In the beginning God created tensor . . . as a picture

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(in particular) quantum measurements can be described in terms of tensor structure alone, merely by making the distinct abilities to copy and delete classical & quantum data explicit.

(in particular)^2 also the classical world and classical data transformations can be described in terms of tensor structure alone.

- **Samson Abramsky** & **BC** ('04) *A categorical semantics for quantum protocols*; IEEE-LiCS; quant-ph/0402130

- **Peter Selinger** ('05) ... *mixed states & CPMs*; his www

- **BC** ('05) *Kindergarten Quantum Mechanics*; quant-ph/0510032

- **BC** ('05) *Introducing Categories to Practicing Physicists*; soon

- **BC** & **Dusko Pavlovic** ('06) *Quantum measurements ...*; soon

Strip-tease of quantum theory, . . . exposing its bare bones

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- **Mathematically: tensors without vectors, . . .**
  - **no *a priori* number field cf. $\mathbb{C}$, $[0, 1]$, . . .**
  - **no *a priori* sums, inner-product, . . .**
  - **no *a priori* algebra of observables, . . .**

Strip-tease of quantum theory, . . . exposing its bare bones

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  - **no *a priori* sums, inner-product, . . .**
  - **no *a priori* algebra of observables, . . .**

- **Physically: "combining systems" as prime concept, . . .**
  - **not probability, statistics, . . .**
  - **not superposition, orthogonality, . . .**
  - **not observation, measurement, tests, . . .**

Strip-tease of quantum theory, . . . exposing its bare bones

So some things vanish, ... **(cf. clothes, make-up)**

Strip-tease of quantum theory, . . . exposing its bare bones

So some things vanish, ... **(cf. clothes, make-up)**

But some things emerge, ... **(cf. skin, organs, and then bones)**

- **operational roots**
- **compositionality**
- **types reflecting kinds**

Strip-tease of quantum theory, . . . exposing its bare bones

These enable interpretation in terms of information-flow, . . .

- **quantum-classical flow (measurement)**
- **quantum-quantum flow (evolution)**
- **classical-quantum flow (control)**
- **classical-classical flow (computation)**
- **and more (entanglement)**

OUTLINE

- **Part I:** General compoundness
- **Part II:** Quantum compoundness
- **Part III:** Measurement & classicality

I

Practising physics as a symmetric monoidal category

I

Practising physics as a picture calculus

Kinds/types of systems:

$$A\,,\ B\,,\ C\,,\ \textbf{...}$$

- **e.g. electron, one qubit, n qubits, classical data, ...**

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Operations/experiments on systems:

$$A \xrightarrow{f} A\,,\; A \xrightarrow{g} B\,,\; B \xrightarrow{h} C\,,\,\ldots$$

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Composition of operations:

$$A \xrightarrow{g\circ f} C \quad := \quad A \xrightarrow{f} B \xrightarrow{g} C$$

'Doing nothing'-operations:

$$A \xrightarrow{1_A} A\,,\; B \xrightarrow{1_B} B\,,\; C \xrightarrow{1_C} C\,,\,\ldots$$

Compoundness/parallel composition:

$$A \otimes B \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D \qquad A \otimes C \xrightarrow{h} B \otimes D$$

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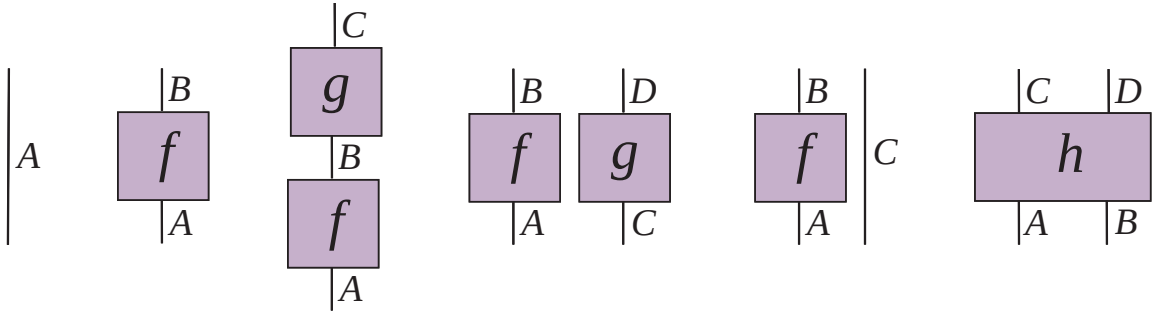
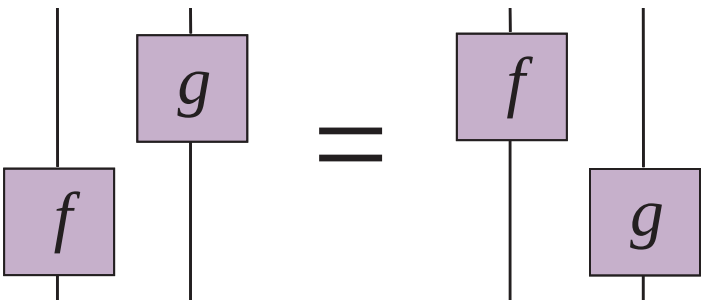
$\otimes$-structure captured by graphical representation:



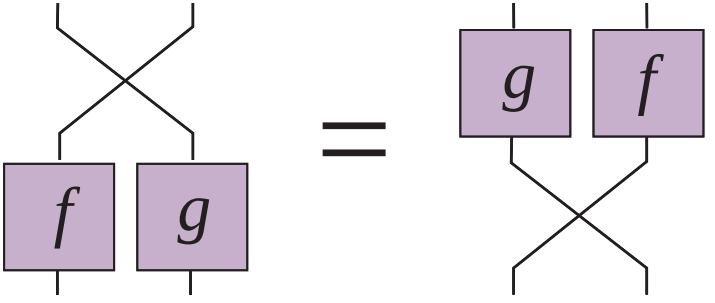
Pseudo-locality of systems/operations

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes 1_{A_2}} & B_1 \otimes A_2 \\ \Big\downarrow{\scriptstyle 1_{A_1} \otimes g} & & \Big\downarrow{\scriptstyle 1_{B_1} \otimes g} \\ A_1 \otimes B_2 & \xrightarrow[f \otimes 1_{B_2}]{} & B_1 \otimes B_2 \end{array}$$



