Geometry of abstraction in quantum computation

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Outline

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Introduction

Categorical quantum mechanics

Q: Why (how) does it work?

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A: ‡-compact/scc categories capture the logically relevant structure of **Hilb**.

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Rational reconstruction of the "logically relevant structure".

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- $\oplus$ — base decompositions

Pro: Need a computational base.
Con: Not preserved on the states.

Proposal: Classical objects

Where do they come from?

Example

$$\frac{\dfrac{f : \Omega \longrightarrow \Omega : x \mapsto f(x)}{f' : \Omega \times \Omega \stackrel{\sim}{\longrightarrow} \Omega \times \Omega : (x, y) \mapsto (x, f(x) \oplus y)}}{U_f : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B} : |x, y\rangle \mapsto |x, f(x) \oplus y\rangle}$$

Example

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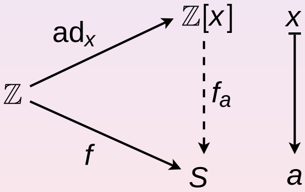
Abstraction in computation

- counterpart of *implementation*:
  - "...whatever x and y might be...
- interface specification
  - denote abstract data by variables: *copiable, deletable*

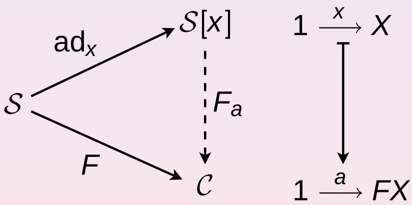
λ-abstraction

$$\frac{\mathbb{Z}^2 \longrightarrow \mathbb{Z}[x] \;:\; (a,b) \mapsto ax^3 + bx + 1}{\mathbb{Z}^2 \longrightarrow \mathbb{Z}^{\mathbb{Z}} \;:\; (a,b) \mapsto \lambda x.\, ax^3 + bx + 1}$$



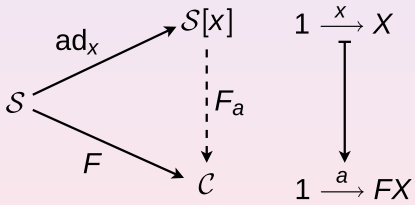
λ-abstraction in cartesian closed categories

$$\frac{A \xrightarrow{f_x} B \quad \text{in } \mathcal{S}[x:X]}{A \xrightarrow{\lambda_x.f_x} B^X \text{ in } \mathcal{S}}$$



λ-abstraction in cartesian closed categories

$$\frac{A \xrightarrow{\langle \varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B}{A \xrightarrow{\varphi} B^X} \qquad \mathcal{S}[x](A,B) \cong \mathcal{S}(A, B^X) \qquad \frac{A \xrightarrow{f_x} B}{A \xrightarrow{\lambda x. f_x} B^X}$$



λ-abstraction in cartesian closed categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \longrightarrow X$.

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Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \to X$.

Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ has a right adjoint $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \; : \; A \mapsto A^X$ and the transpositions

$$
\begin{array}{ccc}
A \xrightarrow{\langle \varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{f_x} B \\
\big\uparrow & \circlearrowleft\;\circlearrowright & \big\downarrow \\
A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x. f_x} B^X
\end{array}
$$

model λ-abstraction and application.

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$$\begin{array}{ccc} A \xrightarrow{\langle\varphi,x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{f_x} B \\ \updownarrow & \updownarrow & \updownarrow \\ A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x.f_x} B^X \end{array}$$

model λ-abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the power monad $(-)^X$.

κ-abstraction in cartesian categories

Theorem (Lambek, Adv. in Math. 79)

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Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ *has a* left adjoint $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \; : \; A \mapsto X \times A$ *and the transpositions*

$$\begin{array}{ccc} A \xrightarrow{\langle x, id \rangle} X \times A \xrightarrow{\varphi} B & \mathcal{S}[x](A, \mathrm{ad}_x B) & A \xrightarrow{f_x} B \\ \Big\uparrow & \circlearrowleft\ \circlearrowright & \Big\downarrow \\ X \times A \xrightarrow{\varphi} B & \mathcal{S}(\mathrm{ab}_x A, B) & X \times A \xrightarrow{\kappa x . f_x} B \end{array}$$

model first order abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the product comonad $X \times (-)$.

