The Finite Length Property of the Rado Graph and Friends

Jingjie Yang
University of Oxford, UK

Mikołaj Bojańczyk
University of Warsaw, Poland

Bartek Klin
University of Oxford, UK

Abstract

An infinite structure has the finite length property (over a given field) if, for each of its finite powers, chains of equivariant subspaces in the corresponding free vector space are bounded in length. Prior work showed that the countable pure set and the countable dense linear order without endpoints have this property. We generalise these results to (a) any structure approximated by finite substructures with few orbits, provided the field is of characteristic zero, and (b) any Fraïssé limit with free amalgamation in a finite vocabulary consisting of unary and binary relations, possibly expanded with a generic total order. As a special case, we deduce the finite length property of the Rado graph using both methods. We also describe some connections with function spaces, weighted register automata, and orbit-finite systems of linear equations.

2012 ACM Subject Classification Theory of computation → Logic; Mathematics of computing → Random graphs; Theory of computation → Automata over infinite objects

Keywords and phrases Rado graph, oligomorphic structure, orbit-finite set, orbit-finitely spanned vector space, equivariant subspace, finite length

Digital Object Identifier 10.4230/LIPIcs.LICS.2026.6

Funding Mikołaj Bojańczyk was supported by the Polish National Science Centre (NCN) grant "Polynomial finite state computation" (2022/46/A/ST6/00072). Jingjie Yang was partially supported by the Polish National Science Centre (NCN) grant "Linear algebra in orbit-finite dimension" (2022/45/N/ST6/03242).

Acknowledgements We thank David M. Evans, Arka Ghosh, and Ehud Hrushovski for helpful discussions. We are very grateful to the anonymous reviewers for their insightful comments.

1 Introduction

This paper is part of a research programme focused on orbit-finite sets and structures. In this programme, one starts with an infinite relational structure $\mathbb{A}$ whose elements are called *atoms*. Based on these, one constructs sets that are called *orbit-finite*. Precise definitions will follow, but the understanding is that elements of an orbit-finite set are constructed using atoms, and there are only finitely many elements up to automorphisms of $\mathbb{A}$. For the theory to make sense, we must assume that $\mathbb{A}$ is *oligomorphic*, which means that $\mathbb{A}^d$ has finitely many orbits for every d. The simplest example of an oligomorphic atom structure is what we refer to as the *equality atoms*; this is the structure with a countably infinite underlying set and no relations except for equality. This structure, like all oligomorphic structures over suitable vocabularies of relations, arises by applying a model-theoretic construction (the Fraïssé limit) to a well-behaved class of finite structures. Figure 1 shows other examples of such structures.

	Class of finite structures	Fraïssé limit
1.	finite sets with equality only	$(\mathbb{N}, =)$
2.	finite orders	$(\mathbb{Q}, <)$
3.	finite graphs	(Rado graph, E)
4.	(subsets of) finite F_2-vector spaces	$\mathsf{F}_2 \oplus \mathsf{F}_2 \oplus \mathsf{F}_2 \oplus \cdots$

Figure 1 Examples of Fraïssé limits

When the underlying atom structure is oligomorphic, the orbit-finite sets have a robust theory, which resembles in some ways the theory of finite sets. This theory was originally developed to describe regular languages over infinite alphabets, by considering orbit-finite versions of various automata models [5, 8], but it has since expanded to cover other models, such as orbit-finite Turing machines [7] or orbit-finite constraint satisfaction problems [36]. There are also programming languages with data structures that can store orbit-finite sets [9, 13], with working implementations [37, 46]. For a survey of the orbit-finite programme, we refer to the lecture notes [6].

Some results about orbit-finite models do not depend on the choice of the atom structure, while others do. An example of the latter case arises for orbit-finite Turing machines [7]. If the atoms are the Fraïssé limit of total orders, as in row 2 of Figure 1, then the orbit-finite version of the $\mathrm{P}\overset{?}{=}\mathrm{NP}$ question has the same answer as the classical version without atoms. On the other hand, for any of the other three rows of the table, $\mathrm{P}\neq\mathrm{NP}$ holds unconditionally in the orbit-finite setting due to problems with choice. (Failure of the axiom of choice in set theory with atoms was the original motivation of Fraenkel and Mostowski in the 1930s to study the first two rows of Figure 1, see e.g. [34, Ch. 4].) The dependence on the underlying atom structure will play a prominent role in this paper.

Vector spaces

One direction of the orbit-finite programme, motivated by the study of orbit-finite weighted automata, is focused on vector spaces [10]. In these spaces (taken over some fixed field), one can take linear combinations and apply automorphisms of the atom structure. We are interested in spaces that have an orbit-finite spanning set, which means that the entire space can be obtained from some finite subset by using atom automorphisms and linear combinations. A prototypical example is the space $\mathrm{Lin}\, X$ that consists of formal linear combinations of elements from some orbit-finite set X. The original application of these spaces was in automata theory, but they have also found applications in the study of orbit-finite linear programming [22], function spaces for orbit-finite sets [12], or the analysis of two-party communication protocols over infinite alphabets [11].

To be useful, the theory of orbit-finitely spanned vector spaces should have certain properties. For example, one would like to be able to represent these spaces in a finite way, or tackle algorithmic problems such as solving systems of linear equations. One rather modest requirement is that such spaces be closed under taking *equivariant* subspaces (i.e. subspaces closed under atom automorphisms as well as linear combinations): an equivariant subspace of an orbit-finitely spanned vector space should itself be orbit-finitely spanned. We do not know if this closure property holds in general, and we see it as an important open problem. It can be equivalently phrased in terms of ascending chains: is it true that every orbit-finitely spanned vector space is *Noetherian*, meaning that one cannot find an infinite ascending chain of its equivariant subspaces? To the best of our knowledge, this question was first recognised by Camina and Evans [14, Q. 2] who identified a sufficient condition for this, namely the existence of a "nice ordering". Using this condition they showed that certain vector spaces are Noetherian: notably, Lin $\mathbb{A}$ over the ordered atoms (row 2 of Figure 1), Lin $\binom{\mathbb{A}}{d}$ over the equality atoms (row 1), and a similar family of spaces over the bit-vector atoms (row 4).

The above question was independently considered in [10, p. 21] where it was conjectured that every oligomorphic structure has the *ascending chain property*, meaning that all orbit-finitely spanned vector spaces over the structure are Noetherian. In such a vector space, if descending chains as well as ascending chains of equivariant subspaces are all finite, by the Jordan–Hölder Theorem, there is some finite upper bound on the length of chains of its equivariant subspaces. In that case, we say that the structure has the *finite length property.* This stronger property has been confirmed for:

- *The equality atoms.* There are three independent proofs of finite length, in three different contexts: model theory [20, Thm. 3.9], representation theory [44, Prop. 6.1.6], and orbit-finite set theory [10, Cor. 4.9]. The result from [44] assumes that the underlying field is the complex numbers, while the other two do not restrict the choice of a field.
- *The ordered atoms.* Here, the finite length property was proved in [10, Thm. 4.8]. There exist alternative methods to show the weaker ascending chain property of this structure. One is to exhibit an AZ enumeration [1, Thm. 2.1] and derive the "nice orderings" from that [17]. Another is based on Hilbert's Basis Theorem [24, Thm. 27], which is independently established in [40, Ex. 3.2(c)], with well-quasi-orders as a key ingredient.

The finite length property is strictly stronger than the ascending chain property. A prime example is the bit-vector atoms (which admit an AZ enumeration [33, p. 143]): even Lin $\mathbb{A}$ does not have finite length over the two-element field, as found independently in [19, Thm. 2.7] and [10, Thm. 4.16]. On the other hand, Evans [18, Rem. 1.3] observed that for any Fraïssé limit over a finite relational vocabulary, failure of the finite length property over some finite field would yield a counterexample to the conjecture of Thomas [47, p. 177] in model theory. The known examples of failure of the finite length property are not good enough for this, since they use an infinite vocabulary (or functions).

Our contributions

Until now, the finite length property has been a theory of two examples: the equality and the ordered atoms. We substantially improve this state of affairs, using two different techniques:

- In Theorem 5.4, we prove the finite length property for those structures $\mathbb{A}$ – e.g. the equality atoms and the bit-vector atoms – which admit what we call oligomorphic approximation, a relaxed version of smooth approximation known from model theory. This is under an additional assumption that the underlying field has characteristic 0.
- In Theorem 7.3, we prove the finite length property for those structures $\mathbb{A}$ – e.g. the ordered atoms – which arise as the generically ordered expansions of the Fraïssé limits of free amalgamation classes, over finite relational vocabularies of arity at most 2. Here we do not restrict the underlying field.

In particular, we may use either of these techniques to deduce the finite length property of the Rado atoms (row 3 in Figure 1), where even the weaker ascending chain property was not known before.

Most of the paper is devoted to introducing the background necessary to understand these results (Sections 2–4) and to proving them (Sections 5, 7 and 8). In Section 6 we discuss some connections with function spaces and weighted automata, and in Section 8.5 we briefly discuss applications to solving orbit-finite systems of equations.

2 Structures

In this section, we briefly recall some basic notions from model theory and describe the main examples of structures that we will consider in this paper.

Let us begin by fixing some terminology. A *vocabulary* is a set of relations, each with a specified positive arity. For example, the vocabulary of graphs contains one binary relation: the edge relation. (We do not include equality in the vocabulary, since it is automatically present.) A *structure* $\mathbb{A}$ over a vocabulary consists of an underlying set, also denoted by $\mathbb{A}$, together with interpretations of relations from the vocabulary as actual relations on that set. An *embedding* between two structures over the same vocabulary is an injective function between their underlying sets that preserves and reflects all relations. An *isomorphism* is a surjective embedding. An *automorphism* of a structure is an isomorphism from the structure to itself; these form a group.

Automorphisms of $\mathbb{A}$ act on tuples in $\mathbb{A}^d$ componentwise. When we speak of orbits in $\mathbb{A}^d$, we mean the orbits under this action of the automorphism group. For example, if $\mathbb{A}$ is a graph, then two pairs (a_1, a_2) and (b_1, b_2) are in the same orbit if and only if there is some automorphism of the graph that maps a_1 to b_1 and a_2 to b_2. In particular, the edge relation must be defined in the same way for both pairs.

All structures considered in this paper will be countable (or finite), and we will always want them to have finitely many orbits in every finite dimension as per the following definition.

▶ **Definition 2.1** (Oligomorphic structure). *A structure $\mathbb{A}$ is* oligomorphic *if $\mathbb{A}^d$ has finitely many orbits for every $d \in \{1, 2, \ldots\}$.*

A relation on a structure $\mathbb{A}$ – i.e. a subset of $\mathbb{A}^d$ – is called *equivariant* if it is invariant under the action of the automorphism group. Equivalently, the relation is a union of orbits. If the structure is oligomorphic, then there are finitely many orbits to consider once the dimension d is fixed, and therefore only finitely many equivariant relations. By the Ryll-Nardzewski Theorem [30, Thm. 7.3.1], if the structure is oligomorphic and countable, then the equivariant relations are exactly those that can be defined (as subsets) in first-order logic – see for instance [6, Cl. 5.9]. In fact, the infinite structures that we consider in this paper will satisfy a stronger property: an equivariant relation will be definable not only by a first-order formula, but even by a quantifier-free one. This will be ensured by the additional homogeneity condition defined below. In the condition, a *substructure* of $\mathbb{A}$ is any structure obtained by restricting $\mathbb{A}$ to some subset of its underlying set; we do not distinguish between substructures and subsets.