Swapping systems/operations

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes g} & B_1 \otimes B_2 \\ \downarrow{\scriptstyle \sigma_{A_1,A_2}} & & \downarrow{\scriptstyle \sigma_{B_1,B_2}} \\ A_2 \otimes A_1 & \xrightarrow[g \otimes f]{} & B_2 \otimes B_1 \end{array}$$



The pictures are

- **not merely an illustration, . . .**
- **not merely an intuitive support, . . .**
- **but an actual formalism:**

diagrammatic proof $\Leftrightarrow$ algebraic proof

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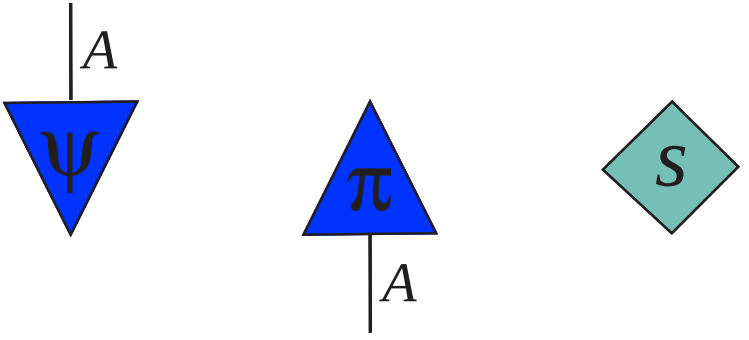
$$\boxed{\textbf{diagrammatic proof} \quad \Leftrightarrow \quad \textbf{algebraic proof}}$$

- **they extend and formalise Dirac's bra-kets in 2-D by distinguishing between sequential & parallel modes.**

Creating/destroying systems

$\mathrm{I} :=$ **'no system'** **i.e.** $A \otimes \mathrm{I} \simeq A \simeq \mathrm{I} \otimes A$

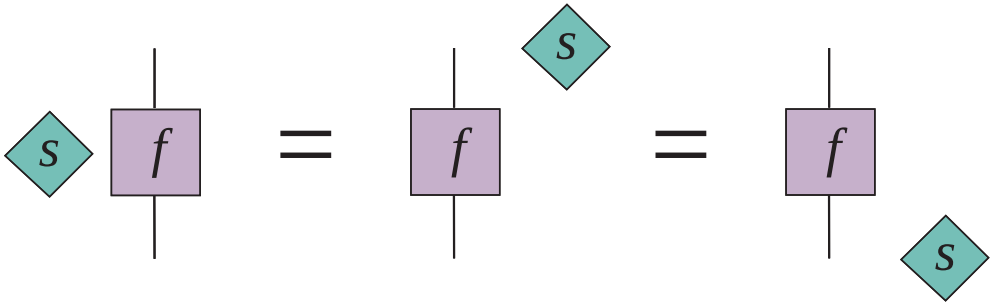


Quantities: probabilistic weights

$$s \bullet f := A \xrightarrow{\simeq} A \otimes \mathrm{I} \xrightarrow{f \otimes s} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$



PRACTICING PHYSICS
Physical System
Physical Operation

PROGRAMMING
Data Types
Programs

LOGIC & PROOF THEORY
Propositions
Proofs

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COOKING
Vegetables, meet, fish, spices, mayonaise
Growing, breeding, catching, cutting, mixing, eating

II

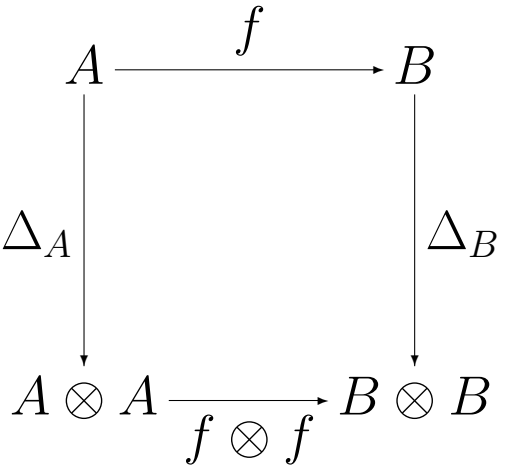
Practising quantum physics as a †-compact category

II

Practising quantum physics as a pictures with U-turns

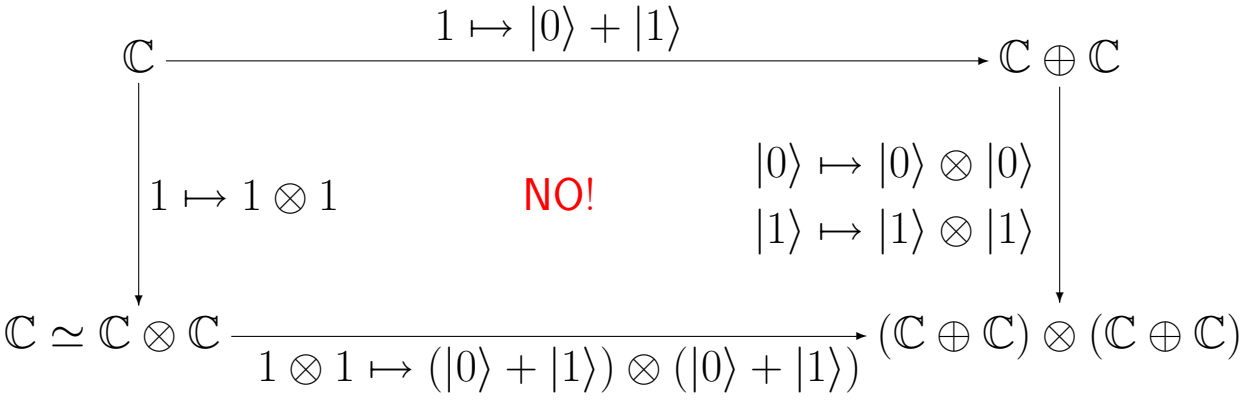
Copying ?

