The Span Construction
Interpretations and Applications

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To Enrique, Arantxa, Daniel and Mireya

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Abstract

The compositional distributional account models meaning of sentences relying on their grammatical structure to guide semantic composition. It has been successfully applied to the theory of conceptual spaces for cognition in [7], using convex algebras that describe the important convex structure of natural concepts. In the literature, metric spaces are also widely used to characterise conceptual spaces, however, they do not have the necessary structure as a category, compact closedness, to fit into the compositional distributional account. There are several alternatives to build a compact closed category from a given one and, in particular, this dissertation investigates the so-called *span construction*.

Various interpretations and applications, including those in the cognition and linguistics setting, are analysed for spans of sets and metric spaces. A model for path composition is first explored to introduce the most important characteristics of this construction. Later, an interpretation for cognition to describe hyper-conceptual spaces results in applications for the phenomena of categorical perception or concept correlation. As a side effect, a connection with algebra of bags is explored and an enrichment over commutative monoids for span categories arises from this setting.

To conclude, it is found that spans of metric spaces are suitable to model several aspects in cognitive science and they have applications in other areas such as algebra of bags.

Contents

Chapter 1

Introduction

Traditionally, quantum mechanics has been described in the setting of finite dimensional Hilbert spaces **FHilb**. However, what makes this category ideal for this application in particular lies in the fact that it features a *dagger compact closed* structure. In fact, the most recent axiomatization of quantum computing takes place in this categorical setting [1], which allows to abstractly describe quantum protocols, such as teleportation and entanglement-swapping, with key elements such as the Born rule arising categorically.

The application of compact closed categories does not reduce exclusively to quantum mechanics. In the linguistics and cognition settings, the compositional distributional model of meaning [10] exploits this kind of structure to guide semantic composition through the syntactic structure of sentences. Is this setting and the important *compact closedness* property that motivates this dissertation.

1.1 The compositional distributional scheme

The syntactic structure of a sentence can be modelled by means of *pregroup grammars* [14]. A pregroup $(P, \leq, \cdot, (-)^l, (-)^r)$ is a *partially ordered monoid* where each element $p \in P$ has a left p^l and right p^r adjoints satisfying certain equations. In this model, words are assigned to elements in the pregroup representing their grammatical type. For example, the sentence *John plays football* is assigned the type

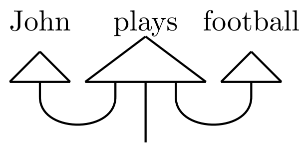
$$\begin{array}{ccc} \textit{John} & \textit{plays} & \textit{football} \\ & n \cdot (n^r s n^l) \cdot n, & \end{array}$$

where both *John* and *football* are assigned to noun n and *plays* is assigned to a compound type of verbs $n^r s n^l$, where the adjoints n^r, n^l encode interactions with the noun subject and direct object of the sentence. *Pregroup reductions* then model the syntactic interactions of the sentence:

$$n \cdot (n^r s n^l) \cdot n \leq (n n^r) \cdot s \cdot (n^l n) \leq 1 \cdot s \cdot (n^l n) \leq s \cdot 1 = s$$

It turns out that as a category, pregroups **Preg** exhibit *compact closed* structure, and here is where the compositional distributional scheme comes into play. Type reductions are regarded as structural morphisms in this *grammar category* defining a *semantic map*, graphically depicted as

The compositional distributional account then takes a *semantic category*, also exhibiting *compact closedness*, and uses functoriality to map the structural morphisms into it. Words are assigned to states in this category and fed into the semantic map,



which reflects the grammatical structure in the semantic category and guides meaning composition.

For example, in the linguistics setting, the traditional distributional model of meaning is embed in the compositional distributional framework in [10]. In this model, the meaning of words is represented in vectors containing information about occurrences of certain close contextual words. The semantic category, in this case, is taken to be finite dimensional Hilbert spaces **FHilb**. In this direction, [6],[17] are examples of recent work that expose the versatility of this abstract categorical framework.

1.2 The cognition setting

Gärdenfors's model for cognition [12] provides an account to describe concepts by means of geometrical structures denoted *conceptual spaces*. For example, the concept of *taste* is described by a 4-dimensional hypercube (See Figure 1.1), where each dimension is one of the four flavour dimensions: *sweet*,*bitter*,*saline*,*sour*. The geometric structure in these spaces is described in terms of notions such as *distance* or *convexity*.

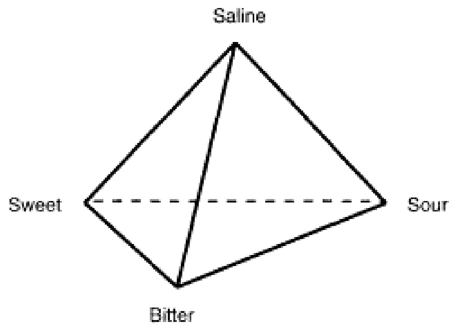


Figure 1.1: Taste hypercube; from [12]

In a model that strongly emphasises convexity of natural concepts, the compositional distributional scheme is successfully applied to Gärdenfors's theory in [7]. In this scenario, convex conceptual spaces are modelled by means of *convex algebras* (A, α), which are essentially a set A together with a *mixing* function α that takes a collection elements in A and creates a *convex mixture* lying in A again. It turns out that convex algebras are Eilenberg-Moore algebras for a monad, and they form a regular category from which a compact closed category of relations can be constructed. The resulting category **ConvexRel** of convex algebras and relations on them is taken as semantic category and determines how meaning is composed.

On the other hand, many of Gärdenfors's ideas involve the notion of distance, which he models mathematically using *metric spaces*. A metric space is a set M together with a distance function d measuring the distance between two points in M. Unfortunately, none of the variants for categories of metric spaces **HMet**, **QMet** and **Met** features *compact closedness* and, consequently, they cannot be regarded as semantic categories for the compositional distributional model of meaning.

In order to use metric spaces with the compositional distributional framework, it is necessary to construct a compact closed category over them. However, the construction of relations over Eilenberg-Moore categories cannot be applied in this scenario and, as a result, it is necessary to resort to other constructions. A very standard one in the literature, but not much investigated in this setting, is the so-called *span construction*, on which this dissertation focuses its attention.

1.3 The span construction

The span construction essentially takes diagrams of the form

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in a base category, regards them as arrows and defines a composition law. Interestingly, the resulting category is compact closed, provided that the original category satisfies a few assumptions.

This is an abstract construction that has not been yet analysed for the cognition setting. For that reason, this dissertation aims to build intuitions on spans of sets and metric spaces to shed some light on the way they behave and analyse their suitability for applications in cognition.

1.4 Outline of the dissertation

The ideas presented in this dissertation unfold along three core chapters. The main contribution is given in Chapters 3 and 4, where all the ideas and results developed are original unless otherwise stated.

As a starting point, John Baez's interpretation of *paths* and *matrix mechanics* for spans of sets [5] is inspected in Chapter 2. This is used as a vehicle to introduce general results about the span construction. A small contribution of the author of this thesis in this part is the introduction of an informal notation, a graphical language, to better describe the path interpretation.

Subsequently, Chapter 3 explores an interpretation of spans of metric spaces for conceptual spaces. This results in a model for *hyper-conceptual spaces*, where spans encode concept-subconcept relations. Several applications, such as the phenomenon of *categorical perception* or the compositional distributional model, are then modelled using this new setting.

Finally, a side connection with *algebra of bags* is investigated in Chapter 4. This results in a categorical framework to describe tables and operations on them, with the most fundamental ones (*projection*, *natural join*, *selection*, *rename* and set-theoretic operations) being described categorically. A good interaction of span composition with set-theoretic operations on them is shown in Theorem 4. Also, this perspective points to an enrichment over commutative monoids for spans of metric spaces (Theorem 2) and, more generally, for spans of locally distributive categories (Theorem 3), with potential applications in the general setting of this dissertation.

Chapter 2

A model for matrix mechanics

This chapter will introduce the span construction and will present the first of the intuitions on spans: John Baez's interpretation of *paths* and *matrix mechanics* for spans of sets [5].

2.1 A bit of category theory

2.1.1 Monoidal categories

A basic knowledge of category theory, such as understanding of products, coproducts and pullbacks, will be assumed in this work and will not be covered; instead, the reader can refer to [4],[2] if necessary. This section will give a brief introduction to the theory of *symmetric monoidal categories* and some of the internal structures that they feature. Further background on the topic can be found in [15] and [1]. Before that, however, some basic notions will be revised.

Lemma 1. *In* **Set***, the pullback of* $X \xrightarrow{f} Z \xleftarrow{g} Y$ *is given by* $X \xleftarrow{\pi_X} X \times_Z Y \xrightarrow{\pi_Y} Y$*, where*

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

and π_X,π_Y *are the projections into the corresponding components.*

Notation. *In the following,* $\Delta_X : X \to X \times X$ *and* $\tau_X : X \to \mathbf{1}$ *will be used to denote the diagonal morphism and the unique terminal arrow from the product structure.*

The main categorical setting on which all the results in this dissertation focus is that of *symmetric monoidal categories*, which are defined as follows:

Definition 1 (Symmetric monoidal category)**.** *A* monoidal category *is given by a structure* $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ *where*

1. **C** *is a category,*

2. $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is a functor; the* tensor product *of the monoidal structure,*

3. I *is a distinguished object in* **C***; the* the monoidal unit*,*

4. α,λ,ρ,σ *are natural isomorphisms, known as the* associator,left unitor*,* right unitor *and* symmetry

$$\begin{aligned}
\alpha_{X,Y,Z} &: X \otimes (Y \otimes Z) \overset{\cong}{\to} (X \otimes Y) \otimes Z \\
\lambda_X &: I \otimes X \overset{\cong}{\to} X \\
\rho_X &: X \otimes I \overset{\cong}{\to} X \\
\sigma_{X,Y} &: X \otimes Y \overset{\cong}{\to} Y \otimes X
\end{aligned}$$

that express associativity of the tensor product, unitality of the monoidal unit with regard to the tensor product and commutativity (symmetry) of the tensor product.

5. *The data above satisfies the* triangle*,* pentagon *and* hexagon equations *[15], which ensure coherence in the monoidal structure.*

Essentially, the coherence conditions ensure that all possible ways of associating, uniting and interchanging tensor-compound types are isomorphic by means of a unique isomorphism built from the axioms of a symmetric monoidal category. For example,

$$(X \otimes Y) \otimes (Z \otimes I) \overset{(\mathrm{id}_X \otimes \sigma_{Y,Z}) \circ (\mathrm{id}_X \otimes (\mathrm{id}_Y \otimes \rho_Z)) \circ \alpha^{-1}_{X,Y,Z \otimes I}}{\longrightarrow} X \otimes (Z \otimes Y),$$

and this is the unique morphism built out of identities, associators, unitors and symmetries. In practice, this means that any rearrangement can be safely ignored and work without the structural morphisms in any formula, since they are implicit.