▶ **Definition 2.2** (Homogeneous structure). *A structure $\mathbb{A}$ is* homogeneous *if every isomorphism between finite substructures of $\mathbb{A}$ (i.e. embedding of a finite substructure of $\mathbb{A}$ into $\mathbb{A}$) extends to an automorphism of $\mathbb{A}$.*

In a homogeneous structure $\mathbb{A}$, an orbit in $\mathbb{A}^d$ consists of tuples that satisfy the same quantifier-free formulas – see for instance [6, Thm. 6.3]. Every homogeneous structure arises via a construction called the *Fraïssé limit* [30, Sec. 7.4], from a class of finite structures that satisfies certain closure properties (a so-called *amalgamation class*). For the Fraïssé limit to be not only homogeneous, but also oligomorphic, we need to assume that the underlying class – consisting precisely of the finite structures that embed in the Fraïssé limit – has only finitely many non-isomorphic structures of each finite size. This assumption is automatically satisfied if the vocabulary contains only finitely many relations of each arity, in particular if the vocabulary is finite.

Here are some of the important structures that we will consider in this paper. They are all homogeneous and oligomorphic.

▶ **Example 2.3** (Equality atoms). In this structure, the underlying set is the natural numbers and there are no relations other than equality. Automorphisms are arbitrary permutations. Two tuples in the same orbit necessarily have the same equality pattern, and conversely this condition is sufficient, which guarantees homogeneity. For instance, if two tuples (a_1, a_2) and (b_1, b_2) both have non-repeating entries, then the mapping $a_1 \mapsto b_1, a_2 \mapsto b_2$ can be extended first to a permutation of $\{a_1, a_2, b_1, b_2\}$, and then to one of the whole structure by mapping other elements identically. There is another orbit in dimension $d = 2$, defined by $x_1 = x_2$. ⌟

▶ **Example 2.4** (Ordered atoms). In this structure, the underlying set is the rational numbers, equipped with the usual order – the vocabulary consists of this binary relation only. Automorphisms are order-preserving permutations. Two tuples are in the same orbit if and only if they have the same order pattern. For instance, for two tuples (a_1, a_2) and (b_1, b_2) such that $a_1 < a_2$ and $b_1 < b_2$, mapping $a_1 \mapsto b_1, a_2 \mapsto b_2$ can be extended to an automorphism by using Cantor's back-and-forth method, or by explicitly defining a monotone piecewise linear bijection. There are two other orbits in dimension $d = 2$, defined by $x_1 = x_2$ and $x_1 > x_2$. ⌟

▶ **Example 2.5** (Vector atoms). Fix some finite field k, and let $\mathbb{A}$ be the vector space of countable dimension over k. This vector space is seen as a structure over an infinite vocabulary containing, for every $d \in \{1, 2, \dots\}$ and coefficients $\lambda_1, \dots, \lambda_d$ in k, the relation

$$\{(a_1, \dots, a_d) \in \mathbb{A}^d \mid \ \lambda_1 a_1 + \cdots + \lambda_d a_d = 0 \ \}.$$

The vocabulary is defined so that automorphisms are the same thing as permutations that are linear maps. Two tuples of the same length are in the same orbit if and only if they have the same linear dependencies. By way of illustration, consider two tuples (a_1, a_2) and (b_1, b_2) each consisting of linearly independent entries. Extend a_1, a_2 to a basis for the subspace V of $\mathbb{A}$ spanned by a_1, a_2, b_1, b_2; do the same for b_1, b_2. This gives us an automorphism of V with $a_1 \mapsto b_1, a_2 \mapsto b_2$, which can then be extended to an automorphism of $\mathbb{A}$ by extending these two bases of V to bases of $\mathbb{A}$. In $\mathbb{A}^2$ there are $2 + |\mathsf{k}|$ other orbits: $x_1 = 0 = x_2$; $x_1 = 0 \neq x_2$; and $x_1 \neq 0, x_2 = \lambda x_1$, where λ ranges over the field k. ⌟

▶ **Example 2.6** (Rado atoms). The Rado graph is the Fraïssé limit of the class of finite undirected graphs. Here, an undirected graph is viewed as a structure with one binary relation that is symmetric and irreflexive. One explicit description of the Rado graph is as follows: its vertices are the natural numbers, and there is an edge between $n < m$ if the n-th least significant bit in the binary representation of m is 1. A more famous characterisation of the Rado graph is that if one randomly selects a graph with a countable set of vertices by independently including each possible edge with probability 1/2, then with probability 1 the resulting graph is isomorphic to the Rado graph.

As a Fraïssé limit, the Rado graph is homogeneous by construction. So two tuples are in the same orbit if and only if they have the same equality and adjacency patterns. In dimension $d = 2$ there are three orbits: the two coordinates can be equal, adjacent hence distinct, or distinct but non-adjacent. ⌟

3 Orbit-finite sets

We shall now briefly explain the concept of orbit-finiteness. We start with a countably infinite oligomorphic structure $\mathbb{A}$, as described in Section 2, whose elements we call *atoms*. These can be used to construct sets that are finite up to the automorphisms of $\mathbb{A}$, such as

$$\underbrace{\mathbb{A}^2}_{\text{pairs}} \qquad \underbrace{\{(a,b) \in \mathbb{A}^2 \mid a \neq b\}}_{\text{non-repeating pairs}} \qquad \underbrace{\binom{\mathbb{A}}{2}}_{\text{unordered pairs}}.$$

There are several equivalent definitions of orbit-finiteness. Of these, we use one that is based on first-order interpretations [30, Sec. 4.3]. To construct an orbit-finite set, we proceed in three steps. In the examples above, this process is as follows: start with all pairs of atoms, as in the first example; restrict to the equivariant subset of non-repeating pairs, as in the second example; and then take the quotient, as in the third example, by the equivariant equivalence relation that identifies two pairs if they differ only in the order of their elements. The general definition is given below.

▶ **Definition 3.1** (Orbit-finite set). *An* orbit-finite set *over an oligomorphic structure* $\mathbb{A}$ *is any set that is obtained as follows:*

1. *Start with a finite power* $\mathbb{A}^d$ *for some* $d \in \{1, 2, \dots\}$*;*
2. *Restrict it to an equivariant subset* $X \subseteq \mathbb{A}^d$*;*
3. *Quotient* X *under an equivariant equivalence relation.*

Let us justify the name "orbit-finite" in this definition. An orbit-finite set is equipped with an action of the automorphism group of the original structure $\mathbb{A}$, namely the action inherited from $\mathbb{A}^d$, suitably extended to the quotient. Under this action, the set has finitely many orbits: $\mathbb{A}^d$ has finitely many orbits by oligomorphicity, and the number of orbits can only go down when restricting to an equivariant subset and quotienting under an equivariant equivalence relation.

There is another equivalent definition of orbit-finite sets, given in Definition 3.2 below, that emphasises the role of the group action. In this definition, we use the following notion of support: if X is a set equipped with an action of the automorphism group of $\mathbb{A}$, then a *support* for an element $x \in X$ is any set $S \subseteq \mathbb{A}$ such that every automorphism π of $\mathbb{A}$ satisfies

$$\underbrace{\forall a \in S : \pi(a) = a}_{\text{action of } \pi \text{ on } \mathbb{A}} \qquad \Longrightarrow \qquad \underbrace{\pi(x) = x}_{\text{action of } \pi \text{ on } X}.$$

▶ **Definition 3.2** (Orbit-finite set, abstractly). *An* orbit-finite set *over an oligomorphic structure* $\mathbb{A}$ *is a set* X *equipped with an action of the automorphism group of* $\mathbb{A}$ *such that:*

(1) *every element has some finite support;*
(2) *there are finitely many orbits under the action.*

Thanks to the underlying atom structure being infinite but oligomorphic, the two definitions above are equivalent in the sense that they describe the same sets, up to equivariant bijections: see [6, Thm. 5.12] and [3, Prop. 2.19]. Both definitions have their uses. Definition 3.1 is more concrete, and it comes with a finite representation, which can be used for algorithms that process orbit-finite sets. On the other hand, some constructions (e.g. disjoint unions, Cartesian products) are more naturally presented using the more abstract Definition 3.2.

4 Orbit-finitely spanned vector spaces

We now introduce the main topic of this paper: vector spaces with an orbit-finite spanning set. We begin with spaces that have an orbit-finite basis; this special case has an elementary definition, and yet it will be the relevant case for almost all results of this paper.

▶ **Definition 4.1** (Orbit-finite-dimensional vector space)**.** $\mathrm{Lin}_{\mathsf{F}}\, X$ *is, given an orbit-finite set* X *and a field* F*, the vector space of finite formal linear combinations of elements in* X *over* F*.*

This definition has two important parameters: the field F and the oligomorphic structure $\mathbb{A}$ over which X is an orbit-finite set. The spaces defined here have two kinds of structure: that of a vector space, and the action of the automorphism group of $\mathbb{A}$, via $\pi(\sum_i \lambda_i x_i) = \sum_i \lambda_i \pi(x_i)$. We will be interested in subsets that preserve both kinds of structure, i.e. those that are closed under taking linear combinations as well as applying automorphisms of $\mathbb{A}$. Such subsets are called *equivariant subspaces*.

▶ **Example 4.2.** Let $\mathbb{A}$ be the equality atoms and let F be an arbitrary field. As explained in [10, Ex. 4.2], the vector space $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ has only three equivariant subspaces: the zero subspace, the full space, and the subspace consisting of vectors whose coefficients add up to zero (e.g. $a - b$ and $a + b - 2c$, with $a, b, c \in \mathbb{A}$ distinct). ⌟

Unfortunately, there is a price to pay for the elementary character of Definition 4.1, and that is the failure of certain closure properties. In particular, these spaces are not closed (even up to equivariant linear bijections) under taking equivariant subspaces, or quotients by such spaces. The problem with subspaces is already apparent in Example 4.2, since the unique non-trivial subspace of $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ does not have any equivariant basis, regardless of the underlying field [10, Ex. 6.1]; this subspace can also be realised as a quotient of $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^2$. We therefore need a more general notion of a vector space, in the style of Definition 3.2.

▶ **Definition 4.3** (Orbit-finitely spanned vector space, abstractly)**.** *An* orbit-finitely spanned vector space *over an oligomorphic structure* $\mathbb{A}$ *is a vector space equipped with an action of automorphisms of* $\mathbb{A}$ *such that:*

(1) *vector addition and scalar multiplication are equivariant,*[1]
(2) *every vector is supported by a finite set of atoms;*[2]
(3) *the space is spanned by some equivariant subset that is orbit-finite.*[3]

The spaces defined above include those of the form $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$, as $\mathbb{A}^d$ is orbit-finite. They are also easily seen to be closed under taking quotients by equivariant subspaces. Closure under taking equivariant subspaces is less obvious: one could imagine that condition (3) above is violated by moving to an equivariant subspace. This closure property turns out to be closely related to the ascending chain property discussed in the introduction:

▶ **Theorem 4.4.** *For any field* F *and oligomorphic structure* $\mathbb{A}$*, the following conditions are equivalent:*

(a) *orbit-finitely spanned vector spaces are closed under taking equivariant subspaces;*
(b) *for every orbit-finitely spanned vector space* V*, there are no infinite ascending chains of equivariant subspaces in* V*;*
(c) *for every* $d \in \{1, 2, \dots\}$*, the vector space* $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ *does not have any infinite ascending chain of equivariant subspaces.*

Furthermore, if these conditions hold, then the orbit-finitely spanned vector spaces are – up to equivariant linear bijections – precisely the vector spaces of the form U/W*, where* $W \subseteq U$ *are equivariant subspaces of* $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ *and* $d \in \{1, 2, \dots\}$*.*

[1] Equivalently, the group action $v \mapsto \pi(v)$ of each automorphism π is linear.
[2] Equivalently, the application $(\pi, v) \mapsto \pi(v)$ is continuous, where the automorphism group is endowed with the topology of pointwise convergence whilst the vector space is endowed with the discrete topology. See [30, Lemma 4.1.5].
[3] Equivalently, the vector space is finitely generated as a module over the group ring of automorphisms.