$$\{\Delta_A : A \to A \otimes A\}_A$$



No-copying of quantum states

$$\{\Delta_{\mathcal{H}} : |\, i \,\rangle \rightarrow |\, i \,\rangle \otimes |\, i \,\rangle\}_{\mathcal{H}}$$

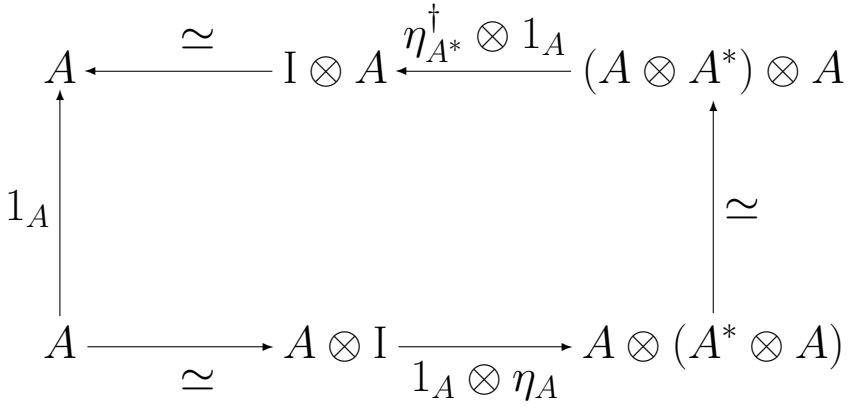


$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \neq (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

Bell-states cause trouble!

Symmetric monoidal category with

- **contravariant $\otimes$-involution adjoint $f_{A\to B} \mapsto f^{\dagger}_{B\to A}$;**
- **involution dual $A \mapsto A^{*}$;**
- **Units $\eta_A : \mathrm{I} \to A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$**



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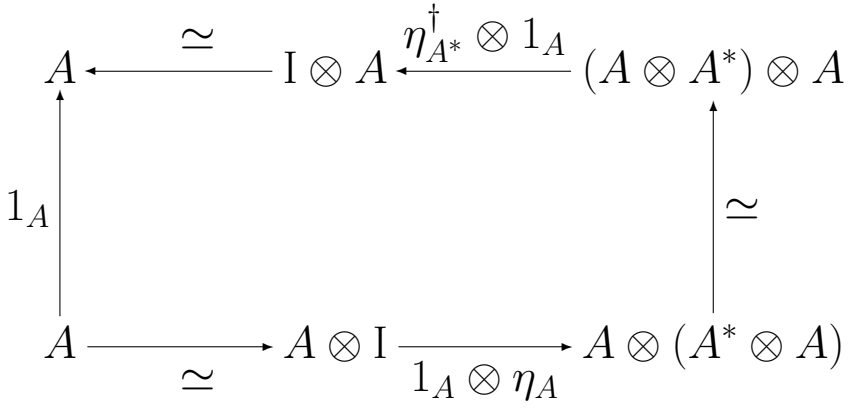

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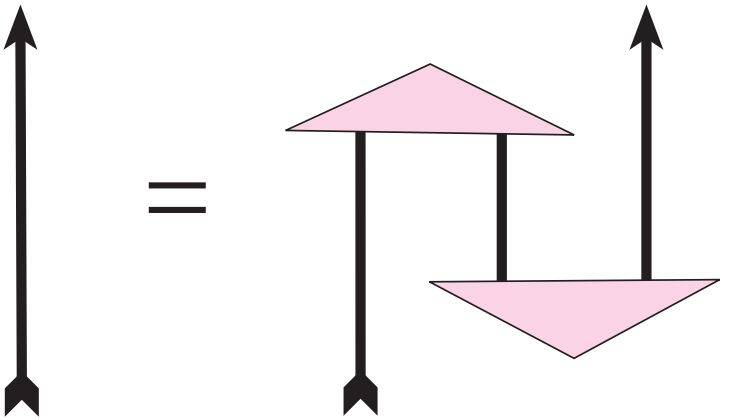

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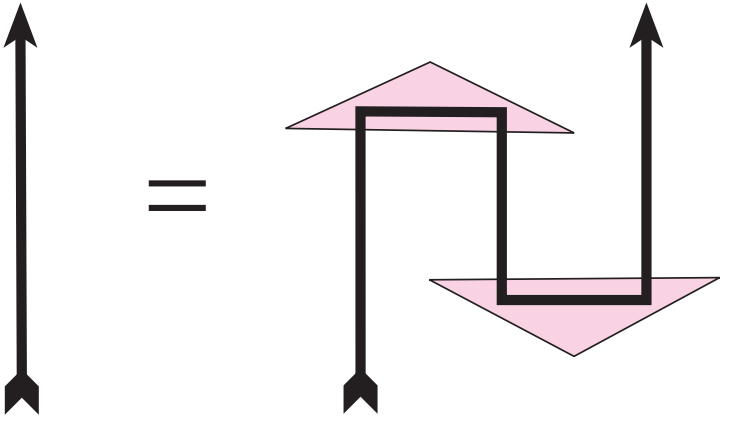