κ-abstraction in *monoidal* categories

Theorem (DP, MSCS 95)

Let $\mathcal{C}$ *be a* monoidal *category, and* $\mathcal{C}[x]$ *the free* monoidal *category generated by* $\mathcal{C}$ *and* $x : 1 \to X$.

Then the strong adjunctions $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *are in one-to-one correspondence with the internal comonoid structures on* X. *The transpositions*

$$\begin{array}{ccc}
A \xrightarrow{x \otimes A} X \otimes A \xrightarrow{\varphi} B & \mathcal{C}[x](A, \mathrm{ad}_x B) & A \xrightarrow{f_x} B \\
\uparrow & \uparrow \quad \downarrow & \downarrow \\
X \otimes A \xrightarrow{\varphi} B & \mathcal{C}(\mathrm{ab}_x A, B) & X \otimes A \xrightarrow{\kappa x.\, f_x} B
\end{array}$$

model action abstraction and application.

$\mathcal{C}[x]$ *is isomorphic with the Kleisli category for the comonad* $X \otimes (-)$, *induced by any of the comonoid structures.*

κ-abstraction in monoidal categories

Task

Extend this to Categorical Quantum Mechanics.

κ-abstraction in monoidal categories

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Problem

Lots of complicated diagram chasing.

κ-abstraction in monoidal categories

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Extend this to Categorical Quantum Mechanics.

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Lots of complicated diagram chasing.

Solution?

What does abstraction mean graphically?

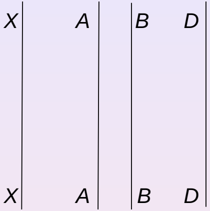
Outline

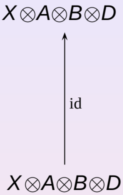
Objects

$$X \otimes A \otimes B \otimes D$$

.

Identities





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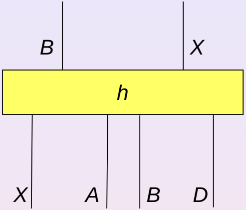
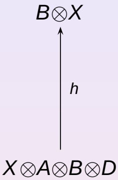
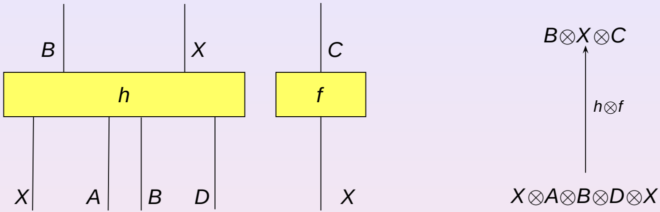
Morphisms

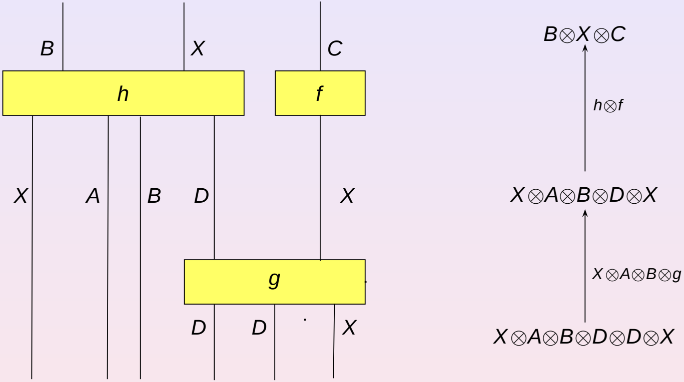




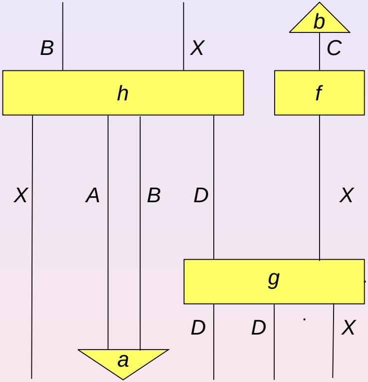
Tensor (parallel composition)



