\rangle, |\sigma\rangle$ can readily be calculated graphically in the ZX-calculus by simply o-composing the ZX-calculus representation of one of the two with its adjoint partner. Amazingly, this generalises to arbitrary density matrices represented via the CPM-construction: Any CPM-diagram representing a density matrix can always be interpreted as a pure state in the ZX-calculus, thus the process of purification is trivially built-in. So we conclude that the propagated gate approximation errors can be calculated almost entirely in the ZX-calculus if expressed as traditional trace distances, and entirely if they are equivalently expressed as fidelities.

3.1.2 Graphical propagation of upper error bounds

Instead of exact error propagation, it can be insightful to just propagate upper error bounds through the diagram. An upper error bound of ϵ on a gate approximation error a means that $a \leq \epsilon$. There are two simple results that define how upper bounds propagate through the diagram: For two subgraphs connected via the \circ -composition:



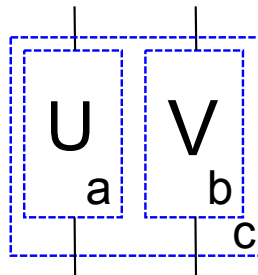
Then

$$c \leq a + b$$

This is suggested in [36] and can be proven as follows.

TODO Give your own proof!!!

For the tensor \otimes -composition:



Then

$$c \leq \max(a, b)$$

This is proven as follows:

TODO Give your own proof!!! (This result is actually a guess so far)

Note that these rules respect $(- \otimes -) \circ (- \otimes -) = (- \circ -) \otimes (- \circ -)$. TODO: Can we form a monoid or so out of those upper-bound rules?

3.1.3 Practicalities

Having established a quantitative notion of gate approximations in the ZX-calculus, we now consider the practicality of calculating propagated gate approximation errors in the ZX-calculus. Clearly, calculating the fidelity between a general quantum state and its gate-approximated alter ego within the ZX-calculus requires the ZX-calculus to be complete for the whole of finite-dimensional quantum mechanics. How such completeness is achieved is still an open question 1.1.

However, in practice it seems that an important part of such graphical simplification is the simplification of circuits containing Clifford gates and $\frac{\pi}{4}$ -gates only, especially if Selinger's gate approximation algorithm is used [41].

As the set of Clifford gates and $\frac{\pi}{4}$ gates is closed under the use of the rule **(new)**, one would hope that application of this rule may help to simplify Selinger-approximated circuits. However, a quick search of some of Selinger's approximation examples [41] actually shows that for none of these, **(new)** can be readily applied. This is of course indicative of an optimal approximation algorithm.

Definition 57. *The map $\|\cdot\|_\epsilon$ takes a ZX-calculus diagram \mathfrak{D} to a ZX-calculus diagram $\mathfrak{D}|_\epsilon$ corresponding to its Selinger approximation.*

Using all ZXZ-XXZ rules containing only Clifford gates and $\frac{\pi}{4}$ gates on Selinger-approximated circuits gives rise to the problem of unclosedness: Application of such rules to the \circ - or \otimes - composite of two Selinger-approximated gates $\mathfrak{A}|_\kappa, \mathfrak{B}|_\lambda$ does not generally result in the Selinger-approximated circuit $(A \circ / \otimes B)|_\mu$. An interesting question for future research is thus: Is there an efficient way of calculating $(A \circ / \otimes B)|_\mu$ from $\mathfrak{A}|_\kappa, \mathfrak{B}|_\lambda$ and can this calculation be performed graphically in the ZX-calculus?

4 | Conclusions

.1 Closed form solution for ZXZ-XXZ conversion

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