Introduction to Formal Proof

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1: Formal Proofs in Propositional Calculus

Introduction

▷ A calculus by which the *validity* (correctness) of propositional conjectures is judged

▷ A propositional conjecture has some *premisses* and a *conclusion*

 \triangleright Example 1:

It is raining **If** I wear a hat **and** it is raining **then** my head stays dry My head is not dry **I therefore conclude that** I am not wearing a hat

 \triangleright Question: is this conjecture valid?

\triangleright Example 2:

It is raining If I wear a hat and it is raining then my head stays dry My head is dry I therefore conclude that I am wearing a hat

Question: is this conjecture valid?

⊳ Example 3:

I conclude (without premisses) that If today is Tuesday then we are in Paris Question: is this conjecture valid?

Propositional Language: propositions

▷ A proposition is a meaningful declarative sentence that may be true or false in a situation.
 ▷ Examples:

- "Socrates is mortal"
- $\circ\,$ "The King's Arms is at the junction of Cornmarket with High Street"
- "I am hungry"
- "Tony Blair is a war-criminal"
- $\circ\,$ "It is raining and my head is wet"
- $\circ\,\,$ "If I wear a hat and it is raining then my head stays dry"

 $\vartriangleright \mathsf{But} \ \mathsf{not}$

- \circ "Do you like green eggs and ham?"
- \circ "Can you catch it in your hat?"
- \circ "Let's go!"
- \circ "Don't mention the war."

Propositional Language: atomic propositions

▷ An *atomic proposition* is a proposition with no logical connectives in it.

 \triangleright Examples:

- \circ "Socrates is mortal"
- "The King's Arms is at the junction of Cornmarket with High Street"
- "I am hungry"
- "Tony Blair is a war-criminal"

 \triangleright But not

- \circ "It is raining and my head is wet" ("... and ...")
- "If I wear a hat and it is raining then my head stays dry" ("if ... and ... then ...")

Symbolic representation

▷ Atomic propositions denoted by letters/identifiers

▷ Propositional connectives written in symbols

It is rainingRIf I wear a hat and it is raining then my head stays dry $(H \land R) \rightarrow D$ My head is not dry $\neg D$ I therefore conclude that $\neg H$

It is not a propositional connective
It is not a propositional connective

Composing Propositions with Logical Connectives

 $\begin{array}{c|c} \triangleright \text{ not } \dots & \neg \phi \\ \triangleright \dots \text{ and } \dots & \phi \land \psi \\ \triangleright \dots \text{ or } \dots & \phi \lor \psi \\ \triangleright \text{ if } \dots \text{ then } \dots & \phi \leftrightarrow \psi \end{array} \end{array} \right\} \text{ where } \phi \text{ and } \psi \text{ are propositions} \\ \phi \leftrightarrow \psi \\ \triangleright \dots \text{ if and only if } \dots & \phi \leftrightarrow \psi \end{array} \right\}$

 \triangleright The connectives are not independent of each other

- \triangleright There are other connectives, but these are the most common
- \triangleright Sometimes other symbols are used for connectives (typically \Rightarrow , \Leftrightarrow for \rightarrow , \leftrightarrow)

Parsing

 \triangleright Priority of connectives is (in descending order) $\neg, \land, \lor, \rightarrow, \leftrightarrow$

 \circ \rightarrow has slightly higher priority on its right than on its left

- \circ Some texts give \wedge the same priority as \vee
- \circ (Jape gives \wedge and \vee slightly higher priority on their left)

▷ If in doubt, parenthesize!

▷ Examples:

$$\circ (\underline{A \land B} \to \underline{C \lor D}) \leftrightarrow A \to \overline{B \to \overline{C \lor D}}$$
$$\circ \underline{\neg \neg A} \to A$$
$$\circ \overline{\overline{A \lor B} \lor C} \lor \underline{D \land E} \land F$$

Presenting a conjecture

Informal: "if you accept *these premisses*¹ then you should accept *this conclusion*"
 Formal: "from *these premisses* we may validly infer *this conclusion*."

