--- Item 4 ---

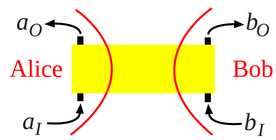
A relativistic universe of interacting quantum processes

Bob Coecke and Ray Lal
Oxford University Computing Laboratory

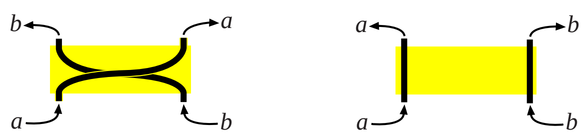
1 A structural inconsistency

1.1 Causality in bipartite physical processes

Consider two parties, Alice and Bob, who each have a device with a binary input and output.



Seen as one joint process, we can denote the pair of input bits as (a_I, b_I) and the pair of output bits as (a_O, b_O). We can now consider the 4-by-4 *possibilistic correlation matrix* that has $f_{a_I,b_I}^{a_O,b_O} = 1$ as entry when for input (a_I, b_I) the output (a_O, b_O) is possible, i.e. may happen with non-zero probability, and $f_{a_I,b_I}^{a_O,b_O} = 0$ otherwise. In contrast, the *probabilistic correlation matrix* [40] would have the actual probabilities as entries. Depending on the internal architecture of such a device, it may or may not be used by Alice and Bob to communicate. Here are simple examples of a device that enables communication and one that doesn't enable communication:



By *possibilistic signaling* we mean that from the possibilistic correlation matrix it can be concluded that the device enables signaling. One easily sees that possibilistic signaling implies the more standard notion of *probabilistic signaling* (e.g. see [6]). Put more precisely, a device enables Alice to signal to Bob if, from his input and his output, Bob can deduce something about Alice's input, i.e., there is at least one input-output pair at Bob's end which reflects something about Alice's input. In terms of the (possibilistic/probabilistic) correlation matrix:

$$\exists(b_I, b_O) : f_{0,b_I}^{0,b_O} + f_{0,b_I}^{1,b_O} \neq f_{1,b_I}^{0,b_O} + f_{1,b_I}^{1,b_O}$$

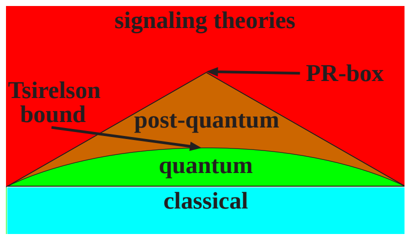
where in the possibilistic case we use Boolean calculus; the sums represent tracing out Alice's output, which is irrelevant to Bob. From this it also easily follows that possibilistic signaling indeed implies probabilistic signaling.

What relativity theory teaches us is the manner in which Nature restricts the ability to signal. While quantum non-locality stretches beyond what can be modeled in classical causal theory (i.e. hidden variable theories), it happens to be still perfectly compatible with relativity, in that it is non-signaling in the above-described sense. (But, is it really? – see §1.3 below)

There has been increasing interest in studying the space of probabilistic *non-signaling* correlation matrices for devices that one may be able to craft with the resources provided by a certain (possibly hypothetical) theory of physics [8, 6, 7, 46]. Typical cases are: theories which admit a local hidden variable theory, quantum theory in which one has entangled states as an additional resource, and Barrett's box-world [8] in which the PR-boxes [41] with respective probabilistic and possibilistic correlation matrix:

$$PR_{\text{probabilistic}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \qquad PR_{\text{possibilistic}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

is a primitive. The polytope of probabilistic correlation matrices decomposes as follows [50]:

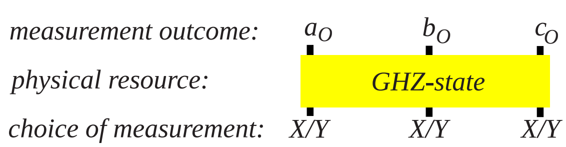


Motivated by the fact that relativistic non-signaling constraint seems to allow for more non-locality, an important current challenge on which there is great activity is to derive the Tsirelson bound from operational and/or information-theoretic principles under the slogan: *"Why isn't nature even more non-local than it is?"* [39, 5, 47, and references therein].

With some exceptions [51, 9], there has been very little study of multipartite processes. But note here that the GHZ-Mermin game (which only relies on impossibilities rather than probabilities) shows that quantum non-locality is already fully encoded in the possibilistic correlation matrix:

$$GHZ_{\text{possibilistic}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

which gives the possible coincidences of outcomes given choices of either X- or Y-measurements.



1.2 Causal orders in relativistic space-times

Many results have been obtained which show that discrete structures are a very powerful tool for understanding a space-time manifold, despite the latter having continuum properties such as differentiable structure. For example, Zeeman [52] showed that the Poincaré group can be defined as the group of automorphisms of Minkowski space that preserve the causal order on points (i.e. space-time events). Here the causal order relation captures signaling (or better, the ability to do so), that is, for events a, b:

- $a \leq b$: a precedes b by either a *time-like* or *light-like* curve;
- $a \not\leq b$: a and b are *space-like separated.*

Moreover, almost the entire structure of many space-times can be reconstructed from this order. Malament [36] showed that for a certain class of spacetimes both the differentiable structure and the conformal metric can be recovered from the causal structure (cf. work by Kronheimer and Penrose [30]). It was also shown that the causal structure is actually an instance of Scott Domains, an important area of research in computer science [37].

