

# Introduction to Formal Proof

## Synopsis of Materials

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### **Abstract**

This is a synopsis of the headings of the slides/notes from which the material presented in our ten lectures will be selected.

There is probably more material than can be covered properly in ten lectures without rushing. The extra material is intended to provide extra reading for those interested in further study of logic and formal proof.

It should be clear from the syllabus and the two tutorial sheets which of the material is **examinable**, and the style in which it will be examined.

## **0: Introduction and Overview**

Preliminary remarks

Course Overview

Case Study: three presentations of an equational proof

Stylized presentation with compressed transitive rules

Inference tree presentation

Linearized presentation

## 1: Formal Proofs in Propositional Calculus

- Propositional Calculus
  - Introduction
  - Propositional Language: propositions
  - Propositional Language: atomic propositions
  - Symbolic representation
  - Composing Propositions with Logical Connectives
  - Parsing
- Natural Deduction in the Propositional Calculus
  - Presenting a conjecture
  - What is the nature of a valid conjecture?
  - What is the purpose of a proof system?
  - Proof Rules for conjunction
  - Proof Rules for disjunction
  - Proof rules as “conjecture transformers”
  - Proof Rules for Implication
  - A Paradox?
  - Rules for iff
  - Proof Rules for Negation
  - Derived Rules
  - A first glance at soundness and completeness

## 2: Proofs *about* Propositional Calculus

- Road Map
- Propositional Semantics
  - Propositional semantics
  - Propositions are a recursive data type
  - Proving things about Propositions
  - Evaluating propositional formulae
  - A lemma about irrelevant atoms
  - Definitions: tautology, satisfiability, entailment
  - Detour: Tautology and Satisfiability Checking
- Soundness of Natural Deduction
  - Soundness 1: Definition
  - Soundness 2: Proofs represented as data structures
  - Soundness 3: a proof checker

- Soundness 4: some observations about subproofs
- Soundness 6: proof of soundness
- Consequence of Soundness
- Completeness of Natural Deduction
  - Statement of the Completeness Theorem for Natural Deduction
  - Reusing proofs and using proofs-about-proofs
  - ASIDE: completeness steps in proofs of admissibility
- From Natural Deduction to Sequent Calculi
  - Reformulating ND as a single-conclusion sequent calculus
- Epilogue
  - Is it essential to represent proofs in Haskell?
  - An alternative approach: valid proofs as a data type
  - Proof procedures and completeness

### **3: Predicate Logic (Semantics)**

- Introduction: Predicate Language
  - Propositional logic has limits
  - Informal predicate language: variables, predicates, quantifiers
  - Informal predicate language: functions
  - Inference systems for predicate logic
- Predicate Calculus Semantics
  - Formal predicate language: grammar
  - Free Variables
  - Substitution
  - Models and Meanings
  - The 7PM Model
  - Evaluation of formulae without variables
  - Evaluation of Quantified Formulae in the 7PM model
  - Formalizing “associate with”
  - Evaluation rules for formulae in a model M with domain Dom
  - Semantic Entailment
  - Satisfiability
  - ASIDE: Partial Functions

## 4: Formal Proofs in Predicate Logic

- Proof Rules for Predicate Calculus
  - Predicate Calculus Proofs
  - Proof Rules for the logical connectives
  - Proof Rules for Quantifiers:  $\forall$ -elimination
  - Proof Rules for Quantifiers:  $\exists$ -introduction
  - Proof Rules for Quantifiers:  $\exists$ -elimination
  - Proof Rules for Quantifiers:  $\forall$ -introduction
  - Freshness is important
  - Summary of the Quantifier Rules
  - Proof Rules for Equality
  - Derived consequences of substitutivity

## 5: Theories

- Extending Predicate Logic
  - Theories
    - Example: elementary group theory
    - Example: theory of Natural Numbers
  - Theories with several types
    - Theories with several types
    - Example: typed theory of natural numbers
    - Example: typed theory of heterogeneous lists
- Indispensability of the logical quantifiers
  - Formal treatment of generalised induction hypotheses