Item 3: Detailed description of proposed research

THE ROAD TO A NEW QUANTUM FORMALISM: CATEGORIES AS A CANVAS FOR QUANTUM FOUNDATIONS

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Contents

1 Failures of the past

It was John von Neumann himself, the creator of the quantum mechanical Hilbert space formalism [58], who three year after publicising his brainchild was the first to denounce it. In 1935, he stated

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more"

in a letter to his colleague Garrett Birkhoff [11, 51]. Shortly thereafter they published "The logic of quantum mechanics" [12]. A new area of research emerged which is still ongoing [32, 37]. However, besides some notable exceptions such as Gleason's theorem, few current textbooks even mention any results which have emerged from 70 years of quantum logic research. von Neumann himself also fairly quickly denounced this research program. Unfortunately, the failure of this program has created a climate in which foundational structural research into quantum theory is not taken very seriously by the broader physics community. It is therefore important to understand why this program failed, and we explore this below. At the same time this will enable us to put forward the motivation behind our approach, outlined in Section 3, as well some preliminaries to it.

Why did this 'quantum logic' program fail? We identify *six key reasons*. While in the narrative we focus on quantum logic, our criticisms also apply to several other attempts at constructing a new quantum formalism.

It was a negative approach. Quantum logic emphasised the failure of a structural property — typically, failure of the distributive law for the lattice of (closed) subspaces of a Hilbert space — rather than laying bear a true *feature* of quantumness and the new *capabilities* it enables, or something that can be used in a *constructive* way. The *non*-distributivity of lattices indeed does not tell us what quantumness actually *is*, but only what it *fails to be*, in terms of *obstructive content*: "You shall not use the distributive law!" This argument in fact applies to many structural approaches to quantum theory, such as *non*-commutativity of algebras, *non*-Kolmogorovianity of probability theories, and so on. Our stance is that structural accounts of quantumness should provide *constructive content*.

Composing systems was not a primitive. The Birkhoff–von Neumann (BvN) approach, as well as many other approaches, take quantum measurement on a single system to be the main primitive. Conceptual analysis of this primitive then results in favouring some mathematical structure, such as lattices in the case of BvN. Given this mathematical structure one then tries to reconstruct as much as possible of quantum theory. But quantum logic notoriously failed to reproduce the quantum mechanical description of compound systems, a (if not *the*) key feature of quantum theory. Moreover, one could argue physically that compoundness is an even more primitive concept than measurement, since the latter requires a compound of the system being measured with the measurement device. Hence, our stance is that *compoundness* should be a primitive when axiomatizing quantum theory.

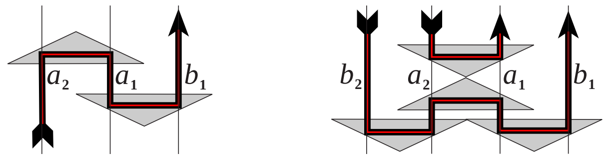
Lack of demonstrated physical phenomena and resulting unambitiousness. The main goal of quantum logic was to obtain a structural understanding of superposition and the resulting probability measures — cf. Gleason's theorem. But probabilistic behaviour under measurement only captures a small part of quantum theory. When we look at a quantum phenomenon such as quantum teleportation [10, 14], only recently experimentally demonstrated, it is clear that much more is going on. In brief, quantum teleportation is a method of transferring a quantum state from party a to party b. Party a possesses two qubits Q_{a_1} and Q_{a_2}, and party b possesses one qubit Q_{b_1}. Qubit Q_{a_2} is in an unknown state $|\psi\rangle$ and Q_{a_1} and Q_{b_1} are jointly in a Bell state. If a now measures Q_{a_1} and Q_{a_2} in the Bell basis, reports the outcome to b, who depending on the outcome, performs a particular unitary operation on Q_{b_1}, then Q_{b_1} will be in state $|\psi\rangle$. Clearly the following are all essential for this phenomenon:

- the initial availability of an entangled state (on systems Q_{a_2} and Q_b);
- measurement against entangled states (the Bell-base measurement);
- the effect of measuring one system (the compound of Q_{a_1} and Q_{a_2}) on another system (system Q_b);
- and last but not least, reliance on the measurement outcomes when picking appropriate unitary operations; in other words, besides the *flow of information from the quantum to the classical* (= measurement) there is also a *flow of information from the classical to the quantum* (= correction operations).

In fact, the probabilistic structure is totally irrelevant for quantum teleportation since the ultimate result, teleporting the state of the qubit, is obtained with certainty, independent from the measurement stochastics. When passing to the slightly more involved phenomenon of entanglement swapping [60] there is even more. Briefly, we do exactly the same as for teleportation with the exception that there is a fourth qubit Q_{b_2}, with also Q_{a_2} and Q_{b_2} jointly in a Bell state. Initially we have entanglement Q_{a_1}— Q_{b_1} and Q_{a_2}— Q_{b_2}, but after carrying out the protocol we will instead have Q_{a_1}— Q_{a_2} and Q_{b_1}— Q_{b_2}. Crucial for entanglement swapping is the fact that:

- the collapsed state is an actual part of the phenomenon.

We recognise an essentially topological structure in these two phenomena [18, 19, 20, 46]:



This topological structure can be analysed independently from the probabilistic structures. Our stance is that any structural approach to quantum theory should account for this essentially topological feature of the theory. In addition, it should also account for the classical-quantum interaction in terms of mutual flows of information, and should be able to describe all the other aspects mentioned above.

The considered structures were purely static, not dynamic. The above examples of quantum teleportation and entanglement swapping clearly indicate that there is a dynamic component to the description of quantum phenomena, even when we do not account for unitary evolution: measurement on one system causes an effect on another system. The passage from a static account of structure to a dynamical one boils down to the passage from *states* to *processes*. Our stance is that quantum phenomena involve not only states, but also processes at the purely descriptive level.

No explicit account of resources. Relatively recent results such as the no-cloning [59], no-deleting [48] and no-broadcasting [8] theorems reveal that quantum theory is a *resource-sensitive* theory: one should explicitly account for the number of times one wishes to use a system's state when specifying a procedure, since one is not able to copy a state and use it as many times as one would like. As we will see in the next paragraph, resource-sensitivity affects mathematical logic in an important way, and at its most basic level. However, quantum logic was formulated much too early to account for these important insights. Also, to even make sense of this important aspect, one obviously needs a good account of compoundness. Our stance is that a quantum formalism should be *resource-sensitive*.

Appropriate mathematical structures and logics did not yet exist. If we want to comply with all the above-stated demands we need to have appropriate structures at hand. The conception of logic and the corresponding models and structures have advanced enormously since the birth of quantum logic. One of the key changes has been the passage 'from *truth* to *provability*', that is, 'from *statics* to *dynamics*'. Even more recently resource-sensitivity has also become an important ingredient in logic, with the discovery of *linear logic* [40]; this has had radical consequences, and has found important applications in computer science. Linear logic allows explicit accounting for computational resources, by restricting the ability to copy and delete premises [40]. Bearing in mind the no-cloning and no-deleting theorems which apply to quantum data, as opposed to the deletability and unlimited copyability of classical data, models of linear logic seem to already incorporate these fundamental informatic constraints of quantum theory.