The Malament result provides part of the motivation for several approaches to quantum gravity, in particular the causal set programme [48].

1.3 A clash!

There is a structural clash at the most foundational level with the two notions of causality put forward above. The striking fact is that this conflict doesn't even require quantum theory, but any non-deterministic theory (i.e. one can model it by a local hidden variable theory) suffices. The fact is mathematically fairly trivial, but to our surprise any leading expert we talked to in the foundations of quantum theory was not aware of this.

The issue is the behavior under *time-reversal* of the fundamental structure in each of the above discussions of causality. Starting with the second one discussed in §1.2, causal orders, these are perfectly time-symmetric, since given a partially ordered set $(P, \leq)$, one again obtains another partial order $(P, \leq')$ by simply setting $a \leq' b \Leftrightarrow b \leq a$. However, the concept of no-signaling discussed in §1.1 is by no means time-symmetric. A counter example is:

$$\begin{aligned} HV &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

where the decomposition in 2 by 2 matrices establishes that this correlation matrix indeed admits a local hidden variable representation, the summation representing the different local states for the four values of the hidden variable — a probabilistic counterpart can easily be constructed too. When performing time reversal, that is, exchanging the roles of inputs and outputs via transposition, we obtain a signaling situation:

$$HV^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{since both} \quad \begin{cases} 1 = f^{0,0}_{0,0} + f^{1,0}_{0,0} \neq f^{0,0}_{1,0} + f^{1,0}_{1,0} = 0 \\ 0 = f^{0,1}_{0,0} + f^{1,1}_{0,0} \neq f^{0,1}_{1,0} + f^{1,1}_{1,0} = 1 \end{cases}$$

Simpler put, if Bob inputs $b_I = 0$, then we have $b_O = a_I$, that is, he receives Alice's input.

The same passage from non-signaling to signaling under time-reversal is also true for the PR matrix given above, while the non-signaling of the GHZ matrix is retained under time-reversal.

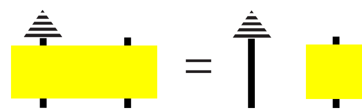
To illustrate how counter-intuitive this passage from non-signaling to signaling under time-reversal is consider the following anecdote of a classical situation:

> A game is going on in which two players, Alice and Bob have an input-output device. They are not supposed to communicate during the game. From the correlations between their inputs and outputs we want to deduce whether they cheated or not. While they may not have been cheating at all, when we record their game and play the movie backwards now exchanging the roles of inputs and outputs, there are cases where we are able to prove that they in fact cheated in this backward reality!

As physicists we are so used to thinking in such a time-symmetric manner about causality while probabilistic reasoning is fundamentally incompatible with it. When quantum theory comes into play, we know that probability is manifestly unavoidable, and we have to adopt a more subtle time-asymmetric of causality. This is what this project is about:

to understand in the most primal manner this time-asymmetric nature of causal structure.

Of great aid here will be the diagrammatic representation of the symmetric monoidal categories in which we can organize composite physical processes [14]. For example, in this language (which we explain below), the no-signaling constraint becomes (= environment):



which directly reflects in an intuitive topological fashion that Alice's input, when we ignore her output, has no affect on the relation between Bob's input and Bob's output — note in particular that this constraint is not at all time-symmetric.

1.4 The even trickier bit

One may wonder why we relied on the 'possibilistic shadow' of processes rather than on the probabilistic matrices themselves. The simple answer is that in that case of probabilities even the concept of time-reversing the process becomes somewhat nonsensical. Indeed, although one can turn a stochastic matrix with I as givens and O as conclusions into one which has O as givens and I as conclusions via Bayesian inversion:

$$P(I|O) = \frac{P(O|I)P(I)}{P(O)}$$

this requires knowing $P(I)$, and since this is an arbitrary choice by the agents, there is no a priori manner of specifying this — e.g. one agent may decide for it to be constant while another may take it to be random. In other words, it makes no sense to speak of time-reversal of a probabilistic process when we do not at the same time fix the behavior of the agents.

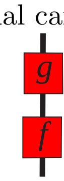
2 Causality in a monoidal category

2.1 Categorical quantum mechanics

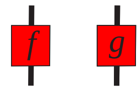
In *categorical quantum mechanics*, initiated in [1], the basic data consists of a collection of named *systems* $A, B, C, ...$, *processes* $f : A \to B$ which may take a *system* of one kind into a system of another kind, and *connectives* which allow the composition of these both 'in parallel' and 'sequentially'. Mathematically, this means that we are dealing with a fairly involved algebraic gadget called a *symmetric monoidal category* (SMC) $(\mathbf{C}, \otimes, \mathrm{I})$ [13].