Some symmetric monoidal categories feature a special internal structure known as *compact closedness*. They are particularly interesting because they can be described by means of a graphical calculus that allows to reason in them using *string diagram calculations*. This graphical calculus will be later introduced in more detail.

Definition 2 (Compact closed category)**.** *A* compact closed category *is a symmetric monoidal category* **C** *in which every object* X *has a dual object* X^* *and there exist* *a* unit *morphism* $\eta_X : I \to X^* \otimes X$ *and a* counit *morphism* $\epsilon_X : X \otimes X^* \to I$ *that satisfy the* snake equations

$$(\epsilon_X \otimes id_X) \circ (id_X \otimes \eta_X) = id_X \qquad (id_{X^*} \otimes \epsilon_X) \circ (\eta_X \otimes id_{X^*}) = id_{X^*}$$

Dual objects are unique up to isomorphism.

On the other hand, symmetric monoidal categories can feature another type of internal structure: a *dagger structure*. They arise from a special type of functor that strongly relates the category with its opposite category while preserving the monoidal structure.

Definition 3 (Dagger functor)**.** *A dagger functor is an involutive, contravariant and identity on objects endofunctor* $\dagger : \mathbf{C^{op}} \to \mathbf{C}$.

Definition 4 (Dagger symmetric monoidal category)**.** A dagger symmetric monoidal category *is a symmetric monoidal category* $\mathbf{C}$ *equipped with a dagger functor* $\dagger : \mathbf{C^{op}} \to \mathbf{C}$ *that preserves the monoidal structure, i.e.* $(X \otimes Y)^\dagger \cong (X^\dagger \otimes Y^\dagger)$ *natural.*

Furthermore, the two previous structures, the *dagger structure* and the *compact closedness*, may interact well with each other, allowing the expression of units in terms of counits, and vice versa, by means of the dagger functor. A category exhibiting this behaviour is known as a *dagger compact closed category.*

Definition 5 (Dagger compact closed category)**.** *A* dagger compact closed category *is a compact closed, dagger symmetric monoidal category in which the compact and dagger structures interact well, i.e. the following diagram commutes for every object:*

$$\begin{tikzcd} I \arrow[r, "\epsilon_X^\dagger"] \arrow[dr, "\eta_X"'] & X \otimes X^* \arrow[d, "\sigma_{X,X^*}"] \\ & X^* \otimes X \end{tikzcd}$$

Important examples of categories with this kind of structure are the category of relations **Rel** and the category of finite dimensional Hilbert spaces **FHilb**.

The well-known mathematical structure of *monoids* and their dual counterpart, *comonoids*, may appear internally in a monoidal category. If they moreover interact well with each other, they give rise to the so-called *Frobenius algebras*, which are typically used to model flows of information and operations on them such as *copying* or *discarding.*

Definition 6 (Frobenius algebra, dagger Frobenius algebra)**.** *A Frobenius algebra in a monoidal category is a quintuple* $(X, \delta, \iota, \mu, \zeta)$ *where*

1. (X, δ, ι) *is an internal comonoid, i.e.* $\delta : X \to X \otimes X$*,* $\iota : X \to I$ *satisfy coassociativity and counitality:*

$$(\delta \otimes id_X) \circ \delta = (id_X \otimes \delta) \circ \delta \qquad (id_X \otimes \iota) \circ \delta = id_X = (\iota \otimes id_X) \circ \delta,$$

2. (X, μ, ζ) *is an internal monoid, i.e.* $\mu : X \otimes X \to X$*,* $\zeta : I \to X$ *satisfy associativity and unitality:*

$$\mu \circ (\mu \otimes id_X) = \mu \circ (id_X \otimes \mu) \qquad \mu \circ (id_X \otimes \zeta) = id_X = \mu \circ (\zeta \otimes id_X),$$

3. *and they moreover satisfy the* Frobenius law*:*

$$(\mu \otimes id_X) \circ (id_X \otimes \delta) = \delta \circ \mu = (id_X \otimes \mu) \circ (\delta \otimes id_X)$$

In a dagger symmetric monoidal category, it is said to be a dagger Frobenius algebra *if* $\dagger(\delta) = \mu$ *and* $\dagger(\iota) = \zeta$*. The maps* δ, μ *are typically referred as* copy *and* uncopy *maps respectively.*

Finally, some categories can exhibit a commutative monoid structure over the hom-set. When it interacts well with composition, it is said that the category is *enriched over commutative monoids.*

Definition 7 (Enrichment over commutative monoids)**.** *A category* $\mathbf{C}$ *is said to be* enriched over commutative monoids *if there is a commutative monoid structure over the hom-set, i.e there exists an operation* $+ : \mathbf{C}(X, Y) \times \mathbf{C}(X, Y) \longrightarrow \mathbf{C}(X, Y)$ *that satisfies*

1. ***Commutativity:*** $f + g = g + f$*,*

2. ***Associativity:*** $(f + g) + h = f + (g + h)$ *and*

3. ***Units:*** *for all* X*,*Y *there exists a* unit *morphism* $u_{X,Y} : X \to Y$ *such that for all* $f : X \to Y$ *it holds that* $f + u_{X,Y} = f$*,*

and that is moreover compatible with composition, i.e

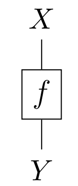
4. ***Addition compatible with composition*** $(f + g) \circ h = (f \circ h) + (g \circ h)$ *and* $h \circ (f + g) = (h \circ f) + (h \circ g)$ *and*

5. ***Units compatible with composition*** $f \circ u_{X,X} = u_{X,Y} = u_{Y,Y} \circ f$*.*

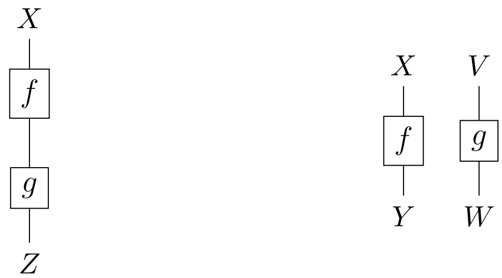
An example of a category enriched over commutative monoids is **FHilb**, where the monoid structure over the hom-set is given by addition of linear maps with unit the constant zero linear map.

2.1.2 Graphical calculus for compact closed categories

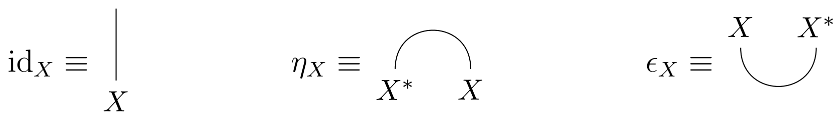
As it was briefly mentioned earlier in the previous section, compact closed categories can be described by a graphical calculus that allows to perform calculations on morphisms. This section is based on the graphical calculus in [9]. In this calculus, arrows $f : X \to Y$ are represented as boxes



with input and output wires for the domain and the codomain respectively. Composition and tensor of arrows are depicted by placing them sequentially or in parallel respectively:



Distinguished morphisms, such as the identity, unit and counit, have special depictions, as they play a fundamental role in this graphical calculus. Their representations are:

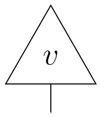


Note that the trivial system I in both the unit and counit, and in general, is not depicted. With this representation, the snake equations receive a characteristic form

that resembles a pulling out, a deformation, of a string. In fact, a very powerful general result [18] for compact closed categories states that

Theorem 1. *An equation between morphisms holds if and only if it holds in the graphical language up to framed isotopy in three dimensions.*

Framed isotopy simply expresses that wires are treated as if they were ribbons in a three dimensional space. This theorem is the cornerstone of the graphical calculus: calculations with morphisms are reduced to deformations of string diagrams. Finally, in some sections, states, which are morphisms of the form $v : I \to X$, will be intensively used and they will be represented in the graphical calculus as:



2.2 The span category

The span construction is an elegant category-theoretic construct that allows to turn any arbitrary category that satisfies some few requirements, such as the existence of pullbacks and finite products, into a richer category with more interesting properties. Specifically, the resulting category will inherit a monoidal structure and will be compact closed, which makes it an attractive candidate for the applications along the lines of this dissertation. This construction relies on some particular diagrams in the original category, which will be the spans, and treats them as arrows. This section is based on [16].

Definition 8 (Span)**.** *Given any category* $\mathbf{C}$*, a span from* X *to* Y *is a diagram of the form*

$$\begin{tikzcd} & S \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & \\ X & & Y \end{tikzcd}$$

S *is the* apex*,* X *and* Y *are the* domain *and* codomain *respectively and the arrows* s_X *and* s_Y *are the* left *and* right legs *of the span.*

Remark. *In many occasions, specially due to the notation adopted for the legs of the spans, different arrows will be given the same name and will be distinguished instead by the type information encoded in their domain and codomain. This choice will increase readability of the ideas presented.*

It turns out that if $\mathbf{C}$ has pullbacks, then spans can be composed. Given two spans $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$, $Y \xleftarrow{s_Y} T \xrightarrow{s_Z} Z$, the pullback of the two apices with respect to the coinciding legs produces a new cone

$$
\begin{tikzcd}
 & & S \times_Y T \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_T"] & & \\
 & S \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & & T \arrow[dl, "s_Y"'] \arrow[dr, "s_Z"] & \\
X & & Y & & Z
\end{tikzcd}
$$

that allows to define a composite span by taking composite legs into the new domain and codomain:

$$
\begin{tikzcd}
 & S \times_Y T \arrow[dl, "s_X \circ \pi_S"'] \arrow[dr, "s_Z \circ \pi_T"] & \\
X & & Z
\end{tikzcd}
.
$$

However, since pullbacks are defined up to isomorphism, a choice of pullbacks will be assumed and instead the span construction will be built over a class of isomorphic spans. To make this idea more precise, first it is necessary to define the notion of *map between spans*.

Definition 9 (2-morphisms)**.** *Let* $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$, $X \xleftarrow{s_X} T \xrightarrow{s_Y} Y$ *be spans in a category* $\mathbf{C}$*. A map between them or* 2-morphism *is given by a morphism* $f : S \longrightarrow T$ *in* $\mathbf{C}$ *such that the following diagram commutes:*

$$
\begin{tikzcd}
 & S \arrow[d, "f"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\
 & T \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\
X & & Y
\end{tikzcd}
$$

The basic idea behind this concept is that both legs of the source span factorise over or can be given in terms of the target span. On the other hand, isomorphic spans are then those such that there exists a 2-isomorphism between them and an isomorphic class of spans is the collection of all spans that are isomorphic to a certain representative. This allows for a suitable definition of span category.