Sequential composition

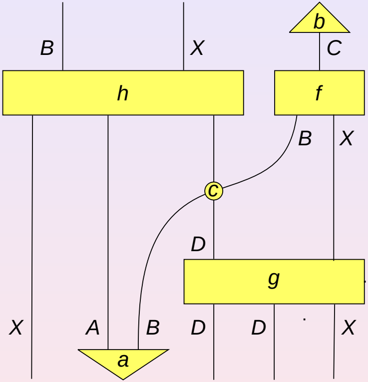


Elements (vectors) and coelements (functionals)



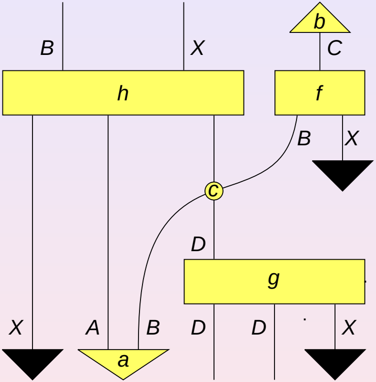
$$
\begin{array}{c}
B\otimes X \\
\uparrow {\scriptstyle B\otimes X\otimes b} \\
B\otimes X\otimes C \\
\uparrow {\scriptstyle h\otimes f} \\
X\otimes A\otimes B\otimes D\otimes X \\
\uparrow {\scriptstyle X\otimes A\otimes B\otimes g} \\
X\otimes A\otimes B\otimes D\otimes D\otimes X \\
\uparrow {\scriptstyle X\otimes a\otimes D\otimes D\otimes X} \\
X\otimes I\otimes D\otimes D\otimes X
\end{array}
$$

Symmetry



$$B\otimes X$$
$$\uparrow B\otimes X\otimes b$$
$$B\otimes X\otimes C$$
$$\uparrow h\otimes f$$
$$X\otimes A\otimes D\otimes B\otimes X$$
$$\uparrow X\otimes A\otimes c\otimes X$$
$$X\otimes A\otimes B\otimes D\otimes X$$
$$\uparrow X\otimes A\otimes B\otimes g$$
$$X\otimes A\otimes B\otimes D\otimes D\otimes X$$
$$\uparrow X\otimes a\otimes D\otimes D\otimes X$$
$$X\otimes I\otimes D\otimes D\otimes X$$

Polynomials



$$
\begin{array}{c}
B\otimes X\\
\uparrow\ {\scriptstyle B\otimes X\otimes b}\\
B\otimes X\otimes C\\
\uparrow\ {\scriptstyle h\otimes f}\\
X\otimes A\otimes D\otimes B\otimes X\\
\uparrow\ {\scriptstyle \mathrm{id}\otimes x}\\
X\otimes A\otimes D\otimes B\otimes I\\
\uparrow\ {\scriptstyle X\otimes A\otimes c\otimes r}\\
X\otimes A\otimes B\otimes D\\
\uparrow\ {\scriptstyle X\otimes A\otimes B\otimes g}\\
X\otimes A\otimes B\otimes D\otimes D\otimes X\\
\uparrow\ {\scriptstyle x\otimes a\otimes D\otimes D\otimes x}\\
I\otimes I\otimes D\otimes D\otimes I
\end{array}
$$

Outline

Abstraction with pictures

Theorem (again)

Let $\mathcal{C}$ be a symmetric monoidal category, and $\mathcal{C}[x]$ the free symmetric monoidal category generated by $\mathcal{C}$ and $x : 1 \to X$.

Then there is a one-to-one correspondence between

- *adjunctions* $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *satisfying*
  1. $\mathrm{ab}_x(A \otimes B) = \mathrm{ab}_x(A) \otimes B$
  2. $\eta(A \otimes B) = \eta(A) \otimes B$
  3. $\eta_I = x$

  and
- *commutative comonoids on X.*

Abstraction with pictures

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  and
- *commutative comonoids on X.*

$\mathcal{C}[x]$ is isomorphic with the Kleisli category for the commutative comonad $X \otimes (-)$, induced by any of the comonoid structures.

