Geometry of abstraction in quantum computation

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- Objective
- λ-abstraction
- λ-abstraction in categories

Graphic notation

Geometry of abstraction
- Abstraction with pictures
- Consequences

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- ‡-monoidal categories
- Quantum objects
- Abstraction in ‡-monoidal categories
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Categorical quantum mechanics

Q: Why (how) does it work?

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Q: Why (how) does it work?

A: ‡-compact/scc categories capture
the logically relevant structure of **Hilb**.

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Task

Rational reconstruction of the "logically relevant structure".

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Rational reconstruction of the "logically relevant structure".

- $\otimes, \ddagger$ — partitions and interactions
- $\oplus$ — base decompositions

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Pro: Need a computational base.

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Con: Not preserved on the states.

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Rational reconstruction of the "logically relevant structure".

- $\otimes, \ddagger$ — partitions and interactions
- $\oplus$ — base decompositions

Pro: Need a computational base.
Con: Not preserved on the states.

Proposal: Classical objects

Where do they come from?

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Example

$$\frac{\dfrac{f : \Omega \longrightarrow \Omega : x \mapsto f(x)}{f' : \Omega \times \Omega \xrightarrow{\sim} \Omega \times \Omega : (x, y) \mapsto (x, f(x) \oplus y)}}{U_f : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B} : |x, y\rangle \mapsto |x, f(x) \oplus y\rangle}$$

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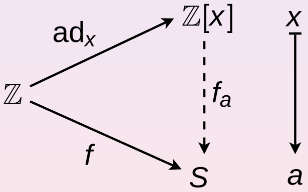
Abstraction in computation

- counterpart of *implementation*:
  - "...whatever x and y might be...
- interface specification
  - denote abstract data by variables: *copiable, deletable*

λ-abstraction

$$\frac{\mathbb{Z}^2 \longrightarrow \mathbb{Z}[x] \;:\; (a,b) \mapsto ax^3 + bx + 1}{\mathbb{Z}^2 \longrightarrow \mathbb{Z}^{\mathbb{Z}} \;:\; (a,b) \mapsto \lambda x.\, ax^3 + bx + 1}$$



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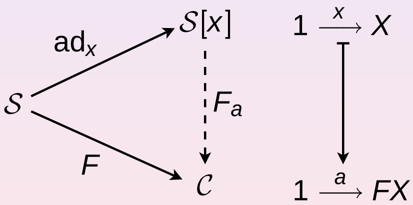
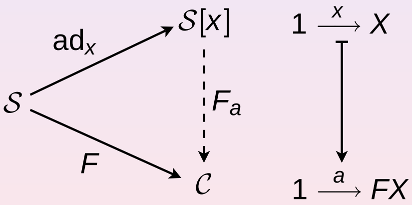
λ-abstraction in cartesian closed categories

$$\frac{A \xrightarrow{f_x} B \quad \text{in } \mathcal{S}[x : X]}{A \xrightarrow{\lambda_x . f_x} B^X \text{ in } \mathcal{S}}$$



λ-abstraction in cartesian closed categories

$$
\begin{array}{ccc}
A \xrightarrow{\langle\varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](A,B) & A \xrightarrow{f_x} B \\
\uparrow & \updownarrow & \downarrow \\
A \xrightarrow{\varphi} B^X & \mathcal{S}(A, B^X) & A \xrightarrow{\lambda x.f_x} B^X
\end{array}
$$



λ-abstraction in cartesian closed categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \longrightarrow X$.

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Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian closed category, and $\mathcal{S}[x]$ the free cartesian closed category generated by $\mathcal{S}$ and $x : 1 \to X$.

Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ has a right adjoint $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \;:\; A \mapsto A^X$ and the transpositions

$$\begin{array}{ccc} A \xrightarrow{\langle\varphi,x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{f_x} B \\ \Big\uparrow & \uparrow\downarrow & \Big\downarrow \\ A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x.f_x} B^X \end{array}$$

model λ-abstraction and application.

λ-abstraction in cartesian closed categories

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$$\begin{array}{ccc} A \xrightarrow{\langle\varphi, x\rangle} B^X \times X \xrightarrow{\epsilon} B & \mathcal{S}[x](\mathrm{ad}_x A, B) & A \xrightarrow{f_x} B \\ \Big\uparrow & \circlearrowleft\ \circlearrowright & \Big\downarrow \\ A \xrightarrow{\varphi} B^X & \mathcal{S}(A, \mathrm{ab}_x B) & A \xrightarrow{\lambda x.\, f_x} B^X \end{array}$$

model λ-abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the power monad $(-)^X$.