The mathematics of proof theory these days is the mathematics of *monoidal categories* — see [5] for a physicist-friendly introduction. Monoidal categories have been extremely successful in computer science as tools to describe semantics because of their generality, their ability to find common structure in many different situations, and their support for compositional modelling: analysing complex systems in terms of how they are built up, using a stock of basic operations of wide applicability, from simpler sub-systems. This leads to an algebraic view of systems which is both elegant and extremely effective in allowing concise descriptions of complex systems, algebraic manipulation of these descriptions, and which provides a basis for distinguishing between different types of systems, such as quantum or classical. Our stance is that these new *logics of interaction* are far more suited to describing quantum theory than what was available at the time of the birth of quantum logic.

Our approach, outlined in Section 3, complies with all the above-stated requirements.

2 But why would one want a new formalism anyway?

There is an important lesson to be learned for the development of computer science in the previous century, which only took place long after the current quantum formalism had been established.

Q: *Why do computer scientists invent new languages, rather than just program in 0s and 1s?*

A: Because otherwise there would be no internet, no mobile phones, no user-friendly operating systems, and so on. It is simply impossible for any human to produce a functioning product of significant complexity if he has to directly construct it bit-by-bit. Modern day programming interfaces are shaped around *high-level concepts* such as data types, compositionality, concurrency and feedback; that is, concepts which are directly connected to *information processing* and *information interaction*, and which are independent of some concrete underlying set such as the Booleans.

Q: *Why was quantum teleportation only discovered 60 years after the birth of the quantum formalism? Why was the no-cloning theorem only discovered after 50 years? Both admit very simple proofs and have important technological applications, in contrast to the structural results of Wigner, Gleason, Piron, Kochen, Specker and others, which are difficult and do not have immediate technological applications, but which were discovered much earlier.*

A: The usual quantum formalism does not tell us these basic principles, nor does it indicate that there is even a question to ask. It is far too *low-level* for that purpose, in the same way that it would be difficult to understand a computer program by staring at its representation in 0s and 1s. The only difference is that rather than arrays of 0s and 1s, we have arrays of complex numbers. We need to solve the following conceptual equation:

$$\frac{???}{\textbf{von Neumann quantum formalism}} \simeq \frac{\textbf{high-level language}}{\textbf{low-level language}}$$

One can express this in more physical terms:

$$\frac{???}{\textbf{quantum physics}} \simeq \frac{\textbf{geometry}}{\textbf{astronomy}}$$

Geometry was indeed crafted with physical space (and later time) as a guide. On the other hand, the complex matrix calculus was not crafted with quantum theory in mind, but as explicitly stated by von Neumann, was used because it was the only thing that was available at the time.

Paradigm shifts witness scientific progress. And a paradigm shift is what quantum informatics stands for: "Quantum weirdness is not a bug but a feature, ready to be exploited." When one truly uses something (here quantum theory), rather than judging from a distance, one starts to appreciate that thing in a very different way. And that is indeed what has happened. New phenomena have been discovered which are radically new and conceptually astonishing, such as quantum teleportation, entanglement swapping, computational speed-up and counterfactual phenomena. But while the paradigm shifts have continued to occur, the formalism and methods have remained the same. A better quantum formalism, one that reflects this new constructive appreciation of the quantum world, is long overdue.

3 High-level structures governing physics: a categorical approach

Recently we initiated an effort towards a new quantum formalism having the above in mind [1, 19, 20, 22, 24, 26, 29, 30, 52, 55]. Key features are an intuitive and purely diagrammatic calculus, which enables pictorial derivation of several protocols as well as computation of the quantum Fourier transform, and comprehension of quantum, classical and mixed data types. The corresponding categorical logic enables automation. We can report substantial initial success, particularly in the domain of applications to quantum informatics, reflected in the fact of having a series of papers accepted to leading computer science conferences which typically have acceptance rates around 1/4. In particular, "A categorical semantics of quantum protocols" [1], a paper jointly written with Abramsky, was the first paper involving quantum theory to ever have been accepted to the prestigious IEEE conference on Logic in Computer Science (LiCS). To this work I owe my faculty position at Oxford University Computing Laboratory. It also gave rise to several substantial research grants and fellowships to further develop the quantum informatic applications, as well as many invited addresses at conferences. "Interacting quantum observables" [24], a paper jointly written with

Duncan, was the first paper involving quantum theory to ever have been accepted to the Logic, Semantics & Theory track of the equally prestigious International Colloquium on Automata, Languages and Programming (ICALP).

Recently, the potential of this approach has also been recognised by the Foundations of Physics community. As a witness to this we mention some recent invited talks: *Deep Beauty: Mathematical Innovation and the Search for an Underlying Intelligibility of the Quantum World*, Princeton University, October 2007; *Operational Approaches to Quantum Theory*, CREA, Paris 5, October 2007; *Structure and Identity*, Royal Academy, Brussels, December 2007; *Symposium on Logic and Physics*, Utrecht University, January 2008; *New Directions in the Foundations of Physics*, American Center for Physics, College Park, April 2008.

The approach reveals high-level concepts which are key to quantum phenomena, while at the same time leaving a large degree of axiomatic freedom. It therefore has the potential to become a unifying framework for a variety of existing approaches to quantum foundations.

This is what this proposal is all about: to establish our approach as a genuine formal foundation for quantum theory, in which we expect to unify several existing approaches and then to go beyond. Compelling evidence for this is provided by our recent presentations of Spekkens' Toy Theory and C*-algebras within this approach, as well as promising developments towards describing BBLW convex theories, algebraic QFTs and Döring-Isham-Butterfield topoi, to name a few. We expect to obtain important insights on all of these approaches as well as to blend the key features of these within new models and axiom systems for quantum reasoning. For this purpose we would like to rely on Jamie Vicary (Imperial College London) and Eric Oliver Paquette (UdMontreal), who are both in the process of finishing their PhDs in Chris Isham's and Gilles Brassard's groups respectively. Both have already made very important contributions to the research program outlined here [26, 27, 30, 55, 56, 57]. In accordance with the FQXi guidelines, they would only work part-time on this project for which we are requesting funding. In the complementary time they will work on applications to quantum information, for which I have already secured funding.

3.1 Structure as phenomenology

In our approach we do not aim to provide a single concrete model describing quantum reality. Rather, we start with a very general structure, to which we adjoin extra *pieces of structure* which in a *positive* manner witness empirically observed physical facts, such as the existence of Bell states, that classical data can be copied and deleted, and that there exist complementary observables. These empirical facts are encoded in a very general fashion, but are sufficiently powerful that we can use them to derive important consequences, as we will see below. We can accommodate a large class of concrete models, which can be thought of as alternative foundations for quantum theory, and we expect to encompass many others, a case for which we provide ample evidence below. One point to stress is that this approach can both accommodate 'processes as operations' as well as 'processes happening in absence of an observer' — the first view is common in quantum informatic communities, while the second is essential for cosmological considerations.