Proof ($\downarrow$)

Given $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$, conditions 1.-3. imply

- $\mathrm{ab}_x(A) = X \otimes A$
- $\eta(A) = x \otimes A$

Proof (↓)

Therefore the correspondence

$$\mathcal{C}\left(\mathrm{ab}_x(A), B\right) \rightleftarrows \mathcal{C}[x]\left(A, \mathrm{ad}_x(B)\right)$$

Proof ($\downarrow$)

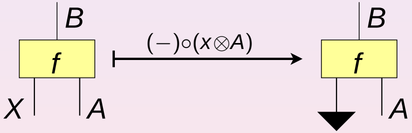
. . . is actually

$$\mathcal{C}(X \otimes A, B) \rightleftarrows \mathcal{C}[x](A, B)$$

Proof (↓)

. . . with

$$\mathcal{C}(X \otimes A, B) \rightleftarrows \mathcal{C}[x](A, B)$$



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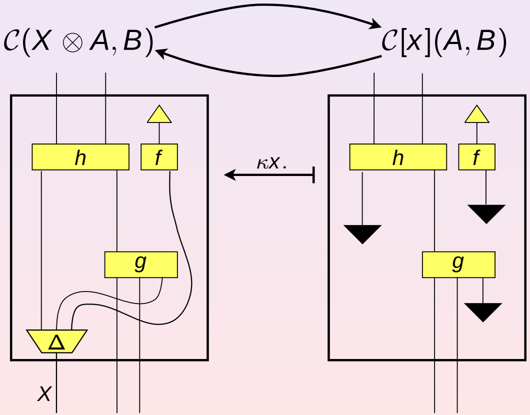


Proof (↓)

The bijection corresponds to the conversion:

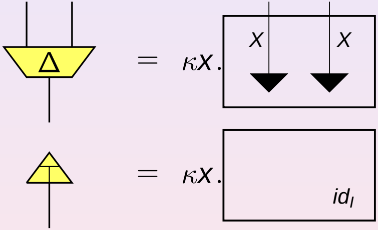
$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\rightleftarrows}} \mathcal{C}[x](A, B) \qquad \cong$$

$$\left(\kappa x.\ \varphi(x)\right) \circ \left(x \otimes A\right) = \varphi(x) \qquad (\beta\text{-rule}$$

$$\kappa x.\ \left(f \circ (x \otimes A)\right) = f \qquad (\eta\text{-rule}$$

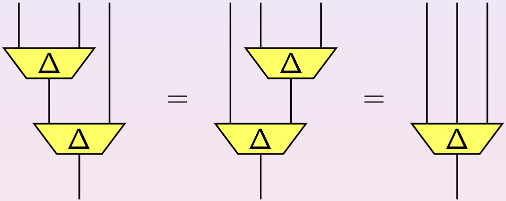
Proof (↓)

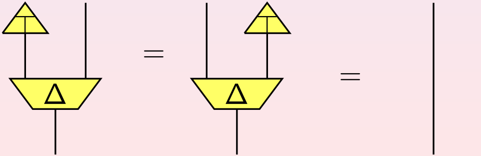
The comonoid structure $(X, \Delta, \top)$ is



Proof (↓)

The conversion rules imply the comonoid laws

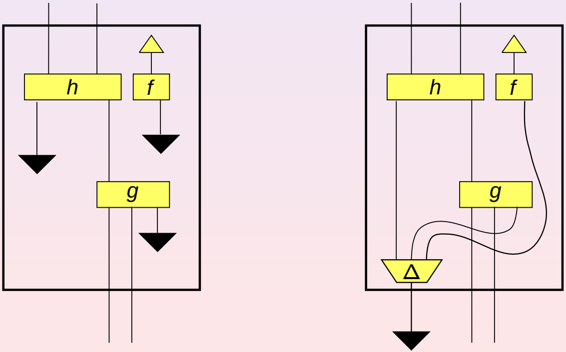




Proof (↑)

Given $(X, \Delta, \top)$, use its copying and deleting power, and the symmetries, to normalize every $\mathcal{C}[x]$-arrow:

$$\varphi(x) \quad = \quad \overline{\varphi} \circ (x \otimes A)$$



Proof (↑)

Then set $\kappa x.\, \varphi(x) = \overline{\varphi}$ to get

$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\cong}} \mathcal{C}[x](A, B)$$

$$\big(\kappa x.\, \varphi(x)\big) \circ \big(x \otimes A\big) = \varphi(x) \qquad\qquad (\beta\text{-rule}$$

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Remark

- $\mathcal{C}[x] \cong \mathcal{C}_{X\otimes}$ and $\mathcal{C}[x, y] \cong \mathcal{C}_{X\otimes Y\otimes}$, reduce the finite polynomials to the Kleisli morphisms.