One of the nice features of SMCs is that they admit an equivalent, very simple, purely diagrammatic calculus in which systems are represented by wires and processes by boxes. In this language the two compositions are:

- The *sequential*, or *dependent*, or *causal*, or *connected* composition of processes $f : A \to B$ and $g : A \to B$ is $g \circ f : A \to C$, which as usual can be depicted as:



- The *parallel*, or *independent*, or *acausal*, or *disconnected* composition of processes $f : A \to B$ and $g : C \to D$ is $f \otimes g : A \otimes C \to B \otimes D$, and is depicted as:



In category-theoretic terms, the key ingredients in categorical quantum mechanics are:

(A) Abramsky and us proposed dagger compact categories as a means to axiomatize bipartite entangled states, map-state duality, bra-ket duality and unitary operations, by endowing compact categories with an identity-on-objects involution, the dagger-functor [1] ;

(B) Selinger proposed a construction which assigns to any dagger compact category $\mathbf{C}$ of pure states and operations another dagger compact category $CPM(\mathbf{C})$ as the abstract counterpart to 'mixed states' and 'completely positive maps'[44]; Perdrix and us axiomatized $CPM(\mathbf{C})$ as a dagger compact category with an 'environment' [20] ;

(C) Pavlovic and us introduced *classical structures* as certain Frobenius algebras, in order to handle quantum measurements and resulting classical data flows [19, 20] ;

(D) Duncan and us axiomatized 'complementary' classical structures and were able to construct from this basic quantum logic gates [15] ; the sum of the above is in [20] .

A substantial portion of the mathematical backbone to this approach is drawn from so-called *Australian category theory*, most notably from work by Kelly-Laplaza [29], Joyal-Street [28] Carboni-Walters [10] and Lack [31]. This abstract mathematical counterpart to the intuitive diagrammatic presentation is crucial in understanding the power of the graphical calculus.

Thm. [29, 44] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy.*

Thm. [45] *An equational statement between expressions in dagger compact symmetric monoidal categorical language holds if and only if it is derivable in the category of finite dimensional Hilbert spaces, linear maps, tensor product, and adjoints.*

More informally, the two together can be stated as follows:

> Any equation involving states, operations, effects, unitarity, adjoints (e.g. self-adjoint), projections, Bell-states/effects, transpose, conjugation, inner-product, trace, Hilbert-Schmidt norm, positivity and complete positivity, holds in quantum theory if and only if it can be derived in the graphical language via homotopy.

This means that an important fragment of quantum theory can be completely done in a sound manner in purely diagrammatic terms, and there is ongoing work to extend this remarkable result to even larger fragments.

This approach meanwhile has had an important impact and new followers in the quantum foundations community, e.g. [27, 16, 11]; Lucien Hardy recently declared:

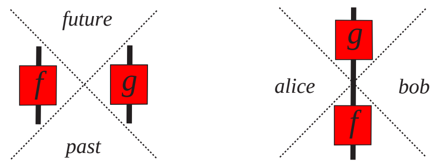
"... we join the ***quantum picturalism*** revolution" [27]

(this term 'quantum picturalism' was coined by us in [13]). The approach has also led to new results in quantum computation and information [17, 23], and supports *automation*; currently a software named `quantomatic` that enables semi-automated theory-exploration is under development.

Categorical quantum mechanics as a 'canvas' for quantum foundations was the subject of our previous FQXi Large grant, which was very successful, as mentioned above. But despite these successes there is a somewhat embarrassing issue that urgently needs addressing! (Cf. §2.3.)

2.2 Implicit 'partial' causal structure

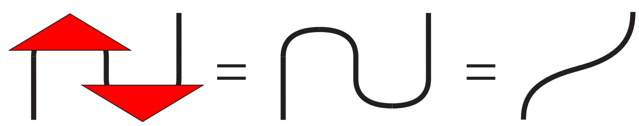
What one could consider as a system in the sense described above is very broad, for example, as in algebraic quantum field theory [24, 25] it could be those observables that we are able to measure within a certain region of space time. Just looking at some protocol in the categorical quantum mechanics language, one tends to think of the 'acausal' composition $\otimes$ as 'spatially' separating, while one thinks of the 'causal' composition $\circ$ as 'time-wise' separating:



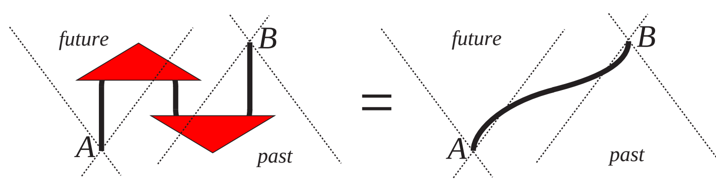
A more detailed discussion of the conceptual understanding of this framework is in [14].

2.3 Incompatibility with a non-postselected causal universe!

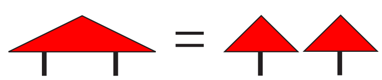
The first achievement of categorical quantum mechanics was deriving physical phenomena such as post-selected quantum teleportation with tiny structural resource [1, 12]:



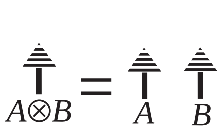
Unfortunately this derivation may be 'abused' to show that quantum theory enables signaling:

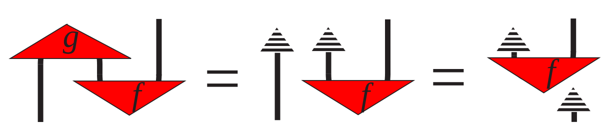


The problem here is *postselection*, that is, the assumed ability to condition on the measurement outcome. If we only consider processes that can be established with certainty, then we will always have:

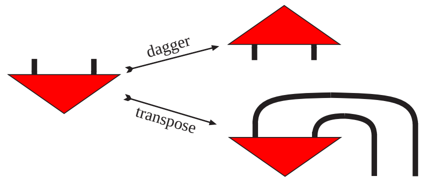


i.e. there are no disentangled effects. This is of course a well-known fact, and one of the subtleties of quantum mechanics, that the non-determinism of measurement avoids allowing entanglement to be a signaling resource. In [11], Chiribella-D'Ariano-Perinotti enforced the disentanglement of effects by axiomatizing the uniqueness of deterministic effects, from which it follows via parallel composition that bipartite effects have to be disentangled (now:  = deterministic effect; which plays the same structural role as the environment):





Unfortunately, this axiom has radical consequences, namely that the key structures of categorical quantum mechanics, which are also central to quantum theory and quantum information, namely the adjoint that turns kets into bras, and the partial transpose which underpins the Choi-Jamiolkowski isomorphism, cannot be retained in a 'causal universe of processes', since both generate entangled effects:



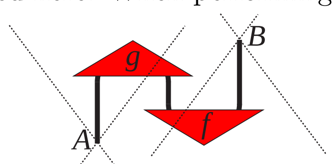
3 Proposed work: a time-asymmetric foundation for causality

In §1.3 we demonstrated that models of causality based on partial orders fail to capture the time-asymmetric aspect which comes into play when we consider fundamentally probabilistic processes. The overall goal of this project is to identify the fundamental structure which underpins a universe of quantum relativistic processes. This will involve taking into account not only causal relationships between processes but also essential aspects of the actual processes that take place. As we argued in §2.3 one such essential aspect is for example whether a process $\Psi : \mathrm{I} \to A \otimes B$ is entangled.

Proof theory and categorical logic have seen a similar passage, from expressing that there *is* a proof which derives predicate B from predicate A, to an explicit account of the *space* of proofs which establish this, these proofs then being the morphisms in some category (see e.g. [32, 2]). The paradigm connecting the ordered structure and the categorical is

$$A \leq B \iff \mathbf{C}(A, B) \neq \emptyset \,.$$

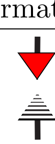
But this paradigm cannot be retained here. When performing a 'physical scenario':



where f and g are arbitrary physical processes, although we argued in §2.3 that g should be disentangled, as a whole this is still a physical process. But the proof-theory paradigm tells us $A \not\leq B \Rightarrow \mathbf{C}(A, B) = \emptyset$, i.e. such a process doesn't even exist! Hence the crucial thing to do is to formally make a clear distinction between:

- the existence of a physical process ; and
- the flow of information enabled by such a process.

While the existence of a process $A \to B$ will indeed imply that $\mathbf{C}(A, B) \neq \emptyset$, the causality assertion $A \leq B$ will stand for the fact that there is a non-zero flow of information from A to B. In the diagrammatic language of symmetric monoidal categories this means that A and B are *connected*, while in the absence of information flow they will be *disconnected*:

information flow	no information flow
	

Hence the proposed work takes a radical departure from standard categorical logic.

Moreover, the categories in categorical quantum mechanics, dagger compact categories, are in fact generalized categories of relations. Therefore, by integrating causal structure and categorical process structure, we axiomatically substantiate the relational approach to space-time (cf. Mach [35], Barbour [4], Rovelli [43]).

As a stepping stone towards a foundational theory of causality that takes time-asymmetry into account at the primal level by incorporating aspects of process structure, we will consider some very concrete subtasks outlined below.

Subtask 1: Causal categories

The first attempt to adjust categorical quantum mechanics in the way we require is proposed in [18]. The key feature of this work is that quantum causality (restricting to normalized processes) and relativistic causality (a global discrete causal structure such as a poset) are unified in a single framework. This is achieved by introducing the notion of a *causal category*, which is a category with the following properties: (i) there is a monoidal product, but it is partial, in the sense that the monoidal product does not exist for all pairs of objects and morphisms in the category; (ii) the hom-set of two objects for which the monoidal product exists contains only disconnected morphisms; (iii) the monoidal unit is terminal. It is important that the paradigm of categorical quantum mechanics can be retained; indeed, causal categories can be seen as the most conservative way of adding causal structure to the existing formalism. One can think of a particular causal category as arising from 'carving out' a causal structure from—i.e. selecting a subcategory of—a dagger symmetric monoidal category of processes.

Milestones for this topic:

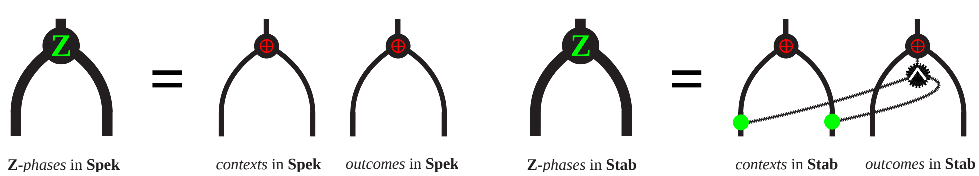
- The definition of a causal category encompasses both causal structure and the restriction to normalized processes; what formal differences arise when considering a category with causal structure but which also contains post-selected measurements?
- An understanding of how physically important classes of causal orders appear in causal categories, and how they interact with the structure of a causal category. For example, the causal order for a Galilean space-time—in which signaling is allowed—can be captured directly by the condition that the relation of space-like separation between points of space-time is transitive.

