CHARACTERIZATIONS OF SIMPLY-CONNECTED FINITE POLYHEDRA IN 3-SPACE

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The following result was discovered by the authors during investigations into the topological problems of three-dimensional pictorial data analysis:

THEOREM 1. *Let Y be a connected finite polyhedron in S^3. Then the following are equivalent:*

(i) *Y is simply-connected*
(ii) *The first (integral) homology group of Y is trivial*
(iii) *The first (integral) Betti number of Y is zero*
(iv) *The first (integral) cohomology group of Y is trivial*
(v) *The Euler Characteristic of Y is equal to the number of components of $S^3 \setminus Y$*
(vi) *Y has one (and hence all six) of the Phragmen–Brouwer properties* [**6**] *(The Phragmen–Brouwer properties are listed in the Appendix.)*

It is known that (ii), (iii), (iv), (v) and (vi) are equivalent, and that they hold if Y is simply-connected. But as far as we know there is no published proof that conditions (ii) to (vi) imply that Y is simply-connected. From the viewpoint of pictorial data analysis by machine the theorem is useful, because the equivalence of (i) and (v) gives us a criterion for simple-connectedness that can be applied *in practice*. Furthermore, the equivalence of (i) and (iii) suggests that it is reasonable to *define* the number of holes in a finite polyhedron X in S^3 to be the first Betti number of X, because attaching a 'solid handle' to any finite polyhedron increases the polyhedron's first Betti number by exactly one. If we adopt this definition then the Euler Characteristic of X is equal to the number of components of X plus the number of cavities in X minus the number of holes in X. This is a natural generalization of the familiar formula for the Euler Characteristic of a handlebody in S^3. The 'number of holes' in a three-dimensional object is a concept which has been of some interest to workers in image processing in recent years (for example, [**3**, **4**, **5**]).

We shall deduce Theorem 1 from the following result, which is essentially due to Alexander [**1**].

LEMMA. *Let S be a finite polyhedron in S^3 with non-empty interior whose boundary is homeomorphic to S^2. Then S is homeomorphic to a closed 3-ball (that is, S is a '3-cell').*

Notation. Let $K = K(Y)$ be a triangulation of S^3 for which there exists a subcomplex L such that $|L| = Y$.

Proof of Theorem 1. Conditions (iii) and (iv) are equivalent since $H^n(Y)$ is the direct sum of the free part of $H_n(Y)$ and the torsion subgroup of $H_{n-1}(Y)$ – and the

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torsion subgroup of $H_0(Y)$ is trivial. Conditions (ii) and (iii) are equivalent because the Alexander Duality Theorem implies tht the torsion subgroup of $H_1(Y)$ is equal to the torsion subgroup of $H_0(S^3\setminus Y)$, and must therefore be trivial. Conditions (iii) and (v) are equivalent because the Euler Characteristic is expressible as the alternating sum of the Betti numbers, and by the Alexander Duality Theorem the second Betti number of Y is exactly one less than the number of components of $S^3\setminus Y$. Condition (iii) is shown to be equivalent to one of the Phragmen–Brouwer properties (unicoherence) in [**2**], and if Y has any one of the Phragmen–Brouwer properties then it has the other five [**6**]. So conditions (ii), (iii), (iv), (v) and (vi) are equivalent. Condition (i) implies condition (ii), for it is well known that $H_1(Y)$ is the abelianization of the fundamental group of Y.

We observe in passing that '(i) implies (ii)', '(ii) implies (iii)', and '(iii), (iv) and (vi) are equivalent' remain true even if Y is not embedded in S^3. We also mention that it is not essential to use *integral* homology and cohomology groups in (ii), (iii) and (iv). For we have already noted that Y is 1-torsion free (in fact Y is n-torsion free for all n), so if G is any abelian coefficient group then $H_1(Y; G) = H^1(Y; G) = G^n$, where n is the first integral Betti number of Y.

To finish the proof we shall show that (v) implies (i). In doing this we may assume that Y is a 3-manifold with boundary. For if Y_1 is the regular neighbourhood of Y in K (this is defined to be the union of all the simplexes of the second barycentric subdivision of K that meet Y) then Y_1 is a 3-manifold with boundary such that $S^3\setminus Y_1$ has the same number of components as $S^3\setminus Y$, and the fundamental group and the Euler Characteristic of Y are the same as the corresponding invariants of Y_1. Moreover, it is only necessary to prove that (v) implies (i) in the case when $S^3\setminus Y$ is connected, for if Y is a 3-manifold with boundary such that $S^3\setminus Y$ is not connected then we may 'bore a thin tunnel' from one of the components of $S^3\setminus Y$ to each of the other components: it is readily confirmed that such tunnelling does not affect the validity of (v), and it is clear that it cannot make the fundamental group of Y trivial if it was not trivial before.