3.1.1 General theories: †-symmetric monoidal categories

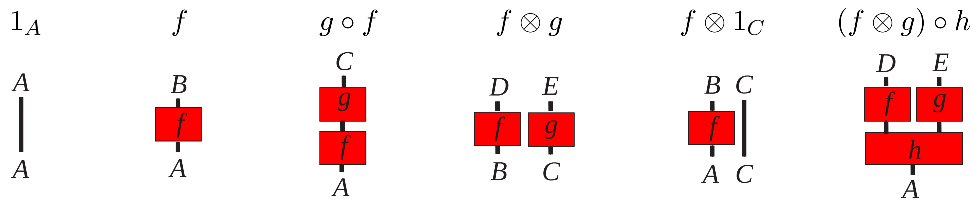
systems: We represent *types* of systems (or 'kinds' if you prefer), such as physical systems like photons and electrons, or classical data types, or combinations thereof (such as the pair consisting of a quantum system together with observed data), by their names $A, B, C, \ldots$. The trivial system is denoted by I.

operations and processes: We represent general operations and processes, such as the evolution of a system between time t_1 and time t_2, or the preparation of a system in a certain state, or a computation which takes data of type A as input and produces data of type B, or a measurement that takes a quantum system A as input, destroys it, and produces an outcome of data type B, by arrows $A \to B$ where A is the input type and B is the output type.

sequential composition: We represent the composite of two operations or processes $f : A \to B$ and $g : B \to C$, which are performed or occur one after the other, by $g \circ f : A \to C$. Doing nothing — or if you prefer, nothing occurs — is described for each system by an 'identity' operation $1_A : A \to A$.

compoundness: The joint system formed from A and B is denoted by $A \otimes B$, and the joint performance of operations $f : A \to C$ and $g : B \to D$ is denoted $f \otimes g : A \otimes B \to C \otimes D$. The symbol $\otimes$ is known as the *tensor*.

All these pieces of data, together with the obvious structural rules which mainly control how sequential and parallel composition interact, canonically make up a symmetric monoidal category [21], or SMC. But rather than stating these rules we will rely on a beautiful feature of this particular mathematical structure: it can be equivalently presented as a purely graphical calculus. This calculus can be traced back to Penrose's work in the 1970s [49], but it took 20 more years for its precise algebraic and topological significance to be settled [45]. Our variant emphasises the connection with the Dirac notation. Operations are represented by *boxes*, types of systems by *wires*, composition by connecting outputs and inputs by wires, and tensor by locating wires or boxes side by side, as depicted here:

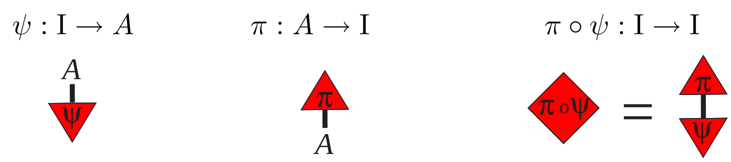


These diagrams should be read from bottom to top, so the second diagram represents the process $f : A \to B$. Typical axioms of the symmetric monoidal structure such as commutation of

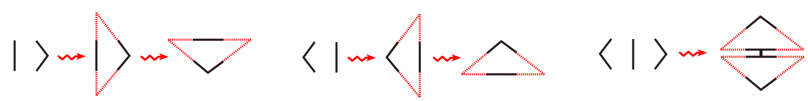
$$\begin{tikzcd} A \otimes C \arrow[r, "1_A \otimes g"] \arrow[d, "f \otimes 1_C"'] & A \otimes D \arrow[d, "f \otimes 1_D"] \\ B \otimes C \arrow[r, "1_B \otimes g"] & B \otimes D \end{tikzcd} \qquad \begin{tikzcd} A \otimes C \arrow[r, "\sigma_{A,C}"] \arrow[d, "f \otimes g"'] & C \otimes A \arrow[d, "g \otimes f"] \\ B \otimes D \arrow[r, "\sigma_{B,D}"] & D \otimes B \end{tikzcd}$$

depict as

Intuitively, we are allowed to 'slide' boxes along the wires, and also along crossings. The void type I — 'nothing' — is represented by 'no wire', which gives rise to triangles and diamonds:



Note that this indeed precisely captures Dirac's *kets*, *bras* and *bra-kets*:

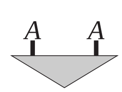


Taking the adjoint of a linear map, which exchanges kets and bras, is captured by picture reversal:

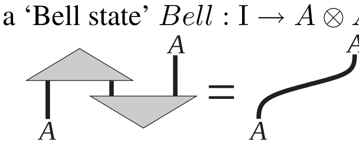
This adjoint allows us to define the inner product, as well as unitarity, positivity, and even *complete positivity* [23].

3.2 Accommodating Bell states: †-compact categories

We will now adjoin additional structure to this basic setting in order to build more specific theories. Firstly, we will assert the existence of Bell states (or the ability to prepare them if you wish) and their ability to realise phenomena such as teleportation. We require a system A to come with a 'Bell state' *Bell* : $\mathrm{I} \to A \otimes A$; that is, a 'triangle'

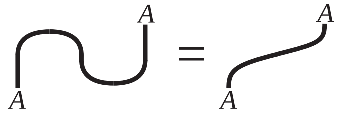


satisfying



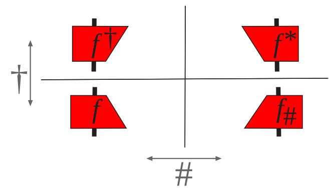
However, when rather than as a triangle we represent this quantum structure as a wire

satisfying

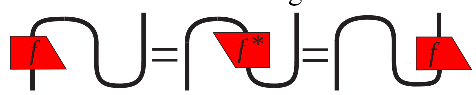


the axiom takes a more lucid form which boils down to ‘yanking a piece of rope’. This is a simple structure, but is already enough to abstractly capture *trace* (e.g. [20]), and also transposition and complex conjugation, respectively:

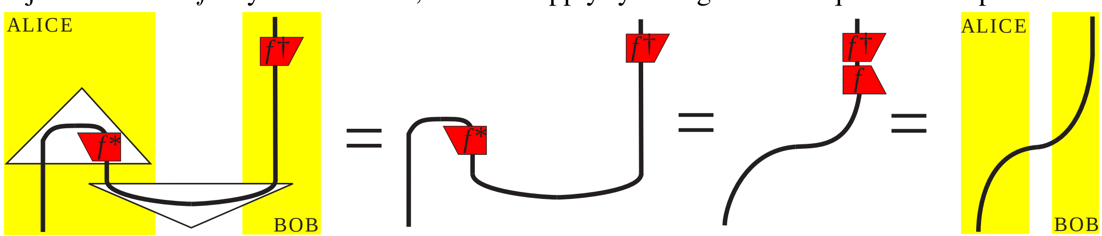
In particular, we have $(f^*)_\sharp = (f_\sharp)^\sharp = f^\dagger$. If we build an asymmetry into the box used to depict a process f, we can depict all of these by giving the box different orientations [52]:



It immediately also follows that we can now ‘slide’ boxes along wires:



To prove this just substitute f^* by its definition, and then apply ‘yanking’. Here is quantum teleportation:



We require f to be unitary, that is, $f^\dagger \circ f = 1_A$ and $f \circ f^\dagger = 1_B$. The required classical information flow is implicit in the dependency of the correction $f^\dagger$ on the effect $Bell^\dagger \circ (1_A \otimes f^*)$.