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- But the extensions $\mathcal{C}[\mathcal{X}]$, where $\mathcal{X}$ is large are also of interest.
  - Cf. $\mathbb{N}[\mathbb{N}]$, Set[Set], and $CPM(\mathcal{C})$.

Interpretation

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In symmetric monoidal categories,
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abstraction applies just to copiable and deletable data.

Definition

A vector $\varphi \in \mathcal{C}(I, X)$ is a *base vector* (or a *set-like element*) with respect to the abstraction operation κx if it can be copied and deleted in $\mathcal{C}[x]$

$$
\begin{aligned}
(\kappa x.x \otimes x) \circ \varphi &= \varphi \otimes \varphi \\
(\kappa x.\mathrm{id}_I) \circ \varphi &= \mathrm{id}_I
\end{aligned}
$$

Interpretation

Upshot

In symmetric monoidal categories,
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Definition

A vector $\varphi \in \mathcal{C}(I, X)$ is a *base vector* (or a *set-like element*) with respect to the abstraction operation κx if it can be copied and deleted in $\mathcal{C}[x]$

$$\begin{aligned} (\kappa x.x \otimes x) \circ \varphi &= \varphi \otimes \varphi \\ (\kappa x.\mathrm{id}_I) \circ \varphi &= \mathrm{id}_I \end{aligned}$$

Proposition

$\varphi \in \mathcal{C}(I, X)$ is a *base vector* with respect to κx if and only if it is a homomorphism for the comonoid structure $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ corresponding to κx.

Outline

Introduction
- Objective
- λ-abstraction
- λ-abstraction in categories

Graphic notation

Geometry of abstraction
- Abstraction with pictures
- Consequences

Geometry of $\ddagger$-abstraction
- $\ddagger$-monoidal categories
- Quantum objects
- Abstraction in $\ddagger$-monoidal categories
- Classical objects
- Consequences

Teleportation through abstraction

‡-monoidal categories

Definitions

A ‡-*category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

$\ddagger$-monoidal categories

Definitions

A $\ddagger$-*category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a $\ddagger$-category $\mathcal{C}$ is called *unitary* if $f^{\ddagger} = f^{-1}$.

‡-monoidal categories

Definitions

A *‡-category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a ‡-category $\mathcal{C}$ is called *unitary* if $f^\ddagger = f^{-1}$.

A (symmetric) monoidal category $\mathcal{C}$ is *‡-monoidal* if its monoidal isomorphisms are unitary.

‡-monoidal categories

Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

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in every ‡-monoidal category we can define

- *abstract inner product*

$$\begin{array}{rcl} \langle - | - \rangle_A \ : \ \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathbb{I} \\ (\varphi, \psi : I \longrightarrow A) & \longmapsto & \left(I \xrightarrow{\varphi} A \xrightarrow{\psi^\ddagger} I \right) \end{array}$$

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$$\begin{array}{rcl} \langle - | - \rangle_{AB} \ : \ \mathcal{C}(AB) \times \mathcal{C}(A) & \longrightarrow & \mathcal{C}(B) \\ (\varphi : I \to A \otimes B, \psi : I \to A) & \longmapsto & \left(I \xrightarrow{\varphi} A \otimes B \xrightarrow{\psi^\ddagger \otimes B} B \right) \end{array}$$

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- *entangled vectors* $\eta \in \mathcal{C}(A \otimes A)$, such that $\forall \varphi \in \mathcal{C}(A)$

$$\langle \eta | \varphi \rangle_{AA} \quad = \quad \varphi$$

‡-monoidal categories

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For every object A in a ‡-monoidal category $\mathcal{C}$ holds
(a) $\iff$ (b) $\Longleftarrow$ (c),