 $\circ \ \textit{In horizontal form:} \ premiss, premiss, premiss, ... \vdash conclusion$

• In vertical form:

| premiss | premiss | premiss | ••• | |
|------------|---------|---------|-----|--|
| conclusion | | | | |

 \triangleright *e.g.* the conjecture:

 $R, H \land R \to D, D \vdash H \qquad \qquad \frac{R \qquad H \land R \to D \qquad D}{H}$

 \triangleright *e.g.* the conjecture:

$$R, H \land R \twoheadrightarrow D, \neg D \vdash \neg H$$

$$\frac{R \qquad H \land R \to D \qquad \neg D}{\neg H}$$

i.e. their truth

What is the nature of a valid conjecture?

▷ Propositional calculus is a formal system that we use to judge the validity of conjectures.

- ▷ The validity of a conjecture is judged *solely from its form*, not on the meanings/interpretations of the atomic propositions.
- \triangleright The validity of $R, H \land R \rightarrow D, \neg D \vdash \neg H$

 \circ is independent of the interpretation H, R, D in the real world.

- *does not* establish the truth of the premisses.
- \circ so should not, on its own, convince you that $\neg H$
- \triangleright An alternative interpretation
 - R "there are roses in my garden"
 - H "there's a hedgehog in my garden"
 - D "I am depressed"

What is the purpose of a proof system?

▷ If you know only that a particular conjecture has been proven:

- \circ When the premisses are all true then you should accept the conclusion
- When some of the premisses are untrue then you need not accept the conclusion

▷ If you know only that a conjecture has not (yet) been proven:

• Then you need not (yet) accept the conclusion, even if all the premisses are true

 \triangleright Example conjecture: commutativity of conjunction: (for any propositions ϕ and ψ)

$$\frac{\psi \land \phi}{\phi \land \psi} \qquad \qquad \psi \land \phi \vdash \phi \land \psi$$

$$\circ$$
 "from $\psi \wedge \phi$ we can infer $\phi \wedge \psi$ "

 \circ "if we have established $\psi \wedge \phi$ then we can infer $\phi \wedge \psi$ "

▷ Intuitive argument

- 1. Take $\psi \wedge \phi$ as a premiss
- 2. Since we have established (by premiss) $\psi \land \phi$ we can infer ϕ
- 3. Since we have established (by premiss) $\psi \land \phi$ we can infer ψ
- 4. Since we established (on the previous two lines) both ϕ and ψ we can infer $\phi \land \psi$

▷ Here we use the words *infer*, *conclude*, and *deduce* more or less interchangeably.

▷ We say that a proposition has been *established* in a proof of a conjecture if it is a premiss or has been inferred / concluded / deduced (directly or indirectly) from the premisses of the conjecture by means of the proof rules that we are using.

Proof Rules for conjunction

- ▷ "and introduction":
 - \circ In a proof in which we have established ϕ and established $\psi,$ we can conclude $\phi \wedge \psi$

$$rac{\phi \quad \psi}{\phi \wedge \psi}$$
 ^-intro

 \triangleright "and elimination"

 \circ In a proof in which we have established $\phi \wedge \psi$, we can conclude ϕ

$$\frac{\phi \land \psi}{\phi} \land \text{-elim-L}$$

 \circ In a proof where we have established $\phi \wedge \psi$, we can conclude ψ

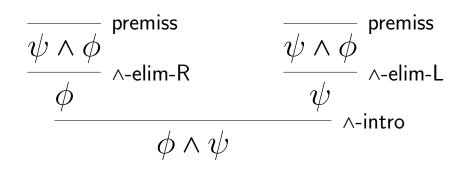
$$\frac{\phi \land \psi}{\psi} \land \text{-elim-R}$$

\triangleright Formal presentations of the proof

 \circ In linear form

| 1: | $\psi \wedge \phi$ | premiss |
|----|--------------------|--------------|
| 2: | ϕ | ∧-elim-R 1 |
| 3: | ψ | ∧-elim-L 1 |
| 4: | $\phi \wedge \psi$ | ∧-intro 2, 3 |