The next step is to make this classical data-flow explicit in purely diagrammatic terms.

3.3 Distinguishing the quantum and the classical: commutative special †-Frobenius algebras

We get traditional logic from linear logic by *adjoining structure which witnesses the ability to copy and delete data*:

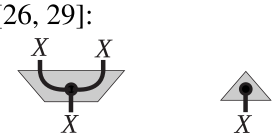
classical logic = linear logic + (copying, deleting)

Similarly we set:

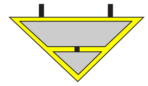
pure or mixed classicality = quantumness + (copying or broadcasting, deleting)

Hence rather than *quantizing a classical theory*, we do the opposite, *classicizing a quantum theory*. If one has a quantum system represented by a Hilbert space $\mathcal{H}$ then specifying a non-degenerate classical context means choosing an orthogonal basis $\{e_i\}$. Hence the resulting classical structure is the pair $(\mathcal{H}, \{e_i\})$, a Hilbert space with additional structure. One should note that this perspective is not that unfamiliar to quantum structures research: given the lattice $\mathbb{L}(\mathcal{H})$ of subspaces of $\mathcal{H}$, specifying a classical context means picking some Boolean algebra B and a monomorphism $\xi : B \to \mathbb{L}(\mathcal{H})$ of ortholattices, resulting in a pair $(\mathbb{L}(\mathcal{H}), \xi)$.

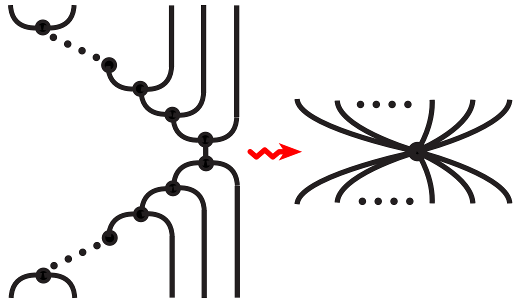
In our approach, a classical context for a quantum structure consists of two operations, $copy : X \to X \otimes X$ and $del : X \to \mathrm{I}$, respectively depicted as follows [26, 29]:



which ‘refine’ the Bell state, in the sense that we can ‘chop’ a big triangle into a small one and a trapezoid:



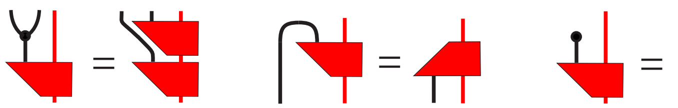
The structural requirements we impose on them are all conceptually very reasonable and translate in mathematical terms as a *commutative special* †*-Frobenius algebra* [29]. In diagrammatic terms these rules boil down to the assertion that any *connected* network involving copying, deleting, Bell states, and their adjoints is equal to a spider-shape [26], for which the only degrees of freedom are the number of inputs and the number of outputs:



A diagram like this is obtained by 'multiplying' the inputs repeatedly using $copy^\dagger : X \otimes X \to X$ until only a single X remains, then copying this repeatedly using $copy : X \to X \otimes X$ to give the correct number of outputs. The connection between these †-Frobenius algebras and classical structures is given by the following theorem [30]:

Theorem. Given a finite-dimensional Hilbert space X, there is a bijective correspondence between orthogonal bases for X and commutative special †-Frobenius algebras on X, where the basis corresponding to a particular algebra is given by those elements which can be perfectly copied by the algebra.

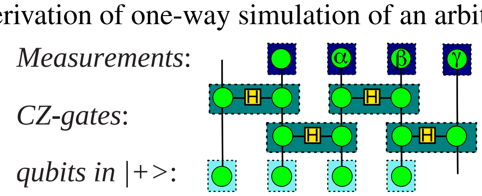
We 'define' measurements as boxes which satisfy the following equations [29]:



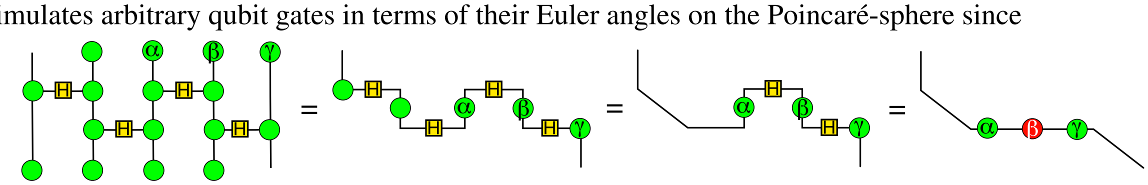
The reader might recognise the first equation to be von Neumann's projection postulate. In the †-symmetric monoidal category of Hilbert spaces, linear maps and the tensor product, this notion of measurement exactly coincides with the usual quantum mechanical one [29]. From a categorical perspective, these measurements are exactly †-Eilenberg-Moore coalgebras for the comonads canonically induced by the †-Frobenius algebra.

3.4 Phase data comes for free: the induced action monoid

Following [24], given a †-Frobenius algebra $(X, copy, del)$ each $\psi : \mathrm{I} \to X$ defines an action $\Lambda(\psi) : copy^\dagger \circ (1_X \otimes \psi)$, and the collection of all these actions constitutes a commutative monoid. We say that $\psi : \mathrm{I} \to X$ is *unbiased* for $(X, copy, del)$ whenever $\Lambda(\psi)$ is unitary. When restricting to unbiased points the actions form a group, which we call the *phase group*. There is a generalisation of the above-mentioned spider-theorem which includes this group [24]. As an example, it allows a very simple derivation of one-way simulation of an arbitrary qubit [24]. The network

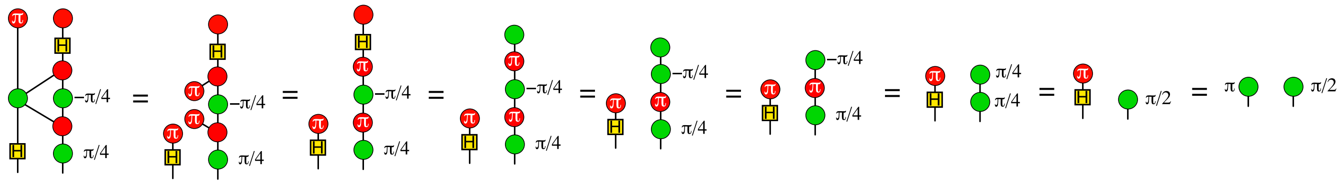


indeed simulates arbitrary qubit gates in terms of their Euler angles on the Poincaré-sphere since



3.5 A positive account of complementarity: scaled †-bialgebras

The most recent step in our program is the categorical characterisation of complementary observables. Formally, this structure is a 'scaled' bialgebra [24]. Making use of this together with the phase group of the previous section, it is possible to perform the *quantum Fourier transform* [24]:



where the two colours green and red distinguish the †-Frobenius algebra of each observables.