Proof sketch. The equivalence of (b) and (a) is a classical result in module theory, and the implication (b) $\Rightarrow$ (c) is immediate. For the "furthermore" part, take some orbit-finitely spanned vector space V. By Definition 3.1, the spanning set can be obtained from some equivariant subset of $\mathbb{A}^d$ by quotienting under an equivariant equivalence relation. This gives us a surjective equivariant linear map to V from an equivariant subspace U of $\operatorname{Lin}_{\mathsf{F}} \mathbb{A}^d$; call its kernel W, so that we have a bijective equivariant linear map $V \leftarrow U/W$.

Finally, the implication (c) $\Rightarrow$ (b) follows from the "furthermore" part: the lack of infinite chains is preserved by taking equivariant subspaces and images under equivariant linear maps. ◀

We say that an atom structure $\mathbb{A}$ has the *ascending chain property over a field* F if any of the equivalent conditions in Theorem 4.4 is satisfied. One interpretation of the theorem is that the ascending chain property is necessary for the theory of vector spaces to be well-behaved. In particular, thanks to the "furthermore" part, we get a result similar to the equivalence of Definitions 3.1 and 3.2 above, i.e. a concrete representation of the orbit-finitely spanned vector spaces that can be used in algorithms (provided we know how to represent equivariant subspaces – see Section 8.5).

We are therefore interested in atom structures that have the ascending chain property. As it turns out, the techniques used in this paper will yield a stronger property, namely a finite bound on the length of chains:

▶ **Definition 4.5** (Finite length property). *An oligomorphic structure* $\mathbb{A}$ *has the* finite length property over a field F *if for every orbit-finite set* X *over* $\mathbb{A}$*, there is a finite upper bound on the length of chains of equivariant subspaces of* $\operatorname{Lin}_{\mathsf{F}} X$*. (The supremum of the chain lengths, finite or not, is called the* length *of* $\operatorname{Lin}_{\mathsf{F}} X$*.)*

In light of Theorem 4.4 we could have used $\mathbb{A}^d$ instead of X in the above definition. As mentioned in the introduction, the finite length property can be strictly stronger than the ascending chain property, as witnessed by $\operatorname{Lin}_{\mathsf{F}} \mathbb{A}$ over the vector atoms from Example 2.5. (Note that there are two fields involved, namely the finite field used to define the vector atoms $\mathbb{A}$ and the field F used to define $\operatorname{Lin}_{\mathsf{F}} \mathbb{A}$. In the counterexample, both fields are the two-element field.) The finite length property was recently studied in [10], where it was shown that the equality atoms (Example 2.3) and the ordered atoms (Example 2.4) have this property over any field.

The main contribution of this paper is to establish the finite length property of more structures. We will use two different techniques for this purpose.

5 Finite length in characteristic zero

In this section, we present the first of our two main results, which is a method for proving the finite length property assuming that the underlying field has characteristic zero. Under this assumption, we will establish the finite length property of the Rado atoms (Example 2.6) and of the vector atoms (Example 2.5). These are new results. We also think that the proof itself, even when applied to get already known results for the equality atoms (Example 2.3), is of independent interest and arguably simpler than previously known proofs. The method that we use will work for all structures satisfying the following condition.

▶ **Definition 5.1** (Oligomorphic approximation). *We say that a homogeneous structure* $\mathbb{A}$ *has* oligomorphic approximation *if, for every* $d \in \{1, 2, \ldots\}$*, there exists a family* $\mathcal{B}$ *of finite substructures of* $\mathbb{A}$ *such that:*

(1) *every finite substructure S of $\mathbb{A}$ is a substructure of some $\mathbb{B}_S \in \mathcal{B}$; and*
(2) *there is a common finite upper bound, for all $\mathbb{B} \in \mathcal{B}$, on the number of orbits in $\mathbb{B}^d$ with respect to automorphisms of $\mathbb{B}$.*

This is related to the notion of *smooth approximation* in model theory introduced by Lachlan, developed in [35] and studied in depth in [15]. There one asks for a family $\mathcal{B}$ that does not depend on d, and instead of (2) one requires that:
(2') $\mathbb{A}$ is oligomorphic, and each $\mathbb{B} \in \mathcal{B}$ is homogeneous.

▶ **Theorem 5.2.** *The following structures have oligomorphic approximation:*
(i) *any smoothly approximated structure;*[4]
(ii) *the equality atoms from Example 2.3;*
(iii) *the vector atoms from Example 2.5, for any finite field* k*;*
(iv) *the Rado atoms from Example 2.6.*

Before proving this, let us note that the ordered atoms (Example 2.4) do not have oligomorphic approximation.

▶ **Non-example 5.3.** Consider the rational numbers with the usual order. The finite substructures in this case are finite total orders, and already for dimension $d = 1$, a finite total order of size n has n orbits. So no family $\mathcal{B}$ satisfies both conditions (1) and (2). ⌟

Proof of Theorem 5.2. For each of these structures, we exhibit a family $\mathcal{B}$ satisfying the two conditions. In fact, in each case the family will not depend on the parameter d.
(i) For a homogeneous structure $\mathbb{A}$ smoothly approximated by $\mathcal{B}$, we just use the family $\mathcal{B}$. We only need to check condition (2), so let $\mathbb{B} \in \mathcal{B}$. It is easy to check that, as observed in [35, (II)], two tuples in $\mathbb{B}^d$ are in the same orbit if and only if they are in the same orbit of $\mathbb{A}^d$. Hence the number of orbits in $\mathbb{B}^d$ is at most that of $\mathbb{A}^d$, which is a finite upper bound because $\mathbb{A}$ is oligomorphic.
(ii) For the equality atoms, we can choose $\mathcal{B}$ to be all finite subsets, which clearly satisfies (1). Each of these is homogeneous: we can extend a bijection between subsets to a permutation. So $\mathcal{B}$ satisfies (2') and hence (2) by the above argument.
(iii) For the vector atoms, we choose $\mathcal{B}$ to be all subspaces of finite dimension. Each of these is homogeneous since we can, by extending a linearly independent set to a basis, extend linear bijections between subspaces to a linear permutation. Again $\mathcal{B}$ satisfies (1) and (2') hence (2). This argument applies to any finite field k.
(iv) The most interesting case is the Rado atoms, which are not smoothly approximated [15, Rem. 2.1.2] (essentially because the finite homogeneous graphs [21, p. 100] are not diverse enough). The witness for oligomorphic approximation is a family of *symplectic graphs*: see [25, Sec. 8.11].[5] For every $n \in \{1, 2, \ldots\}$ define a finite graph as follows. The set of vertices is the vector space over the two-element field with basis

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}.$$

Since the field has two elements, we can view vertices as subsets of this basis. In this graph, there is an edge between vertices v and w if and only if the sets

$$\{i \in \{1, \ldots, n\} \mid \ e_i \in v \text{ and } f_i \in w\ \}$$
$$\{i \in \{1, \ldots, n\} \mid \ f_i \in v \text{ and } e_i \in w\ \}$$

[4] We work over the canonical vocabulary if the structure is not already homogeneous; see [35, p. 443]. This does not change the automorphism group or, consequently, the finite length property.
[5] We are grateful to Ehud Hrushovski for drawing our attention to this construction.

have different sizes modulo two. These graphs (embedded in all possible ways in the Rado graph) satisfy condition (1) from Definition 5.1, i.e. every finite graph embeds in some symplectic graph [25, Thm. 8.11.2]. It is also not difficult – using techniques from [2, pp. 75–83] – to prove condition (2), i.e. that the number of orbits of d-tuples in symplectic graphs is uniformly bounded by a function of d only. ◀

The main result of this section is the following theorem.

▶ **Theorem 5.4.** *If $\mathbb{A}$ has oligomorphic approximation, then it has the finite length property over any field of characteristic* 0*.*

Combining Theorems 5.2 and 5.4, we can get the following results, both old and new, concerning the finite length property.

▶ **Corollary 5.5.** *Over any field of characteristic* 0*, the following structures have the finite length property: (i) all smoothly approximated structures; (ii) the equality atoms; (iii) the vector atoms; (iv) the Rado atoms.*

As mentioned in the introduction, the finite length property was already known for the equality atoms – even over arbitrary fields. The results for the vector atoms and the Rado atoms are new. The assumption on characteristic zero is important, at least in the case of the vector atoms where the finite length property is known to fail over finite fields: see [10, Sec. 4.4]. Later on in this paper, we will prove the result for the Rado atoms again using a different method that works for any field.

The rest of this section is devoted to proving Theorem 5.4.

Proof of Theorem 5.4. Fix a structure $\mathbb{A}$ with oligomorphic approximation and a field of characteristic zero. Since the field is fixed, we omit the field subscript and write $\operatorname{Lin} X$ for linear combinations of elements in X that use coefficients from that field. Fix some power $d \in \{1, 2, \dots\}$. Our goal is to show that $\operatorname{Lin} \mathbb{A}^d$ has the finite length property. For technical reasons, we apply the assumption on oligomorphic approximation not to d, but to $2d$, yielding some class $\mathcal{B}$ of finite structures that satisfies Definition 5.1.

We will now proceed in three steps. For convenience, we summarise these steps in Figure 2.

▶ **Lemma 5.6.** *For every $d \in \{1, 2, \dots\}$ we have*

$$\textit{length of } \operatorname{Lin} \mathbb{A}^d \quad \leq \quad \sup_{\mathbb{B} \in \mathcal{B}} \textit{length of } \operatorname{Lin} \mathbb{B}^d.$$

Proof. Consider some chain of equivariant subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n \subseteq \operatorname{Lin} \mathbb{A}^d,$$

where equivariance is with respect to automorphisms of $\mathbb{A}$. For each $i \in \{1, \dots, n\}$, choose some vector that is in V_i but not in V_{i-1}. Let S be the (finite) set of atoms that appear in these chosen vectors, and use (1) to find some $\mathbb{B} \in \mathcal{B}$ that contains all atoms from S. Define

$$W_i = V_i \cap \operatorname{Lin} \mathbb{B}^d.$$

By homogeneity, every automorphism of $\mathbb{B}$ extends to an automorphism of $\mathbb{A}$, and therefore the space $W_i \subseteq \operatorname{Lin} \mathbb{B}^d$ is equivariant with respect to automorphisms of $\mathbb{B}$. The chain of W_i's continues to be strictly growing, since it contains vectors that witness the growth of the original chain. Hence, the new chain forces the length of $\operatorname{Lin} \mathbb{B}^d$ to be at least n. ◀

$$
\begin{array}{cl}
\text{length of } \operatorname{Lin} \mathbb{A}^d & \\
\wedge\!| & \text{(Lemma 5.6)} \\
\sup_{\mathbb{B} \in \mathcal{B}} \text{length of } \operatorname{Lin} \mathbb{B}^d & \\
\wedge\!| & \text{(Lemma 5.8)} \\
\sup_{\mathbb{B} \in \mathcal{B}} \text{dimension of } (\operatorname{Lin} \mathbb{B}^d \underset{\text{lineq}}{\longrightarrow} \operatorname{Lin} \mathbb{B}^d) & \\
\| & \text{(Lemma 5.9)} \\
\sup_{\mathbb{B} \in \mathcal{B}} \text{number of orbits in } \mathbb{B}^{2d} & \\
\wedge & \text{(Definition 5.1)} \\
\infty &
\end{array}
$$

Figure 2 Summary of the proof of Theorem 5.4

Thanks to the above lemma, it remains to show that the length of $\operatorname{Lin} \mathbb{B}^d$ is bounded by some number that depends only on d. In our proof, this bound will be the number of orbits in $\mathbb{B}^{2d}$. What we have gained by moving from $\mathbb{A}$ to $\mathbb{B} \in \mathcal{B}$ is that our vector spaces now have finite (albeit unbounded) linear dimension, and are acted upon by finite (albeit unbounded) groups. This will let us leverage a well-known result from representation theory called Maschke's Theorem (see e.g. [16, Thm. 6.3]). In order to be more precise, recall that we write $V = V_1 \oplus V_2$ if each $v \in V$ is equal to $v_1 + v_2$ for some unique $(v_1, v_2) \in V_1 \times V_2$.