Subtask 2: Non-locality (or contextuality) in pictures

In [16] Edwards, Spekkens and us showed that non-locality of (the stabilizer fragment of qubit) quantum theory (**Stab**, as a dagger compact category) versus that of Spekkens' toy theory [49] (**Spek**, as a dagger compact category) can be traced back to a difference in *phase group*, these respectively being Z_4 and $\mathsf{Z}_2 \times \mathsf{Z}_2$; in categorical quantum mechanics, the phase group is a group that for general abstract reason exists for any classical structure [15]. Elsewhere [3], Anders and Browne's analysis showed that entangled quantum resources can boost the power of a classical computer by providing an AND-gate. This AND-gate turns out to be exactly the 'difference' between the Z_4 and $\mathsf{Z}_2 \times \mathsf{Z}_2$ phase groups. Concretely, one relies on:

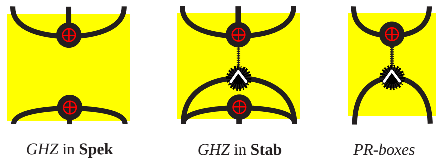
Thm. [15] If (*green*, *red*) is a pair of coherent bialgebraic structures then under minor conditions the eigenstates O_{red} of *red* form a subgroup of the phase group U_{green} of *green*.

We call the Abelian group O_{red} the *outcome group* and the corresponding Abelian quotient group of cosets $C_{g,r} = U_{green}/O_{red}$ the *context group*. The multiplication of the phase group itself can then be depicted in terms of the multiplication of the outcome group and the multiplication of the context group, explicitly giving rise to the AND-gate:

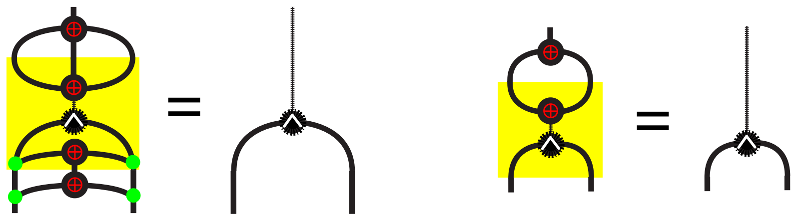


Note here that the AND-'interface' computes from the choice of the measurements of two parties the outcome of the third party, so in fact in a sense it takes place 'out of space-time'.

In terms of input-output physical processes as in §1.1 we have as internal structures:



and by relying on the 'red-green'-calculus of [15] we extract the AND-gate as follows:



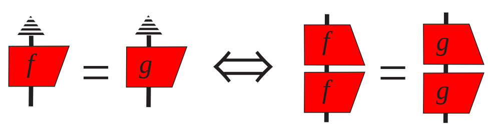
in this manner recovering the Anders-Browne result in a purely diagrammatic manner.

Milestones for this topic:

- An extension of the above-discussed results beyond qubits.
- The above exposes that an essential part of non-locality is a computation (of an AND-gate) 'outside of space-time'. What is the equivalent computation 'outside of space-time' for other states?
- How can we depict this non-local flow of information 'outside of space-time' within the foundational structure of causality that we wish to craft within this project.

Subtask 3: Interaction with the environment axiom

The concept of *environment* is rarely explicitly accounted for in axiomatic theories. However, for any theory that is not considered to be complete, or in a theory which can describe open systems, the mathematical representation of the environment is a very useful gadget for assigning 'all we don't know or can consider'. In [20] the environment is axiomatized (also with a graphical element) as part of an abstraction of the CPM construction, with the key axiom being the following:



This allows for the comprehensive formalization of the interaction between classical and quantum information flow.

Formally the setting is close to that of causal categories, which can allow a rich interplay between the two concepts.

Milestones for this topic:

- The environment axiom efficiently axiomatizes features that we have shown are key to the notion of causality in a quantum setting, such as normalization and whether information flow occurs in a process. Since the environment axiom also allows for other features, such as the derivation of various protocols, initially the question is what is the optimal way of combining it with a causal structure, in particular with the definition of a causal category.
- Can the environment axiom be used to define a setting in which causality is satisfied from the point of view of quantum theory (i.e. post-selection is disallowed) but in which the global causal structure required from relativity is not present? How would this relate to causal categories?
- The environment axiom allows in particular for the distinction between classical and quantum information to be characterized in terms of broadcastibility: can this be incorporated into causal categories, to structurally understand how classical and quantum information differ in their interaction with causal structure?

Subtask 4: Relation to Hardy's and Chiribella-D'Ariano-Perinotti's work

In [11] the fact that every mixed state in quantum theory can be described as pure state in a larger system is elevated to an axiom for the derivation of information-theoretic features. This is developed in an operational setting that implicitly assumes much of the formal structure of categorical quantum mechanics. It is also important in this approach that post-selection is disallowed, i.e. it is imposed that there is a unique deterministic effect. Along with the purification principle, this allows for the derivation of various features of quantum theory without assuming the usual Hilbert-space structure, e.g. the existence of entangled states. However, many of the proofs use methods other than the graphical calculus that could be used in view of the underlying monoidal structure.