3.6 Other consequences and results

Other important results include the characterisation of completely positive maps and Choi's theorem due to Selinger [52], the characterisation of positive operator-valued measurements and Naimark's extension theorem due to Paquette and myself [26] and a categorical presentation of the quantum harmonic oscillator due to Vicary [55].

4 Task I: Unifying quantum foundations approaches

There are many different approaches to the foundations of quantum theory. We hope to cast many of these within our categorical framework. We have clear evidence that this has successfully been done in many cases, such as for Spekkens' toy quantum theory (Edwards and myself [25]) and C*-algebras (Vicary [56]). We have achieved substantial progress with others, namely BBLW convex theories (Barnum and Wilce have set themselves a goal to axiomatise their approach categorically [9], and are in contact with us) and quantaloidal quantum logic (e.g. [31]).

4.1 Spekkens' Toy Model

Spekkens' toy quantum theory [53, 54] has received ample attention from the foundations of physics community. Important in [53] is the fact that the theory arises from a principle of the 'observer's necessarily incomplete information', which he refers to as 'epistemic restriction'. Initially this aspect is less important for us than the structure of the model resulting from it; once that is established we will address this interpretational aspect in a formal manner. We are interested in the model because, while being discrete, it does produce many quantum-like phenomena.

4.1.1 Success obtained so far

The category **Spek** defined in [25] produces the same ingredients of Spekkens' toy theory. Many, if not all, of its quantum-like properties and phenomena then follow for the general reasons outlined in sections 3.2–3.7.

Definition. Let $\mathrm{I} = \{*\}$ and let $\mathrm{IV} = \{1, 2, 3, 4\}$. The objects of **Spek** are I, or of the form $\mathrm{IV} \times \ldots \times \mathrm{IV}$; for convenience we enforce the congruence $\mathrm{IV} \times \mathrm{I} = \mathrm{I} \times \mathrm{IV} = \mathrm{IV}$ and strictness of associativity. The morphisms of **Spek** are all relations generated by composition, cartesian product of relations, and relational converse from:

- all permutations $\{\sigma : \mathrm{IV} \to \mathrm{IV}\}$ on four elements;
- a (copying) relation $\delta_Z : \mathrm{IV} \to \mathrm{IV} \otimes \mathrm{IV}$ defined by:

$$1 \sim \{(1,1),(2,2)\} \qquad 2 \sim \{(1,2),(2,1)\} \qquad 3 \sim \{(3,3),(4,4)\} \qquad 4 \sim \{(3,4),(4,3)\}\,;$$

- a (corresponding deleting) relation $\epsilon_Z : \mathrm{IV} \to \mathrm{I} :: \{1,3\} \sim *\,.$

Hence the whole structure is generated from a copying operation, a deleting operation and a symmetry group, which radically simplifies the description as compared to the one in [53]. Two important comments need to be made here:

The compositional closure principle. The simple 'physical' principle that generates the whole model is the fact that if we have two operations (in the general sense, that is, also including states and effects) which can be composed, then their sequential composite should also exist in the model, as should their parallel (c.f. tensor) composite. For example, the presence of a Bell state $\delta_Z \circ \epsilon_Z^\dagger : \mathrm{I} \to \mathrm{IV} \times \mathrm{IV}$ together with its adjoint, the corresponding Bell effect, guarantees that this category is †-compact.[1]

No biproduct structure. Monoidal categories with biproducts come with a canonical notion of basis, namely the set of injections $\{\iota_i : \mathrm{I} \to \mathrm{I} \oplus \ldots \oplus \mathrm{I}\}$; for example, see the literature on Tannakian categories [35]. In $\mathbf{FdHilb}$ these bases are in bijective correspondence with the ones of section 3.4, by [30]. This is not the case anymore in $\mathbf{FRel}$: there are more bases in the sense of 3.4 than there are Tannakian bases [25]. These extra bases are the ones which play a role in [25], not the Tannakian ones. In particular, $\mathbf{Spek}$ does not inherit biproducts from $\mathbf{FRel}$! Hence it is radically different from categories of vector spaces, semimodules over rigs or anything semi-abelian.

4.1.2 Program, methodology and technical issues

We aim to produce an alternative construction of $\mathbf{Spek}$ which explicitly exposes the role of the epistemic restriction in [53]. This would provide insights on how essential this principle is for generating this kind of structure, and for producing similar models arising from the same principle. Our simple presentation of $\mathbf{Spek}$ in terms of generators would also allow us to easily adjoin extra features, such as the W-state which is not present in Spekkens' original model. We are in contact with Rob Spekkens and expect to collaborate with him on this topic.

4.2 The C*-algebra approach

Many see C*-algebras as an axiomatic framework for quantum theory (for example, [16, 41]). Given a Hilbert space representing the pure states of a quantum system, the algebra of observables on the Hilbert space generates a C*-algebra, and many questions about the quantum system can be completely phrased in terms of this algebra. Some argue that the algebra of observables is in fact more fundamental than the Hilbert space of states, an idea at the heart of algebraic quantum field theory [41]. There are also several important theorems, such as the spectral theorem and the Stone–von Neumann theorem, which are crucial for our modern understanding for quantum theory, and which are naturally described in terms of C*-algebras.

4.2.1 Success obtained so far

It has been demonstrated that, in finite dimensions, C*-algebras are equivalent to a certain class of †-Frobenius algebras [56]. This is closely related to the equivalence between certain *commutative* †-Frobenius algebras and bases for finite-dimensional Hilbert spaces, as described in section 3.4. This is significant, as it demonstrates that finite-dimensional C*-algebras can be completely axiomatised within the approach described in section 3 of this proposal.

There has also been success in describing the spectral theorem in a categorical fashion [56]. This sheds further light on the relationship between the categorical approach and the emergence of classicality.

4.2.2 Program, methodology and technical issues

Although we know how to formulate finite-dimensional C*-algebras and the spectral theorem in a categorical fashion, it is still not known how to prove the spectral theorem or other important theorems in a purely categorical way. Understanding what extra categorical properties are required to achieve this will be an important area of research. It will also be important to explore ways to overcome the finite-dimensional restriction.

We intend to cast the proof of CBH in purely categorical terms, and hence identify the information-theoretic constraints required for the Hilbert space representation within our framework.

[1] In [53] Spekkens does not explicitly 'derive' effects as part of his model, but in [54] he admits they should be there and should be in bijective correspondence with states. Having both a Bell state and Bell effect yields this correspondence for free.