 \circ As a tree

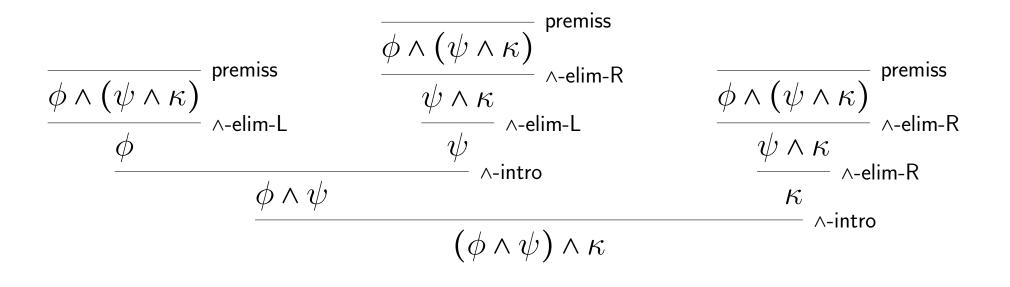


▷ The proof tree is complete because its root is the conclusion of the conjecture and each leaf is a premiss of the conjecture.

 \rhd The proof rules are parameterized by ϕ and ψ

 \triangleright View them as functions that construct proofs from proofs

 \triangleright Example: proof that $\phi \land (\psi \land \kappa) \vdash (\phi \land \psi) \land \kappa$



▷ The proof tree is complete because its root is the conclusion of the conjecture and each leaf is a premiss of the conjecture.

▷ Same proof (linear presentation)

T

| 1: | $\phi \wedge (\psi \wedge \kappa)$ | premiss |
|----|------------------------------------|----------------------|
| 2: | ϕ | ∧-elim-L 1 |
| 3: | $\psi \wedge \kappa$ | ∧-elim-R 1 |
| 4: | ψ | ∧-elim-L 3 |
| 5: | $\phi \wedge \psi$ | ∧-intro 2 4 |
| 6: | κ | ∧-elim-R 3 |
| 7: | $(\phi \land \psi) \land \kappa$ | \wedge -intro 5, 6 |

 \triangleright In this proof the pattern for each rule is matched in more than one way

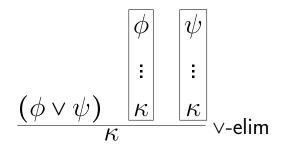
Proof Rules for disjunction

 \triangleright Introduction rules are straightforward

$$rac{\phi}{\phi \lor \psi}$$
 v-intro-L

$$\frac{\psi}{\phi \lor \psi} \lor \text{-intro-R}$$

▷ Elimination rule captures the idea of case analysis



 \triangleright We can conclude κ in a proof in which we have established $\phi \lor \psi$ and in which we have

- (a) established κ by assuming ϕ , and
- (b) established κ by assuming ψ

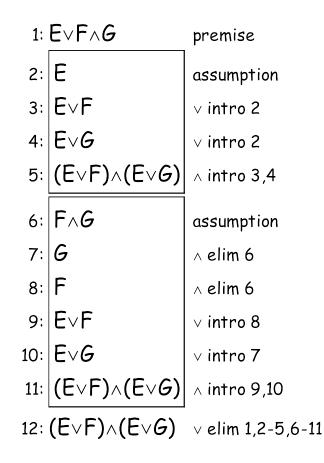
 \triangleright We have established $(\phi \lor \psi)$, *i.e.* that at least one of ϕ and ψ hold, but not which of them \triangleright Having *both* proof (a) and proof (b) means it doesn't matter which

 $\triangleright \kappa$ means: *this particular instance of* α cannot be referenced outside the supbroof of κ

 α

÷

\triangleright Case study: proof of $E \lor (F \land G) \vdash (E \lor F) \land (E \lor G)$



What makes this proof **formal** is that it doesn't depend on the **meanings** of E, F, or G or of the premiss or the conclusion, it just depends on the **syntactic forms** of the premiss and the conclusion and the propositions (formulæ) that arise in the course of the proof.

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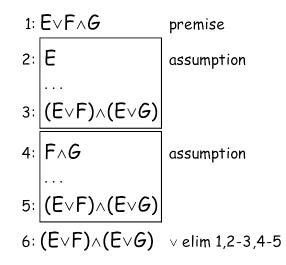
Proof rules as "conjecture transformers"

 \triangleright Q: But how did I go about *finding* the proof of $E \lor (F \land G) \vdash (E \lor F) \land (E \lor G)$?

A: At each stage I used a proof rule to transform a conjecture (the goal) to the set of conjectures that need to be proved in order for it to hold (the subgoals).
 A subgoal that's an assumption (or premiss) requires no further work.

