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LOCAL SYMMETRY AND TRIANGLE LAWS ARE SUFFICIENT FOR METRISABILITY

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There is a long history in which attempts have been made to weaken the conditions defining a metric function whilst retaining metrisability. In this note, we survey some of our recent results in this context, deduce some well-known metrisation theorems, and present some related unsolved problems.

Some of the more important early papers on this topic were by E. W. Chittenden (see, for example [2]). But perhaps the most interesting of early theorems is the following.

Theorem 1 (V. W. Niemytzki [10]). *In order that a Hausdorff space X is metrisable it is necessary and sufficient that there exist a real-valued function d on $X \times X$ generating the topology on X and satisfying*

(1) $d(x, y) \geqslant 0$, $d(x, y) = 0$ *if and only if* $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) *given* $x \in X$, $\epsilon > 0$, *there is* $\delta > 0$ *such that* $d(x, z) < \epsilon$

whenever $d(x, y) < \delta$, $d(y, z) < \delta$ *('local triangle law').*

(T is open in the topology generated by d if, for each $x \in T$, there is $\epsilon > 0$ with $\{y: d(x, y) < \epsilon\} \subseteq T$.)

In [3], we show that Theorem 1 may be strengthened to yield the following result.

Theorem 2. *A* T_0*-space is* X *metrisable if and only if there is a real-valued function* d *on* $X \times X$ *generating the topology on* X *and satisfying* (1), (3) *of Theorem* 1 *together with the 'local symmetry law':*

(2′) *given* $x \in X$, $\epsilon > 0$, *there is* $\delta > 0$ *such that* $d(y, x) < \epsilon$ *whenever* $d(x, y) < \delta$.

Equivalent to Theorem 2 is the next result which expresses metrisability in terms of conditions on local neighbourhood bases. (Neighbourhoods here need not be open.)

Theorem 3 (see [3]). *In order that a* T_0*-space* X *be metrisable it is necessary and sufficient that, for each* x *in* X, *there be a countable local neighbourhood basis* $\{V(n, x): n = 1, 2, \ldots\}$ *at* x *satisfying*

(A) *for each* $x \in X$ *and integer* n, *there exists open* $V_1 \ni x$ *such that* $x \in V(n, y)$ *whenever* $y \in V_1$,

(B) *for each* $x \in X$ *and open* $U \ni x$, *there exist an integer* r *and open* $V_2 \ni x$ *such that* $V(r, y) \subseteq U$ *whenever* $y \in V_2$.

The spaces with local bases satisfying (A) can be shown (see [3]) to be precisely the semi-metric spaces, whereas those satisfying (B) are the γ-spaces of R. E. Hodel [6].

We now show that the 'Moore Metrisation Theorem' is an immediate deduction from Theorem 3. For a covering $\mathscr{U}$ of X and $A \subseteq X$, set $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U}: U \cap A \neq \phi\}$ and denote $\mathrm{St}(\{x\}, \mathscr{U})$ by $\mathrm{St}(x, \mathscr{U})$ when $x \in X$.

Theorem 4 (see [1], [8], [12]). *In order that a* T_0*-space* X *be metrisable it is necessary and sufficient that there exist a sequence of open coverings* $\{G(n): n = 1, 2, \ldots\}$ *such that, if* U *is a neighbourhood* *of* x *in* X, *there is a neighbourhood* V *of* x *and an integer* n *with* $\mathrm{St}(V, G(n)) \subseteq U$.

Proof of sufficiency. Set $V(n, x) = \mathrm{St}(x, G(n))$ and apply Theorem 3 directly.

Recalling that a development for X is a sequence $\{G(n): n = 1, 2, \ldots\}$ of open coverings of X for which $\{\mathrm{St}(x, G(n)): n = 1, 2, \ldots\}$ is a neighbourhood basis at x for each x. Theorem 4 can be re-stated as follows.

Theorem 5. *A* T_0*-space* X *is metrisable if and only if* X *has a development* $\{G(n): n = 1, 2, \ldots\}$ *for which the neighbourhoods* $V(n, x) \equiv \mathrm{St}(x, G(n))$ *satisfy* (B) *of Theorem* 3.

Since the existence of a development $\{G(n): n = 1, 2, \ldots\}$ always implies that the neighbourhoods $V(n, x) \equiv \mathrm{St}(x, G(n))$ satisfy (A), and since L.F. McAuley's 'bow-tie space' [7] provides an example of a non-developable (collectionwise normal) space for which there are neighbourhoods satisfying (A), Theorem 3 may be regarded as strengthening the Moore Theorem.

Another theorem which is equivalent to Theorem 2 is the following result in [3].

Theorem 6. *In order that a* T_1*-space* X *be metrisable it is necessary and sufficient that, for each* x *in* X, *there be a countable decreasing local neighbourhood basis* $\{W(n, x): n = 1, 2, \ldots\}$ *at* x *satisfying*

(C) *if* $x \in U$ *and* U *is open, then there exist an integer* s *and an open set* $V \ni x$ *such that* $x \in W(s, y) \subseteq U$ *whenever* $y \in V$.

We now show that the Nagata–Smirnov Theorem follows quickly from Theorem 6.

Theorem 7 (see [9], [11]). *If a regular* T_1*-space* X *has a* σ*-locally finite basis, then it is metrisable.*

Proof. Suppose that $\bigcup\{G(n): n = 1, 2, \ldots\}$ is a basis for X made up of locally finite families $G(n)$ for which, with no loss of generality,

$G(n) \subseteq G(n+1)$ for each n. Putting $W(n, x) = \bigcap \{\bar{G}: x \in G \in G(n)\}$ and $V(n, x) = \bigcap \{G: x \in G \in G(n)\} - \bigcup \{\bar{G}: G \in G(n),\ x \notin \bar{G}\}$, one can readily verify that $x \in W(n, y) \subseteq W(n, x)$ whenever $y \in V(n, x)$ and hence that the conditions of Theorem 6 are met.

In Theorem 6, the integer s of (C) was dependent on the point x and the open set U only. If s also depends on y (as in (D) below), we do not in general have a metric space. Suppose the space X has countable decreasing local neighbourhood bases $\{W(n, x): n = 1, 2, \ldots\}$ satisfying

(D) if $x \in Y$ and U is open, then there exists open $V \ni x$ such that, whenever $y \in V$, there is an integer $s = s(x, y, U)$ with $x \in W(s, y) \subseteq U$.

X need not then be metrisable as the 'bow-tie space' again shows us. However, if each $W(n, x)$ may be chosen to be open, X must be metric (as has recently been shown by the authors, G.M. Reed and M.E. Rudin [4]).

In connexion with this work we should like to have answers to the following questions:

(i) Is there a convenient characterisation of collectionwise normality in terms of local bases in a first countable space?

(ii) Can one characterise the class of those (first countable) spaces for which monotone and collectionwise normality are equivalent? (Monotone normality is in the sense of R.W. Heath, D.J. Lutzer, P.L. Zenor [5].)

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