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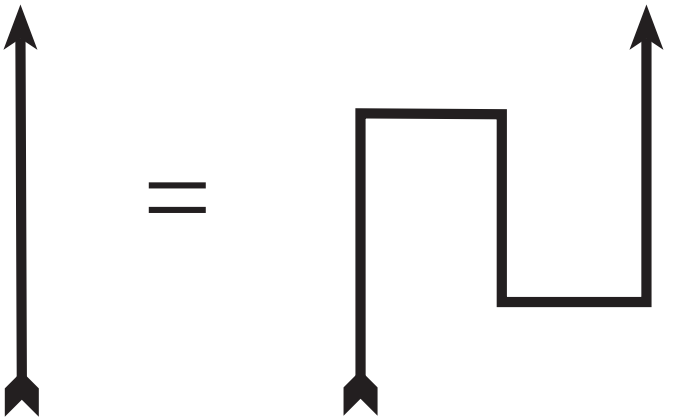
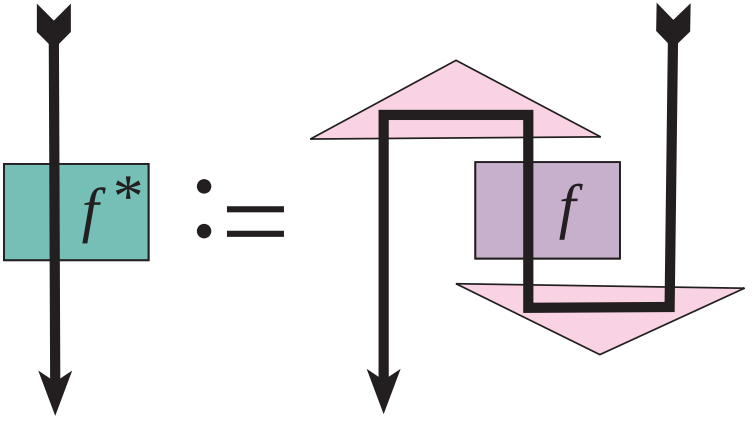


The "contravariant" involution

$$f : A \to B \quad \mapsto \quad f^* : B^* \to A^*$$

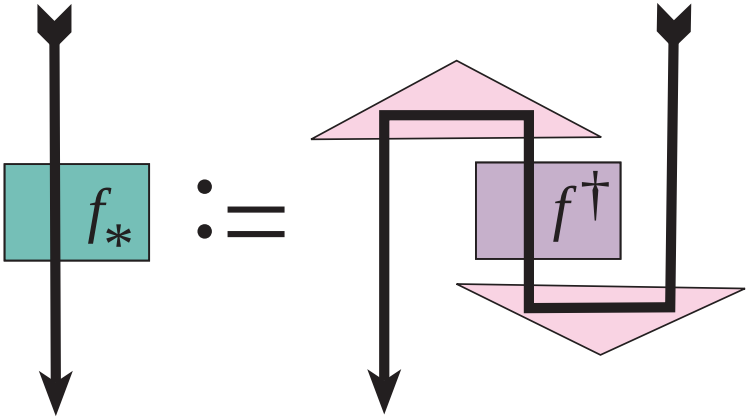
arises as



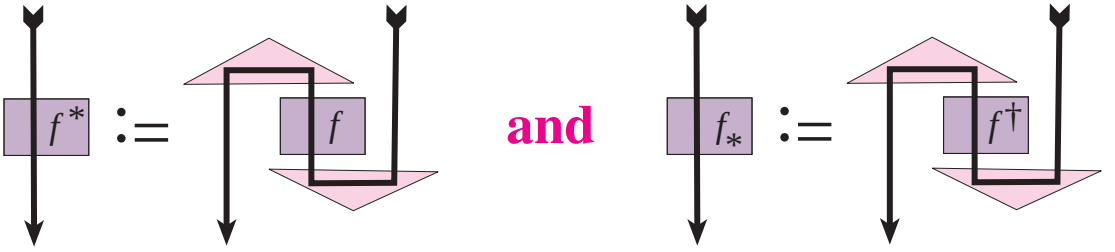
The "covariant" involution

$$f : A \to B \quad \mapsto \quad f_* : A^* \to B^*$$

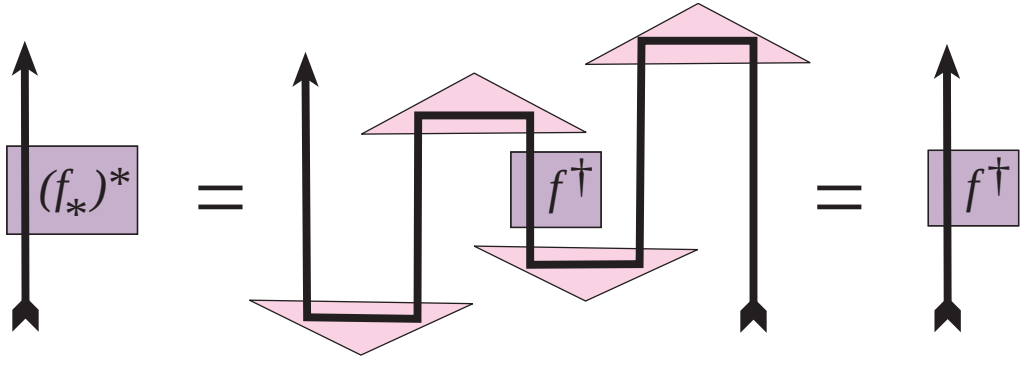
arises as



From



follows



and analogous we can prove that $(f^*)_* = f^\dagger$

The adjoint decomposes:

$$f^\dagger = (f^*)_* = (f_*)^*$$

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$$f^\dagger = (f^*)_* = (f_*)^*$$

For Hilbert spaces:

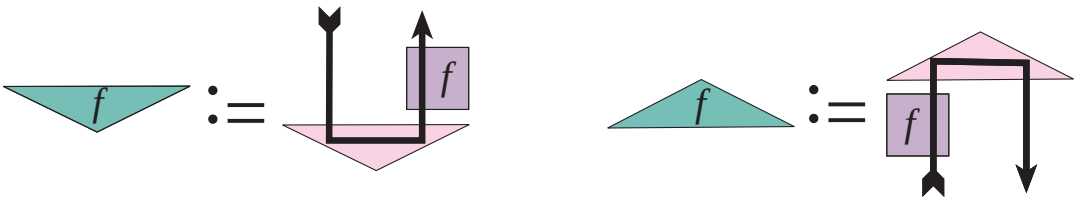
$(-)^*$ **:= transposition**

$(-)_*$ **:= complex conjugation**

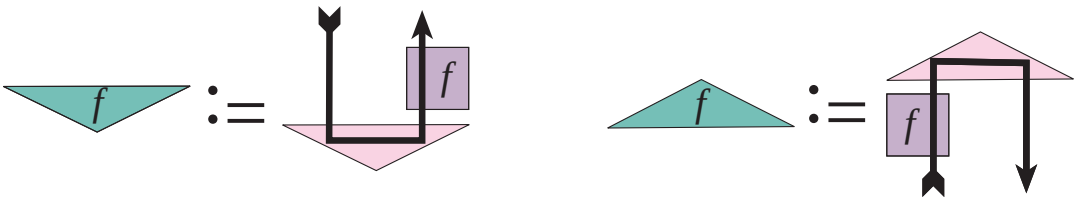
Others:

- **Unitaries**
- **Projectors**
- **Bipartite projectors**
- **Hilbert-Schmidt inner-product**
- **Identification of projective structure**
- **Mixed states and completely positive maps**