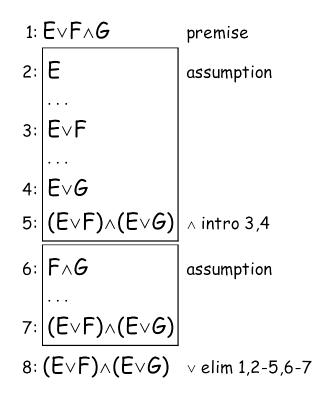
▷ The starting goal (the original conjecture) is:

- 1: $E \lor (F \land G)$ premiss 2: $(E \lor F) \land (E \lor G)$
- We guess from the form of the premiss that we can finish the proof with v-elim
 Using this rule transforms the starting goal into two subgoals



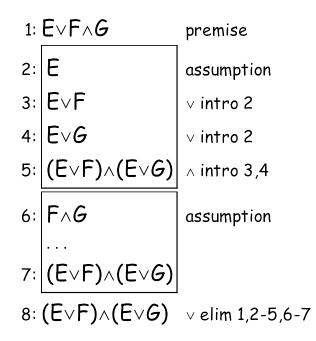
(alternate guess is that we can finish the proof with *^*-intro)

 \triangleright Working on the first subgoal: we guess we can finish with \land -intro



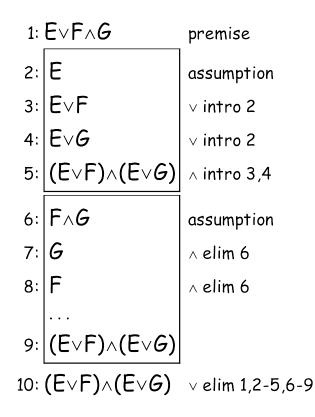
This yields two nested subgoals (2...3) and (2...4) – one for each conjunct

\triangleright After two v-intro steps we have completed the first subgoal



▷ Working on the second subgoal (the bottom box)

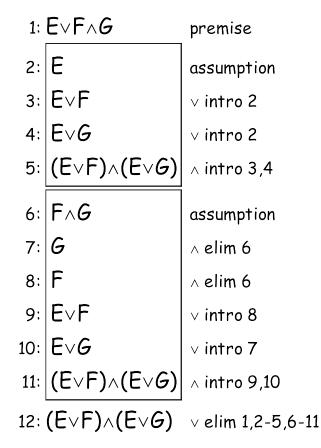
• we can see we are going to need both conjuncts so we take two A-elim steps



 \triangleright We see that we can finish this subproof with $\wedge\textsc{-intro}$

| 1: | EvFAG | premise |
|-----|----------------------|-------------------|
| 2: | E | assumption |
| 3: | E∨F | v intro 2 |
| 4: | E∨G | v intro 2 |
| 5: | (E∨F)∧(E∨ <i>G</i>) | ∧ intro 3,4 |
| 6: | F∧G | assumption |
| 7: | G | ∧ elim 6 |
| 8: | F | ∧ elim 6 |
| | | |
| 9: | E∨F | |
| 10: | E∨G | |
| 11: | (E∨F)∧(E∨ <i>G</i>) | ∧ intro 9,10 |
| 12: | (E∨F)∧(E∨G) | ∨ elim 1,2-5,6-11 |

 \triangleright The two resulting subgoals are closed by $\lor\-intro$ rules



▷ Notice that assumption 2 is not used outside of 2-5, nor is 6 used outside of 6-11.
Exercise: Could we have started the proof search by using ∧-intro?

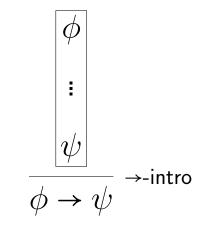
Proof Rules for Implication

▷ Elimination rule (a.k.a *modus-ponens*) is straightforward

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi} \rightarrow \text{-elim}$$

 \triangleright Concrete example: proof of $H, R, H \land R \rightarrow D \vdash D$

\triangleright Introduction rule



To prove $\phi \rightarrow \psi$ assume ϕ and prove ψ from it.