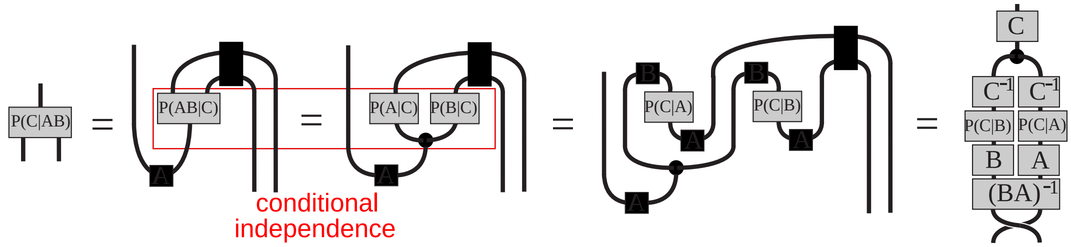
In a similar vein, Hardy has recently developed a formalization of general probabilistic theories, along with a graphical calculus, that is 'formalism-local' [27]. This refers to a formalism in which the calculations for a particular space-time region need only use mathematical objects that the theory has assigned to that region. Hardy's approach develops a mathematical device called *duotensors*.

Milestones for this topic:

- The condition of a unique deterministic effect is powerful; how much of this power be attributed graphically (especially in relation to the environment axiomatization above)?
- An account of the extent to which the results of the purification approach can be improved and extended in the categorical setting.
- In Hardy's duotensor approach, the aim of incorporating spatiotemporal information in a probabilistic theory is similar to the reason for using causal categories; what is the relation between these constructions?

Subtask 5: Causal correlations and probabilistic causality

In §1.4 we discussed the problem with genuine probabilistic data versus possibilistic data. In [21] Spekkens and us, building further on ideas from Leifer-Poulin [33], crafted a general diagrammatic framework for Bayesian reasoning which encompasses both classical and quantum Bayesian reasoning, as well as entropic calculus. Here is an example of a computation:



Milestones for this topic:

- We wish to obtain an understand correlations in space-time in terms of the graphical calculus for Bayesian updating.
- Is there a 'probabilistic categorical logic' that can be drived from the graphical framework for classical and quantum Bayesian updating in the light of our discussion at the beginning of this project's section?
- In [26] in the *causaloid formalism* Hardy proposes an operational strategy to construct a probabilistic theory of general relativity ProbGR. Compare the foundational structure of causality as we intend to construct it here with the strategy proposed by Hardy; in particular, is it compatible with the causaloid approach?

Subtask 6: Time asymmetry in generalized probabilistic theories

We have shown how the interaction between causality and time-symmetry provides a structural difference between relativity and quantum theory. However there is a subtlety in this issue, in the sense that although probabilistic correlations will not be causally time-symmetric, quantum theory exhibits a kind of 'formal' time-symmetry which needs to be understood better. This arises when considering different no-signaling theories, in particular the difference between *boxworld*, the theory containing all possible no-signaling correlations, and quantum theory. By using a convex operational approach to study these theories in a common framework, Barrett [8] and Short and Barrett [46] have shown that although boxworld allows more nonlocal correlations than quantum theory, it has fewer possible measurements. Indeed boxworld lacks the possibility of entangled joint measurements, leading to very restricted dynamics.

This implies that boxworld lacks the state-effect bijection present in the usual formalism of quantum theory, and which is axiomatized in categorical quantum mechanics in the form of the dagger functor. Since the dagger functor can be thought of as imposing time-symmetry on processes in the category, the breakdown of state-effect symmetry in boxworld indicates that in some sense quantum theory 'optimizes' the balance between the amount of nonlocality provided by states and the richness of the dynamics. A precise development of this idea would not only enhance our understanding of the relation between time-symmetry and quantum theory, but it may also lead to an understanding of the origin of Tsirelson's bound.

Hence structural insight into the compatibility between quantum theory and relativity will require understanding the role of time-symmetry on two levels. Firstly, in the direct sense that the causal structure of relativity is time-symmetric but non-local correlations are not. Secondly, and pulling in the other direction, amongst non-local non-signaling theories, the underlying formalism of quantum theory exhibits a kind of time-symmetry which boxworld does not.

Milestones for this topic:

- We will formulate no-signaling theories categorically, and by a constructive encoding of the isomorphism between states and effects (using compact structure), we will investigate how close the resulting set of correlations is to the quantum boundary (which is limited by Tsirelson's bound).

- McKague [38] has shown that PR boxes are allowed by a quantum-like formalism that uses quaternionic instead of complex scalars. This presents an intriguing relation to categorical quantum mechanics, since quaternionic matrices seem to instantiate the notion of *premonoidal categories* [42], which generalize monoidal categories by allowing a breakdown of bifunctoriality. We wish to understand to what extent premonoidal categories can be used to understand no-signaling theories abstractly and more efficiently.

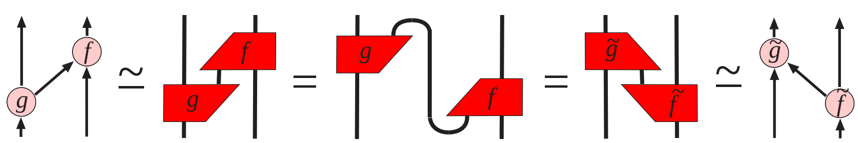
Subtask 7: Varying causal structure via compactness

We discussed in §3 that in the causaloid formalism a notion of probabilistic causality structure is sought [26], which is motivated by considerations of quantum gravity, in which causal structure may be indefinite at a fundamental level. In most approaches it is not clear how best to implement this, however with a category-theoretic formalism this can be achieved in a simple way as follows [14].