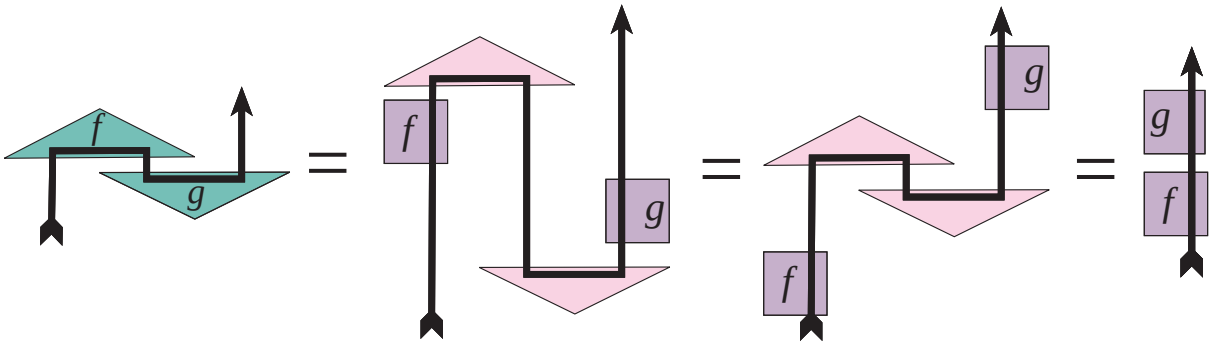
When setting



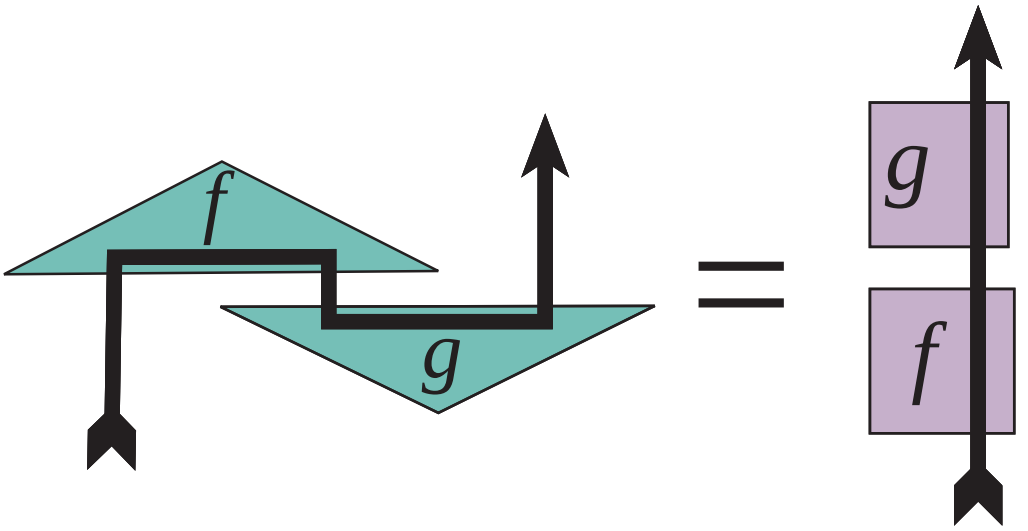
When setting



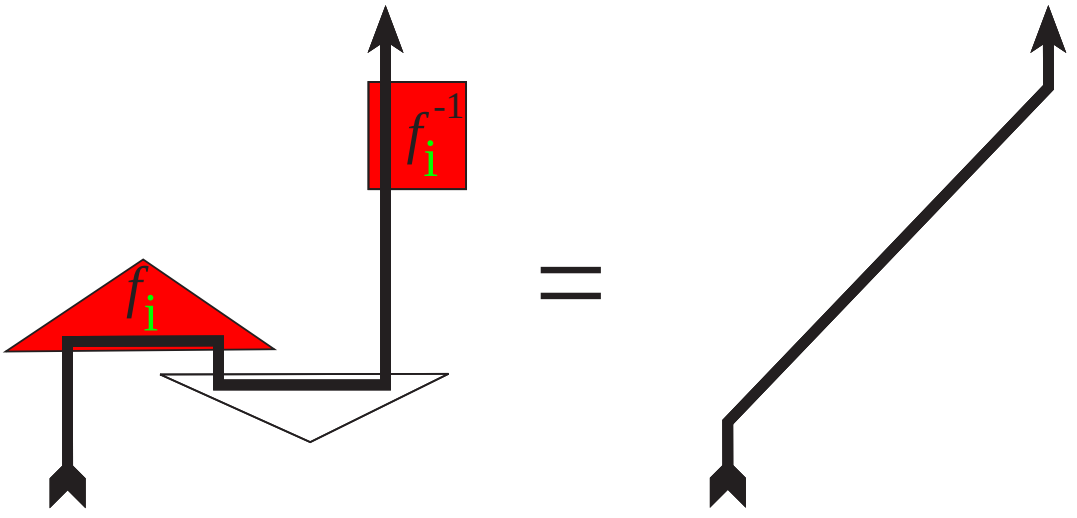
we obtain



Hyper-compositionality



OUR description and proof



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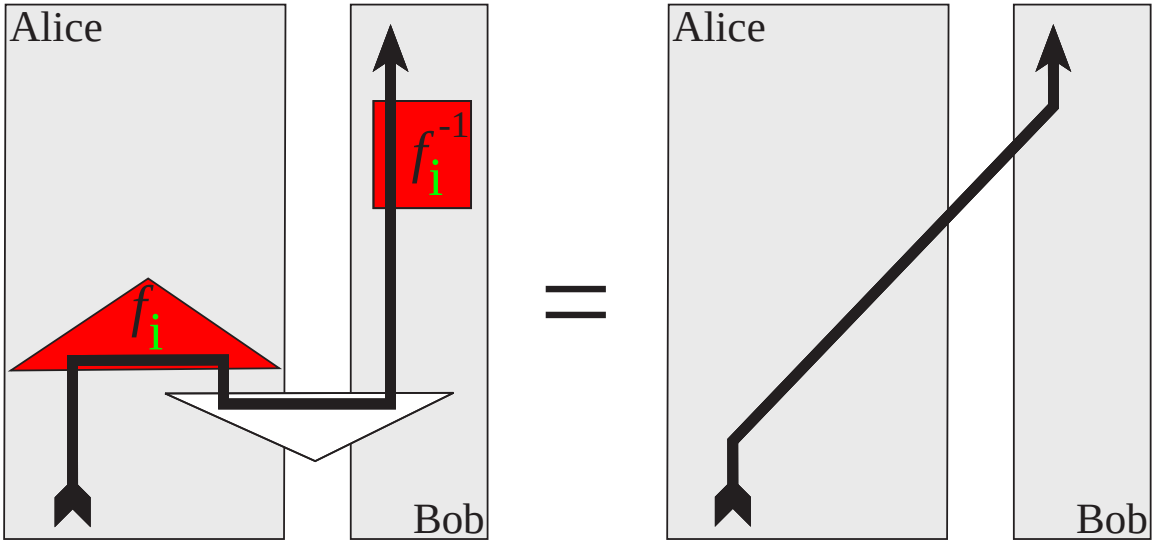


III

Quantum measurement as an Eilenberg-Moore coalgebra

III

Quantum measurement as a non-linear picture

Quantum measurement type:

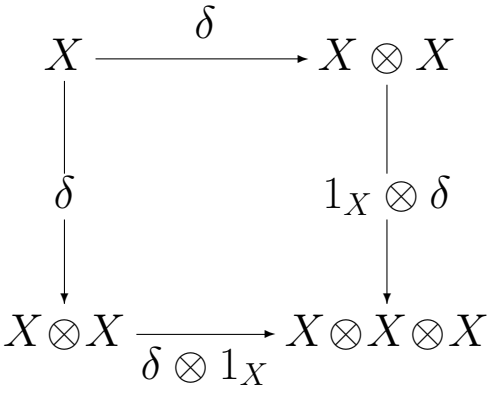
Quantum measurement type:

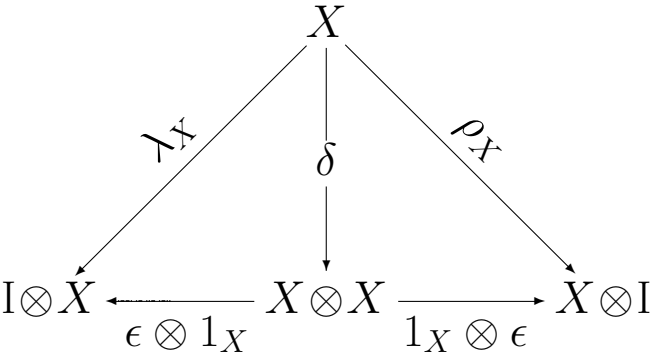
$$A \xrightarrow{\mathcal{M}} X \otimes A$$

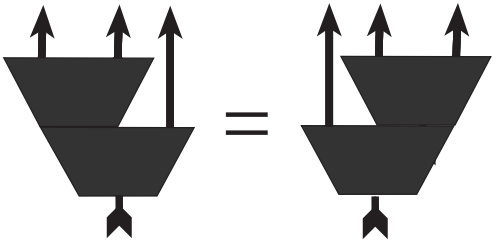
Classical data type:

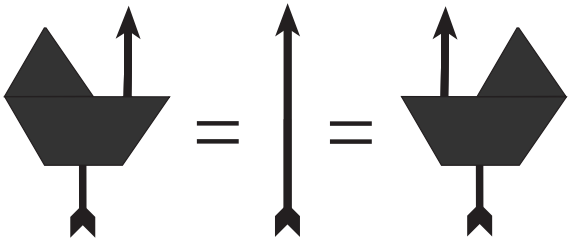
$$(X, \delta, \epsilon)$$

$$X \otimes X \xleftarrow{\delta} X \xrightarrow{\epsilon} \mathrm{I}$$









+ commutativity + $\delta = \delta_*$ **+ . . .** **(capture behaviour)**

Proposition. **Internal commutative comonoid structures over X are in 1-1 correspondence with commutative comonad structures on $X \otimes - : \mathrm{C} \to \mathrm{C}$.**