▶ **Fact 5.7** (Maschke's Theorem). *Let V be a finite-dimensional vector space over a field of characteristic zero, equipped with a linear action of a finite group G. Then V can be decomposed as*

$$V = V_1 \oplus \cdots \oplus V_n,$$

where each V_i is an equivariant subspace (with respect to the action of G) that has length 1, i.e. the only equivariant subspaces of V_i are the zero space and the full space V_i.

We will use this theorem to bound the lengths of the vector spaces $\operatorname{Lin} \mathbb{B}^d$ for $\mathbb{B} \in \mathcal{B}$. For two vector spaces V and W equipped with a linear action of the same group G, let us write

$$(V \underset{\text{lineq}}{\longrightarrow} W) = \left\{ \psi : V \to W \;\middle|\; \psi \text{ is linear and } \forall g \in G, \forall v \in V : g(\psi(g^{-1}(v))) = \psi(v) \right\}$$

for the set of all those linear maps from V to W that are equivariant with respect to the action of G. This set is closed under taking linear combinations, and therefore it can be seen as a vector space. In particular, it is meaningful to talk about the dimension of this space.

▶ **Lemma 5.8.** *Let V be a finite-dimensional vector space over a field of characteristic zero, equipped with a linear action of a finite group G. Then*

$$\textit{length of } V \quad \leq \quad \textit{dimension of } (V \underset{\text{lineq}}{\longrightarrow} V).$$

Proof. Apply Maschke's Theorem, yielding a decomposition

$$V = V_1 \oplus \cdots \oplus V_n,$$

where the subspaces $V_1, \ldots, V_n$ are equivariant and of length 1, with respect to the action of the group G. The length is additive with respect to direct sums (see e.g. [16, Prop. 3.17]), i.e.

$$\text{length of } V = \text{length of } V_1 + \cdots + \text{length of } V_n,$$

so the length of V is equal to n.

We will now consider the right-hand side of the inequality in the statement of the lemma. For every $i \in \{1, \ldots, n\}$, there is an equivariant linear map from V to itself that

- is the identity on V_i; and
- maps vectors from the other components to zero.

This gives us at least n equivariant linear maps from V to itself. None of these maps can be spanned by the others, and hence the dimension is no less than n. ◀

Thanks to Lemmas 5.6 and 5.8, the length of the vector space $\mathrm{Lin}\,\mathbb{A}^d$ is bounded by the dimensions of the vector spaces

$$(\mathrm{Lin}\,\mathbb{B}^d \underset{\text{lineq}}{\longrightarrow} \mathrm{Lin}\,\mathbb{B}^d), \qquad (*)$$

where $\mathbb{B}$ ranges over the family $\mathcal{B}$. To complete the proof of the theorem, it still remains to show that these dimensions are bounded by some number that depends only on d. We do so now.

▶ **Lemma 5.9.** *For every $\mathbb{B} \in \mathcal{B}$, the dimension of the vector space in $(*)$ is precisely the number of orbits in $\mathbb{B}^{2d}$.*

Proof. A linear map ψ in the F-vector space $(*)$ can be seen as a square matrix indexed by $\mathbb{B}^d$, i.e. a function

$$\mathbb{B}^d \times \mathbb{B}^d \to \mathsf{F}; \quad \text{defined by } (x, y) \mapsto \lambda_{x,y} \text{ where } \psi(1 \cdot x) = \sum_z \lambda_{x,z} \cdot z.$$

This function must be equivariant with respect to automorphisms of $\mathbb{B}$, i.e. for any automorphism g and $x, y \in \mathbb{B}^d$ we must have $\lambda_{g(x),g(y)} = \lambda_{x,y}$. So to define such a function, we need to choose one element of F for each orbit of $\mathbb{B}^d \times \mathbb{B}^d$. The dimension of the space in $(*)$ is therefore equal to the number of orbits of $\mathbb{B}^{2d}$. ◀

This number of orbits has a finite upper bound that depends only on d, by condition (2) of Definition 5.1. This completes the proof of Theorem 5.4. ◀

The inequalities shown in Figure 2 give us upper bounds on the length of $\mathrm{Lin}\,\mathbb{A}^d$. In the case of the equality atoms, this bound is the $2d$-th Bell number. In the case of the vector atoms from Example 2.5, the bound is the number of linear dependency patterns for $2d$-tuples over a finite field. Such a pattern is described by: (a) indicating a subset of the coordinates which is a basis for the tuple; and (b) indicating the basis decompositions for the remaining coordinates. This can be done in at most $2^{(2d)^2}$ ways. A similar upper bound can be obtained for the Rado graph by an analysis of orbits in symplectic graphs.

6 Function spaces and weighted automata

The original motivation to introduce orbit-finitely spanned vector spaces in [10] was the study of weighted orbit-finite automata. In this section, we recall this motivation and discuss how it relates to our new results. This discussion also involves the issue of function spaces, arguably more fundamental, so we begin with that.

6.1 Function spaces

Given two orbit-finitely spanned vector spaces V and W over the same atom structure, there are two natural ideas for a function space: the space of all linear maps from V to W, and the subspace consisting of equivariant linear maps. As it turns out, the most relevant function space lies between them.

▶ **Definition 6.1** (Finitely supported function space). *For two orbit-finitely spanned vector spaces V and W, we define their* finitely supported function space*, denoted by*

$$(V \underset{\text{linfs}}{\longrightarrow} W),$$

to be the vector space of linear maps f that satisfy the following finite support condition: there is some finite set of atoms $S \subset \mathbb{A}$ such that, for every automorphism π of $\mathbb{A}$,

$$\forall a \in S : \pi(a) = a \quad \Longrightarrow \quad \forall v \in V : \pi(f(v)) = f(\pi(v)).$$

This notion of finite support is the same as the one used in Section 3, here applied to the space of linear maps from V to W. It is also the standard restriction used in the study of nominal sets [42, Thm. 2.19]. As argued in [6, Sec. 8.3] on category-theoretic grounds, the finitely supported function space is the "right" function space. We hence consider:

▶ **Definition 6.2** (Function space property). *An atom structure $\mathbb{A}$ has the* function space property *if orbit-finitely spanned vector spaces over $\mathbb{A}$ are closed under taking finitely supported function spaces.*

The only structure known to have this property is the equality atoms [6, Thm. 8.16]. If we only require function spaces of the form

$$(V \underset{\text{linfs}}{\longrightarrow} \mathsf{F})$$

to be orbit-finitely spanned, we accordingly speak of the *dual space property*. This weaker property is also satisfied by the ordered atoms [10, Cor. 6.8] and, following [43, Thm. 3.7], by all ω-stable oligomorphic structures – a strict subclass [35, p. 440] of the smoothly approximated structures mentioned after Definition 5.1. In contrast, the Rado atoms (which are not smoothly approximated but have the finite length property) fail to have even the dual space property, as we explain below.

▶ **Example 6.3.** Assume that $\mathbb{A}$ is the Rado atoms and the field F is arbitrary – in particular, a field of characteristic zero as considered in Section 5. (A variant of this example for the two-element field was shown in [10, Ex. 6.9], when the finite length property had not been established yet.) We will show that

$$(\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A} \underset{\text{linfs}}{\longrightarrow} \mathsf{F})$$

is not orbit-finitely spanned. This function space is easily seen to be in a linear equivariant bijection with the vector space

$$(\mathbb{A} \underset{\text{fs}}{\longrightarrow} \mathsf{F}) \tag{1}$$

consisting of functions (not linear maps) from $\mathbb{A}$ to F that are finitely supported in the sense of Definition 6.1. We will show that this space is not orbit-finitely spanned.

For a finite set $S \subset \mathbb{A}$ of atoms, define a function $f_S : \mathbb{A} \to \mathsf{F}$ by

$$a \mapsto \begin{cases} 1 & \text{if } a \text{ is a neighbour of all atoms in } S \\ 0 & \text{otherwise.} \end{cases}$$

Define V to be the subspace of (1) that is spanned by the functions f_S, where S ranges over finite sets of atoms. The spanning set $\{f_S\}_S$ is not orbit-finite, and it is not difficult to check that no orbit-finite spanning set can be found for this subspace. But being orbit-finitely spanned is closed under taking subspaces, which follows from Theorem 4.4 and the fact that the Rado atoms have the ascending chain property (by Theorem 5.4 for fields of characteristic zero, and by the upcoming Corollary 7.5 for arbitrary fields). Hence (1) cannot be orbit-finitely spanned. ⌟

6.2 Weighted automata

We now explain how the issues with function spaces have an impact on the theory of weighted automata. There are several ways of defining these; we choose one viewing them as deterministic automata, in which the states are endowed with a vector space structure.

▶ **Definition 6.4.** *An* orbit-finite weighted automaton *is given by:*

- Σ*, an orbit-finite set (the alphabet);*
- Q*, an orbit-finitely spanned vector space (of states);*
- q_0*, an equivariant element of* Q *(the initial state);*
- δ*, an equivariant function of type* $Q \times \Sigma \to Q$ *(the transition function) that becomes a linear map from* Q *to itself after fixing any letter from* Σ*;*
- F*, an equivariant linear map of type* $Q \to \mathsf{F}$ *(the output map).*

Such an automaton computes a function of type $\Sigma^* \to \mathsf{F}$*, defined in the same way as for deterministic automata.*

From the finite length property of the underlying atom structure, we can deduce decidability results about orbit-finite weighted automata. (Though, to be able to effectively represent the maps δ and F, we should assume that Q has an orbit-finite basis.) For example, a nondeterministic algorithm for non-equivalence is derived in [10, Sec. 5]. In light of our present results, such an algorithm exists if we use the Rado graph as the atoms – or any other structure with the finite length property. Also, weighted automata can be minimized [10, Sec. 7]; the bound for chains allows us to decide whether two input words are syntactically congruent.

However, certain other results on weighted automata depend on the function space property. Let us give one such example. Our definition of weighted automata is deterministic in the left-to-right direction. One could also imagine a model where the input word is processed right-to-left. Are these equivalent? If the atoms have the function space property and the ascending chain property, then one can introduce a symmetric model based on monoids (or more precisely, unital associative F-algebras) to show that the left-to-right and right-to-left variants of weighted automata are equivalent: see, essentially, [10, Thm. 7.4]. However, as we show in the following example, this equivalence can fail without the function space property.

▶ **Example 6.5.** Consider the Rado atoms and any field. We shall prove that the left-to-right and right-to-left variants of Definition 6.4 are not equivalent. The counterexample is the characteristic function of the language "the first letter is adjacent to all later ones", i.e. the function f defined by

$$a_1 \cdots a_n \in \mathbb{A}^* \quad \mapsto \quad \begin{cases} 1 & \text{if } a_1 \text{ is adjacent to all of } a_2, \ldots, a_n \\ 0 & \text{otherwise (e.g. if } n = 0). \end{cases}$$

We will show that this function is computed by a left-to-right orbit-finite weighted automaton, but not by a right-to-left one.