4.3 Quantaloidal quantum logic

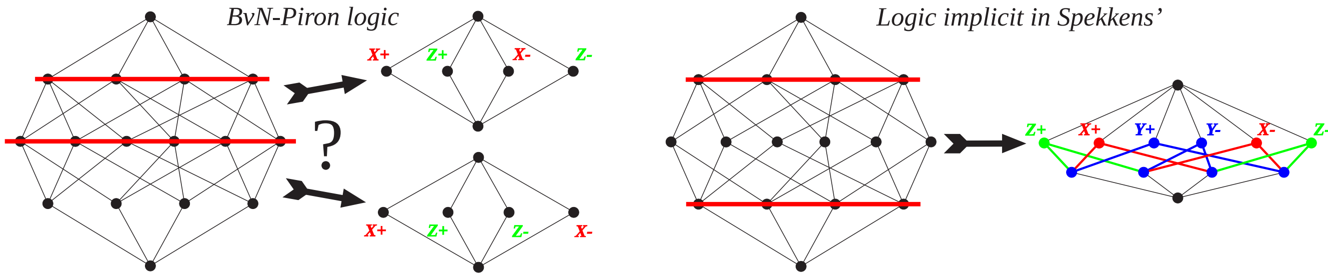
Building further on [38], quantaloidal quantum logic embeds traditional BvN quantum logic within a dynamical setting [31, 33, and references therein]. Our new approach, which in its basic setup is much less constrained than quantum logic, should be able to reproduce BvN quantum logic as well as the quantaloidal extension.

4.3.1 Success obtained so far

Intuitively, quantaloids stand to the category of relations as complete lattices stand to powersets. They emphasize the local (in a categorical sense) ordering in the category of relations. In [28] we showed that from each †-SMC we can extract, in terms of †-Frobenius algebras, a 'Cartesian bicategory of relations' in the sense of Carboni and Walters [15]. Harding has shown that †-compact categories give rise to proper orthomodular posets of projectors [42].

4.3.2 Program and methodology and technical issues

From the arguments in [31, 38] we expect to obtain a better conceptual interpretation of the dagger in †-SMCs. We intend to perform a conceptual derivation of the BvN quantum logic structure as in [50] within a general †-SMC. One important insight gained from constructing the category $\mathbf{Spek}$ is the fact that *epistemic restriction* provides a way to generate non-distributive lattices (cf. Quantaloids) which canonically come with an orthocomplementation:



We intend to extract from this a general construction producing 'orthoquantaloids'. Let us mention that recently Crane [34] also proposed quantaloids as models of quantum gravity. We would investigate the connection between this approach and ours.

4.4 BBLW convex theories

An important aspect of [6, 7] is to study 'virtual' universes other than ours, not subject to quantum theory but to 'something else', which is encoded in terms of generalised convex probabilistic structure. Barnum, Barrett, Leifer and Wilce then investigates which constraints have to be imposed on these universes to end up with ours.

4.4.1 Success obtained so far.

It is not clear whether this approach can be comprehensively captured by our categorical framework outlined in 3.2–3.7, or whether we should relax compactness to, for example, $*$-autonomy. But is seems that when one wishes to model the quantum phenomenon of entanglement swapping it seems to us that necessarily the convex models need to satisfy our axiomatics. In categorical terms, recent unpublished results by Barnum, Barrett, Leifer and Wilce indicate that for these protocols one needs linearly distributive categories [17].

4.4.2 Program, methodology and technical issues

We are currently in the process, in collaboration with Barnum and Wilce, of producing a categorical axiomatization of the convex approach. It would in particular be interesting to see how 'straight generalised convexity' relates to the more abstract counterparts to convexity arising from Selinger's CPM-construction [23, 52], or from 'mixing relative to †-Frobenius structures' [28].

5 Task II: Axiomatic quantumness

After having provided category-theoretic models of the above approaches 4.1–4.4 — that is, having obtained *concrete categorical models* that reproduce them — we would like to see them translated as *abstract categorical structure*, further refining the structures discussed in sections 3.2–3.6. These categorical structures would then axiomatically articulate the intentions of these various approaches. In turn, it would then be possible to generate alternative models for each approach. For example, one could consider variations on Spekkens' toy model which does not depend on a concrete underlying set of 'ontic states', but which still reflects in a structural manner the idea of epistemic restriction.

There are some other aspects which deserve an axiomatic account in their own right. An example of this is the structural role which the complex numbers play within quantum theory, in particular in the passage from the concrete Hilbert space model to abstract categorical structure. The complex conjugation operation $(-)_*$ discussed in section 3.3 is an important part of the categorical structure, and so this aspect of the complex numbers is built into the formalism. The phase group of section 3.5 explicitly introduces the abstract equivalent to the unit circle of $\mathbb{C}^2$. Vicary also showed in [57] that it is possible to axiomatise the complex numbers in a categorical way, by describing categorical properties of a monoidal †-category which imply that the scalars are a subrig of the complex numbers.

We will study similar important features of quantum theory within the resulting axiomatic scheme, such as information-theoretic constraints in terms of the Tsirelson's bound, no-signalling constraints, connections of these with informatic orderings which canonically arise in the categorical setting [28] and so on.

6 Task III: Beyond quantum theory

If time permits it is also our intention to go beyond the domain of conventional quantum theory. Of all the arguments that standard quantum theory may not ultimately be correct, those arising from quantum gravity [44] are perhaps the most persuasive. Robust heuristic arguments suggest that the continuum is not an appropriate mathematical representation of space and time at a fundamental level, but quantum theory makes intrinsic use of the continuum in the form of complex vector spaces, probabilities, the position observables and the time parameter.

A strength of the categorical approach described here is that the task of finding generalisations of quantum theory reduces to the task of finding novel models for a particular set of categorical axioms. If a novel model can be found which reduces to conventional quantum theory in some limit, but which avoids any fundamental use of the complex numbers, then it would be of great importance for the study of quantum gravity. While this goal will be difficult to achieve within the duration of this project, it at least sets a clear path to guide further development of the formalism.

We list below some models from the literature that go beyond conventional quantum theory, and describe how the categorical approach might relate to them. We are in direct contact with the proponents of most of these approaches and have obtained several initial results already.

6.1 Algebraic quantum field theory

Algebraic quantum field theory (e.g. [41]), crafted by Doplicher, Haag, Roberts and several others, is an algebraic counterpart to ordinary QFT which is particularly appropriate for the study of foundations. It formally articulates connections between operator algebras of observables and spatio-temporal causal structure.

6.1.1 Success obtained so far

Recently Abramsky, Blute, Porter and myself recast AQFT in terms of †-SMCs [2]. Vicary's theorem [56] on the connection between C*-algebras and †-SMCs relates these more general AQFTs to the usual ones. Of particular interest would be models as in [39] which arose from Joyal's theory of mathematical species.

6.1.2 Program, methodology and technical issues

We intend to interpret more features of ordinary AQFT within our more general setting. Of particular interest would be combinatorial models of these which would not explicitly rely on a continuum.

6.2 Baez's approach

Baez has emphasised the structural connection between categories of Hilbert spaces and of space-time manifolds [3, 4]. This connection suggests that the common structures might be the ones one needs to focus on for crafting a theory of quantum gravity.

6.2.1 Program, methodology and technical issues

We will investigate how the structures discussed 3.2–3.7 in appear in categories of space-time manifolds. We expect not to find any complementary observables. Therefore one would like to augment categories of space-time manifolds to include non-trivial complementarity.

6.3 Döring-Isham-Butterfield Topoi

Döring and Isham have suggested a new way to construct theories of physics, in which the space of states of a physical system is an object in a topos [36]. When applied to quantum theory, this topos is constructed via the lattice of abelian subalgebras of the von Neumann algebra of observables on the system — essentially, the lattice of all classical contexts. Taking the category of sheaves over this gives a topos, in which statements about the system itself can be formulated using the topos' internal language.