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Proposition

For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

(a) $\eta \in \mathcal{C}(A \otimes A)$ is entangled

(b) $\varepsilon = \eta^\ddagger \in \mathcal{C}(A \otimes A, I)$ internalizes the inner product

$$\varepsilon \circ (\psi \otimes \varphi) \quad = \quad \langle \varphi | \psi \rangle$$

‡-monoidal categories

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For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

(a) $\eta \in \mathcal{C}(A \otimes A)$ is entangled

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(c) (η, ε) realize the self-adjunction $A \dashv A$, in the sense

$$A \xrightarrow{\eta \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes \varepsilon} A \quad = \quad \mathrm{id}_A$$

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The three conditions are equivalent if I generates $\mathcal{C}$.

‡-monoidal categories

Proposition in pictures

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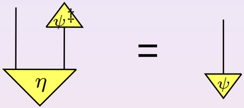
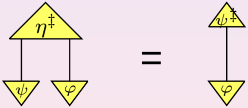
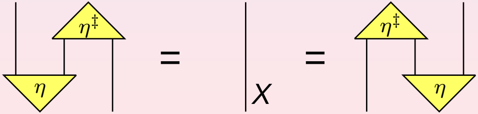


(b)



(c)



Quantum objects

Definition

A *quantum object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Quantum objects

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Remark

The subcategory of quantum objects in any ‡-monoidal category is ‡-compact (strongly compact) — with all objects self-adjoint.

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ *be a ‡-monoidal category, and* $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces* $\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

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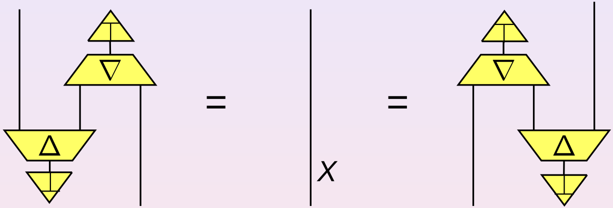
(c) $(X \otimes \nabla) \circ (\Delta \otimes X) = \Delta \circ \nabla = (\nabla \otimes X) \circ (X \otimes \Delta)$

where $X \otimes X \xrightarrow{\nabla} X \xleftarrow{\bot} I$ *is the induced monoid*

$$\begin{aligned} \nabla &= \Delta^{\ddagger} \\ \bot &= \top^{\ddagger} \end{aligned}$$

Abstraction in ‡-monoidal categories

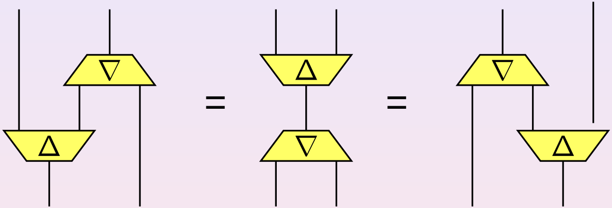
Theorem in pictures

(b) 

Abstraction in ‡-monoidal categories

Theorem in pictures

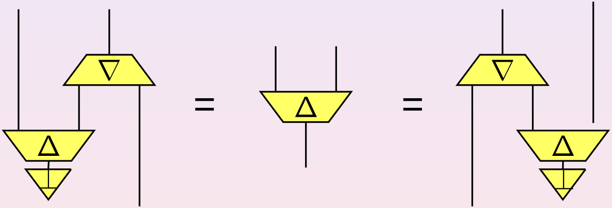
(c)



Proof of (b)$\Longrightarrow$(c)

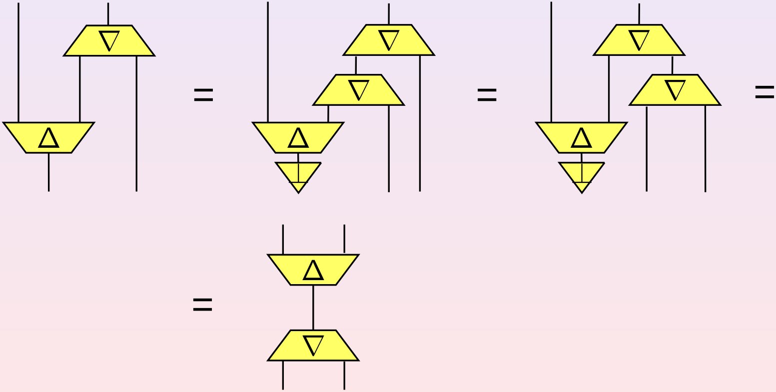
Lemma 1

If (b) holds then



Proof of (b)$\Longrightarrow$(c)