Definition 10 (Span category)**.** *Given a category* $\mathbf{C}$ *with pullbacks, the span category* $\mathsf{Span}(\mathbf{C})$ *has objects of* $\mathbf{C}$ *as objects and classes of isomorphic spans as arrows.*

To start appreciating some interesting properties of this construction, take $\mathsf{Span}(\mathbf{Set})$ and consider two binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$. They can be made into spans by taking the projections into their domain and codomain $X \xleftarrow{\pi_X} R \xrightarrow{\pi_Y} Y$, $Y \xleftarrow{\pi_Y} S \xrightarrow{\pi_Z} Z$. Composing them results in the span

$$\begin{tikzcd} & R \times_Y S \arrow[dl, "\pi_X \circ \pi_R"'] \arrow[dr, "\pi_Z \circ \pi_S"] & \\ X & & Z \end{tikzcd}$$

where

$$\begin{aligned} R \times_Y S =& \{(r, s) \in R \times S : \pi_Y(r) = \pi_Y(s)\} \\ =& \{(r, s) = ((x, y), (y', z)) \in R \times S : \pi_Y(x, y) = y = y' = \pi_Y(x', y')\} \\ =& \{((x, y), (y, z)) \in R \times S\}, \end{aligned}$$

i.e. if there is some shared $y \in Y$, then elements $(x, y) \in R$ are related with elements $(y, z) \in S$. In other words, the composite span encodes binary relation composition and, in fact, there is a 2-morphism into the actual composite relation

$$R; S = \{(x, z) : \exists \quad y (x, y) \in R, (y, z) \in S\},$$

given by $f : R \times_Y S \to R; S : ((x, y), (y, z)) \mapsto (x, z)$, that disregards the extra information, the shared element, that the composite span carries. It is easy to observe that the diagram

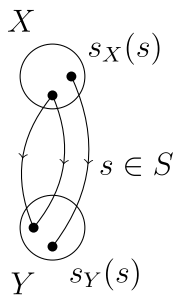
$$\begin{tikzcd} & R \times_Y S \arrow[ddl, "\pi_X \circ \pi_R"'] \arrow[d, "f"] \arrow[ddr, "\pi_Z \circ \pi_S"] & \\ & R; S \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Z"'] & \\ X & & Z \end{tikzcd}$$

commutes, since the right projections are involved.

2.3 A model for matrix mechanics

This section will present John Baez's matrix mechanics and path interpretation [5] for spans of sets and will use it as a vehicle to introduce general results about span categories.

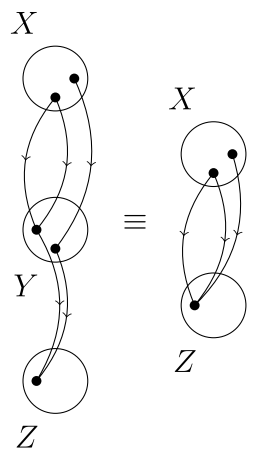
Surprisingly, $\mathsf{Span}(\mathbf{Set})$ can be interpreted as a model for path composition. In this interpretation, a span $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ describes a collection of paths together with their starting and ending points. The sets X and Y are collections of starting and ending points respectively; whereas the set S is the collection of paths. The left and right legs of the span associate each path to the point where it starts and the point where it ends.



Span composition then models path composition. Given two collections of paths $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ and $Y \xleftarrow{s_Y} T \xrightarrow{s_Z} Z$, their composite

$$\begin{array}{ccc} & S \times_Y T = \{(s,t) : s_Y(s) = s_Y(t)\} & \\ \swarrow^{s_X \circ \pi_S} & & \searrow^{s_Z \circ \pi_T} \\ X & & Z \end{array}$$

joins a path $s \in S$ and a path $t \in T$ if they meet in the middle and preserves the starting point of s and the ending point of t.



Tightly connected with this path interpretation, there is another interpretation that emphasises the fact that there is some sort of movement from a source to a target. Letting S_y^x be the set of paths or ways to go from point x to y, a span $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ can be interpreted as a matrix $(S_y^x)_{y \in Y}^{x \in X}$, where X,Y are the rows and columns and the two legs assign each element in S to a cell.

To get from a point $x \in X$ to a point $z \in Z$ through an intermediate location Y, any $y \in Y$ is taken as a commuting point, so the collection of paths from x to z

is given by $(\bigsqcup_{y \in Y} S^x_y \times T^y_z)$. Span composition, under the matrix interpretation, is matrix multiplication

$$(S^x_y)^x_y \,;(T^y_z)^y_z = (\bigsqcup_{y \in Y} S^x_y \times T^y_z)^x_z$$

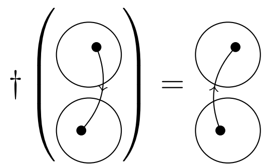
in the semiring $(\mathbf{Set}, \sqcup, \times)$; a categorified semiring in which the elements are sets and the addition and product operations are the categorical coproduct and product. As a result, $\mathsf{Span}(\mathbf{Set})$ models movements or *mechanics* by means of matrices: it is also a model for *matrix mechanics*.

The inherent regularity of spans provides the span construction with a dagger functor. Each span can be identified with a symmetric version that has both domain and codomain interchanged. Moreover, the very same symmetry makes the operation an involution.

Proposition 1 ($\mathsf{Span}(\mathbf{C})$ is a dagger category)**.** $\mathsf{Span}(\mathbf{C})$ *is a dagger category when equipped with the dagger functor that takes each span to its mirror image:*

$$\dagger\left(\begin{tikzcd} & S \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & \\ X & & Y \end{tikzcd}\right) = \begin{tikzcd} & S \arrow[dl, "s_Y"'] \arrow[dr, "s_X"] & \\ Y & & X \end{tikzcd}$$

This functor satisfies the axioms of a *dagger functor*: it is contravariant, involutive and the identity on objects. In $\mathsf{Span}(\mathbf{Set})$, interpreting spans as collection of paths, the dagger functor turns their orientation around:



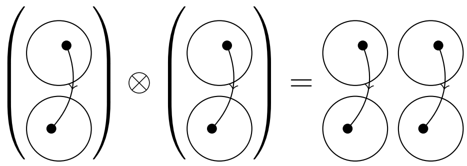
On the other hand, interpreted as matrices, the dagger functor takes each matrix to its transpose $\dagger[(S^x_y)^x_y] = (S^y_x)^x_y$.

It turns out that if $\mathbf{C}$ moreover has finite products, $\mathsf{Span}(\mathbf{C})$ inherits a monoidal structure that makes it into a monoidal category. In this setting, binary products in $\mathbf{C}$ are used to glue together two spans into a combined one, as if it were a realisation of both of them at the same time. The monoidal unit is taken to be the terminal object in $\mathbf{C}$.

Proposition 2. *If* $\mathbf{C}$ *is a category with finite limits, then* $\mathsf{Span}(\mathbf{C})$ *is a symmetric monoidal category with a tensor product given by:*

$$\left(\begin{tikzcd} & S \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & \\ X & & Y \end{tikzcd}\right) \otimes \left(\begin{tikzcd} & S' \arrow[dl, "s_{X'}"'] \arrow[dr, "s_{Y'}"] & \\ X' & & Y' \end{tikzcd}\right) = \begin{tikzcd} & S \times S' \arrow[dl, "s_X \times s_{X'}"'] \arrow[dr, "s_Y \times s_{Y'}"] & \\ X \times X' & & Y \times Y' \end{tikzcd}$$

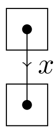
In $\mathsf{Span}(\mathbf{Set})$, a path in the tensor product comprises two paths that take place in *parallel*, with independent starting and ending points.



In addition, in the matrix mechanics counterpart, the tensor product of two set-valued matrices is a generalisation of the usual tensor product:

$$(S_y^x)_y^x \otimes (T_w^v)_w^v = (S_y^x \times T_w^v)_{y,w}^{x,v}.$$

Hereafter, a more diagrammatic representation for paths will be adopted in order to illustrate equations involving them in a more insightful manner. This notation will only be used for certain distinguished spans and will reflect properties of the span that it is being represented. Notice, however, that this is only an informal representation and it is not meant to be a rigorous system to prove statements. To begin with, the identity arrow span, which is given by $\mathrm{id}_X \equiv X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$, will be depicted as


,

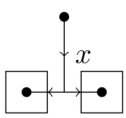
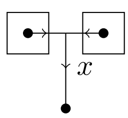
where the index x ranges over the collection of paths X. Informally, this means that there is a path for each $x \in X$ that starts at a homonymous point x in a location X and ends at a homonymous point x in a location X as well.

The same product structure in $\mathbf{C}$ allows for a richer compact closed structure on $\mathsf{Span}(\mathbf{C})$. In this case, an object X is both left and right dual of itself.

Proposition 3. *If* $\mathbf{C}$ *is a category with finite limits, then* $\mathsf{Span}(\mathbf{C})$ *is a compact closed category with unit and counit given by:*

$$\eta \equiv \begin{tikzcd} & X \arrow[dl, "\tau_X"'] \arrow[dr, "\Delta_X"] & \\ \mathbf{1} & & X \times X \end{tikzcd} \qquad \epsilon \equiv \begin{tikzcd} & X \arrow[dl, "\Delta_X"'] \arrow[dr, "\tau_X"] & \\ X \times X & & \mathbf{1} \end{tikzcd}$$

In $\mathsf{Span}(\mathbf{Set})$, the *unit* is a collection of paths that are originated at the same fixed point and end in mirror points in two different locations. On the other hand, the *counit* is a collection of paths that start at mirror points in two different locations and end at the same fixed point.