To prove this, for an input word $w \in \mathbb{A}^*$, we define two functions of type $\mathbb{A}^* \to \mathsf{F}$ as follows:

$$\underbrace{u \mapsto f(wu)}_{\text{left derivative}} \qquad \text{and} \qquad \underbrace{u \mapsto f(uw)}_{\text{right derivative}}.$$

Using the usual Myhill–Nerode construction, one can show that a function is computed by a left-to-right orbit-finite weighted automaton if and only if (the space spanned by) the set of its left derivatives is orbit-finitely spanned [11, Thm. 6.19], and similarly for its right derivatives.

The set of left derivatives is already orbit-finite: there is one left derivative for each $a \in \mathbb{A}$, plus f itself and one extra derivative for the always-zero function. Hence the set of left derivatives is certainly orbit-finitely spanned. On the other hand, the set of right derivatives restricted to $\mathbb{A}$ is precisely the spanning set of the vector space V from Example 6.3; that space is not orbit-finitely spanned. ⌟

7 Finite length from free amalgamation with a generic order

As we saw in Non-example 5.3, the ordered atoms do not have oligomorphic approximation. Nonetheless they do have the finite length property over any field [10, Thm. 4.8], not just over those of characteristic zero. In this section, we shall generalise that result to a wide class of structures. In particular, we will deduce the finite length property of the Rado atoms and its variants.

The structures that we consider here are Fraïssé limits satisfying certain conditions, which are defined by the underlying amalgamation classes. We shall now state our assumptions and results, with proofs in Section 8.

Graph vocabulary Consider a finite relational vocabulary σ_0 consisting of unary and binary relations only. This allows us to talk about, in essence, directed graphs with coloured vertices and edges.

Free amalgamation class Let $\mathcal{C}_0$ be a *free* amalgamation class of finite σ_0-structures. The precise definition may be found in [41, Sec. 2.1], but informally it means that when we perform amalgamation, we do not need to glue together any new elements or introduce new relations. Here is a useful characterisation [45, Lem. 4.5] of when an amalgamation class $\mathcal{C}_0$ is free. Knowing that $\mathcal{C}_0$ is closed under substructures and isomorphisms, we see that it consists of all the finite σ_0-structures which do not embed any structure from $\mathcal{F}$, where $\mathcal{F}$ consists of all the minimal (with respect to substructures) finite σ_0-structures that do not belong to $\mathcal{C}_0$; we write

$$\mathcal{C}_0 = \mathrm{Forb}(\mathcal{F}).$$

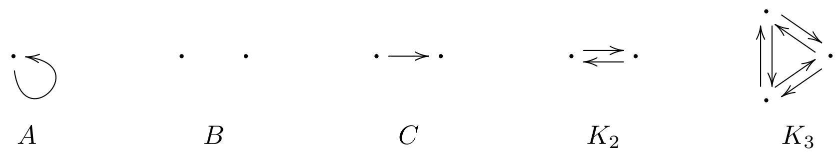
That $\mathcal{C}_0$ is a free amalgamation class means that in each $F \in \mathcal{F}$, every two elements x, y are either equal or satisfy at least one of $R(x, y)$ and $R(y, x)$ for some binary relation

$R \in \sigma_0$; in that case we will say that x and y are *related*. Conversely, and conveniently, given a family $\mathcal{F}$ of finite σ_0-structures where every two elements are related, the class $\text{Forb}(\mathcal{F})$ of finite σ_0-structures is a free amalgamation class.

▶ **Example 7.1.** Let σ_0 be empty. Then $\mathcal{C}_0 = \text{Forb}(\{\})$ is a free amalgamation class consisting of all finite pure sets. ⌟

▶ **Example 7.2.** Let σ_0 consist of a single binary relation E. A (i) *graph*, (ii) *digraph*, (iii) *tournament* is a σ_0-structure where E is irreflexive and (i) symmetric, (ii) antisymmetric, (iii) antisymmetric and total. For instance, the ordered atoms can be viewed as a tournament once we interpret E as the order relation.

For more examples, consider σ_0-structures:



Then:

- $\text{Forb}(\{A, C\})$ is a free amalgamation class and consists of all finite graphs;
- $\text{Forb}(\{A, C, K_3\})$ is a free amalgamation class and consists of all finite triangle-free graphs;
- $\text{Forb}(\{A, K_2\})$ is a free amalgamation class and consists of all finite digraphs;
- $\text{Forb}(\{A, B, K_2\})$ consists of all finite tournaments. It is an amalgamation class, but not a free one: in the minimal forbidden structure B, the two elements are not related. ⌟

Irreflexivity For technical reasons, we restrict attention to classes $\mathcal{C}_0$ where all structures are *irreflexive*, i.e. such that if $R(x, y)$ then $x \neq y$ for each binary relation R. This does not lose generality, since we can replace every binary relation R in the vocabulary with its irreflexive fragment $R_{\neq}(x, y)$ and a unary relation $R_{=}(x)$; see [29, Sec. 2.4] or [45, p. 121]. All examples considered here are already irreflexive.

Generically ordered expansion Let σ consist of σ_0 together with a new binary relation $<$. Consider the class $\mathcal{C}$ of σ-structures obtained from $\mathcal{C}_0$ by interpreting $<$ in any σ_0-structure there as any total order. Note that $\mathcal{C}$ is an amalgamation class. Indeed, let X, Y_1, Y_2 be σ-structures in $\mathcal{C}$ with $X \subseteq Y_1 \cap Y_2$. Then we can amalgamate Y_1, Y_2 over X as σ_0-structures and as $\{<\}$-structures, both using the disjoint union of Y_1, Y_2 over X as the underlying set. Superposing these relations gives a σ-structure in $\mathcal{C}$, which is the desired amalgamation. (Because of the total order, this amalgamation in $\mathcal{C}$ is not free except in trivial cases.) Denote the Fraïssé limit of $\mathcal{C}$ by $\mathbb{A}$.

Our main result of this section is that this ordered structure $\mathbb{A}$ has the finite length property over any field. For clarity, we summarise our assumptions:

▶ **Theorem 7.3.** *Consider:*

- σ_0 *a finite, at-most-binary relational vocabulary;*
- $\mathcal{C}_0$ *a free amalgamation class of finite, irreflexive* σ_0*-structures;*
- $\mathcal{C}$ *all total orderings of all structures in* $\mathcal{C}_0$*, over the vocabulary* $\sigma_0 \uplus \{<\}$*;*
- $\mathbb{A}$ *the Fraïssé limit of* $\mathcal{C}$ *– a homogeneous and oligomorphic* $(\sigma_0 \uplus \{<\})$*-structure.*

Then the totally ordered structure $\mathbb{A}$*, even with finitely many constants named (i.e. added as unary relations to the vocabulary), has the finite length property over any field.*

▶ Remark 7.4. We do not know how to drop the arity restriction from our proofs, so it is interesting that Evans recently developed a Ramsey-theoretic approach in [18] which could be used to show that $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^2$, for free amalgamation classes over vocabularies of arbitrarily high arity, has finite length. We do not know how to combine these results.

We also do not know whether the finite length property of a general structure $\mathbb{A}$ implies the finite length property of the same structure with finitely many constants fixed. There is a quick, positive answer for the equality and the ordered atoms [10, Thm. 4.10], but that argument relies on a property that these atoms have [3, Lem. 2.21] and the Rado atoms lack [48]. ⌟

Before we prove Theorem 7.3 in Section 8, let us state an easy consequence of it and give some examples. Note that the Fraïssé limit $\mathbb{A}_0$ of $\mathcal{C}_0$ is a first-order reduct of $\mathbb{A}$. To see this, notice that $\mathbb{A}$, when viewed as a σ_0-structure, embeds the same finite σ_0-structures as $\mathbb{A}_0$ – namely, $\mathcal{C}_0$. Moreover, it follows from a back-and-forth argument [30, Lem. 7.1.4] that the σ_0-structure $\mathbb{A}$ is also homogeneous and therefore isomorphic to $\mathbb{A}_0$. So we may assume that $\mathbb{A}$ and $\mathbb{A}_0$ have the same underlying set; we then have $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(\mathbb{A}_0)$, where $\mathrm{Aut}(X)$ denotes the group of automorphisms of X. We thus call $\mathbb{A}$ the *generically ordered expansion* of $\mathbb{A}_0$, or simply the *ordered* $\mathbb{A}_0$.

▶ **Corollary 7.5.** *The reduct $\mathbb{A}_0$ of $\mathbb{A}$, even with finitely many constants fixed, has the finite length property over any field.*

Proof. A chain of subspaces in $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}_0^d = \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ each supported by $S \subset \mathbb{A}_0$ is also a chain of subspaces supported by $S \subset \mathbb{A}$; the latter has a bounded length by the theorem above. ◀

▶ **Example 7.6.** Continuing from Example 7.1, the ordered atoms (Example 2.4) are the generically ordered expansion of the equality atoms (Example 2.3). ⌟

▶ **Example 7.7.** Continuing from the first two items of Example 7.2, we obtain generically ordered expansions of the Rado graph and of *Henson's triangle-free graph.* The ordered Rado graph was studied in [4] along with its first-order reducts (e.g. the Fraïssé limit of all finite tournaments). The ordered triangle-free graph will be studied in an extended example in Section 8.4. ⌟

▶ **Corollary 7.8.** *The following structures have the finite length property over any field:*
- **(i)** *the ordered atoms and the equality atoms;*
- **(ii)** *all three [38] countable homogeneous tournaments;*
- **(iii)** *all of the countably many [39] countable homogeneous graphs, including the Rado graph and Henson's triangle-free graph;*
- **(iv)** *uncountably many [28, Thm. 2.4] countable homogeneous digraphs obtained by forbidding tournaments.*

8 How the cogs turn: proof of Theorem 7.3

To prove Theorem 7.3 we will proceed in several steps, with the general idea similar to [10, Sec. 4.1], but with significant new complications arising from the presence of non-trivial relations in $\mathbb{A}_0$. Throughout we fix $\mathbb{A}$ from the statement of Theorem 7.3.

8.1 Orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\dots,d\}}$. More generally, it will be convenient to consider $\mathbb{A}^I$ for a finite totally ordered indexing set I (say contained in $\mathbb{Q}$, so that we may take unions later). Fix a finite support $S \subset \mathbb{A}$. We shall say that a tuple $a \in \mathbb{A}^I$ is *(S-)ordered* if $a_i \notin S$ for all i, and $a_i < a_j$ whenever $i < j$. Then the orbit $\mathcal{O} = \mathrm{Aut}(\mathbb{A}/S) \cdot a$ only contains S-ordered tuples, and we shall call the orbit S-ordered as well.[6] If a is not S-ordered, by removing the entries that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant bijection from $\mathcal{O}$ to an S-ordered orbit.

To study the lengths of orbit-finitely spanned spaces, we may focus on a single ordered orbit at a time:

▷ Claim 8.1. The following are equivalent for every finite $S \subset \mathbb{A}$:

(a) For each $d \in \{1, 2, \dots\}$, there is some number l_d such that chains of $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces in $\mathrm{Lin}_\mathsf{F}\, \mathbb{A}^d$ have length at most l_d;

(b) For each $d \in \{1, 2, \dots\}$ and every S-ordered orbit $\mathcal{O} \subseteq \mathbb{A}^d$, $\mathrm{Lin}_\mathsf{F}\, \mathcal{O}$ has finite length with respect to $\mathrm{Aut}(\mathbb{A}/S)$.