6.3.1 Program, methodology and technical issues

The categorical formulation of C*-algebras described in section 4.2 gives a way to approach this directly, at least for finite-dimensional systems. Given a C*-algebra represented in the category, its the poset of its abelian $*$-subalgebras is analogous to the lattice of abelian subalgebras of the von Neumann algebra of observables considered by Döring and Isham. Taking the presheaf topos over this would then yield a theory of physics in their generalised sense.

This provides an explicit bridge between the approaches, and allows the tools developed by Döring and Isham to be brought to bear on the categorical models for quantum theory that we have developed. In particular, we would be able to ask whether a generalised Kochen-Specker theorem holds.

6.4 Hardy's causaloids

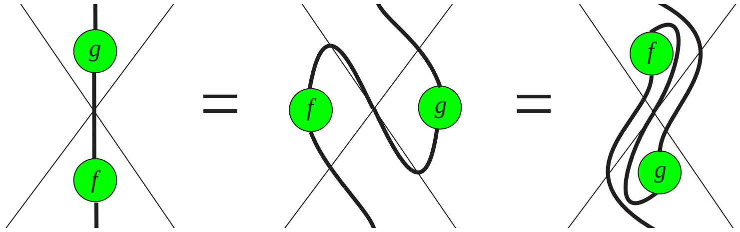
Hardy's theory of causaloids [43] accommodates variations of the causal structure by treating spatial and temporal separation on the same footing. A key structure is the so-called causaloid product.

6.4.1 Program, methodology and technical issues

At first sight, as recently pointed out to us by Hardy, the SMC structure seems to imply a clear distinction between spatial and temporal separation cf., respectively

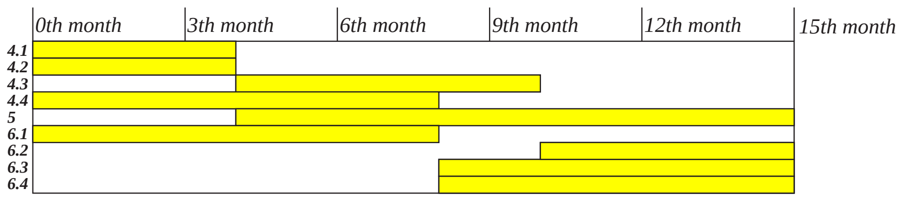
$$f \otimes \ldots \otimes g \qquad vs. \qquad f \circ \ldots \circ g,$$

for processes f and g. However, by adjoining more structure we can actually *relax* constraints, softening the structural distinction between spatial and temporal separation. The way one could achieve this is to interpret compact structure not as Bell states but purely as structural ingredients at the same level as the symmetry natural isomorphism — in the case of compact structure this requires the notion of *dinaturality* [13]. We then have:



We intend to cast as much as possible of Hardy's construction of causaloids in terms of †-compact SMCs.

7 Diagrammatic workplan



References

[1] S. Abramsky and B. Coecke (2004) *A categorical semantics of quantum protocols*. In: Proc. 19th IEEE conf. on Logic in Computer Science, pages 415–425. IEEE Press. arXiv:quant-ph/0402130.

[2] S. Abramsky, R. Blute, B. Coecke and T. Porter (2008) *Dagger nets and algebraic QFT*. In: Proc. 2007 Clifford Lectures, in preparation.

[3] J. C. Baez (2006) *Quantum quandaries: a category-theoretic perspective*. In: Structural Foundations of Quantum Gravity, D. Rickles et al, pages 240–266. Oxford UP. arXiv:quant-ph/0404040

[4] J. C. Baez and J. Dolan (1995) *Higher-dimensional algebra and topological quantum field theory*. Journal of Mathematical Physics **36**, 6073–6105. arXiv:q-alg/9503002

[5] J. C. Baez and M. Stay (2008) *Physics, topology, logic and computation: A Rosetta stone*. In: New Structures for Physics, B. Coecke, Ed, to appear, Springer Lecture Notes in Physics. http://golem.ph.utexas.edu/category/2008/03/physics_topology_logic_and_com.html

[6] H. Barnum, J. Barrett, M. Leifer and A. Wilce (2006) *Cloning and broadcasting in generic probabilistic theories*. arXiv:quant-ph/0611295

[7] H. Barnum, J. Barrett, M. Leifer and A. Wilce (2007) *A generalized no-broadcasting theorem*. arXiv:0707.0620

[8] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher (1996) *Noncommuting mixed states cannot be broadcast*. Physical Review Letters, **76**, 2818–2821. arXiv:quant-ph/9511010

[9] H. Barnum and A. Wilce (2008) *Teleportation protocols for (categories of) abstract state spaces*. In: Proc. Quantum Physics and Logic/Development of Computational Models (QPL-DCM). Electronic Notes in Theoretical Computer Science, to appear.

[10] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. Wooters (1993) *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*. Physical Review Letters **70**, 1895–1899.

[11] G. Birkhoff (1958) *von Neumann and lattice theory*. Bulletin of the American Mathematical Society **64**, 50–56.

[12] G. Birkhoff and J. von Neumann (1936) *The logic of quantum mechanics*. Annals of Mathematics **37**, 823–843.

[13] R. Blute (1993) *Linear logic, coherence and dinaturality*. Theoretical Computer Science **115**, 3–41.

[14] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, A. Zeilinger (1997) *Experimental quantum teleportation*. Nature **390**, 575.

[15] A. Carboni and R. F. C. Walters (1987) *Cartesian bicategories I*. Journal of Pure and Applied Algebra **49**, 11–32.

[16] R. Clifton, J. Bub and H. Halvorson (2003) *Characterizing quantum theory in terms of information-theoretic constraints*. Foundations of Physics **33**, 1561–1591. arXiv:quant-ph/0211089

[17] J. R. B. Cockett and R. A. G. Seely (1997) *Weakly Distributed Categories*. Journal of Pure and Applied Algebra **114**, 133–173

[18] B. Coecke (2003) *The logic of entanglement. An invitation*. Oxford University Computing Laboratory Research Report PRG-RR-03-12. arXiv:quant-ph/0402014 (8 page short version) web.comlab.ox.ac.uk/oucl/publications/ tr/rr-03-12.html (full 160 page version)

[19] B. Coecke (2005) *Quantum information-flow, concretely, and axiomatically*. In: Proc. Quantum Informatics 2004, Y. Ozhigov (ed), pages 15–29. Proc. SPIE **5833**. arXiv:quant-ph/0506132

[20] B. Coecke (2005) *Kindergarten quantum mechanics — lecture notes*. In: Quantum Theory: Reconsiderations of the Foundations III, pages 81–98. AIP Press. arXiv:quant-ph/0510032

[21] B. Coecke (2006) *Introducing categories to the practicing physicist*. In: What is Category Theory? pages 45–74. Advanced Studies in Mathematics and Logic **30**, Polimetrica Publishing.