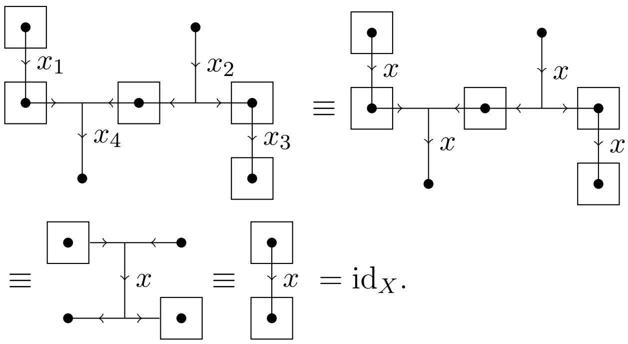
$\eta \equiv$  $\qquad \epsilon \equiv$ 

In terms of matrices, the unit and counit are the column vector $\eta = (\delta_{x,x'})_{x,x'}$ and row vector $\epsilon = (\delta_{x,x'})^{x,x'}$ respectively, where

$$\delta_{x,x'} = \begin{cases} \{x\} & \text{if } x = x' \\ \emptyset & \text{otherwise} \end{cases}$$

is a generalised *Kronecker delta.* The snake equations can be verified under both interpretations:

$$(\epsilon \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \eta) =$$



The first step is effectively using path composition: only when the paths are $x_1 = x_2 = x_3 = x_4$ they meet in all the middle points; and it is merely an intermediate step that leads to the span in the next equivalence. The last equivalence uses that $\mathbf{1} \times X \cong X \cong X \times \mathbf{1}$. On the other hand, using the matrix mechanics interpretation, the remaining snake equation can be verified.

$$\begin{aligned}
(\mathrm{id}_X \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_X) &= [(\delta_{x_1,x_1'})^{x_1}_{x_1'} \otimes (\delta_{x_2,x_2'})^{x_2,x_2'}] \circ [(\delta_{x_3,x_3'})_{x_3,x_3'} \otimes (\delta_{x_4,x_4'})^{x_4}_{x_4'}] \\
&= [(\delta_{x_1,x_1'} \times \delta_{x_2,x_2'})^{x_1,x_2,x_2'}_{x_1'}] \circ [(\delta_{x_3,x_3'} \times \delta_{x_4,x_4'})^{x_4}_{x_3,x_3',x_4'}] \\
&= (\bigsqcup_{x_1 \in X, x_2 \in X, x_2' \in X} \delta_{x_1,x_1'} \times \delta_{x_2,x_2'} \times \delta_{x_1,x_2} \times \delta_{x_4,x_2'})^{x_4}_{x_1'} \\
&= (\delta_{x_1',x_4})^{x_4}_{x_1'} \\
&= (\delta_{x_1,x_1'})^{x_1}_{x_1'} = \mathrm{id}_X
\end{aligned}$$

The first three equalities use definition of identity, unit and counit; definition of tensor product and definition of composition. The last equality uses that it must be $x_1' = x_1 = x_2 = x_2' = x_4$ so the product is not empty.

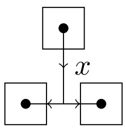
Finally, both the compact and dagger structures are compatible and there exist Frobenius algebras for all objects.

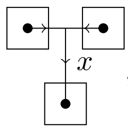
Proposition 4. *If* $\mathbf{C}$ *is a category with finite limits,* $\mathsf{Span}(\mathbf{C})$ *is a dagger compact category.*

Proposition 5. *If* $\mathbf{C}$ *is a category with finite limits, then* $\mathsf{Span}(\mathbf{C})$ *has dagger Frobenius algebras* $(X, \delta, \iota, \mu, \zeta)$ *for all objects* X *given by*

$$\delta = \dagger(\mu) \equiv \begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[dr, "\Delta_X"] & \\ X & & X \times X \end{tikzcd} \qquad \iota = \dagger(\zeta) \equiv \begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[dr, "\tau_X"] & \\ X & & \mathbf{1} \end{tikzcd}$$

In the path interpretation, δ and μ communicate three mirror points in three different locations

$\delta \equiv$ 

$\mu \equiv$ 

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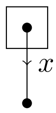
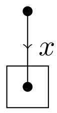
while ι connects every point with a single fixed point in a different location and ζ links a fixed point with every point in another location.

$\iota \equiv$ 

$\zeta \equiv$ 

.

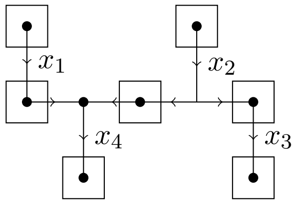
On the other hand, in terms of matrices, the morphisms of this Frobenius algebra are given by $\delta = (\delta_{x,x',x''})^{x}_{x',x''}$, $\mu = (\delta_{x,x',x''})^{x,x'}_{x''}$, $\iota = (x)_x$, $\zeta = (x)^x$, where

$$\delta_{x_1,x_2,\cdots,x_n} = \begin{cases} \{x\} & \text{if } x_1 = x_2 = \cdots = x_n \\ \emptyset & \text{otherwise} \end{cases}$$

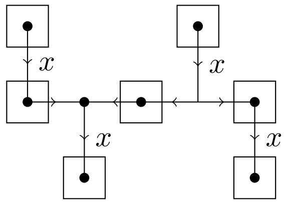
is a generalised multiparameter *Kronecker delta*.

The fact that $(X, \delta, \iota, \mu, \zeta)$ as defined earlier is a dagger Frobenius algebra can be readily verified using the graphical language of paths. As an instance, the Frobenius law unfolds as follows:

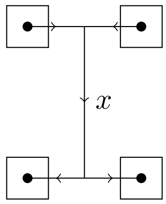
$(\mu \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \delta) =$



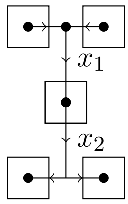
$\equiv$



$\equiv$

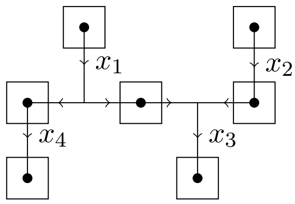


$\equiv$



$= \delta \circ \mu$

$\equiv$



$= (\mathrm{id}_X \otimes \mu) \circ (\delta \otimes \mathrm{id}_X).$

The first step uses path composition and leads to the resulting span in the second equivalence. The remaining equivalences are different compositions of spans that result in the same canonical form. Observe how, in the path interpretation, the different components of the Frobenius law are just rearrangements of ways to go from an initial location to a final one.

Chapter 3

A model for hyper-conceptual spaces

This chapter will present interpretations for spans of metric spaces. As a first step, it will briefly introduce a slight modification to the *path interpretation* developed in Chapter 2 to account for metrics. Afterwards, the main interpretation for conceptual spaces will be given: a model for *hyper-conceptual spaces* that describes extension of concepts with correlation. Finally, this interpretation will be applied to the phenomenon of *categorical perception* and *the compositional distributional model of meaning* in the cogintion setting.

3.1 Metric spaces and conceptual spaces

3.1.1 Categories of metric spaces

Hitherto, spans of sets have been analysed and possible interpretations for them have been discussed. However, the main focus of this dissertation is on metric spaces and, therefore, a further step will be taken, exploring the slightly different setting in which spans are constructed over metric spaces. This section is based on [13], where the reader is referred to for further information.

These spaces add some additional structure on sets. Namely, they come with a *distance* map that allows to talk about the concept of *proximity* between two points in a set. Depending on the conditions imposed on this map, three different notions of *metric* arise.

Definition 11 (Hemi-metric,Quasi-metric,Metric). *A* hemi-metric *d on a set X is a* *map* $d : X \times X \to [0, \infty]$ *such that*

$$d(x, x) = 0 \text{ for all } x \in X$$
$$d(x, y) \leq d(x, z) + d(y, z) \text{ for all } x, y, z \in X.$$

A quasi-metric *is a hemi-metric* d *that is moreover* T_0*, i.e.*

$$d(x, y) = d(y, x) = 0 \implies x = y.$$

And a metric *is a hemi-metric* d *that is additionally symmetric, i.e*

$$d(x, y) = d(y, x).$$

Hemi-metric is the weakest notion of the three of them. It requires the two essential axioms that capture the concept of a distance: non-negativity and the triangular inequality. Notice that the symmetry condition that later appears in the definition of a metric is not present. This may be interesting in situations where there is a different cost or effort between two points depending on the direction. A *Quasi-metric*, on the other hand, satisfies the T_0 axiom, which is adequate in settings where if two points are at zero distance from each other regardless of the direction then they are treated as the same.

Finally, a *metric* embodies the traditional axiomatisation of the concept of distance. In this case, distances are symmetric, so the proximity of two points does not take into account the direction of the movement, which is closer to the natural intuition of distance. In particular, the T_0 condition is direct consequence of the axioms of a metric, which makes this kind of map into a stronger version of a quasi-metric. Moreover, combined with the symmetry prerequisite, T_0 implies

$$d(x, y) = 0 \iff x = y,$$

i.e. points at zero distance are regarded as the same.

An important remark is that this definition of distance allows for infinity distances. It is not unusual to find a definition of distance that does not allow for infinity in the literature. For the purposes of this dissertation, infinity distances are more suitable for the categories that will be investigated.

The three notions of distance above then give rise to three different conceptions of metric spaces when paired together with a set.

Definition 12. *A* (hemi-,quasi-) metric space (X, d) *is a set* X *together with a (hemi-,quasi-) metric* d *on it.*

In addition, there is a notion of morphism between these spaces that allows to consider transformations between them. In this dissertation, only the so-called *weak contractions* will be acknowledged, as they are the canonical choice to build categories upon in the literature.

Definition 13 (1-Lipschitz map)**.** *A* 1-Lipschitz map *or* weak contraction *is a map between (hemi-,quasi-) metric spaces* $f : (X, d) \to (Y, d')$ *such that* $d'(f(x), f(x')) \leq d(x, x')$ *for all* $x, x' \in X$*.*

Definition 14 (**HMet**,**QMet**,**Met**)**.** **HMet**,**QMet**,**Met** *are the categories of hemi-metric, quasi-metric, metric spaces respectively and 1-Lipschitz maps between them.*

The most general theory developed along this thesis can be described regardless of the choice of category. However, in some particular scenarios and applications, it will be necessary to resort to the weakest of the three of them, i.e. *hemi-metric* spaces, and it will be because it is the only scenario that allows *metric 0* on arbitrary sets:

$$0 : X \times X \longrightarrow [0, \infty] : (x, x) \mapsto 0(x, x) = 0.$$

The T_0 axiom on both **QMet** and **Met** would collapse all points, which are at distance 0 from each other in both directions, into the same point and, therefore, this metric does not exist there.