κ-abstraction in cartesian categories

Theorem (Lambek, Adv. in Math. 79)

Let $\mathcal{S}$ be a cartesian category, and $\mathcal{S}[x]$ the free cartesian category generated by $\mathcal{S}$ and $x : 1 \to X$.

Then the inclusion $\mathrm{ad}_x : \mathcal{S} \longrightarrow \mathcal{S}[x]$ *has a* left *adjoint* $\mathrm{ab}_x : \mathcal{S}[x] \longrightarrow \mathcal{S} \ : \ A \mapsto X \times A$ *and the transpositions*

$$\begin{array}{ccc} A \xrightarrow{\langle x, id\rangle} X \times A \xrightarrow{\varphi} B & \mathcal{S}[x](A, \mathrm{ad}_x B) & A \xrightarrow{f_x} B \\ \uparrow & \uparrow\downarrow & \downarrow \\ X \times A \xrightarrow{\varphi} B & \mathcal{S}(\mathrm{ab}_x A, B) & X \times A \xrightarrow{\kappa x. f_x} B \end{array}$$

model first order abstraction and application.

$\mathcal{S}[x]$ is isomorphic with the Kleisli category for the product comonad $X \times (-)$.

κ-abstraction in *monoidal* categories

Theorem (DP, MSCS 95)

Let $\mathcal{C}$ *be a* monoidal *category, and* $\mathcal{C}[x]$ *the free* monoidal *category generated by* $\mathcal{C}$ *and* $x : 1 \to X$.

Then the strong adjunctions $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *are in one-to-one correspondence with the internal comonoid structures on* X. *The transpositions*

$$\frac{A \xrightarrow{x \otimes A} X \otimes A \xrightarrow{\varphi} B}{X \otimes A \xrightarrow{\varphi} B} \qquad \frac{\mathcal{C}[x](A, \mathrm{ad}_x B)}{\mathcal{C}(\mathrm{ab}_x A, B)} \qquad \frac{A \xrightarrow{f_x} B}{X \otimes A \xrightarrow{\kappa x.f_x} B}$$

model action abstraction and application.

$\mathcal{C}[x]$ *is isomorphic with the Kleisli category for the comonad* $X \otimes (-)$, *induced by any of the comonoid structures.*

κ-abstraction in monoidal categories

Task

Extend this to Categorical Quantum Mechanics.

κ-abstraction in monoidal categories

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Extend this to Categorical Quantum Mechanics.

Problem

Lots of complicated diagram chasing.

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Task

Extend this to Categorical Quantum Mechanics.

Problem

Lots of complicated diagram chasing.

Solution?

What does abstraction mean graphically?

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Objects

$$X \otimes A \otimes B \otimes D$$

.

Identities

.

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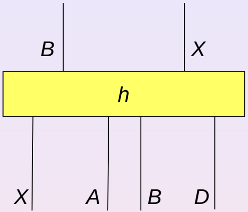
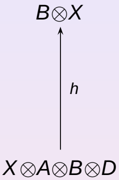
Morphisms





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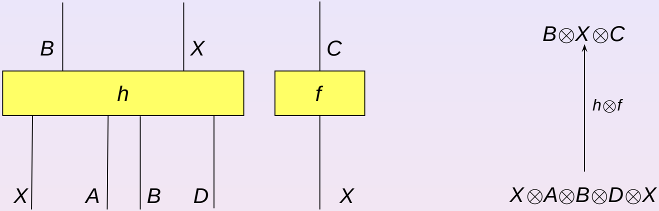
Tensor (parallel composition)



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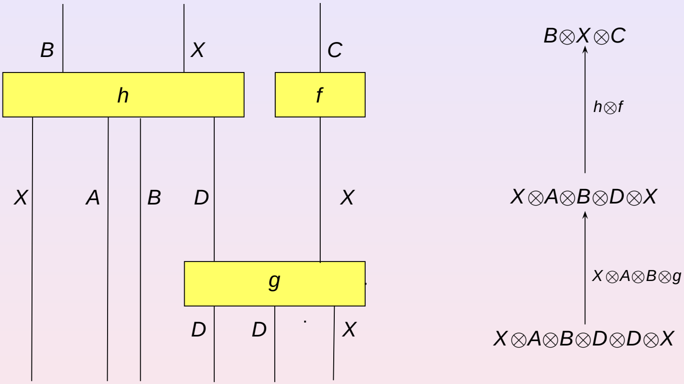
Sequential composition