The box means "don't refer to the assumed occurence of ϕ outside of the nested subproof"

▷ Concrete example: "discovering" a proof of $E \to (F \to G) \vdash (E \to F) \to (E \to G)$ ▷ We know that the proof is eventually going to look like this

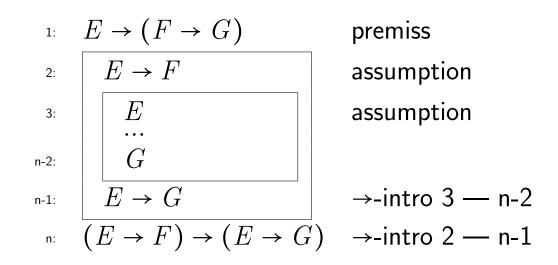
1:
$$E \rightarrow (F \rightarrow G)$$
 premiss
...
n: $(E \rightarrow F) \rightarrow (E \rightarrow G)$

▷ We cannot do anything immediately with the premiss (→-elim is not applicable) But we *could* start a new hypothetical subproof using →-intro

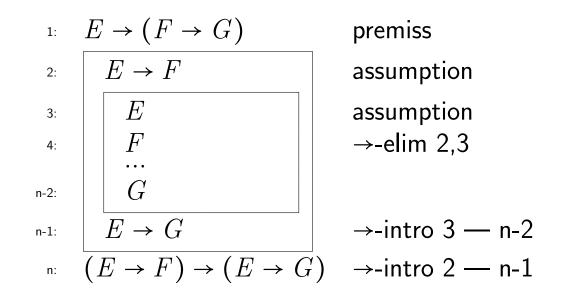
1:
$$E \rightarrow (F \rightarrow G)$$
 premiss
2: $E \rightarrow F$ assumption
n-1): $E \rightarrow G$
n: $(E \rightarrow F) \rightarrow (E \rightarrow G)$ \rightarrow -intro 2 — n-1

In fact we were *forced* to do this! (Why?)

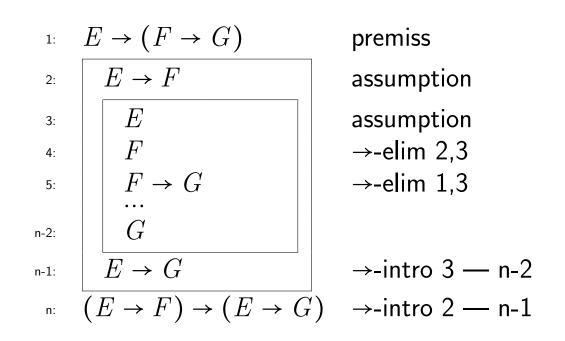
 \triangleright Exactly the same consideration holds for the subproof 2 — (n-1), leaving us with



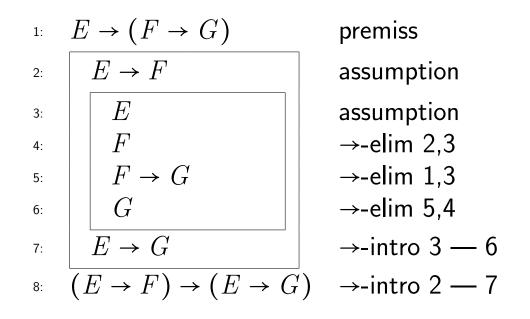
 \triangleright Now we can start to use \rightarrow -elim



 \triangleright and again ...



 \triangleright and it just takes one more \rightarrow -elim to close the (gap in the) proof



 \triangleright EXERCISE: Use this sequence to explain why the "boxed assumption" restriction of \rightarrow -intro is satisfied by this proof.

> Here's the proof in tree form (with the origins of assumptions labelled):

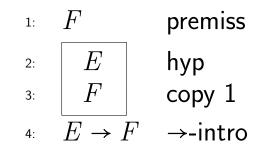
$$\frac{\overline{E} \rightarrow (F \rightarrow G)}{F \rightarrow G} \xrightarrow{\text{premiss}} \overline{E} \xrightarrow{\text{hyp}_3} \rightarrow \text{-elim}} \frac{\overline{E} \rightarrow F}{F} \xrightarrow{\text{hyp}_2} \overline{E} \xrightarrow{\text{hyp}_3} \rightarrow \text{-elim}} \xrightarrow{P \rightarrow \text{-elim}} \xrightarrow{P \rightarrow \text{-elim}} \xrightarrow{F \rightarrow \text{-elim}}$$

▷ EXERCISE: Use this tree to explain why the "boxed assumption" restriction of →-intro is satisfied by this proof.