Finally, the most important limits for metric spaces have the following form:

Lemma 2 (Product)**.** *The* product *in* **Met** *(resp.* **HMet**,**QMet***) of two metric spaces* (X, d_X)*,* (Y, d_Y) *is given by*

$$(X, d_X) \overset{\pi_X}{\leftarrow} (X \times Y, d_X \times d_Y) \overset{\pi_Y}{\rightarrow} (Y, d_Y),$$

where $(d_X \times d_Y) : (X \times Y) \times (X \times Y) \to [0, \infty]$ *is the product metric, defined as*

$$(d_X \times d_Y)((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\},$$

and π_X*,* π_Y *are the projections into the respective components.*

Lemma 3 (Coproduct)**.** *The* coproduct *in* **Met** *(resp.* **HMet**,**QMet***) of two metric spaces* (X, d_X)*,* (Y, d_Y) *is given by*

$$(X, d_X) \overset{i_X}{\rightarrow} (X \sqcup Y, d_X \sqcup d_Y) \overset{i_Y}{\leftarrow} (Y, d_Y),$$

where $(d_X \sqcup d_Y) : (X \sqcup Y) \times (X \sqcup Y) \to [0, \infty]$ *is defined as*

$$(d_X \sqcup d_Y)((z,i),(z',i')) = \begin{cases} d_X(z,z') & \text{if } i = i' = 1 \\ d_Y(z,z') & \text{if } i = i' = 2 \\ \infty & i \neq i', \end{cases}$$

and i_X*,* i_Y *are the injections into the disjoint union.*

Lemma 4 (Pullback)**.** *The* pullback *in* **Met** *(resp.* **HMet**,**QMet***) of a pair of arrows* $(X, d_X) \xrightarrow{f} (Z, d_Z) \xleftarrow{g} (Y, d_Y)$ *is given by*

$$(X, d_X) \xleftarrow{\pi_X} (X \times_Z Y, d_X \times d_Y) \xrightarrow{\pi_Y} (Y, d_Y),$$

where $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$*,*

$(d_X \times d_Y) : (X \times_Z Y) \times (X \times_Z Y) \to [0, \infty] : ((x,y),(x',y')) \mapsto \max\{d_X(x,x'), d_Y(y,y')\}$,

is the product metric and π_X*,* π_Y *are the projections into the respective components.*

3.1.2 Conceptual spaces

The interpretation of spans in this chapter is mainly inspired by Gärdenfors's cognitive framework for concept representation and aims to establish connections with some of his ideas developed in [12]. In particular, the idea of conceptual space plays a key role in this account and will be central to the span perspective in this thesis. The following is a humble introduction to his framework [12, Ch. 1,3,4] [1].

Before introducing the notion of conceptual space, some other ideas need to be clarified. The first of these notions is *quality dimension*. A quality dimension represents an attribute of an object. Examples of these dimensions are height, width, depth (spatial dimensions), bitterness, sweetness (flavour dimensions), hue, brightness or chromaticness (colour dimensions).

These quality dimensions can be *integral* or *separable*. Integral dimensions are sets of quality dimensions that need to be assigned a value simultaneously in order to make sense. For instance, hue cannot be given a value without giving also brightness and chromaticness values, so they describe a particular colour. Separable dimensions, on the contrary, are those that are independent from each other, such as hue and bitterness.

Collections of integral quality dimensions are grouped together into *domains*. A domain is a set of integral dimensions that are separable from any other quality

[1] All images in this section and the following subsection were taken from [12]

dimension. For example, *colour* is a domain comprised of the integral quality dimensions of hue, brightness and chromaticness; and *tone* is another domain that can be described with the integral quality dimensions of *pitch* and *loudness*.

A conceptual space is then defined as a collection of one or more domains. Both *colour* and *tone* are examples of conceptual spaces with only one domain, however, the juxtaposition of both of them can also be considered as a conceptual space *colour-tone* that realises all the attributes of colour and tone at the same time.

Moreover, Gärdenfors puts a strong emphasis in the geometrical structure of these conceptual spaces, which is explained in terms of several notions such as *betweeness* or *distance*. In this regard, elements or stimuli in a conceptual space are points or vectors representing each of their corresponding quality dimensions, which are then said to be *in between* other points or *at some distance of* other point. For example, the conceptual space of colour is represented as a *colour spindle*, where each point in the spindle represents a particular colour and where there is a clear geometrical structure (See Figure 3.1).

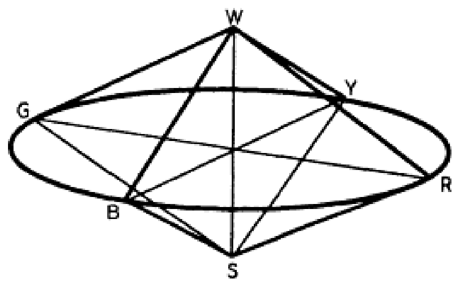


Figure 3.1: Colour spindle

This work will mainly focus on the distance approach to model the geometrical structure of conceptual spaces, which uses metric spaces as the primordial mathematical tool. A conceptual space will be a set of points representing the different stimuli together with a distance function (X, d) that encodes concept similarity and confers the aforementioned geometrical structure.

Two important conceptual spaces in this dissertation are the conceptual spaces of *colour* and *taste*. As pointed out earlier, *colour* is comprised of the quality dimensions hue, brightness and chromaticness. The first one, hue, is represented by means of the so-called colour circle (See Figure 3.2), which corresponds to the metric space of a circumference $(\mathcal{S}^1, d_\rho)$, with a distance function that measures proximity in polar terms. On the other hand, brightness and chromaticness can be described by an interval of the real line $([0, 1], d_e)$ and Euclidean distance representing the extent of lightness and intensity of the colour.

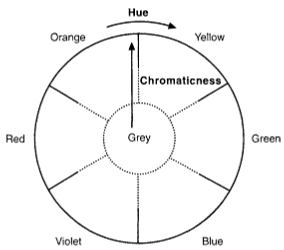


Figure 3.2: Colour circle

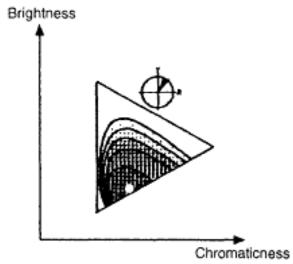


Figure 3.3: Triangle representation

However, there is some correlation between the dimensions of brightness and chromaticness, namely, the more intense the colour is (chromaticness), the less choices of brightness there are. This correlation is depicted by means of a triangular representation (See Figure 3.3). For that reason, when the three colour dimensions are integrated together, they give rise to the characteristic shape of the conceptual space of colour (See Figure 3.1). The distance function on this space is polar in the hue dimension and Euclidean in the remaining ones.

On the other hand, the conceptual space of *taste* is given by the four integral flavour dimensions: salt, sweet, bitter and sour. Each of these components is given by the interval $([0,1], d_e)$ of the real line, with Euclidean metric on them and they are combined into the higher conceptual space of taste $([0,1]^4, d_e)$, which is 4-dimensional hypercube with Euclidean metric (See Figure 3.4).

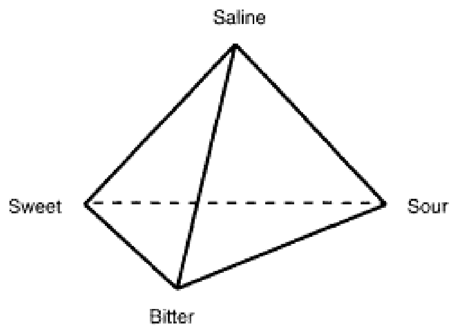


Figure 3.4: Taste hypercube

Concepts as cognitive entities are represented in this account according to the following criterion:

> CRITERION C A natural concept is represented as a set of regions in a number of domains together with an assignment of salience weights to the domains and information about how the regions in different domains are correlated. [12, p. 105]

As a result, concepts can be regarded as metric subspaces of a conceptual space comprised of one or more domains. The importance or salience of each of the domains can be expressed in terms of metrics. To see that, consider the conceptual space of taste (T, d_T) and colour (C, d_C) as described earlier. The conceptual space realising both domains at the same time is given by the product metric space $(C \times T, d_C \times d_T)$, where the product metric is giving equal importance to both taste and colour. However, it can be instead considered a weighted combination of the two metrics, where each of the weights adjust the specific salience of each domain

$$(C \times T, w_C d_C \times w_T d_T),$$

where

$$(w_C d_C \times w_T d_T)((c,t),(c',t')) = \max\{w_C d_C(c,c'), w_T d_T(t,t')\}.$$

Finally, a basic notion of correlation between domains will be subsequently explored in the span interpretation for conceptual spaces.

3.1.2.1 Prototype theory and the phenomenon of categorical perception

Gärdenfors also reviews *prototype theory* [12, Ch. 3] from the perspective of conceptual spaces and tries to tie it to the notion of metric spaces. In prototype theory, stimuli in a conceptual space are grouped together in certain categories representing concepts, as if describing a process of mental categorisation or conceptualisation. Some specific stimulus within a category are regarded as more representative than others and they are referred as *prototypes*. For instance, the conceptual space of *colour* can be categorised in seven major colours, *yellow*, *green*, *blue*, *violet*, *red*, *orange* and *grey*, according to Figure 3.2.

Gärdenfors uses *Voronoi tesselations* as a mathematical tool to link prototype theory with metric spaces. Given a metric space and a set of distinguished points, a Voronoi tesselation is a partition of that space in which each element is assigned to its closest (in terms of the metric) distinguished element. As a result, a Voronoi tesselation of a conceptual space based on a collection of representatives conceptualises the space in regions described by a representative and its most similar stimuli.

A process of mental categorisation occurs with the so-called *stop consonants* /*p*/,/*b*/,/*t*/,/*d*/. These consonants can be accurately described by means of two articulatory parameters that correspond to two integral dimensions: the *voiced-unvoiced* dimension and the *labial-dental-velar* dimension, where the latter controls the position of the tongue while making the sound. Those two dimensions can be described by the interval $([0,1], d_e)$ of the real line, which expresses the degree to which the dimension occurs, and they are combined into the conceptual space of *articulatory parameters* $([0,1]^2, d_e)$; a rectangle in the plane with Euclidean metric. A Voronoi tesselation of this space taking the stop consonants as prototypes yields the discretisation in Figure 3.5.

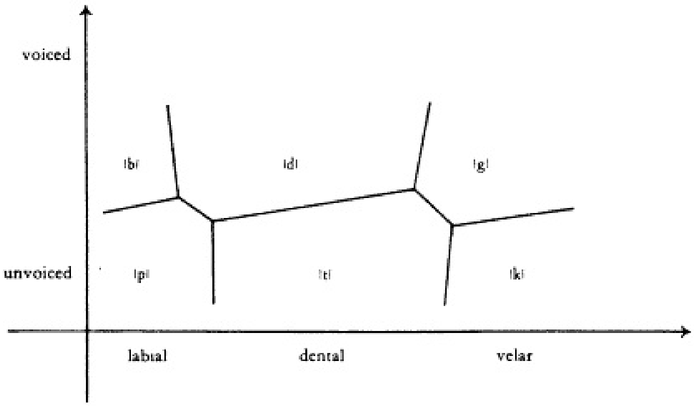


Figure 3.5: Stop consonants categorisation

Although the conceptual space of *articulatory parameters* is a comprised of infinitely many stimuli, it is perceived by the auditory system in a categorical way, i.e. there is a noticeable difference between two elements in different categories. Although these stimulus may be close in their conceptual space representations, their perceived dissimilarity is significant. This is known as the phenomenon of *categorical perception*. Specifically, in the example of the stop consonants, /p/ and /b/ could be very close in terms of their articulatory parameters, however, they would be perceived as different consonants.