Proof. Recall that $\mathbb{A}$ is oligomorphic: given any S and d, we know that $\mathbb{A}^d$ is in an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant bijection with a finite disjoint union $\biguplus_i \mathcal{O}_i$ of S-ordered orbits (possibly in lower dimensions). Hence the length of $\mathrm{Lin}_\mathsf{F}\, \mathbb{A}^d$, under the action of $\mathrm{Aut}(\mathbb{A}/S)$, equals

$$\text{length of } \mathrm{Lin}_\mathsf{F}(\biguplus_i \mathcal{O}_i) = \text{length of } \bigoplus_i \mathrm{Lin}_\mathsf{F}\, \mathcal{O}_i = \sum_i \text{length of } \mathrm{Lin}_\mathsf{F}\, \mathcal{O}_i,$$

which is finite if and only if each summand is finite. ◁

So fix S and an S-ordered orbit $\mathcal{O} = \mathrm{Aut}(\mathbb{A}/S) \cdot o \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J$ we mean the restriction of $o : I \to \mathbb{A}$ to $J \subseteq I$; particularly, we will often write $o|^{-i}$ instead of $o|^{I \setminus \{i\}}$. The image $\mathcal{O}|^J$ of $\mathcal{O}$ under this projection agrees with $\mathrm{Aut}(\mathbb{A}/S) \cdot o|^J$ and is still ordered. The function $(-)|^J$ lifts to a linear $\mathrm{Aut}(\mathbb{A}/S)$-equivariant map

$$(-)|^J : \mathrm{Lin}_\mathsf{F}\, \mathcal{O} \to \mathrm{Lin}_\mathsf{F}\, \mathcal{O}|^J, \quad \text{defined by} \quad v \mapsto \sum_{a \in \mathcal{O}|^J} \sum_{b \in \mathcal{O}, b|^J = a} v(b) \cdot a.$$

Many cancellations can occur under $(-)|^J$. Following the terminology used in [10, Eq. (4)], by the *balanced vectors* of $\mathrm{Lin}_\mathsf{F}\, \mathcal{O}$ we mean the $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace below:

$$\mathrm{BLin}_\mathsf{F}\, \mathcal{O} = \bigcap_{i \in I} \ker (-)|^{-i}.$$

▷ Claim 8.2. The following are equivalent for every finite $S \subset \mathbb{A}$:

(a) $\mathrm{Lin}_\mathsf{F}\, \mathcal{O}$ has finite length with respect to $\mathrm{Aut}(\mathbb{A}/S)$ for every S-ordered orbit $\mathcal{O}$;

(b) $\mathrm{BLin}_\mathsf{F}\, \mathcal{O}$ has finite length with respect to $\mathrm{Aut}(\mathbb{A}/S)$ for every S-ordered orbit $\mathcal{O}$.

Proof. That (a) implies (b) is clear, as $\mathrm{BLin}_\mathsf{F}\, \mathcal{O} \subseteq \mathrm{Lin}_\mathsf{F}\, \mathcal{O}$.

To prove the other implication, assume (b) and let $\mathcal{O} \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then $\mathcal{O}$ must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_\mathsf{F}\, \mathcal{O}$ has no non-trivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\mathrm{Lin}_\mathsf{F}\, \mathcal{O} \to \bigoplus_{i \in I} \mathrm{Lin}_\mathsf{F}\, \mathcal{O}|^{-i}, \quad \text{defined by} \quad v \mapsto (v|^{-i})_{i \in I},$$

---

[6] Here and in the following, $\mathrm{Aut}(\mathbb{A}/S)$ is the (sub)group of those automorphisms of $\mathbb{A}$ that fix every element of S. So "$\mathrm{Aut}(\mathbb{A}/S)$-equivariant" is synonymous with "supported by S".

whose kernel is precisely $\mathrm{BLin}_{\mathsf{F}}\,\mathcal{O}$. We thus have

$$\text{length of } \mathrm{Lin}_{\mathsf{F}}\,\mathcal{O} - \text{length of } \mathrm{BLin}_{\mathsf{F}}\,\mathcal{O} \le \sum_{i\in I} \text{length of } \mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}|^{-i},$$

which shows that the length of $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$ is finite from the assumptions. ◁

As we will see shortly, there exist plenty of balanced vectors.

8.2 Cogs

From now on we will use a lightweight notation for combining tuples of atoms: for disjoint indexing sets I and J (contained in $\mathbb{Q}$), if $a \in \mathbb{A}^I$ and $b \in \mathbb{A}^J$ are both ordered, then $ab \in \mathbb{A}^{I\cup J}$ will denote their obvious combination. We only use this notation if this ab is ordered. As an obvious example, for any S-ordered $a \in \mathbb{A}^I$ and $J \subseteq I$, we have $a|^{I\setminus J}a|^J = a$.

▶ **Definition 8.3.** *Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-ordered orbit. An* $\mathcal{O}$-duo $a \parallel b$ *consists of tuples $a, b \in \mathcal{O}$ such that:*

(1) *$a_i < b_i$ for all $i \in I$;*
(2) *$b_i < a_j$ for all $i < j \in I$;*
(3) *for any binary R in σ_0 (which we assumed to be irreflexive) and $i, j \in I$:*

$$R(a_i, b_j) \iff R(a_i, a_j) \overset{a,b\in\mathcal{O}}{\iff} R(b_i, b_j) \iff R(b_i, a_j).$$

▶ Remark 8.4. Conditions (1) and (2) specify a total order on the $2|I|$ atoms in a duo. Moreover, thanks to irreflexivity, each a_i is unrelated to its counterpart b_i. Further, given any $J \subseteq I$, the combined tuple $a|^{I\setminus J}b|^J$ satisfies the same relations as a (and b), so it lies in $\mathcal{O}$. In particular, taking $J = \{i\}$, there is an automorphism π_i that sends a_i to b_i and fixes all the other elements of a, b, and S. ⌟

For the special case of the ordered atoms (Examples 2.4 and 7.6) the following construction was studied in [10, p. 11], and it already appeared as early as in [26, p. 125] under the name of "polytabloids".

▶ **Definition 8.5.** *Given an $\mathcal{O}$-duo $a \parallel b$, the corresponding* $\mathcal{O}$-cog *is the vector*

$$a \between b = \sum_{J\subseteq I} (-1)^{|J|}(a|^{I\setminus J}b|^J)$$

in $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$. The linear span of all $\mathcal{O}$-cogs is denoted by $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$.

Note that, for a fixed S-ordered orbit $\mathcal{O}$, all $\mathcal{O}$-duos (hence all $\mathcal{O}$-cogs) are in the same $\mathrm{Aut}(\mathbb{A}/S)$-equivariant orbit. As a result, $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$ is an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$ and it is generated by any single cog.

▷ Claim 8.6. $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O} \subseteq \mathrm{BLin}_{\mathsf{F}}\,\mathcal{O}$.

Proof. Let $\mathcal{O} \subseteq \mathbb{A}^I$, let $a \parallel b$ be an $\mathcal{O}$-duo, and take any $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where $J \subseteq I \setminus \{i\}$. The two tuples $a|^{I\setminus J}b|^J$ and $a|^{I\setminus(J\cup\{i\})}b|^{J\cup\{i\}}$ are present in $a \between b$ with the opposite coefficients, and they differ only on the i-th entry. Therefore they cancel out under $(-)|^{-i}$; hence $(a \between b)|^{-i} = 0$. ◁

It would be remiss not to address the fact that so far, we have not shown that $\mathcal{O}$-duos and $\mathcal{O}$-cogs even exist in general. Let us rectify this by showing that they can be found in every non-zero $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$.

8.3 Finding cogs

We begin with a technical lemma, which combines the free amalgamation in $\mathcal{C}_0$ and the generic order of $\mathbb{A}$.

▶ **Lemma 8.7.** *Let $X, Y, \{z\} \subset \mathbb{A}$ be disjoint and finite. Then there is an automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*
(1) *τ fixes every $x \in X$;*
(2) *$\tau(z)$ is unrelated to all $y \in Y$ and to z;*
(3) *$\tau(z) > z$.*

Proof. Form the free amalgam in $\mathcal{C}_0$ of $X \cup Y \cup \{z\}$ and $X \cup \{z\}$ over the common part X. The resulting structure can be presented as $X \cup Y \cup \{z, z'\}$ for some new element z'. To make this a σ-structure, inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$, as well as $z' < a$ whenever $z < a$ for $a \in X \cup Y$. By homogeneity, $X \cup Y \cup \{z, z'\}$ embeds into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$x \in X \mapsto x, \quad z \mapsto f(z')$$

to some automorphism τ that satisfies (1), (2), and (3). ◀

▶ **Lemma 8.8.** *Suppose $a \parallel b$ is an $\mathcal{O}$-duo, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-ordered. Given $z \in S$,*
- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $az \in \mathbb{A}^{I \cup \{j\}}$ – thus $\mathcal{O}' = \mathrm{Aut}(\mathbb{A}/S') \cdot az \subseteq \mathbb{A}^{I \cup \{j\}}$ – is S'-ordered;*
- *let $X \subset \mathbb{A}$ be any finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z.*

Denote $z' = \tau(z)$, where $\tau \in \mathrm{Aut}(\mathbb{A}/X)$ is afforded by Lemma 8.7 (with an arbitrary Y). Then $az \parallel bz'$ is an $\mathcal{O}'$-duo.

Proof. First, notice that $bz' \in \mathcal{O}'$ and that we have the required order relations with z and z'. The remaining condition (3) of Definition 8.3, for any binary R in σ_0, splits into the following cases (and their symmetric counterparts):
- $R(a_i, b_{i'}) \iff R(a_i, a_{i'})$ since $a \parallel b$ is an $\mathcal{O}$-duo;
- $R(a_i, z') \iff R(a_i, z)$ since τ is an automorphism that fixes all a_i;
- $R(a_i, z) \iff R(b_i, z)$ since $a, b \in \mathcal{O}$ and $z \in S$;
- $R(z, z')$ and $R(z, z)$ are both false: z' is unrelated to z by Lemma 8.7, and R is irreflexive. ◀

Starting from an empty duo, we may apply Lemma 8.8 inductively and obtain:

▶ **Lemma 8.9.** *Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-ordered orbit. Then any $a \in \mathcal{O}$ can be extended to an $\mathcal{O}$-duo $a \parallel b$.*

As some $a \in \mathcal{O}$ always exists, it follows that Claim 8.6 was not vacuous: we now know $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$, and hence $\mathrm{BLin}_{\mathsf{F}}\,\mathcal{O}$, is non-zero – but barely so. Indeed, as the result below shows, $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$ admits no non-trivial $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces.