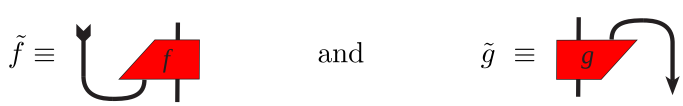
The key feature is that we can use compact structure to turn an input of a process to an output, and vice versa. Compact structure has so far been used in categorical quantum mechancs to structurally isolate the resource that a theory requires to contain a teleportation protocol [1]. Graphically, this corresponds to the existence of a cup and a cap:

subject to the following 'yanking' equations:

Now, given morphisms $g : C \otimes A \to B$ and $f : A \to B \otimes C$, consider the causal structure defined by the composite morphism $(1_C \otimes f) \circ (g \otimes 1_B)$: the process g occurs 'before' the process f. But by using the 'yanking rule', this composite can be seen to induce a different causal structure:



where the morphisms $\tilde{f}$ and $\tilde{g}$ are defined as:



Milestones for this topic:

- A systematic treatment of indefinite causal structure using compactness.

References

[1] S. Abramsky and B. Coecke (2004) *A categorical semantics of quantum protocols.* In: Proceedings of 19th IEEE conference on Logic in Computer Science, pages 415–425. IEEE Press. arXiv:quant-ph/0402130. Revised version (2009): *Categorical quantum mechanics.* In: Handbook of Quantum Logic and Quantum Structures, K. Engesser, D. M. Gabbay and D. Lehmann (eds), pages 261–323. Elsevier. arXiv:0808.1023

[2] S. Abramsky and N. Tzevelekos (2010) *Introduction to categories and categorical logic.* In: New Structures for Physics, B. Coecke (ed), pages 3–94. Lecture Notes in Physics 813. Springer-Verlag.

[3] J. Anders and D. E. Browne (2009) *Computational power of correlations.* Physical Review Letters **102**, 050502. arXiv:0805.1002

[4] J. Barbour (1999) *The End of Time.* Oxford University Press, New York.

[5] H. Barnum, J. Barrett, L. O. Clark, M. Leifer, R. Spekkens, N. Stepanik, A. Wilce and R. Wilke (2010) *Entropy and information causality in general probabilistic theories.* New Journal of Physics **12**, 033024. arXiv:0909.5075

[6] H. Barnum, J. Barrett, M. Leifer and A. Wilce (2006) *Cloning and broadcasting in generic probabilistic theories.* arXiv:quant-ph/0611295

[7] H. Barnum, J. Barrett, M. Leifer and A. Wilce (2007) *A generalized no-broadcasting theorem.* Physical Review Letters **99**, 240501. arXiv:0707.0620

[8] J. Barrett (2007) *Information processing in generalised probabilistic theories.*Physical Review A **75** 032304. arXiv:quant-ph/0508211

[9] D. E. Browne and M. J. Hoban (2010) *Bell inequalities for (measurement based quantum) computation.* Talk at: Pictures, Quanta, Probability; August 19, Oxford University Computing Laboratory.

[10] A. Carboni and R. F. C. Walters (1987) *Cartesian bicategories I.* Journal of Pure and Applied Algebra **49**, 11–32.

[11] G. Chiribella, G. M. D'Ariano and P. Perinotti (2010) *Probabilistic theories with purification.* Physical Review A **81**, 062348. arXiv:0908.1583

[12] B. Coecke (2005) *Kindergarten quantum mechanics.* In: Quantum Theory: Reconsiderations of the Foundations III, pages 81–98. AIP Press. arXiv:quant-ph/0510032

[13] B. Coecke (2010) *Quantum picturalism.* Contemporary Physics **51**, 59–83. arXiv:0908.1787

[14] B. Coecke (2010) *A Universe of Processes and Some of its Guises.* In: Deep Beauty: Understanding the Quantum World through Mathematical Innovation. H. Halvorson, Ed., Cambridge University Press. arXiv:1009.3786

[15] B. Coecke and R. W. Duncan (2008) *Interacting quantum observables.* In: Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP), pp. 298–310, Lecture Notes in Computer Science 5126, Springer-Verlag. Extended version (2009): *Interacting quantum observables: categorical algebra and diagrammatics.* arXiv:0906.4725

[16] B. Coecke, B. Edwards and R. W. Spekkens (2010) *Phase groups and the origin of non-locality for qubits.* Electronic Notes in Theoretical Computer Science, to appear. arXiv:1003.5005

[17] B. Coecke and A. Kissinger (2010) *The compositional structure of multipartite quantum entanglement*. In: Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP), pp. 297–308, Lecture Notes in Computer Science 6199, Springer-Verlag. arXiv:1002.2540

[18] B. Coecke and R. Lal (2010) *Causal categories*. Proc. Quantum Physics and Logic VII.