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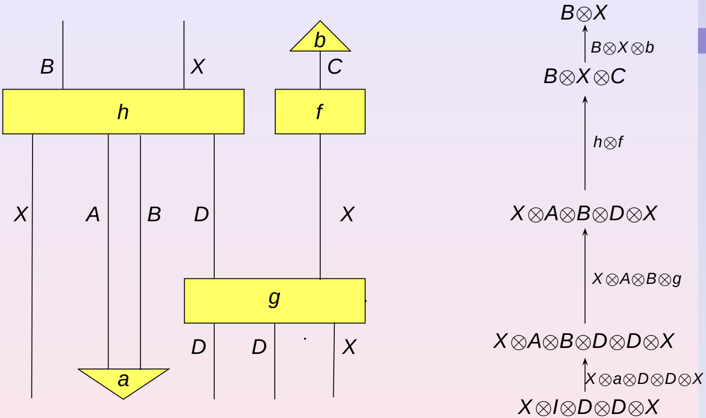
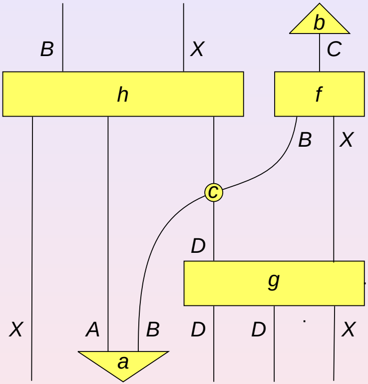
Elements (vectors) and coelements (functionals)



Symmetry



$$B \otimes X$$

$$\uparrow B \otimes X \otimes b$$

$$B \otimes X \otimes C$$

$$\uparrow h \otimes f$$

$$X \otimes A \otimes D \otimes B \otimes X$$

$$\uparrow X \otimes A \otimes c \otimes X$$

$$X \otimes A \otimes B \otimes D \otimes X$$

$$\uparrow X \otimes A \otimes B \otimes g$$

$$X \otimes A \otimes B \otimes D \otimes D \otimes X$$

$$\uparrow X \otimes a \otimes D \otimes D \otimes X$$

$$X \otimes I \otimes D \otimes D \otimes X$$

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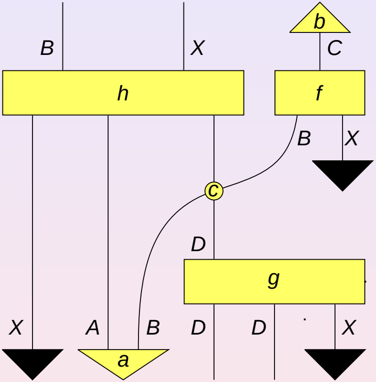
Polynomials



$$
\begin{array}{c}
B \otimes X \\
\uparrow B\otimes X\otimes b \\
B\otimes X\otimes C \\
\uparrow h\otimes f \\
X\otimes A\otimes D\otimes B\otimes X \\
\uparrow \mathrm{id}\otimes x \\
X\otimes A\otimes D\otimes B\otimes I \\
\uparrow X\otimes A\otimes c\otimes r \\
X\otimes A\otimes B\otimes D \\
\uparrow X\otimes A\otimes B\otimes g \\
X\otimes A\otimes B\otimes D\otimes D\otimes X \\
\uparrow x\otimes a\otimes D\otimes D\otimes x \\
I\otimes I\otimes D\otimes D\otimes I
\end{array}
$$

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Abstraction with pictures

Theorem (again)

Let $\mathcal{C}$ be a symmetric monoidal category, and $\mathcal{C}[x]$ the free symmetric monoidal category generated by $\mathcal{C}$ and $x : 1 \to X$.

Then there is a one-to-one correspondence between

- *adjunctions* $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *satisfying*
  1. $\mathrm{ab}_x(A \otimes B) = \mathrm{ab}_x(A) \otimes B$
  2. $\eta(A \otimes B) = \eta(A) \otimes B$
  3. $\eta_I = x$

  and
- *commutative comonoids on X.*

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  and
- *commutative comonoids on X.*

$\mathcal{C}[x]$ is isomorphic with the Kleisli category for the commutative comonad $X \otimes (-)$, induced by any of the comonoid structures.