▷ ASIDE: it can be quite challenging to keep track of assumptions made during the process of discovering a proof that you are recording in tree form.

A Paradox?

 \triangleright One consequence of accepting the \rightarrow -intro rule is the theorem $F \vdash E \rightarrow F$



- We have proved that if F holds anyway, then (for any proposition E) that $E \rightarrow F$, the natural language interpretation of which is: "if E then F"
- But in natural language "if E then F" is sometimes taken to suggest that E is, in some sense, *relevant to*, or a *causal factor* in F.
- There is no real paradox here: just take $E \rightarrow F$ to mean "F holds in every situation in which E holds."

T

Rules for iff

 \triangleright If we take $\phi \leftrightarrow \psi$ as an abbreviation for " $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ " we get the rules:

$$\frac{\phi \to \psi \quad \psi \to \phi}{\phi \leftrightarrow \psi} \text{ abb-} \leftrightarrow \text{-intro} \qquad \frac{\phi \leftrightarrow \psi}{\phi \to \psi} \text{ abb-} \leftrightarrow \text{-elim-r} \qquad \frac{\phi \leftrightarrow \psi}{\psi \to \phi} \text{ abb-} \leftrightarrow \text{-elim-l}$$

which capture the essence of the abbreviation; but mention an additional connective (\rightarrow)

 \triangleright The following rules are of equivalent logical power; and they mention *only* \leftrightarrow

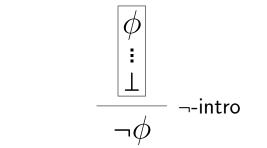
$$\begin{array}{c|c}
\psi & \phi \\
\vdots & \psi \\
\phi & \psi \\
\hline \psi & \leftrightarrow -\text{elim-r} \\
\hline \psi & \phi & \leftrightarrow \psi \\
\hline \phi & \leftrightarrow -\text{elim-l}
\end{array}$$

Proof Rules for Negation

 \triangleright Informal meaning of \neg is captured by

- \circ "If you believe ϕ then you shouldn't believe $\neg\phi$ "
- \circ "If you believe $\neg\phi$ then you shouldn't believe ϕ "

▷ The rules for ¬ must demonstrate that ϕ and ¬ ϕ contradict each other. ▷ We use the symbol \bot to mean *contradiction*. \triangleright Introduction: if ϕ leads to a contradiction, then believe $\neg\phi$



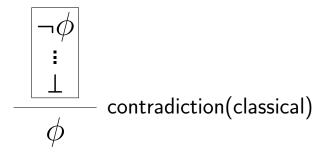
 \triangleright Elimination: believing both ϕ and $\neg\phi$ is contradictory

$$\frac{\phi \qquad \neg \phi}{\bot} \neg \text{-elim}$$

 \vartriangleright Contradiction-elimination: $\frac{1}{\phi}\,{}^{\bot-\text{elim}}$

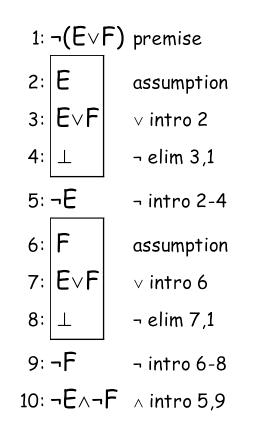
 \triangleright Double-negation-elimination: $\frac{\neg\neg\phi}{\phi}\,\neg\neg\text{-elim}$

▷ An important consequence of these rules – called classical contradiction or *reductio ad* absurdam (RAA) – is: if $\neg \phi$ leads to a contradiction, then believe ϕ



 \triangleright Exercise: "prove" the classical contradiction rule

▷ A straightforward proof using both negation rules



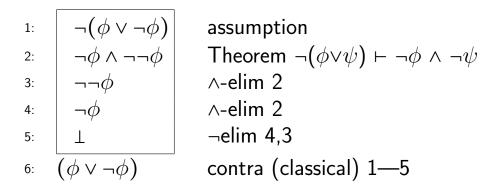
▷ Proof discovery:

 \circ the goal consequent matches a rule consequent.

 \circ then for each conjunct goal we looked for a way of eliminating not from the premiss

 \rhd Law of the Excluded Middle: $\vdash \phi \lor \neg \phi$

 \triangleright This theorem has no premisses.