▶ **Theorem 8.10.** *Any non-zero $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace $V \subseteq \mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$ contains $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$.*

Proof. Pick any $v \in V$ and $a \in \mathcal{O}$ with $v(a) \neq 0$; it is enough to show that V contains $a \between b$ for some $b \in \mathcal{O}$. Define:

$$S^* = S \cup \{c_i \mid v(c) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $\mathcal{O}^* = \mathrm{Aut}(\mathbb{A}/S^*) \cdot a \subseteq \mathcal{O}$ – then $\mathcal{O}^*$ is S^*-ordered. By Lemma 8.9, we can find $b \in \mathcal{O}^*$ such that $a \parallel b$ is an $\mathcal{O}^*$-duo and *a fortiori* an $\mathcal{O}$-duo. Take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from Remark 8.4, where $i_1, \ldots, i_d$ enumerate I. Now define $v^{(0)} = v$ and

$$v^{(k)} = v^{(k-1)} - \pi_{i_k} v^{(k-1)}.$$

Then each $v^{(k)}$ is in V. By induction on k, we have:

$$v^{(k)} = \sum_{c \in C_k} \sum_{J \subseteq \{i_1, \ldots, i_k\}} (-1)^{|J|} v(c) \left(\prod_{j \in J} \pi_j c \right),$$

where

$$C_k = \{c \mid \ v(c) \neq 0 \text{ and } \{c_{i_1}, \ldots, c_{i_k}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_k}\} \ \}.$$

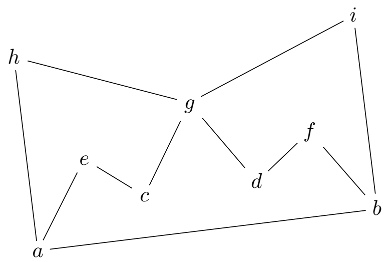
But $\{c_{i_1}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_d}\}$ means that $c = a$, so $C_d = \{a\}$ and $\frac{1}{v(a)} v^{(d)} = a \between b$. ◀

▶ **Corollary 8.11.** $\mathrm{Cog}_{\mathsf{F}}\, \mathcal{O}$ *has length* 1*.*

In light of Claims 8.1, 8.2 and 8.6, for the finite length property it is enough to prove that $\mathrm{BLin}_{\mathsf{F}}\, \mathcal{O} \subseteq \mathrm{Cog}_{\mathsf{F}}\, \mathcal{O}$. In words, we need to show that every balanced vector in $\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}$ is a linear combination of $\mathcal{O}$-cogs. Before stating this (as Theorem 8.12), let us illustrate the key ideas of a proof on an example.

8.4 Spanning by cogs: an extended example

Let $\mathbb{A}_0$ be the universal triangle-free (undirected) graph, introduced by Henson [27], and let $\mathbb{A}$ be its generically totally ordered version. Consider nine atoms $\{a, \ldots, i\}$, ordered by $<$ alphabetically, with the edge relation as shown here:

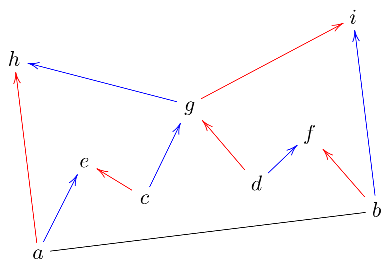


This graph is drawn so that the total order of the atoms corresponds to the vertical order.

Putting $S = \emptyset$ and $|I| = 2$, let $\mathcal{O}$ be the ordered orbit of pairs of atoms which are adjacent. Consider the following vector:

$$v = ah - ae + ce - cg + dg - df + bf - bi + gi - gh \in \mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}.$$

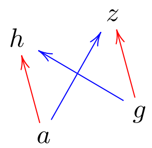
This can be graphically presented as the following graph:



where edges with coefficient $+1$ are marked as red, and with -1 as blue. The arrows on the chosen edges remind us that the pairs in $\mathcal{O}$ are ordered, but this is mere decoration: the definition of $\mathcal{O}$ means that all arrows must point upwards.

Note that v is balanced. Graphically, this means that every atom has as many outgoing red edges as outgoing blue edges, and as many incoming red edges as incoming blue edges.

It is easy to draw $\mathcal{O}$-cogs in this way. Given some additional atom $z > h$ which is adjacent to a and g but not to h, the $\mathcal{O}$-cog $ah \between gz$ can be drawn as:



We would like to present v as a sum of such $\mathcal{O}$-cogs. Some additional atoms must be used for that, as no four atoms among the original nine form an $\mathcal{O}$-duo. It would be very convenient to introduce a single new atom z to form all the $\mathcal{O}$-duos that we will use. (Such a new atom is not necessary if $\mathbb{A}$ is the ordered atoms, as then we can simply take z to be the largest atom present in the vector; see also the proof of [10, Cl. 4.7].) We can naïvely require z to be:

- larger than every atom in v,
- adjacent to every atom that is a source of a directed edge in v (equivalently: that occurs as the first component of a pair in v), and
- not adjacent to any atom that is a target of a directed edge in v (equivalently: that occurs as the second component of a pair in v).

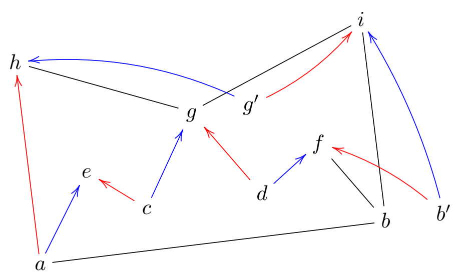
However, such a z does not exist in the ordered triangle-free graph $\mathbb{A}$. There are two problems:

- The atom g occurs both as the first and as the second component in pairs present in v. Our specification of whether z is adjacent to g is therefore inconsistent.
- Atoms a and b both occur as first components in v, and they are adjacent in $\mathbb{A}$. As a result, an atom z as prescribed would create a triangle abz in $\mathbb{A}$, which is forbidden.

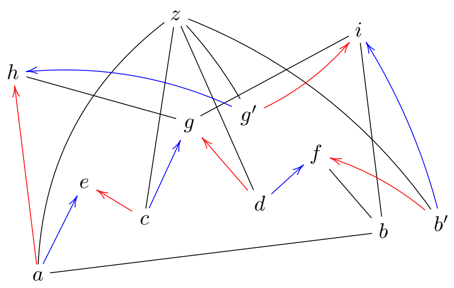
We resolve such *conflicts* by considering auxiliary atoms $g' > g$ and $b' > b$, with just enough edges to make $gh \parallel g'i$ and $bf \parallel b'i$ valid $\mathcal{O}$-duos. Specifically, we postulate edges $E(g', h), E(g', i), E(b', f), E(b', i)$ and no more. Such atoms exist by Lemma 8.7. We then define:

$$v' = v + (gh \between g'i) - (bf \between b'i)$$

which can be drawn as:



Now an atom z as postulated above does not create any triangles:



and it is easy to calculate:

$$v' = (ah \parr g'z) - (ae \parr cz) - (cg \parr dz) + (b'f \parr dz) - (b'i \parr g'z)$$

which presents $v = v' - (gh \parr g'i) + (bf \parr b'i)$ as a linear combination of $\mathcal{O}$-cogs.

8.5 All those equivariant subspaces

Inspired by the above example, we shall now assert in somewhat greater generality:

▶ **Theorem 8.12.** *Every $v \in \mathrm{BLin}_{\mathsf{E}}\,\mathcal{O}$ can be written as*

$$v = \sum_{a \| b} \lambda_{a\|b} \cdot a \parr b$$

with $\lambda_{a\|b} \in \mathsf{E}$, where E is any subspace of some F^n.

▶ Remark 8.13. We must tread carefully here, as E is not a field. For example E can be $\{(\kappa, -\kappa) \mid \kappa \in \mathsf{F}\}$ in F^2. Then $\mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$ is an equivariant subspace of $\mathrm{Lin}_{\mathsf{F}^2}\,\mathcal{O}$; the latter is more traditionally viewed as $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O} \oplus \mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$.

Given $\lambda \in \mathsf{E}$ and $a \in \mathcal{O}$, we need to understand $\lambda \cdot a$ as a formal expression for an element of $\mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$, rather than the result of a scalar multiplication. Accordingly, we need to redefine $\mathrm{Cog}_{\mathsf{E}}\,\mathcal{O}$ to be spanned by formal expressions $\lambda \cdot (a \parr b)$ for $\lambda \in \mathsf{E}$. We still have $\mathrm{Lin}_{\mathsf{F}^n}\,\mathcal{O} \supseteq \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O} \supseteq \mathrm{BLin}_{\mathsf{E}}\,\mathcal{O} \supseteq \mathrm{Cog}_{\mathsf{E}}\,\mathcal{O}$ as equivariant spaces over F. ⌟

The proof of Theorem 8.12 follows the lines of the example in Section 8.4, by carefully removing atom conflicts from a vector before inventing a fresh atom that creates enough duos and cogs. Technical details are deferred to an extended version of this paper.

Putting $\mathsf{E} = \mathsf{F}$ in Theorem 8.12, we have – as discussed at the end of Section 8.3 – established Theorem 7.3. But by allowing E to be finite-dimensional vector spaces, we can now describe all equivariant subspaces of an orbit-finite-dimensional vector space: we need only look at local sums of coefficients.

▶ **Theorem 8.14** (Compare with [26, Cor. 3.17], [31, Thm. 15], [32, Thm. 3.4], [23, Sec. 6], [18, Thm. 1.4])**.** *Let $\mathbb{A}$ be as in Theorem 7.3 and fix $d \in \{1, 2, \dots\}$. Then there exists a finite family of equivariant linear maps of the form*

$$\restriction_i \colon \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d \to \mathrm{Lin}_{\mathsf{F}^{n_i}}\,\mathcal{O}_i,$$

where $\mathcal{O}_i$ is an equivariant ordered orbit of $\mathbb{A}^{d_i}$ with $d_i \le d$, such that every equivariant subspace $W \subseteq \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$ is equal to

$$\{v \in \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d \mid \ \forall i, \forall a \in \mathcal{O}_i : v\restriction_i(a) \in \mathsf{E}_i\ \}$$

with the finite-dimensional subspaces $\mathsf{E}_i \subseteq \mathsf{F}^{n_i}$ *given by*

$$\mathsf{E}_i = \{w{\restriction_i}(b) \mid \ w \in W, b \in \mathcal{O}_i \ \}.$$

▶ **Example 8.15.** Let $\mathbb{A}$ be the ordered atoms. For $d = 1$ there are two maps, ${\restriction_1}\colon \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A} \to \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}$ and ${\restriction_2}\colon \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A} \to \mathsf{F}$, given by

$$v{\restriction_1}(a) = v(a), \qquad v{\restriction_2}() = \sum_x v(x).$$

For an equivariant vector space $V \subseteq \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}$, the subspace $V{\restriction_1}(\mathbb{A}) \subseteq \mathsf{F}$ is either $\{0\}$ or F.

- In the first case, V must be the zero space.
- In the second case, we similarly distinguish two cases for $V{\restriction_2}() \subseteq \mathsf{F}$:
  - if it is the entire F, then V must be the full space $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}$ by Theorem 8.14;
  - if it is $\{0\}$, then V must be the zero-sum space (which exists and is spanned by $\{a - b \mid a, b \in \mathbb{A}\}$) again using Theorem 8.14.

So $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}$ has the same structure as the space over the equality atoms in Example 4.2.

For $d = 2$, there are three maps, one of which is of the form ${\restriction_2}\colon \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^2 \to \mathrm{Lin}_{\mathsf{F}^5}\,\mathbb{A}$. By considering subspaces of F^5, when the field is infinite we can find infinitely many equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^2$. Note that the length of $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^2$ is still finite (and equal to $2^1 + 2^2 + 2^2$, as we will see below). ⌟

Going through the finite-dimensional subspaces of the F^{n_i}'s gives a complete list of the equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$, by Theorem 8.14, but note that it is far from being irredundant. In particular, to bound the length of $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$ from below, we still need to exhibit a long chain of equivariant subspaces. It is fortunately straightforward to adapt [10, Cor. 4.12] and establish an exact length:

▶ **Corollary 8.16.** *The length of* $\mathrm{Lin}_{\mathsf{F}}\,\mathcal{O}$*, where* $\mathcal{O} \subseteq \mathbb{A}^d$ *is an ordered orbit, is precisely* 2^d*.*

As described in [24, Sec. 9], results like Theorem 8.14 allow us to decide the solvability of orbit-finite systems of linear equations. We briefly repeat the argument. Checking whether the system $\mathbf{Ax} = \mathbf{b}$ admits a solution amounts to checking whether $\mathbf{b}$ is spanned by the columns of $\mathbf{A}$. In the orbit-finite setting, we assume that the rows and columns are indexed by $\mathbb{A}^d$ and $\mathbb{A}^{d'}$, that each column has finitely many non-zero entries, and that the definition $j \mapsto \mathbf{A}_{-,j}$ of columns is equivariant. That is, we ask whether $\mathbf{b} \in \mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$ lies in the span of the $\mathbf{A}_{-,j}$'s. By Theorem 8.14, it suffices to check whether $\mathbf{b}{\restriction_i}(\mathcal{O}_i) \subseteq \mathbf{A}_{-,j_1}{\restriction_i}(\mathcal{O}_i) + \cdots + \mathbf{A}_{-,j_r}{\restriction_i}(\mathcal{O}_i)$ for all i, in the finite-dimensional spaces F^{n_i}, having chosen orbit representatives $j_1, \ldots, j_r \in \mathbb{A}^{d'}$.