[19] B. Coecke and D. Pavlovic (2007) *Quantum measurements without sums*. In: Mathematics of Quantum Computing and Technology, G. Chen, L. Kauffman and S. Lamonaco (eds), pages 567–604. Taylor and Francis. arXiv:quant-ph/0608035

[20] B. Coecke and S. Perdrix (2010) *Environment and classical channels in categorical quantum mechanics*. In: Proceedings of the 19th EACSL Annual Conference on Computer Science Logic (CSL), Lecture Notes in Computer Science 6247, Springer-Verlag. arXiv:1004.1598

[21] B. Coecke and R. W. Spekkens (2010) *Picturing classical and quantum Bayesian inference*. Synthese, to appear.

[22] L. Dixon, R. W. Duncan, A. Merry and A. Kissinger (2010) `quantomatic` software. http://dream.inf.ed.ac.uk/projects/quantomatic/.

[23] R. Duncan and S. Perdrix (2010) *Rewriting measurement-based quantum computations with generalised flow*. In: Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP), Lecture Notes in Computer Science 6199, Springer-Verlag.

[24] R. Haag (1992) *Local Quantum Physics: Fields, Particles, Algebras*. Springer-Verlag.

[25] H. Halvorson and M. Müger (2006) *Algebraic quantum field theory*. In: *Handbook of Philosophy of Physics*, J. Butterfield and J. Earman (eds), pages 731–922, Elsevier. arXiv:math-ph/0602036

[26] L. Hardy (2007) *Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure*. Journal of Physics A **40**, 3081–3099. arXiv:gr-qc/0608043

[27] L. Hardy (2010) *A formalism-local framework for general probabilistic theories including quantum theory*. arXiv:1005.5164

[28] A. Joyal and R. Street (1991) *The Geometry of tensor calculus* I. Advances in Mathematics **88**, 55–112.

[29] G. M. Kelly and M. L. Laplaza (1980) *Coherence for compact closed categories*. Journal of Pure and Applied Algebra **19**, 193–213.

[30] E. H. Kronheimer and R. Penrose (1967) *On the structure of causal spaces*. Mathematical Proceedings of the Cambridge Philosophical Society **63**, 481–501.

[31] S. Lack (2004) *Composing PROPs*. Theory and Applications of Categories **13**, 147–163.

[32] J. Lambek and P. J. Scott (1986) *Higher Order Categorical Logic*. Cambridge University Press.

[33] M. S. Leifer and D. Poulin (2008) *Quantum graphical models and belief propagation*. Annals of Physics **323**, 1899–1946. arXiv:0708.1337

[34] E. R. Livine and D. R. Terno (2007) *Quantum causal histories in the light of quantum information*. Physical Review D **75**, 084001.

[35] E. Mach (1960) *The science of mechanics: A critical and historical account of its development*. Open Court, La Salle.

[36] D. B. Malament (1977) *The class of continuous timelike curves determines the topology of spacetime*. Journal of Mathematical Physics, **18**, 1399–1404.

[37] K. Martin and P. Panangaden (2006) *A domain of spacetime intervals in general relativity.* Communications in Mathematical Physics **267**, 563–586. arXiv:gr-qc/0407094

[38] M. McKague (2009) *Quaternionic quantum mechanics allows non-local boxes.* arXiv:0911.1761

[39] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter and M. Zukowski (2009) *Information causality as a physical principle.* Nature **461**, 1101. arXiv:0905.2292

[40] I. Pitowski (1989) *Quantum Probability, Quantum Logic.* Springer-Verlag, Heidelberg.

[41] S. Popescu and D. Rohrlich (1994) *Quantum nonlocality as an axiom.* Foundations of Physics **24** 379.

[42] J. Power (1997) *Premonoidal categories and notions of computation.* Mathematical Structures in Computer Science **7**(5), 453–468.

[43] C. Rovelli (1997) *Half way through the woods.* In: The Cosmos of Science, J. Earman and J.D. Norton (eds), University of Pittsburgh Press.

[44] P. Selinger (2007) *Dagger compact closed categories and completely positive maps.* Electronic Notes in Theoretical Computer Science **170**, 139–163.

[45] P. Selinger (2010) *Finite dimensional Hilbert spaces are complete for dagger compact closed categories.* Electronic Notes in Theoretical Computer Science, to appear.

[46] A. J. Short and J. Barrett (2010) *Strong nonlocality: a trade-off between states and measurements.* New Journal of Physics **12** 033034. arXiv:0909.2601

[47] A. J. Short and S. Wehner (2010) *Entropy in general physical theories.* New Journal of Physics **12**, 033023. arXiv:0909.4801

[48] R. D. Sorkin (2005) *Causal sets: Discrete gravity (Notes for the Valdivia Summer School).* In A. Gomberoff and D. Marolf, editors, Lectures on Quantum Gravity (Proceedings of the Valdivia Summer School), Plenum. arXiv:gr-qc/0309009

[49] R. W. Spekkens (2007) *Evidence for the epistemic view of quantum states: A toy theory.* Physical Review A **75**, 032110. arXiv:quant-ph/0401052

[50] B. S. Tsirelson (1980) *Quantum generalizations of Bell's inequality.* Letters in Mathematical Physics **4**, 93–100.

[51] R. F. Werner and M. M. Wolf (2001) *All-multipartite Bell-correlation inequalities for two dichotomic observables per site.* Physical Review A **64**, 032112. arXiv:quant-ph/0102024

[52] E. C. Zeeman (1964) *Causality implies the Lorentz group.* Journal of Mathematical Physics, **5**, 490–493.