▷ Proof discovery:

 \circ the goal consequent is a disjunction, but

 \circ using an \lor -intro rule would require us to choose one of the disjuncts to prove

 \circ so we structure the proof as a proof by contradiction

 \triangleright Exercise: prove the theorem cited on line 2

Derived Rules

- \triangleright Exercise: prove $\phi \land \psi \rightarrow \theta \vdash \phi \rightarrow (\psi \rightarrow \theta)$ (call this proof IC(ϕ, ψ, κ))
- $\triangleright \text{ Exercise: prove } \phi \rightarrow (\psi \rightarrow \theta) \vdash \phi \land \psi \rightarrow \theta \text{ (call this proof Cl}(\phi, \psi, \kappa))$
- Q: Can these proofs become part of the proof of $\vdash E \rightarrow (F \rightarrow G) \leftrightarrow E \land F \rightarrow G$?
- A: Imagine just substituting the proof trees at the appropriate point

$$\begin{array}{c}
\left(E \to (F \to G)\right) \\
\vdots & CI(E, F, G) \\
E \land F \to G
\end{array} \rightarrow \operatorname{intro} \qquad \left(E \land F \to G\right) \\
\left(E \to (F \to G)\right) \to (E \land F \to G) \rightarrow \operatorname{intro} \qquad \left(E \land F \to G\right) \rightarrow (E \to (F \to G)) \\
E \to (F \to G) \leftrightarrow E \land F \to G
\end{array} \rightarrow \operatorname{intro} \leftrightarrow \operatorname{-intro} \leftrightarrow \operatorname{-intro} \qquad \left(E \to (F \to G)\right) \rightarrow \left(E \to (F \to G)\right) \rightarrow \operatorname{intro} \leftrightarrow \operatorname{-intro} + \operatorname{intro} \left(E \to (F \to G) \to (F \to G)\right) \rightarrow \operatorname{intro} + \operatorname{intro} +$$

▷ This justifies the notion that (substitution instance of) a proven conjecture (a.k.a theorem) that has been named can be used within another proof *as if it were a proof rule*.

▷ Denying the Conseqent: (a.k.a *Modus Tollens*)

$$\frac{\phi \to \psi \qquad \neg \psi}{\neg \phi} \text{ MT}$$

 \triangleright Proof

1:
$$\phi \rightarrow \psi$$
premiss2: $\neg \psi$ premiss3: ϕ assumption4: ψ \rightarrow -elim 1, 35: \bot \neg -elim 4, 26: $\neg \phi$ \neg -intro 3-5

A first glance at soundness and completeness

\triangleright If I find a proof of $R, H \land R \rightarrow D, \neg D \vdash \neg H$

... then what should I do if I am wearing a hat and it is raining and my head is wet?

 \triangleright If I find a proof of $R, H \land R \rightarrow D, D \vdash H$

... then what should I do if it is raining and my head is dry and I am not wearing a hat?

 \triangleright What if we cannot find a proof of " $R, H \land R \rightarrow D, D \vdash H$ "?

- \circ is it because the conjecture is invalid?
- is it because we are insufficiently clever?
- \circ is it because the proof rules we have given so far are inadequate or wrong?
- ▷ More generally, we can ask questions *about the proof rules*:
 - Completeness: is there a proof of every valid conjecture of the form " $P_1, P_2, ... P_n \vdash Q$ "?
 - Soundness: if we can find a proof for " $P_1, P_2, ..., P_n \vdash Q$ " then is it valid?

 \triangleright But to answer these questions we need

 \circ an independent characterization of the notion of *validity*.

• a way of conducting rigorous proofs *about proofs*!

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Note 1:

"Calculus" used in this logical context signifies a systematic (*i.e.* rule-based rather than intuitive) method of reasoning by calculation.

Note 2:

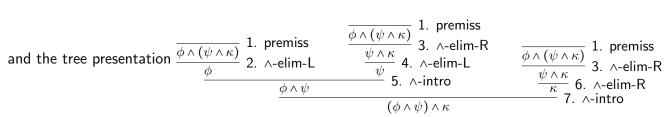
We use "rules of inference", "inference rules", and "proof rules" interchangeably in this course.

Note 3: Linear Proofs represent DAGs

We emphasise that the tree and linear presentations are presentations of the same underlying proof structure. They are not different proofs.