3.1.3 Compositional distributional model of cognition

Compositional distributional models for natural language [10] provide a method for interpreting language semantics in a compositional manner, guided by the grammatical structure of the sentences. In this framework, a grammar category, such as pregroup grammars [14], is used to describe the syntactic structure of an expression; while a semantic category determines how meaning should be composed. The model then leverages a shared abstract structure, that of compact closedness, between the grammar category and the semantic category to mediate between the two of them; the structural morphisms mimicking the syntactic structure lead meaning composition in the semantic category.

The traditional distributional model of meaning for natural language uses vectors of co-ocurrence statistics extracted from a corpus to give semantic significance to words. Syntactic structure can be incorporated in this model by embedding it into the compositional distributional account with finite dimensional Hilbert spaces **FHilb** as the semantic category. Furthermore, Gärdenfors's model for cognition can be described using the semantic category of *convex algebras* and *convex relations* **ConvexRel** [7], which puts emphasis on the convex structure of natural properties. This dissertation will, however, analyse the suitability of spans of metric spaces, which are compact closed, as a semantic category.

As far as the grammar category is concerned, the choice in this thesis will be pregroups [14]. A pregroup is defined as follows:

Definition 15 (Pregroup)**.** *A* pregroup *is a tuple* $(P, \leq, \cdot, (-)^l, (-)^r)$ *where* $(P, \leq, \cdot)$ *is a partially ordered monoid and* $(-)^l, (-)^r : P \to P$ *are endofunctions onto* P*, known as the* left *and* right adjoints *respectively, that satisfy the following properties:* $p^l \cdot p \leq 1 \leq p \cdot p^l$ *and* $p \cdot p^r \leq 1 \leq p^r \cdot p$.

Each grammatical role in a sentence will be associated with an element in a *pregroup grammar*, which is a pregroup freely generated over a set of some basic types. In the simplest scenario, this pregroup grammar is generated by a *noun* type n and a *sentence* type s, i.e. the set $\mathcal{B} = \{n, s\}$, and all other types are composite types built upon these basic ones. For example, adjectives are assigned the type $n \cdot n^l$, while intransitive and transitive verbs are assigned the type $n^r \cdot s$ and $n^r \cdot s \cdot n^l$ respectively.

Pregroup reductions reveal grammatical interactions between the words in a sentence. For example, *John plays football*, which is represented by the composite type $n \cdot (n^r s n^l) \cdot n$, exhibits the following grammatical reduction into the sentence type:

$$n \cdot (n^r s n^l) \cdot n = (n n^r) \cdot s \cdot (n^l n) \leq 1 \cdot s \cdot (n^l n) \leq s \cdot 1 = s.$$

Similarly, the sentence *red apple*, typed as $(n n^l) \cdot n$, can be reduced to the noun type by means of the following reduction:

$$(n n^l) \cdot n = n \cdot (n^l n) \leq n \cdot 1 = n.$$

A pregroup can also be seen as a monoidal category; objects are elements of the pregroup, morphisms are type reductions and tensor is given by the monoidal product on objects. Moreover, it is a compact closed category with counits $\epsilon^l : p^l \otimes p \to 1 \equiv p^l \cdot p \leq 1$, $\epsilon^r : p \otimes p^r \to 1 \equiv p \cdot p^r \leq 1$ and units $\eta^l : 1 \to p \otimes p^l \equiv 1 \leq p \cdot p^l$,

$\eta^r : 1 \to p^r \otimes p \equiv 1 \leq p^r \cdot p$. Pregroup reductions are then interpreted equivalently in this category; the previous example would now be:

$$\begin{aligned} n \otimes (n^r \otimes s \otimes n^l) \otimes n &\to (n \otimes n^r) \otimes s \otimes (n^l \otimes n) \\ &\to 1 \otimes s \otimes (n^l \otimes n) \\ &\to s \otimes 1 \to s, \end{aligned}$$

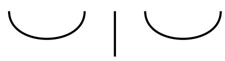
which corresponds to the following final morphism

$$\epsilon_n^r \otimes \mathrm{id}_s \otimes \epsilon_n^l : n \otimes (n^r \otimes s \otimes n^l) \otimes n \to s.$$

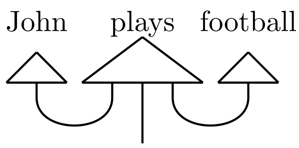
These morphisms encoding the grammatical structure are used to guide composition of meaning in the semantic category. A monoidal functor F : **Grammar** $\longrightarrow$ **Semantic** from the grammar category to the semantic category is defined to bridge both aspects. The resulting morphism in the semantic category is finally applied to the semantic input in order to obtain the composite meaning. In the previous example, the morphism

$$F(\epsilon_n^r \otimes \mathrm{id}_s \otimes \epsilon_n^l) : F(n) \otimes (F(n)^r \otimes F(s) \otimes F(n)^l) \otimes F(n) \to F(s)$$

would give a *meaning map* to interpret the meaning of the sentence. Graphically (See Section 2.1.2, [9]), this morphism is depicted as



As far as the semantic input is concerned, words are mapped into states in the semantic category, which are then fed into the semantic map. These states can be regarded as preparation of data that is then transformed by the meaning map encoding the grammatical structure. The resulting state is the semantic output. Graphically,



3.2 The path interpretation for spans of metric spaces

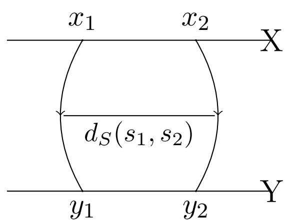
The path interpretation from Chapter 2 can be enhanced in order to take into account metrics. Both the starting and ending locations, as well as the path space, can be refined with an additional metric structure. A span of metric spaces is now given by a diagram

$$\begin{tikzcd} & (S, d_S) \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & \\ (X, d_X) & & (Y, d_Y) \end{tikzcd}$$

in **HMet** (**QMet**,**Met**), where s_X and s_Y are weak contractions. It can be interpreted as a collection of paths together with their ending and starting points, in which there is an additional requirement, given by the weak contractions assumption, on them:

$$d_X(s_X(s_1), s_X(s_2)) \leq d_S(s_1, s_2)$$
$$d_Y(s_Y(s_1), s_Y(s_2)) \leq d_S(s_1, s_2)$$

This conditions signify that the distance between two paths must take into account the distance between their extreme points, in the sense that *two paths are at least as far apart as their initial and ending points are.*



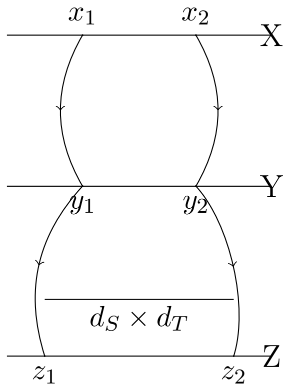
Span composition is again path composition. Given two collections of paths $(X, d_X) \xleftarrow{s_X} (S, d_S) \xrightarrow{s_Y} (Y, d_Y)$ $(Y, d_Y) \xleftarrow{s_Y} (T, d_T) \xrightarrow{s_Z} (Z, d_Z)$, composition joins them in the middle and preserves extreme points

$$(X, d_X) \xleftarrow{s_X \circ \pi_S} (S \times_Y T, d_S \times d_T) \xrightarrow{s_Z \circ \pi_T} (Z, d_Z).$$

The metric structure in both X and Z is preserved, however, there is a new metric on the resulting paths; the distance between two composite paths is the maximum distance between the original paths

$$d_S \times d_T\ ((s_1, t_1), (s_2, t_2)) = \max\{d_S(s_1, s_2), d_T(t_1, t_2)\}.$$

In other words, *two composite paths are as far apart as their components jointly are.*



It goes beyond the aim of this dissertation to discuss further the path interpretation. Instead, it is more relevant an interpretation for conceptual spaces, which will be presented in the next section.

3.3 Interpretation for conceptual spaces

3.3.1 A model for hyperconceptual spaces

A span in **HMet** (**QMet**,**Met**) for conceptual spaces can be interpreted as describing a hyperconcept - hypoconcept relation, i.e. a relation between a general conceptual space and a more specific one. The head of the span is the super-conceptual space, whereas the domain and codomain are two different sub-conceptual spaces. Each concept in (S, d_S) is associated with its respective sub-concepts in two different domains (X, d_X) and (Y, d_Y).

$$\begin{tikzcd}
 & (S, d_S) \arrow[dl, "s_X"'] \arrow[dr, "s_Y"] & \\
(X, d_X) & & (Y, d_Y)
\end{tikzcd}$$

For instance, consider the conceptual space of *taste* $\mathbf{T} = (T, d_T) = ([0,1]^4, d_e)$ and the corresponding projections into the sub-conceptual spaces *bitterness* $\mathbf{B} = ([0,1], d_e)$ and *sweetness* $\mathbf{S} = ([0,1], d_e)$. Each taste concept, described by the four flavour dimensions $t = (t_{salt}, t_{sweet}, t_{bitter}, t_{sour})$, is projected into its subconcepts t_{bitter} and t_{sweet} by means of weak contractions.

$$\begin{tikzcd}
 & (T, d_T) \arrow[dl, "s_B"'] \arrow[dr, "s_S"] & \\
\mathbf{B} = ([0,1], d_e) & & \mathbf{S} = ([0,1], d_e)
\end{tikzcd}$$

More generally, a span symbolises a hyperconcept-hypoconcept relation of (S, d_S) with regards to the product space $(X \times Y, d_X \times d_Y)$. Due to the universal mapping property of the categorical product, the following diagram commutes

$$\begin{tikzcd}
 & (S, d_S) \arrow[dl, "s_X"'] \arrow[d, "\langle s_X, s_Y \rangle"] \arrow[dr, "s_Y"] & \\
 & (X \times Y, d_X \times d_Y) \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
(X, d_X) & & (Y, d_Y)
\end{tikzcd}$$

and the span can be seen as a projection of super-concepts in (S, d_S) into sub-concepts in $(X \times Y, d_X \times d_Y)$

$$\begin{tikzcd}
 & (S, d_S) \arrow[d, "\langle s_X, s_Y \rangle"] & \\
 & (X \times Y, d_X \times d_Y) \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
(X, d_X) & & (Y, d_Y)
\end{tikzcd}.$$

It will be useful, however, to treat the two legs of the span independently for different schemes of composition.