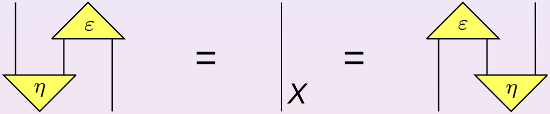
Then (c) also holds because



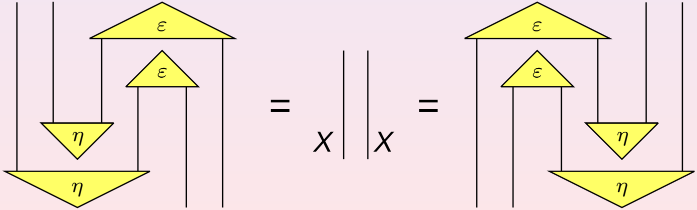
Proof of Lemma 1

Lemma 2

If

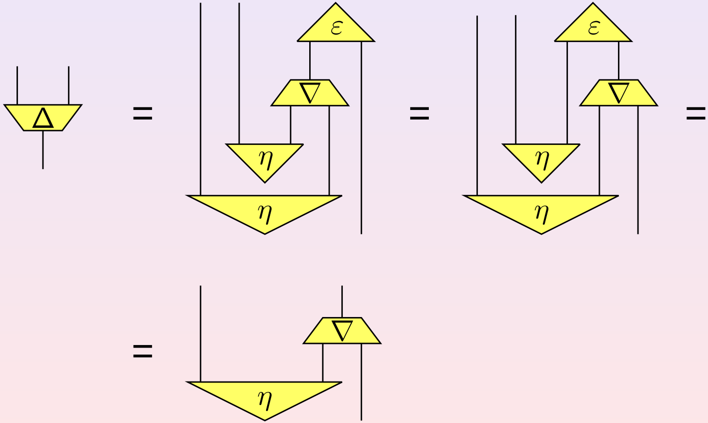


then



Proof of Lemma 1

Using Lemma 2, and the fact that (b) implies $\nabla = \Delta^{\ddagger} = \Delta^{*}$, we get



The message of the proof

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There is more to categories than just diagram chasing.

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There is more to categories than just diagram chasing.

There is also **picture chasing.**

Classical objects

Definition

A *classical object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Consequences

Upshot

The Frobenius condition (c) assures the preservation of the abstraction operation under $\ddagger$.

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The Frobenius condition (c) assures the preservation of the abstraction operation under $\ddagger$.

This leads to entanglement.

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Definition

Two vectors $\varphi, \psi \in \mathcal{C}(A)$ in a ‡-monoidal category are *orthonormal* if their inner product is idempotent:

$$\langle \varphi \mid \psi \rangle \quad = \quad \langle \varphi \mid \psi \rangle^2$$

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Proposition

Any two base vectors are orthonormal.
In particular, any two variables in a polynomial category are orthonormal.

Consequences

Definition

A classical object X is *standard* if it is generated by its base vectors

$$\mathcal{B}(X) \quad = \quad \{\varphi \in \mathcal{C}(X) | \ (\kappa x.\, x \otimes x)\varphi = \varphi \otimes \varphi\}$$

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Proposition

There are classical objects with no base vectors.

Consequences

Example

In $(\mathbf{Rel}, \times, 1, \ddagger = \mathrm{Id})$, take any $A > 3$ and

$$X = \{\{a, b\} \mid a, b \in A\}$$

Define $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ by

$$\begin{array}{ccc} \{a, b\} & \Delta & (\{a, c, \}, \{b, c\}) \\ \{a\} & \top & \{*\} \end{array}$$

Then $(\kappa x.\, x \otimes x)\varphi$ is entangled for every φ.

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The example lifts to **Hilb** as $X = A \underset{S}{\otimes} A$.

Outline

Variables in teleportation

This was not presented