The correspondence between the linear presentation

| 1: | $\phi \land (\psi \land \kappa)$ | premiss |
|----|------------------------------------|--------------|
| 2: | ϕ | ∧-elim-L 1 |
| 3: | $\psi \wedge \kappa$ | ∧-elim-R 1 |
| 4: | ψ | ∧-elim-L 3 |
| 5: | $\phi \wedge \psi$ | ∧-intro 2 4 |
| 6: | κ | ∧-elim-R 3 |
| 7: | $(\phi \wedge \psi) \wedge \kappa$ | ∧-intro 5, 6 |



is made explicit here by labelling each node of the tree with its corresponding line number.

In general the linear presentation of a proof represents a DAG whose traversal yields a tree presentation.

Note 4:

For a simple concrete example suppose we want to show that 2n is always even, we can go about it in this way:

$$\frac{(even(n) \lor odd(n))}{(even(n) \lor odd(n))} \xrightarrow{even(2n)} odd(n) = even(2n) \lor -elim$$

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Note 5: Natural Deduction

- \triangleright The rules we have presented so far arguably formalize "natural" ways of reasoning about propositions formed with \land and \lor without taking a position in advance about the relationship between these two connectives
- > The natural deduction style of presenting a logical system or a calculus characterizes each construct by
 - an introduction rule (or rules)

(showing how to establish composite predicates from simpler ones)

- an elimination rule (or rules)
 - (showing how to use parts of composite predicates)

Natural Deduction is one of many systems used in the formalization of logic. It arose out of dissatisfaction with more austere forms of formalizing logic, such as Hilbert's. The Wikipedia article on Natural Deduction is a good place to follow the story if you are interested.

If you enjoyed functional programming then you may also be interested in the work I did with James J. Leifer intended to build a bridge between Hilbert Systems and Natural Deduction systems.

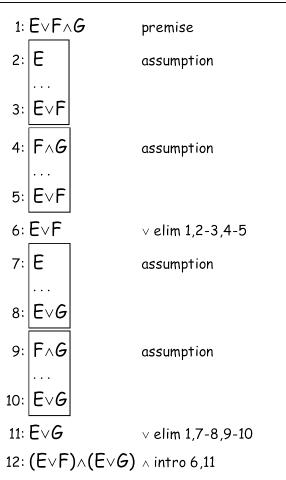
These two papers are available on the web.

- 1. Deduction for functional programmers by James J. Leifer and Bernard Sufrin. Journal of Functional Programming, volume 6, number 2, 1996.
- 2. Formal logic via functional programming by James J. Leifer (June 1995). Was James Leifer's final year dissertation: a greatly expanded version of the JFP paper.

Note 6:

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Here's an outline of the proof of $E \lor (F \land G) \vdash (E \lor F) \land (E \lor G)$ that we would have ended up with if we had decided to use \land -intro as our first goal-transforming step.



In this case the \vee -elim rule is used twice: once to establish the conjunct on line 6, and once to establish the conjunct on line 11. Compare this with the two uses of \wedge -intro in our original proof, once in each of the colateral subproofs required by the (single) use of \vee -elim.

Note 7: Equivalent logical power

When we say that two (collections) of rules are of equivalent logical power, we mean that any proof that can be constructed using only one of the collections of rules can also be constructed using the other collection. This does not mean that the structures of the proofs using one collection will be identical to those using the other collection.

To convince ourselves that two collections are of equivalent power (in the context of an otherwise-fixed set of rules), we need only prove the rules of the first collection from the second (and the otherwise-fixed set of rules), and vice-versa.

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Note 8: Typographical conventions

- Many texts follow very strict typographical conventions when discussing logic and, in particular, proving general theorems. We are much less meticulous in this part of the course, though we have a mild tendency to use Greek capitals to stand for arbitrary propositions in proof rules and very general theorems, while using Roman capitals to stand for specific atomic propositions.
- ▷ But because our arguments are strictly formal, and do not depend on the interpretations of atomic propositions, a proof done in our notes using Roman letters is as valid as one that would appear typographically more general if we were enforcing a typographical convention.
- ▷ For example: earlier we showed part of a proof that ∨ distributes through ∧ using Roman capitals to stand for the propositions involved; and here is a proof of the law of the excluded middle that uses a Roman letter for the proposition.