9 Conclusion

With Theorems 5.4 and 7.3 and their Corollaries 5.5 and 7.8, we have extended the finite length property far beyond equality and ordered atoms, the two examples considered in [10].

We finish the paper by reiterating some general questions that nonetheless remain open:

- [14, Q. 2] Does every oligomorphic structure have the ascending chain property?
- [18, Q. 1.4] Does every structure that is homogeneous over a finite relational vocabulary have the finite length property?

We also ask a new one, prompted by the limited but growing stock of examples:

- Does every oligomorphic structure have the finite length property over fields of characteristic 0?

We do not even know the answer to these questions for some concrete and well-studied Fraïssé limits, notably the universal partial order or the countable atomless Boolean algebra.

References

1 Michael Albert and Ambar Chowdhury. The rationals have an AZ-enumeration. *Journal of the London Mathematical Society*, 59(2):385–395, 1999. doi:10.1112/S0024610799007140.

2 Michael Aschbacher. *Finite group theory*. Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, 2nd edition, 2000. doi:10.1017/CBO9781139175319.

3 Manuel Bodirsky, Bertalan Bodor, and Paolo Marimon. Taking model-complete cores, 2025. arXiv:2512.21278.

4 Manuel Bodirsky, Michael Pinsker, and András Pongrácz. The 42 reducts of the random ordered graph. *Proceedings of the London Mathematical Society*, 111(3):591–632, July 2015. doi:10.1112/plms/pdv037.

5 Mikołaj Bojańczyk. Nominal Monoids. *Theory of Computing Systems*, 53(2):194–222, 2013. doi:10.1007/s00224-013-9464-1.

6 Mikołaj Bojańczyk. Slightly infinite sets. Lecture notes available at https://mimuw.edu.pl/~bojan/paper/atom-book (accessed: July 3, 2025), 2025.

7 Mikołaj Bojańczyk, Bartek Klin, Sławomir Lasota, and Szymon Toruńczyk. Turing Machines with Atoms. In *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013*, pages 183–192. IEEE Computer Society, 2013. doi:10.1109/LICS.2013.24.

8 Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata theory in nominal sets. *Logical Methods in Computer Science*, 10(3), 2014. doi:10.2168/lmcs-10(3:4)2014.

9 Mikołaj Bojańczyk, Laurent Braud, Bartek Klin, and Sławomir Lasota. Towards nominal computation. In *Proceedings of the 39th annual ACM SIGPLAN-SIGACT symposium on Principles of programming languages*, pages 401–412, 2012. doi:10.1145/2103656.2103704.

10 Mikołaj Bojańczyk, Joanna Fijalkow, Bartek Klin, and Joshua Moerman. Orbit-finite-dimensional vector spaces and weighted register automata. *TheoretiCS*, Volume 3, May 2024. doi:10.46298/theoretics.24.13.

11 Mikołaj Bojańczyk, Aliaume Lopez, Rafał Stefański, and Omid Yaghoubi. Regularity as seen by Alice and Bob. To appear, 2026.

12 Mikołaj Bojańczyk, Lê Thành Dung Nguyên, and Rafał Stefański. Function spaces for orbit-finite sets. In Karl Bringmann, Martin Grohe, Gabriele Puppis, and Ola Svensson, editors, *ICALP*, volume 297 of *LIPIcs*, pages 130:1–130:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPIcs.ICALP.2024.130.

13 Mikołaj Bojańczyk and Szymon Toruńczyk. Imperative programming in sets with atoms. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2012)*, pages 4–15. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2012. doi:10.4230/LIPIcs.FSTTCS.2012.4.

14 Alan R. Camina and David M. Evans. Some infinite permutation modules. *The Quarterly Journal of Mathematics*, 42(1):15–26, January 1991. doi:10.1093/qmath/42.1.15.

15 Gregory L. Cherlin and Ehud Hrushovski. *Finite structures with few types*. Annals of mathematics studies ; no. 152. Princeton University Press, Princeton ; Oxford, 2003.

16 Karin Erdmann and Thorsten Holm. *Algebras and Representation Theory*. Springer Undergraduate Mathematics Series. Springer International Publishing, 2018. doi:10.1007/978-3-319-91998-0.

17 David M. Evans. Some more infinite permutation modules. Unpublished notes, October 2025.

18 David M. Evans. Permutation modules for Ramsey structures, 2026. arXiv:2603.29606.

19 David M. Evans and Darren G. D. Gray. Kernels and cohomology groups for some finite covers. In J. M. Larrazabal, D. Lascar, and G. Mints, editors, *Logic Colloquium '96: Proceedings of the Colloquium held in San Sebastián, Spain, July 9–15, 1996*, Lecture Notes in Logic, page 79–100. Cambridge University Press, 2017. doi:10.1017/9781316716816.004.

20 David M. Evans and Osama A. Rashwan. Bounds in the theory of finite covers. *Journal of Algebra*, 250(2):757–777, 2002. doi:10.1006/jabr.2001.9110.

21 A. Gardiner. Homogeneous graphs. *Journal of Combinatorial Theory, Series B*, 20(1):94–102, 1976. doi:10.1016/0095-8956(76)90072-1.

22 Arka Ghosh, Piotr Hofman, and Sławomir Lasota. Orbit-finite linear programming. *J. ACM*, 72(1):1:1–1:39, 2025. doi:10.1145/3703909.

23 Arka Ghosh, Piotr Hofman, and Sławomir Lasota. Solvability of orbit-finite systems of linear equations. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '22, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3531130.3533333.

24 Arka Ghosh and Sławomir Lasota. Equivariant ideals of polynomials. In Pawel Sobocinski, Ugo Dal Lago, and Javier Esparza, editors, *Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2024, Tallinn, Estonia, July 8-11, 2024*, LICS '24, pages 38:1–38:14, New York, NY, USA, 2024. ACM. doi:10.1145/3661814.3662074.

25 Christopher D. Godsil and Gordon F. Royle. *Algebraic graph theory*. Graduate texts in mathematics. Springer, 1st edition, 2001. doi:10.1007/978-1-4613-0163-9.

26 Darren G.D. Gray. The structure of some permutation modules for the symmetric group of infinite degree. *Journal of Algebra*, 193(1):122–143, 1997. doi:10.1006/jabr.1996.6963.

27 C. Ward Henson. A family of countable homogeneous graphs. *Pacific Journal of Mathematics*, 38(1):69–83, July 1971. doi:10.2140/pjm.1971.38.69.

28 C. Ward Henson. Countable homogeneous relational structures and $\aleph_0$-categorical theories. *The Journal of Symbolic Logic*, 37(3):494–500, 1972. doi:10.2307/2272734.

29 Bernhard Herwig. Extending partial isomorphisms for the small index property of many ω-categorical structures. *Israel Journal of Mathematics*, 107(1):93–123, 1998. doi:10.1007/BF02764005.

30 Wilfrid Hodges. *Model Theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 1993. doi:10.1017/CBO9780511551574.

31 Piotr Hofman, Jérôme Leroux, and Patrick Totzke. Linear combinations of unordered data vectors. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, LICS '17, pages 1–11. IEEE Computer Society, 2017. doi:10.1109/LICS.2017.8005065.

32 Piotr Hofman and Jakub Różycki. Linear equations for unordered data vectors in $[\mathcal{D}]^k \to \mathbb{Z}^d$. *Log. Methods Comput. Sci.*, 18(4), December 2022. doi:10.46298/lmcs-18(4:11)2022.

33 Ehud Hrushovski. Totally categorical structures. *Transactions of the American Mathematical Society*, 313(1):131–159, 1989. doi:10.2307/2001069.

34 Thomas J Jech. *The axiom of choice*. Studies in Logic and the Foundations of Mathematics. Elsevier, 1973.

35 W. M. Kantor, Martin W. Liebeck, and H. D. Macpherson. $\aleph_0$-categorical structures smoothly approximated by finite substructures. *Proceedings of the London Mathematical Society*, s3-59(3):439–463, 1989. doi:10.1112/plms/s3-59.3.439.

36 Bartek Klin, Eryk Kopczyński, Joanna Ochremiak, and Szymon Toruńczyk. Locally finite constraint satisfaction problems. In *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 475–486. IEEE, 2015. doi:10.1109/LICS.2015.51.

37 Eryk Kopczyński and Szymon Toruńczyk. LOIS: an application of SMT solvers. In Tim King and Ruzica Piskac, editors, *Proceedings of the 14th International Workshop on Satisfiability Modulo Theories affiliated with the International Joint Conference on Automated Reasoning, SMT@IJCAR 2016, Coimbra, Portugal, July 1-2, 2016*, volume 1617 of *CEUR Workshop Proceedings*, pages 51–60. CEUR-WS.org, 2016. URL: https://ceur-ws.org/Vol-1617/paper5.pdf.

38 A. H. Lachlan. Countable homogeneous tournaments. *Transactions of the American Mathematical Society*, 284(2):431–461, 1984. doi:10.1090/s0002-9947-1984-0743728-1.

39 A. H. Lachlan and Robert E. Woodrow. Countable ultrahomogeneous undirected graphs. *Transactions of the American Mathematical Society*, 262(1):51–94, 1980. doi:10.1090/S0002-9947-1980-0583847-2.

40 Robert P. Laudone and Andrew Snowden. Systems of ideals parametrized by combinatorial structures, 2023. `arXiv:2304.03686`.

41 Dugald Macpherson. A survey of homogeneous structures. *Discrete Mathematics*, 311(15):1599–1634, 2011. `doi:10.1016/j.disc.2011.01.024`.

42 Andrew M. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*, volume 57 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2013.

43 Michał R. Przybyłek. A note on Stone–Čech compactification in ZFA, 2024. `arXiv:2304.09986`.

44 Steven V Sam and Andrew Snowden. Stability patterns in representation theory. *Forum of Mathematics, Sigma*, 3, June 2015. `doi:10.1017/fms.2015.10`.

45 Daoud Siniora and Sławomir Solecki. Coherent extension of partial automorphisms, free amalgamation and automorphism groups. *J. Symb. Log.*, 85(1):199–223, 2020. `doi:10.1017/jsl.2019.32`.

46 Michał Szynwelski. *A functional programming language for sets with atoms*. PhD thesis, University of Warsaw, 2022. URL: `https://www.mimuw.edu.pl/media/uploads/doctorates/thesis-michal-szynwelski.pdf`.

47 Simon Thomas. Reducts of the random graph. *The Journal of Symbolic Logic*, 56(1):176–181, 1991. `doi:10.2307/2274912`.

48 Szymon Toruńczyk. The Rado graph admits only one oligomorphic action of its automorphism group. `https://atoms.mimuw.edu.pl/?p=1240`, 2016. Blog post on Atompress.