[22] B. Coecke (2007) *De-linearizing linearity: projective quantum axiomatics from strong compact closure*. Electronic Notes in Theoretical Computer Science **170**, 47–72. arXiv:quant-ph/0506134

[23] B. Coecke (2007) *Complete positivity without positivity and without compactness*. Oxford University Computing Laboratory Research Report PRG-RR-07-05. web.comlab.ox.ac.uk/oucl/publications/tr/rr-07-05.html
[24] B. Coecke and R. Duncan (2008) *Interacting quantum observables*. In: Proc. 35th Int. Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science, to appear. On arXiv early May.
[25] B. Coecke and B. Edwards (2008) *Toy quantum categories*. In: Proc. Quantum Physics and Logic/Development of Computational Models (QPL-DCM). Electronic Notes in Theoretical Computer Science, to appear.
[26] B. Coecke and E. O. Paquette (2006) *POVMs and Naimark's theorem without sums*. Electronic Notes in Theoretical Computer Science (To appear). arXiv:quant-ph/0608072
[27] B. Coecke, E. O. Paquette and S. Perdrix (2008) *Bases in diagramatic quantum protocols*. In: Proc. Mathematical Foundations for Programming Semantics XXIV, to appear.
[28] B. Coecke, E. O. Paquette and D. Pavlovic (2008) *Classical structures from tensorial quantum structures*. In: New Structures for Physics II: approaches, B. Coecke, Ed, to appear, Springer Lecture Notes in Physics.
[29] B. Coecke and D. Pavlovic (2007) *Quantum measurements without sums*. In: Mathematics of Quantum Computing and Technology, G. Chen et al (eds), pages 567–604. Taylor and Francis. arXiv:quant-ph/0608035.
[30] B. Coecke, D. Pavlovic, and J. Vicary (2008) *Commutative †-Frobenius algebras in* $\mathbf{FdHilb}$ *are bases*. In preparation.
[31] B. Coecke, D. J. Moore and I. Stubbe (2001) *Quantaloids describing causation and propagation of physical properties*. Foundations of Physics Letters **14**, 133–145 (2001). arXiv:quant-ph/0009100
[32] B. Coecke, D. J. Moore and A. Wilce (2000) *Operational quantum logic: An overview*. Introduction to *Current Research in Operational Quantum Logic: Algebras, Categories and Languages*. Fundamental Theories of Physics **111**, Springer. arXiv:quant-ph/0008019.
[33] B. Coecke and S. Smets (2004) *The Sasaki hook is not a [static] implicative connective but induces a backward [in time] dynamic one that assigns causes*. International Journal of Theoretical Physics **43**, 1705–1736. arXiv:quant-ph/0111076
[34] L. Crane (2008) *Relational topology and quantum gravity*. Talk given at the 1st workshop on Categories, Logic and Foundations of Physics, Imperial College, Jan. 9, 2008. http://155.198.140.124/CLAP/CLAP-LouisCrane.mp4
[35] P. Deligne (1990) *Catégories tannakiennes*. In: The Grothendieck Festschrift Volume II, Birkhäuser.
[36] A. Döring and C. Isham (2008) *'What is a thing?': Topos theory in the foundations of physics*. In: New Structures for Physics II: approaches, B. Coecke, Ed, to appear, Springer Lecture Notes in Physics. arXiv:0803.0417
[37] K. Engesser, D. Gabbay, D. Lehmann, Eds. (2007) *Handbook of Quantum Logic and Quantum Structures*. Elsevier.
[38] CL.-A. Faure, D. J. Moore and C. Piron (1995) *Deterministic evolutions and Schrödinger flows*. Helvetica Physica Acta **68**, 150–157.
[39] M. Fiore (2007) *An axiomatics and a combinatorial model for creation/annihilation operators and differential structure*. Talk given at the workshop on Categorical Quantum Logic, Oxford University, Jul. 2007. Slides available at http://se10.comlab.ox.ac.uk:8080/FOCS/CQL program en.html
[40] J.-Y. Girard (1987) *Linear logic*. Theoretical Computer Science **50**, 1–102.
[41] H. Halvorson and M. Müger (2006) *Algebraic quantum field theory*. In: *Handbook of Philosophy of Physics*, 731-922, Elsevier
[42] J. Harding (2007) *Some quantum logic and a few categories*. Talk given at the workshop on Categorical Quantum Logic, Oxford University, Jul. 2007. Slides available at http://se10.comlab.ox.ac.uk:8080/FOCS/CQL program en.html
[43] L. Hardy (2005) *Probability theories with dynamic causal structure*. arXiv:gr-qc/0509120
[44] C. Isham (2002) *Some reflections on the status of conventional quantum theory when applied to quantum gravity*. Talk presented at Stephen Hawking's 60th birthday symposium. arXiv:quant-ph/0206090
[45] A. Joyal and R. Street (1991) *The geometry of tensor calculus I*. Advances in Mathematics **88**, 55–112.
[46] L. H. Kauffman (2005) *Teleportation topology*. Optics and Spectroscopy **99**, 227–232. arXiv:quant-ph/0407224
[47] G. M. Kelly and M. L. Laplaza (1980) *Coherence for compact closed categories*. Journal of Pure and Applied Algebra **19**, 193–213.
[48] A. K. Pati and S. L. Braunstein (2000) *Impossibility of deleting an unknown quantum state*. Nature **404**, 164–165.
[49] R. Penrose (1971) *Applications of negative dimensional tensors*. In: Combinatorial Mathematics and its Applications, D.J.A. Welsh, Ed, pages 221–244. Academic Press.
[50] C. Piron (1976) *Foundations of Quantum Physics*. W. A. Benjamin.
[51] M. Rédei (1997) *Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead).* Studies in History and Philosophy of Modern Physics **27**, 493–510.
[52] P. Selinger (2007) *Dagger compact categories and completely positive maps*. Electronic Notes in Theoretical Computer Science **170**, 139–163. http://www.mathstat.dal.ca/∼selinger/papers.html♯dagger
[53] R. Spekkens (2007) *Evidence for the epistemic view of quantum states: A toy theory*. Physical Review A **75**, 032110.
[54] R. Spekkens (2007) *Axiomatization through foil theories*. Talk given at the FOILS workshop: Operational probabilistic theories as foils to quantum theory, University of Cambridge, July 5.
[55] J. Vicary (2007) *A categorical framework for the quantum harmonic oscillator*. International Journal of Theoretical Physics (To appear). arXiv:0706.0711
[56] J. Vicary (2008) *Categorical formulation of C*-algebras*. In: Proc. Quantum Physics and Logic/Development of Computational Models (QPL-DCM). Electronic Notes in Theoretical Computer Science, to appear.
[57] J. Vicary (2008) *Complex numbers in monoidal categories.* In preparation.
[58] J. von Neumann (1932) *Mathematische Grundlagen der Quantenmechanik*. Springer-Verlag.
[59] W. Wootters and W. Zurek (1982) *A single quantum cannot be cloned*. Nature **299**, 802–803.
[60] M. Żukowski, A. Zeilinger, M. A. Horne and A. K. Ekert (1993) *'Event-ready-detectors' Bell experiment via entanglement swapping*. Physical Review Letters **71**, 4287–4290.