On the other hand, recall that both legs of the span are weak contractions that impose the conditions

$$\begin{aligned}
d_X(s_X(s_1), s_X(s_2)) &\leq d_S(s_1, s_2) \\
d_Y(s_Y(s_1), s_Y(s_2)) &\leq d_S(s_1, s_2)
\end{aligned}$$

on the conceptual spaces. This constraint can be read as: *if two concepts are close, then their respective sub-concepts are also close.* At the same time, it can also be read as *if two sub-concepts are far apart, then their respective super-concepts must be at least that far apart.*

Consider the more abstract conceptual space of *dogs* $\mathbf{Dog} = (D, d_D)$ with some metric d_D on it. Both *skin colour* and *height* are sub-conceptual spaces that also describe any dog $\mathbf{SkinColour} \longleftarrow \mathbf{Dog} \longrightarrow \mathbf{Height}$. If two dogs are similar then they will be similar both in height and skin colour. Also, if two dogs are dissimilar in height or skin colour to some extent, they will be at least dissimilar to that extent.

Moreover, weak contractions are hinting a loss of information from the hyperconcept to the hypoconcepts. The distance between two points in the head of the span is bigger because there is more refined information to compare. This loss of information can be represented by means of *partial vectors*, where some information of the hyperconcept is known, the hypoconcepts, while possibly some other is not.

Definition 16 (Partial vector)**.** *A* partial vector *is a representation of a concept $s \in S$ in a span $(X, d_X) \xleftarrow{s_X} (S, d_S) \xrightarrow{s_Y} (Y, d_Y)$ that emphasises the idea that $s_X(s)$, $s_Y(s)$ are sub-concepts describing the super-concept to some extent:*

$$s = (s_X(s), \cdots, s_Y(s))$$

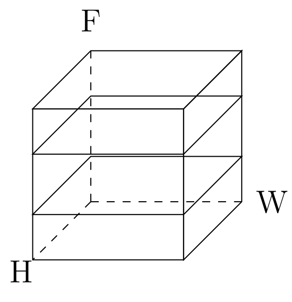
Notice that the weak contractions constraints follow naturally from this interpretation: *the distance between two partial vectors must be bigger than the distance of their components*:

$$d_S[(s_X(s_1), \cdots, s_Y(s_1)), (s_X(s_2), \cdots, s_Y(s_2))] \geq d_X(s_X(s_1), s_X(s_2))$$
$$d_S[(s_X(s_1), \cdots, s_Y(s_1)), (s_X(s_2), \cdots, s_Y(s_2))] \geq d_Y(s_Y(s_1), s_Y(s_2)).$$

Remark. *The idea of loss of information will be central to the intuitions developed in this and the following chapter. Essentially, it captures the concept that both domain and codomain objects are parts of a greater whole, the apex of the span.*

Additionally, each 1-Lipschitz map is partitioning the upper space into regions. Each sub-concept $x \in X$ (resp. $y \in$Y) can be associated with its inverse image $\pi_X^{-1}(x)$ (resp. $\pi_Y^{-1}(y)$), which is a metric subspace of the hyperconceptual space. Recall that, by criterion C, regions in a conceptual space are concepts of this conceptual space and, thus, each element in the domain and codomain is assigned to a concept in (S, d_S).

Take as example a conceptual space of dogs $\mathbf{D}$ described by sub-concepts *ferociousness*, *height* and *weight*. This conceptual space is determined by a three-dimensional cube with Euclidean metric $\mathbf{D} = ([0, 1]^3, d_e)$ where each of the components represents a sub-conceptual dimension. The projection into the sub-concept *ferociousness* $\mathbf{D} \xrightarrow{\pi_F} ([0, 1], d_e)$ creates a partition of this three-dimensional space in planes corresponding to different concepts describing different grades of *ferociousness*, such as *ferocious-0.3* or *ferocious-0.7*.



3.3.2 Concept extension with correlation

Span composition in this setting is *concept extension with correlation.* Given two hyperconceptual spaces with their respective hypo-concepts $(X, d_X) \xleftarrow{s_X} (S, d_S) \xrightarrow{s_Y} (Y, d_Y)$,$(Y, d_Y) \xleftarrow{s_Y} (T, d_T) \xrightarrow{s_Z} (Z, d_Z)$, composition joins two concepts if they agree on their common hypoconceptual space:

$$\begin{tikzcd}[column sep=small] & (S \times_Y T = \{(s,t) \in S \times T : s_Y(s) = s_Y(t)\}, d_S \times d_T) \arrow[dl, "s_X \circ \pi_S"'] \arrow[dr, "s_Z \circ \pi_T"] & \\ (X, d_X) & & (Z, d_Z) \end{tikzcd}$$

The composition creates a bigger conceptual space $(S \times_Y T, d_S \times d_T)$ composed of separable domains S, T, where there exists correlation through the shared hypo-conceptual space (Y, d_Y). Notice that the taxicab distance $d_S \times d_T$ is the best choice to join separable domains according to Gärdenfors's ideas [12, Sec. 1.8]. Moreover, the product metric guarantees that *two concepts are close if the compound concepts are also close in their original conceptual spaces.*

As an example, take into account the positive correlation between *sweetness* $\mathbf{S} = ([0,1], d_e)$ in the taste domain (T, d_T) and *sugar level* $\mathbf{L} = ([0,1], d_e)$ in the *nutrition* domain (N, d_N) [12, Sec. 4.2.1]. Span composition of $(T, d_T) \xleftarrow{\text{id}} (T, d_T) \xrightarrow{s_S} \mathbf{S} = ([0,1], d_e)$ and $\mathbf{L} = ([0,1], d_e) \xleftarrow{s_L} (N, d_N) \xrightarrow{\text{id}} (N, d_N)$ will create a higher conceptual space $(T, d_T) \xleftarrow{\pi_T} (T \times_S N, d_T \times d_N) \xrightarrow{\pi_N} (N, d_N)$ integrating both *taste* and *nutrition* where high levels of *sweet* correspond to high levels of *sugar content.* Moreover, negative correlations can also be modelled; taking $1 - s_S$ in the taste span instead of s_S, which is still a weak contraction, would link high values of sweetness with low values of sugar content.

As another example, take the conceptual spaces of **Dog** and **Cat** and their respective sub-concepts *skin colour* and *height.* Composing $\mathbf{SkinColour} \xleftarrow{s_S} \mathbf{Dog} \xrightarrow{s_H} \mathbf{Height}$ and $\mathbf{Height} \xleftarrow{s_H} \mathbf{Cat} \xrightarrow{s_C} \mathbf{SkinColour}$ produces a higher conceptual space $\mathbf{Dog} \times_{\mathbf{Height}} \mathbf{Cat}$ composed of exemplars of dogs and cats that agree on their height.

In partial vector notation, every vector of type $s = (x, \cdots, y)$ in (S, d_S) is joined with every vector of type $t = (y, \cdots, z)$ in (T, d_T) to obtain a vector of the form $(s,t) = (x, \cdots, z)$. Notice that the composite object implicitly records information about the sub-concept in (Y, d_Y), since the projection $s_Y \circ \pi_S = s_Y \circ \pi_T : (S \times_Y T) \to (Y, d_Y)$ holds, and the concept could be regarded as $(s,t) = (x, \cdots, y, \cdots, z)$; however, the composite span disregards that projection and the information is lost.

Finally, *concept extension with correlation* can be used to model the phenomena of categorical perception, which will be explored later on.

3.3.3 Concept extension without correlation

Span tensoring in this scenario is *concept extension without correlation.* Given two spans $(X, d_X) \xleftarrow{s_X} (S, d_S) \xrightarrow{s_Y} (Y, d_Y)$,$(X', d_{X'}) \xleftarrow{s_{X'}} (S', d_{S'}) \xrightarrow{s_{Y'}} (Y', d_{Y'})$ carrying two hyper-conceptual spaces with corresponding hypo-conceptual spaces, their composite

$$\begin{tikzcd}
& (S \times S', d_S \times d_{S'}) \arrow[dl, "s_X \times s_{X'}"'] \arrow[dr, "s_Y \times s_{Y'}"] & \\
(X \times X', d_X \times d_{X'}) & & (Y \times Y', d_Y \times d_{Y'})
\end{tikzcd}$$

is a higher conceptual space $S \times S'$ that consists of two separable domains S,S' interrelated by means of the L1-norm, or taxi-cab distance, $d_S \times d_{S'}$, which preserves the original relations. This product metric again guarantees that *two composite concepts are close to each other if their sub-concepts are close in their original conceptual spaces.*

The conceptual spaces of *colour* (C, d_C) and *taste* (T, d_T) together with respective subconcepts *hue*, *brightness* $\mathbf{H} \xleftarrow{s_H} (C, d_C) \xrightarrow{s_{Br}} \mathbf{Br}$ and *sweetness*, *bitterness* $\mathbf{S} \xleftarrow{s_S} (T, d_T) \xrightarrow{s_{Bi}} \mathbf{Bi}$ can be extended to a higher conceptual space of *food experience* while preserving the original relations $\mathbf{H} \times \mathbf{S} \xleftarrow{s_H \times s_S} (C \times T, d_C \times d_T) \xrightarrow{s_{Br} \times s_{Bi}} \mathbf{Br} \times \mathbf{Bi}$.

Lastly, using partial vector notation, every concept of the form $s = (x, \cdots, y)$ is joined with every concept of the form $s' = (x', \cdots, y')$ to give rise a concept $(s, s') = ((x, x'), \cdots, (y, y'))$.

3.3.4 Trivial relations

The distinguished morphisms from the compact closed structure and Frobenius algebras

$$\begin{array}{rr}
\textit{identity} & (X, d_X) \xleftarrow{\mathrm{id}_X} (X, d_X) \xrightarrow{\mathrm{id}_X} (X, d_X) \\
\textit{unit} & (\{*\}, 0) \longleftarrow (X, d_X) \xrightarrow{\Delta_X} (X \times X, d_X \times d_X) \\
\textit{counit} & (X \times X, d_X \times d_X) \xleftarrow{\Delta_X} (X, d_X) \longrightarrow (\{*\}, 0) \\
\textit{copy} & (X, d_X) \xleftarrow{\mathrm{id}_X} (X, d_X) \xrightarrow{\Delta_X} (X \times X, d_X \times d_X) \\
\textit{uncopy} & (X \times X, d_X \times d_X) \xleftarrow{\Delta_X} (X, d_X) \xrightarrow{\mathrm{id}_X} (X, d_X)
\end{array}$$

represent trivial hyper-concept - hypo-concept relations, where each concept in the super-conceptual space is paired with zero, one or two copies of the same concept in the domain and codomain. Notice the trivial sub-concept $*$ in $(\{*\}, 0)$, which is sub-concept of every concept.