Proof ($\downarrow$)

Given $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$, conditions 1.-3. imply

- $\mathrm{ab}_x(A) = X \otimes A$
- $\eta(A) = x \otimes A$

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Proof (↓)

Therefore the correspondence

$$\mathcal{C}\big(\mathrm{ab}_x(A), B\big) \rightleftarrows \mathcal{C}[x]\big(A, \mathrm{ad}_x(B)\big)$$

Proof ($\downarrow$)

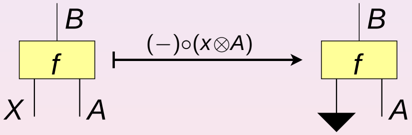
. . . is actually

$$\mathcal{C}(X \otimes A, B) \rightleftarrows \mathcal{C}[x](A, B)$$

Proof (↓)

...with

$$\mathcal{C}(X \otimes A, B) \rightleftarrows \mathcal{C}[x](A, B)$$



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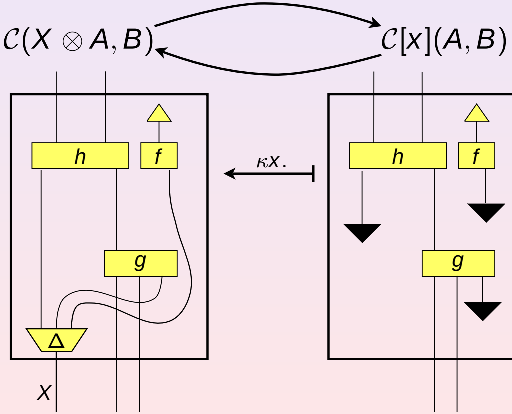
Proof (↓)

. . . and



Proof (↓)

The bijection corresponds to the conversion:

$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\rightleftarrows}} \mathcal{C}[x](A, B) \qquad \cong$$

$$\left(\kappa x.\ \varphi(x)\right) \circ \left(x \otimes A\right) = \varphi(x) \qquad (\beta\text{-rule}$$

$$\kappa x.\ \left(f \circ (x \otimes A)\right) = f \qquad (\eta\text{-rule}$$

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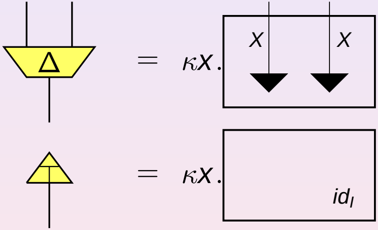
Proof (↓)

The comonoid structure $(X, \Delta, \top)$ is



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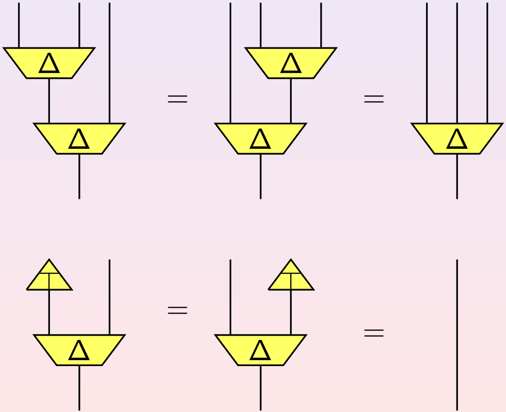
Proof (↓)

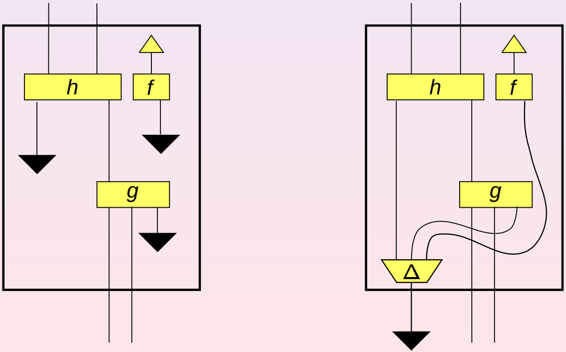
The conversion rules imply the comonoid laws



Proof (↑)

Given $(X, \Delta, \top)$, use its copying and deleting power, and the symmetries, to normalize every $\mathcal{C}[x]$-arrow:

$$\varphi(x) \quad = \quad \overline{\varphi} \circ (x \otimes A)$$



Proof ($\uparrow$)