So suppose that Y is a 3-manifold with boundary, that $S^3\setminus Y$ is connected and that the Euler Characteristic of Y is one. We must show that Y is simply-connected: in fact we shall show that Y is a 3-cell. Since Y is a polyhedral 3-manifold its frontier Fr Y is a polyhedral 2-manifold. Since (v) and (vi) are equivalent S^3 has the third Phragmen–Brouwer property, so Fr Y is connected; thus Fr Y is a connected 2-manifold.

Now it is known that the Euler Characteristic of a finite polyhedron which is a 3-manifold with boundary is exactly one half of the Euler Characteristic of its boundary. (For we can 'glue' two copies X_1, X_2 of such a polyhedron together by identifying their boundaries to produce a 3-manifold (without boundary). It is well-known that the Euler Characteristic of any odd dimensional triangulated manifold (without boundary) is zero – this fact can be proved by dual cell decomposition, and it is a corollary of the Poincaré Duality Theorem. Thus the Euler Characteristic of $X_1 \cup X_2$ is zero, whence the Euler Characteristic of $\mathrm{Fr}\, X_1 = X_1 \cap X_2$ is the sum of the Euler Characteristics of X_1 and X_2, which are equal.) So since the Euler Characteristic of Y is one, the Euler Characteristic of Fr Y must be two, whence by the Classification Theorem for Surfaces Fr Y is a polyhedral 2-sphere. Hence Y is a 3-cell by the above lemma.

We are grateful to the referee for pointing out an alternative proof of Theorem 1. He shows that (iii) implies (i) by taking a regular neighbourhood N of Y,

and using the Lefschetz Duality Theorem, the Exact Homology Sequence of $(N, \partial N)$, the Classification Theorem for Surfaces, and the Seifert–van Kampen Theorem.

COROLLARY. *If Y is a connected finite polyhedron in S^3 then Y is simply-connected if and only if each component of $S^3 \setminus Y$ is simply-connected.*

Proof. We may assume that Y is a 3-manifold with boundary, for the regular neighbourhood Y_1 of Y in K is a 3-manifold with boundary, Y is a deformation retract of Y_1, and if Y_2 denotes the regular neighbourhood of Y_1 in the second barycentric subdivision of K then the closure of $S^3 \setminus Y_2$ is a deformation retract both of $S^3 \setminus Y$ and of $S^3 \setminus Y_1$.

Now each component D of $S^3 \setminus Y$ has the same fundamental group and the same first homology group as its closure $\mathrm{cl}\, D$ – for the union of all the simplexes of the second barycentric subdivision of K that are contained in D is a deformation retract both of D and of $\mathrm{cl}\, D$. Hence if D is a component of $S^3 \setminus Y$ then (on applying Theorem 1 to the finite polyhedron $\mathrm{cl}\, D$) we deduce that D is simply-connected if and only if the rank of $H_1(D)$ is zero. The Alexander Duality Theorem implies that the rank of $H_1(Y)$ is zero if and only if the rank of $H_1(D)$ is zero for every component D of $S^3 \setminus Y$. Theorem 1 implies that Y is simply-connected if and only if the rank of $H_1(Y)$ is zero.

Appendix – The Phragmen–Brouwer Properties

The Phragmen–Brouwer properties are defined for a topological space S as follows:

1. If A and B are disjoint closed subsets of S and $x, y \in S$ are such that neither A nor B separates x and y in S then $(A \cup B)$ does not separate x and y in S.
2. If neither of the disjoint closed subsets A and B of S separates S then $A \cup B$ does not separate S.
3. If M is a closed connected subset of S and C is a component of $S \setminus M$ then the frontier of C is connected.
4. (Unicoherence) If A and B are closed connected sets such that $A \cup B = S$ then $A \cap B$ is connected.
5. If F is a closed subset of S and C_1, C_2 are disjoint components of $S \setminus F$ which have the same frontier B then B is connected.
6. If A and B are disjoint closed subsets of S, $a \in A$ and $b \in B$, then there exists a closed connected subset C of $S \setminus (A \cup B)$ which separates a and b.

It is proved in **[6]** that if a connected and locally connected metric space has any one of these properties then it has the other five. (It is also shown that in such spaces property 6 still holds if 'disjoint closed' is replaced by 'mutually separated', and property 3 can be generalized to assert that the intersection of any component of a closed set M with the frontier of any component of $S \setminus M$ is connected.)

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