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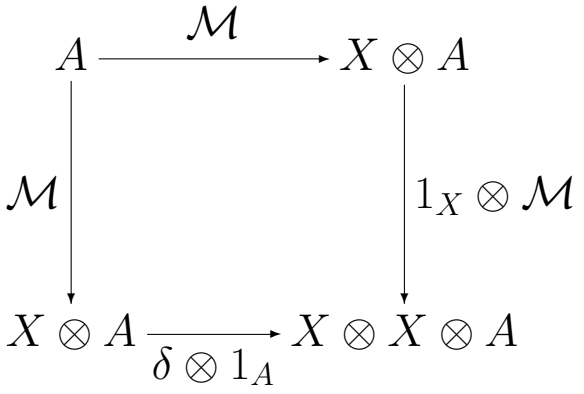
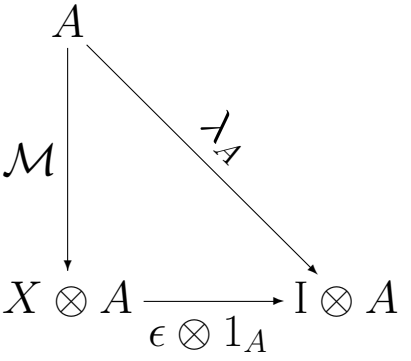
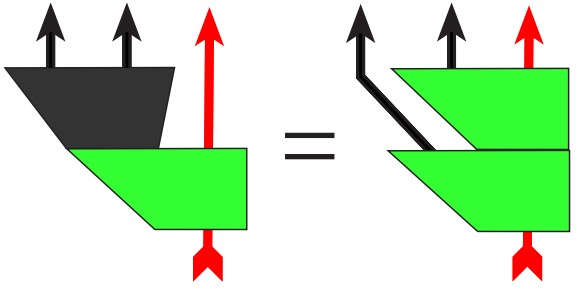
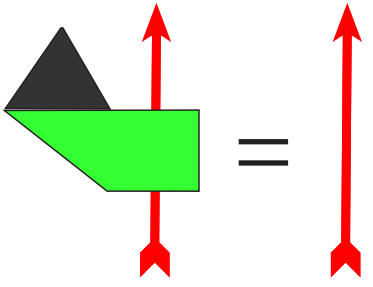
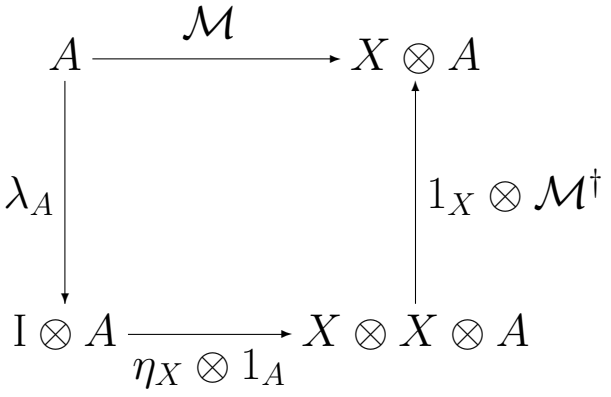
Question. **What are Eilenberg-Moore coalgebras for the comonad $X \otimes -$ given a classical object (X, δ, ϵ) ?**

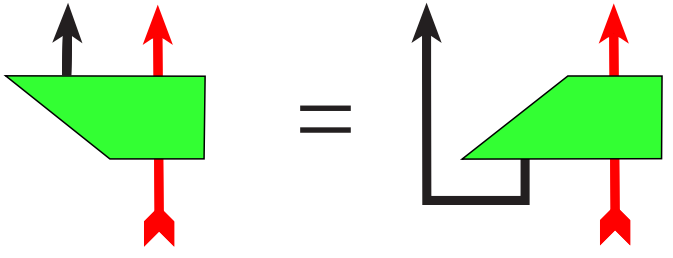

A
$\mathcal{M}$
$X \otimes A$
$\mathcal{M}$
$1_X \otimes \mathcal{M}$
$X \otimes A$
$\delta \otimes 1_A$
$X \otimes X \otimes A$


A
$\mathcal{M}$
ϵ_A
$X \otimes A$
$\epsilon \otimes 1_A$
$I \otimes A$






A
$\mathcal{M}$
$X \otimes A$
λ_A
$1_X \otimes \mathcal{M}^\dagger$
$\mathrm{I} \otimes A$
$\eta_X \otimes 1_A$
$X \otimes X \otimes A$



Thm. Self-adjoint Eilenberg-Moore coalgebras for

$$X \otimes - : \mathbf{FdHilb} \to \mathbf{FdHilb}$$

are in 1-1 correspondence with $\mathsf{dim}(X)$**-outcome quantum measurements where** $X := \bigoplus_i \mathbb{C}$ **and** $\delta := \Delta_X^{\otimes}$**.**

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Coalg-square $\Rightarrow$

idempotence	$\mathrm{P}_i^2 = \mathrm{P}_i$
mutual orthogonality	$\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$

Coalg-triangle $\Rightarrow$

Completeness of spectrum	$\sum_i \mathrm{P}_i = 1_{\mathcal{H}}$

Self-adjointness $\Rightarrow$

Orthogonality of projectors	$\mathrm{P}_i^\dagger = \mathrm{P}_i$

PROJECTOR
SPECTRUM

The classical world

Thm. [Tom Fox, 1976] The category $\mathrm{C}_\times$ of commutative comonoids and corresponding morphisms of a symmetric monoidal category with the forgetful functor $\mathrm{C}_\times \to \mathrm{C}$, is final among all cartesian categories with a monoidal functor to C, mapping the cartesian product to the monoidal tensor.