Then set $\kappa x.\, \varphi(x) = \overline{\varphi}$ to get

$$\mathcal{C}(X \otimes A, B) \underset{\kappa x.}{\overset{(-)\circ(x\otimes A)}{\rightleftarrows}} \mathcal{C}[x](A, B) \qquad \cong$$

$$\big(\kappa x.\, \varphi(x)\big) \circ \big(x \otimes A\big) = \varphi(x) \qquad (\beta\text{-rule}$$

$$\kappa x.\, \big(f \circ (x \otimes A)\big) = f \qquad (\eta\text{-rule}$$

Remark

- $\mathcal{C}[x] \cong \mathcal{C}_{X\otimes}$ and $\mathcal{C}[x,y] \cong \mathcal{C}_{X\otimes Y\otimes}$, reduce the finite polynomials to the Kleisli morphisms.

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- But the extensions $\mathcal{C}[\mathcal{X}]$, where $\mathcal{X}$ is large are also of interest.

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- $\mathcal{C}[x] \cong \mathcal{C}_{X\otimes}$ and $\mathcal{C}[x,y] \cong \mathcal{C}_{X\otimes Y\otimes}$, reduce the finite polynomials to the Kleisli morphisms.
- But the extensions $\mathcal{C}[\mathcal{X}]$, where $\mathcal{X}$ is large are also of interest.
  - Cf. $\mathbb{N}[\mathbb{N}]$, Set[Set], and $CPM(\mathcal{C})$.

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Interpretation

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Upshot

In symmetric monoidal categories,
abstraction applies just to copiable and deletable data.

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Upshot

In symmetric monoidal categories,
abstraction applies just to copiable and deletable data.

Definition

A vector $\varphi \in \mathcal{C}(I, X)$ is a *base vector* (or a *set-like element*) with respect to the abstraction operation κx if it can be copied and deleted in $\mathcal{C}[x]$

$$(\kappa x.x \otimes x) \circ \varphi = \varphi \otimes \varphi$$
$$(\kappa x.\mathrm{id}_I) \circ \varphi = \mathrm{id}_I$$

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$$\begin{aligned} (\kappa x. x \otimes x) \circ \varphi &= \varphi \otimes \varphi \\ (\kappa x. \mathrm{id}_I) \circ \varphi &= \mathrm{id}_I \end{aligned}$$

Proposition

$\varphi \in \mathcal{C}(I, X)$ is a *base vector* with respect to κx if and only if it is a homomorphism for the comonoid structure $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ corresponding to κx.

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Definitions

A ‡-*category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

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‡-monoidal categories

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A ‡*-category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a ‡-category $\mathcal{C}$ is called *unitary* if $f^{\ddagger} = f^{-1}$.

‡-monoidal categories

Definitions

A ‡-*category* $\mathcal{C}$ comes with ioof $\ddagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$.

A morphism f in a ‡-category $\mathcal{C}$ is called *unitary* if $f^\ddagger = f^{-1}$.

A (symmetric) monoidal category $\mathcal{C}$ is ‡-*monoidal* if its monoidal isomorphisms are unitary.

‡-monoidal categories

Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

‡-monoidal categories

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- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

in every ‡-monoidal category we can define

- *abstract inner product*

$$\begin{array}{rcl} \langle - | - \rangle_A \; : \; \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathbb{I} \\ (\varphi, \psi : I \longrightarrow A) & \longmapsto & \left(I \xrightarrow{\varphi} A \xrightarrow{\psi^{\ddagger}} I \right) \end{array}$$

‡-monoidal categories

Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

in every ‡-monoidal category we can define

- *abstract inner product*

$$\begin{array}{rcl} \langle - | - \rangle_A \; : \; \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathbb{I} \\ (\varphi, \psi : I \longrightarrow A) & \longmapsto & \left(I \xrightarrow{\varphi} A \xrightarrow{\psi^\ddagger} I \right) \end{array}$$

- *partial inner product*

$$\begin{array}{rcl} \langle - | - \rangle_{AB} \; : \; \mathcal{C}(AB) \times \mathcal{C}(A) & \longrightarrow & \mathcal{C}(B) \\ (\varphi : I \to A \otimes B, \psi : I \to A) & \longmapsto & \left(I \xrightarrow{\varphi} A \otimes B \xrightarrow{\psi^\ddagger \otimes B} B \right) \end{array}$$

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Using the monoidal notations for:

- vectors: $\mathcal{C}(A) = \mathcal{C}(I, A)$
- scalars: $\mathbb{I} = \mathcal{C}(I, I)$

in every ‡-monoidal category we can define

- *abstract inner product*

$$\begin{array}{rcl} \langle - | - \rangle_A \;:\; \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathbb{I} \\ (\varphi, \psi : I \longrightarrow A) & \longmapsto & \left(I \xrightarrow{\varphi} A \xrightarrow{\psi^\ddagger} I \right) \end{array}$$

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- *entangled vectors* $\eta \in \mathcal{C}(A \otimes A)$, such that $\forall \varphi \in \mathcal{C}(A)$

$$\langle \eta | \varphi \rangle_{AA} = \varphi$$

‡-monoidal categories

Proposition

For every object A in a ‡-monoidal category $\mathcal{C}$ holds
(a) $\iff$ (b) $\Longleftarrow$ (c),

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For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

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‡-monoidal categories

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(a) $\eta \in \mathcal{C}(A \otimes A)$ is entangled

(b) $\varepsilon = \eta^\ddagger \in \mathcal{C}(A \otimes A, I)$ internalizes the inner product

$$\varepsilon \circ (\psi \otimes \varphi) \quad = \quad \langle \varphi | \psi \rangle$$

‡-monoidal categories

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$$\varepsilon \circ (\psi \otimes \varphi) \quad = \quad \langle \varphi | \psi \rangle$$

(c) (η, ε) realize the self-adjunction $A \dashv A$, in the sense

$$A \xrightarrow{\eta \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes \varepsilon} A \quad = \quad \mathrm{id}_A$$

$$A \xrightarrow{A \otimes \eta} A \otimes A \otimes A \xrightarrow{\varepsilon \otimes A} A \quad = \quad \mathrm{id}_A$$

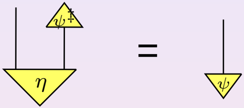
The three conditions are equivalent if I generates $\mathcal{C}$.

‡-monoidal categories

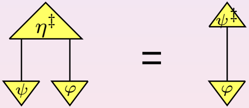
Proposition in pictures

For every object A in a ‡-monoidal category $\mathcal{C}$ holds (a) $\iff$ (b) $\Longleftarrow$ (c), where

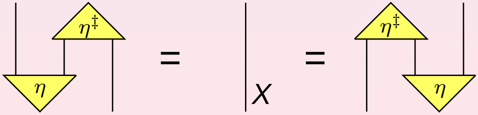
(a)



(b)



(c)



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Definition

A *quantum object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

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Definition

A *quantum object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Remark

The subcategory of quantum objects in any ‡-monoidal category is ‡-compact (strongly compact) — with all objects self-adjoint.

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Theorem

Let $\mathcal{C}$ be a ‡-monoidal category,
and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ a comonoid that induces
$\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ be a ‡-monoidal category,
and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces*
$\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ *be a ‡-monoidal category,*
and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces*
$\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

(a) $\mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *creates* $\ddagger : \mathcal{C}[x]^{op} \longrightarrow \mathcal{C}[x]$
such that $\langle x | x \rangle = x^{\ddagger} \circ x = \mathrm{id}_I$.

Abstraction in ‡-monoidal categories

Theorem

Let $\mathcal{C}$ *be a* ‡*-monoidal category, and* $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces* $\mathrm{ab}_x \dashv \mathrm{ad}_x : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

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(b) $\eta = \Delta \circ \perp$ *and* $\varepsilon = \eta^{\ddagger} = \nabla \circ \top$ *realize* $X \dashv X$.

Abstraction in $\ddagger$-monoidal categories

Theorem

Let $\mathcal{C}$ be a $\ddagger$-monoidal category, and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ a comonoid that induces $\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

(a) $\mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *creates* $\ddagger : \mathcal{C}[x]^{op} \longrightarrow \mathcal{C}[x]$ *such that* $\langle x|x\rangle = x^\ddagger \circ x = \mathrm{id}_I$.

(b) $\eta = \Delta \circ \bot$ *and* $\varepsilon = \eta^\ddagger = \nabla \circ \top$ *realize* $X \dashv X$.

(c) $(X \otimes \nabla) \circ (\Delta \otimes X) = \Delta \circ \nabla = (\nabla \otimes X) \circ (X \otimes \Delta)$

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Theorem

Let $\mathcal{C}$ *be a* ‡*-monoidal category,*
and $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ *a comonoid that induces*
$\mathrm{ab}_X \dashv \mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$.

Then the following conditions are equivalent:

(a) $\mathrm{ad}_X : \mathcal{C} \longrightarrow \mathcal{C}[x]$ *creates* $\ddagger : \mathcal{C}[x]^{op} \longrightarrow \mathcal{C}[x]$
such that $\langle x|x\rangle = x^{\ddagger} \circ x = \mathrm{id}_I$.

(b) $\eta = \Delta \circ \bot$ *and* $\varepsilon = \eta^{\ddagger} = \nabla \circ \top$ *realize* $X \dashv X$.

(c) $(X \otimes \nabla) \circ (\Delta \otimes X) = \Delta \circ \nabla = (\nabla \otimes X) \circ (X \otimes \Delta)$

where $X \otimes X \xrightarrow{\nabla} X \xleftarrow{\bot} I$ *is the induced monoid*

$$\begin{aligned} \nabla &= \Delta^{\ddagger} \\ \bot &= \top^{\ddagger} \end{aligned}$$

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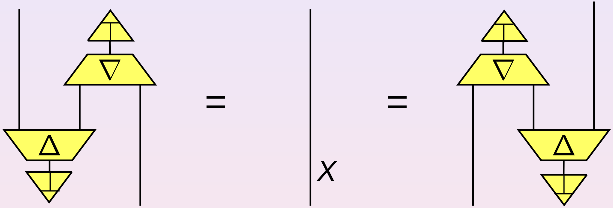
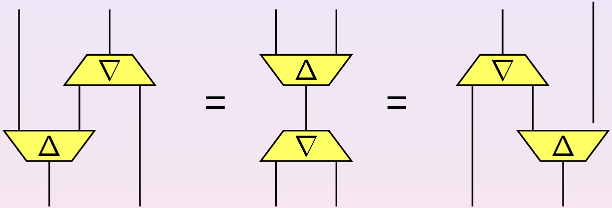
Theorem in pictures

(b) 

Abstraction in ‡-monoidal categories

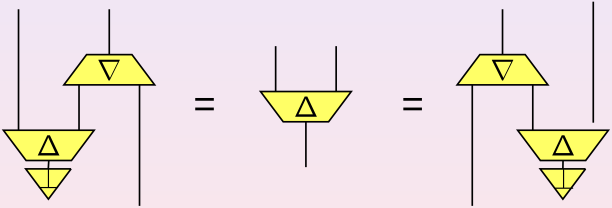
Theorem in pictures

(c) 

Proof of (b)⟹(c)

Lemma 1

If (b) holds then



Geometry of quantum abstraction

Dusko Pavlovic

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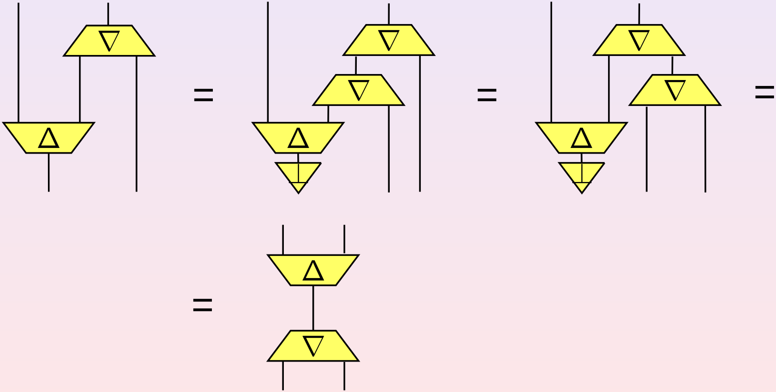
Proof of (b)$\Longrightarrow$(c)