- $\mathrm{FdHilb}_\times = \mathrm{FSet}$

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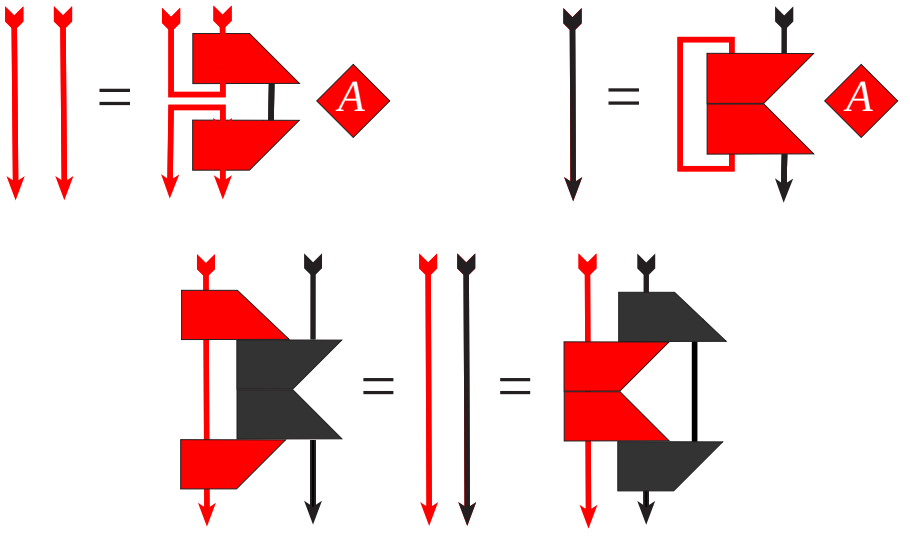
- $\mathrm{FdHilb}_\times = \mathrm{FSet}$
- **ϵ-morphisms $\equiv$ stochastic maps**
- $\mathrm{C}_\times \hookrightarrow \mathrm{C}_{stoch} \hookrightarrow \mathrm{C}$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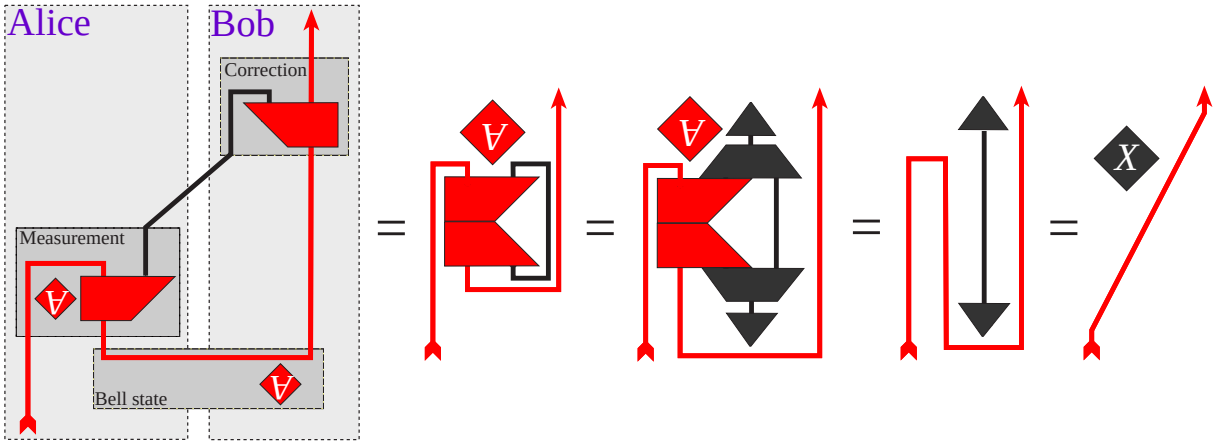
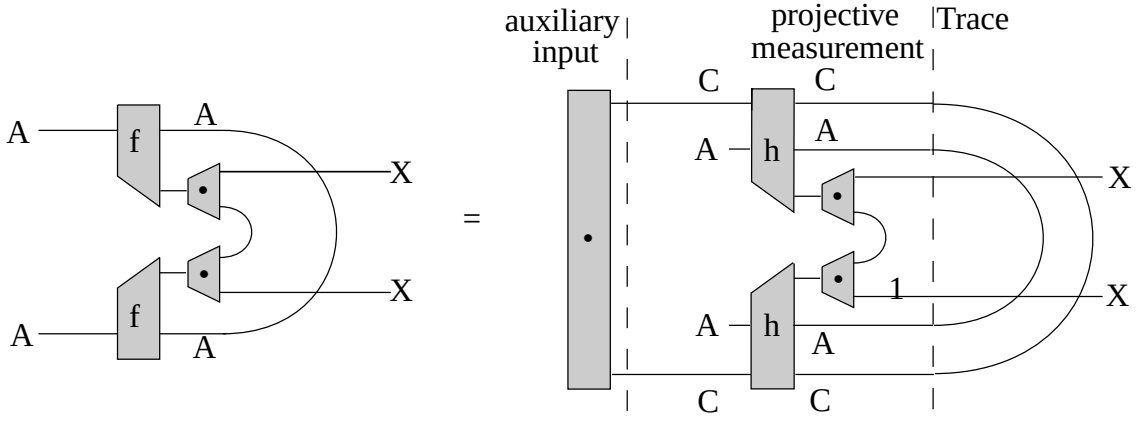
Full quantum teleportation



POVMs and Naimark's theorem



B.C. & Eric Paquette (2006) ***POVMs & Naimark's thm without sums*****.**

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since $\sum_i |i\rangle|i\rangle$ diverges, ...

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What is dimension anyway?

Do there exit ∞-Bell states mr. Popper?

Capturing infinite behaviours via recursive types, ...