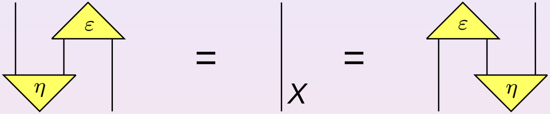
Then (c) also holds because



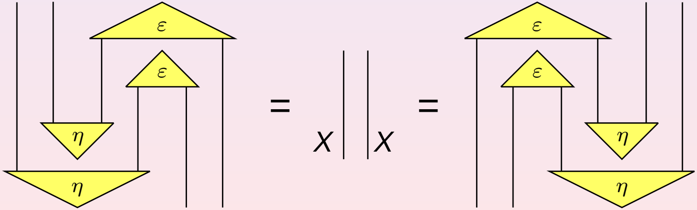
Proof of Lemma 1

Lemma 2

If

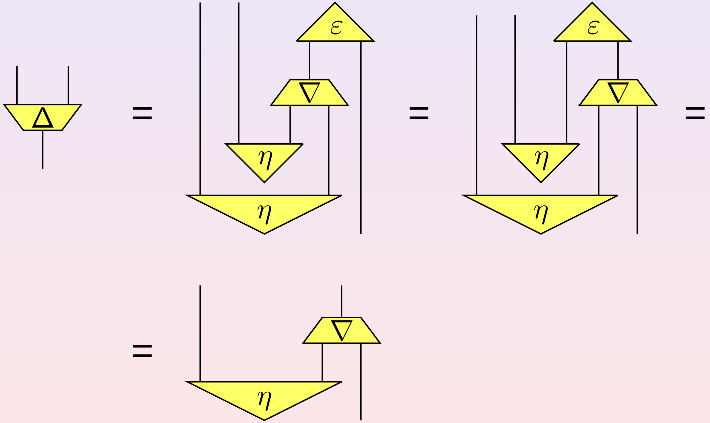


then



Proof of Lemma 1

Using Lemma 2, and the fact that (b) implies $\nabla = \Delta^{\ddagger} = \Delta^{*}$, we get



The message of the proof

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The message of the proof

There is more to categories than just diagram chasing.

The message of the proof

There is more to categories than just diagram chasing.

There is also **picture chasing.**

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Definition

A *classical object* in a ‡-monoidal category is an object equipped with the structure from the preceding proposition.

Consequences

Upshot

The Frobenius condition (c) assures the preservation of the abstraction operation under $\ddagger$.

Consequences

Upshot

The Frobenius condition (c) assures the preservation of the abstraction operation under ‡.

This leads to entanglement.

Consequences

Definition

Two vectors $\varphi, \psi \in \mathcal{C}(A)$ in a $\ddagger$-monoidal category are *orthonormal* if their inner product is idempotent:

$$\langle \varphi \mid \psi \rangle \quad = \quad \langle \varphi \mid \psi \rangle^2$$

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Two vectors $\varphi, \psi \in \mathcal{C}(A)$ in a $\ddagger$-monoidal category are *orthonormal* if their inner product is idempotent:

$$\langle \varphi \mid \psi \rangle \quad = \quad \langle \varphi \mid \psi \rangle^2$$

Proposition

Any two base vectors are orthonormal.
In particular, any two variables in a polynomial category are orthonormal.

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Definition

A classical object X is *standard* if it is generated by its base vectors

$$\mathcal{B}(X) \quad = \quad \{\varphi \in \mathcal{C}(X) | \ (\kappa x.\, x \otimes x)\varphi = \varphi \otimes \varphi\}$$

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Dusko Pavlovic

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Definition

A classical object X is *standard* if it is generated by its base vectors

$$\mathcal{B}(X) \quad = \quad \{\varphi \in \mathcal{C}(X) | \ (\kappa x.\ x \otimes x)\varphi = \varphi \otimes \varphi\}$$

in the sense that

$$\forall f, g \in \mathcal{C}(X, Y).\ (\forall \varphi \in \mathcal{B}(X).\ f\varphi = g\varphi) \Longrightarrow f = g$$

Consequences

Definition

A classical object X is *standard* if it is generated by its base vectors

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in the sense that

$$\forall f, g \in \mathcal{C}(X, Y).\; (\forall \varphi \in \mathcal{B}(X).\, f\varphi = g\varphi) \Longrightarrow f = g$$

Proposition

There are classical objects with no base vectors.

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Example

In $(\mathbf{Rel}, \times, 1, \ddagger = \mathrm{Id})$, take any $A > 3$ and

$$X = \{\{a, b\} \mid a, b \in A\}$$

Define $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ by

$$\begin{array}{ccc} \{a, b\} & \Delta & (\{a, c,\}, \{b, c\}) \\ \{a\} & \top & \{*\} \end{array}$$

Then $(\kappa x.\, x \otimes x)\varphi$ is entangled for every φ.

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Define $X \otimes X \xleftarrow{\Delta} X \xrightarrow{\top} I$ by

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Then $(\kappa x.\, x \otimes x)\varphi$ is entangled for every φ.

The example lifts to **Hilb** as $X = A \underset{S}{\otimes} A$.

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Variables in teleportation

This was not presented