In partial vector notation the relations are expressed as $x = (x, \cdots, x)$, $x = (*, \cdots, (x, x))$, $x = ((x, x), \cdots, *)$, $x = (x, \cdots, (x, x))$, $x = ((x, x), \cdots, x)$ respectively. Observe how each concept is trivially described by copies of the same concept, which proves useful for different correlation schemes.

3.4 Applications

3.4.1 The phenomenon of categorical perception

In this section, a connection between the phenomenon of categorical perception (See Section 3.1.2.1) and the interpretation herein developed will be explored. Recall that in this phenomenon, stimuli are perceived in a categorical way and, as a result, distances between stimuli in different categories may be regarded as larger than they are.

Stop consonants constitute a valuable instance of the latter phenomenon. They are precisely described by the conceptual space of *articulatory parameters*, however, they are perceived in a categorical way and, for that reason, this conceptual space lacks the necessary information to explain the larger distances between different categories. As a result, a stop consonant belongs to some higher conceptual space of *articulated consonants* $\mathbf{A} = (A, d_A)$ where both *articulatory parameters* $\mathbf{P} = (P, d_P)$ and *perceived consonants* $\mathbf{C} = (C, d_C)$ are sub-concepts.

$$\begin{tikzcd} & (A, d_A) \arrow[dl, "s_P"'] \arrow[dr, "s_C"] & \\ (P, d_P) & & (C, d_C) \end{tikzcd}$$

The conceptual space of *perceived consonants* is given by the set of phonemes together with a distance function characterising their perceived dissimilarity $\mathbf{C} = (\{/p/, /b/, /t/, /d/\}, d_C)$.

In partial vector notation, an articulated consonant in this higher conceptual space is given by $a = (p, \cdots, c)$ and now the metric takes into account both the articulatory parameters and the perceived distance:

$$d_A(a = (p, \cdots, c), a' = (p', \cdots, c') \geq d_P(p, p'), d_C(c, c'))$$

Consequently, even if two different stop consonants are articulated in a similar way, $d_P(p, p') \approx 0$, they will be easily distinguished due to categorical perception $d_C(c, c')$.

The richer conceptual space of *articulated consonants* can be built upon the spaces of *articulatory parameters* and *perceived consonants* bridging them by means of concept extension with correlation. These two spaces are correlated through the common sub-subconcept that is given by the consonant they represent. Thereby, letting $\mathbf{S} = (\{p, b, t, d\}, 0)$ be the conceptual space of the different categories with no metric interpretation on them (metric 0), $(P, d_P) \xleftarrow{\mathrm{id}_P} (P, d_P) \xrightarrow{s_C} (\{p, b, t, d\}, 0)$ an articulatory parameters span with s_C the 1-Lipschitz map taking every articulatory parameter in (P, d_P) to its corresponding consonant ; and $(\{p, b, t, d\}, 0) \xleftarrow{s_C} (C, d_C) \xrightarrow{\mathrm{id}_C} (C, d_C)$ a perceived consonant span where s_C takes each phoneme into its category, its composite creates a higher conceptual space embedding the two spaces and taking into account the existing correlation.

$$\begin{tikzcd}
 & (P \times_S C, d_P \times d_C) \arrow[dl, "\pi_P"'] \arrow[dr, "\pi_C"] & \\
(P, d_P) & & (C, d_C)
\end{tikzcd}$$

Notice that metric 0 in $\mathbf{S}$ is necessary in order to obtain a weak contraction. Since stimuli in different categories in (P, d_P) can be arbitrarily close to each other, the sub-concept must also be that close and, therefore, they result in distance 0. In general, metric 0 imposes no interpretation in a conceptual space and it is useful to correlate different conceptual spaces where the underlying sub-conceptual structure is equivalent but the metrics are not compatible. Notice also that, if the conceptual space has more than two points, metric 0 can only be used in hemi-metric spaces; otherwise the T_0 condition would collapse them into a single point.

3.4.2 Compositional distributional model of cognition

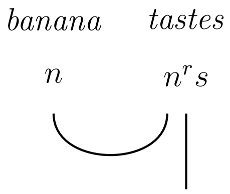
Many of the notions introduced by Gärdenfors implicitly rely on the mathematical tool of metric spaces. This section will try to connect them with the compositional distributional model of cognition, in order to guide concept composition through the grammatical structure of the sentence. As categories of metric spaces are not compact closed categories, spans of metric spaces, which embeds metric spaces, will be taken as the semantic category instead.

This section will primarily focus on intransitive sentences; sentences composed by a subject and an intransitive verb that does not accept a direct object. Nouns will be described as concepts in the conceptual space of *food experience*, with *colour* $\mathbf{C} = (C, d_C)$ and *taste* $\mathbf{T} = (T, d_T)$ as the domains composing it: $\mathbf{N} = (N, d_N) =$

$\mathbf{C} \times \mathbf{T} = (C \times T, d_C \times d_T)$. An *apple* is some region $N_{apple} \subseteq C \times T$ comprising all possible combinations of colour and taste stimuli that an apple can be. Also, a *banana* is some other region $N_{banana} \subseteq C \times T$ possibly sharing stimuli with the concept of apple, as in a yellow sweet apple and a yellow banana.

Verbs, on the other hand, essentially encode actions and Gärdenfors suggests the use of differential equations on some conceptual space of shapes to describe movement dynamics [12, Sec. 3.10.3]. As there is not much research yet on this area, verbs will be concepts in some abstract conceptual space of actions $\mathbf{A} = (A, d_A)$. For instance, both *to taste* and *to shine* will be regions in this conceptual space of actions $A_{taste} \subseteq A$, $A_{shine} \subseteq A$. Moreover, verbs contextualise nouns, in the sense that they highlight the importance of some integral domains over some others. For example, in *banana tastes*, the domain of *taste* is given more importance; while in *apple shines*, it is the *colour* dimension the one that is given more attention.

The last two sentences are instances of intransitive sentences. Their grammatical structure can be modelled in the pregroup grammar **Preg** generated by $\{n, s\}$ as $n \cdot (n^r s)$, which is reduced into the sentence type by means of the morphism $\epsilon_n \otimes \mathrm{id}_s : n \otimes n^r \otimes s \to s$. Graphically,



This constitutes the meaning map that will guide meaning composition.

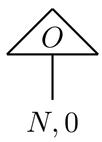
Spans of hemi-metric spaces is the choice for the semantic category in this scenario. The monoidal functor $F : \mathbf{Preg} \to \mathsf{Span}(\mathbf{Met})$ interrelating the grammar category with the semantic category will act on objects as

$$F(n) = F(n^r) = F(n^l) = (N, 0)$$
$$F(s) = F(s^r) = F(s^l) = (N \times A, 0) = (S, 0)$$
$$F(x \otimes y) = F(x) \otimes F(y).$$

Notice the choice of no metric interpretation for the target conceptual spaces. This will prove useful to correlate through the full conceptual structure of the hyper-conceptual spaces, regardless of their metric interpretation. Notice, also, the choice of $(N \times A, 0)$ for the sentence space $(S, 0)$. This is because the resulting sentence concept will summarise all the visual, taste and action dynamics stimuli taking place in the sentence, after processing implicit interactions. On the other hand, nouns and their adjoints are mapped to nouns as space of stimuli. As a word of caution, recall that metric 0 on non-singleton sets can only be used in the setting of hemi-metric spaces.

Since **Preg** is the pregroup freely generated over the set $\{n, s\}$, the only morphisms are those built out of identities and structural morphisms (units and counits). These are necessarily mapped to identities and structural morphisms in $\mathsf{Span}(\mathbf{HMet})$. Functoriality and monoidality follow from construction of the functor.

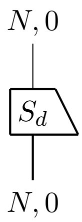
The main idea underlying the following use of spans as a semantic category will be to describe concepts as hyperconceptual spaces with relevant noun or action sub-conceptual structures and use specific morphisms to manipulate the salience or metric of each of the concepts. In order to apply the semantic map, the concepts represented by the different words are prepared into states of $\mathsf{Span}(\mathbf{HMet})$. A noun $O \subseteq C \times T$ will be regarded as a hyper-conceptual space



$$\begin{tikzcd} & (O,0) \arrow[dl, "!"'] \arrow[dr, hook, "i"] & \\ (\{*\},0) & & (N,0) \end{tikzcd},$$

with an underlying sub-conceptual structure of noun $(N, 0)$, where i is the inclusion map. Note that no metric was given to the concept, $(O, 0)$, as it will be later contextualised by the verb in the sentence. In the previous example, *apple* corresponds to the state $(\{*\}, 0) \xleftarrow{!} (N_{apple}, 0) \xhookrightarrow{i} (N, 0)$; while *banana* corresponds to the state $(\{*\}, 0) \xleftarrow{!} (N_{banana}, 0) \xhookrightarrow{i} (N, 0)$.

The metric interpretation on the hyperconceptual space embodied by the noun will be adjusted by the corresponding verb in the sentence and it will be done by means of a *salience morphism*:



$$\begin{tikzcd} & (N,d) \arrow[dl, hook', "i"'] \arrow[dr, hook, "i"] & \\ (N,0) & & (N,0) \end{tikzcd},$$

where d is any particular metric for the conceptual space of nouns N and i is the identity on stimuli. This morphism correlates through the noun sub-conceptual structure and effectively changes the metric when composed to any noun state. To see that, notice that composing any concept state $(\{*\}, 0) \xleftarrow{!} (O, 0) \xhookrightarrow{i} (N, 0)$ with a salience morphism $(N,0) \xhookleftarrow{i} (N,d) \xhookrightarrow{i} (N,0)$ results in $(\{*\},0) \xleftarrow{!} (O \times_N N, 0 \times d) \xhookrightarrow{i\circ\pi_N} (N,0)$, where

$$O \times_N N = \{(o,n) \in O \times N : i(o) = o = n = i(n)\}.$$

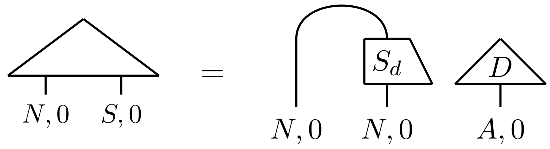
This composite span is equivalent to the concept state with the desired metric, since there exists a 2-isomorphism $(O,d) \xrightarrow{\cong} (O \times_N N, 0 \times d) : o \mapsto (o,o)$ that makes the following diagram commute

$$\begin{tikzcd}
 & (O,d) \arrow[ddl, "!"'] \arrow[d, "\cong"] \arrow[ddr, "i"] & \\
 & (O \times_N N, 0 \times d) \arrow[dl, "!"] \arrow[dr, "i\circ\pi_N"'] & \\
(\{*\},0) & & (N,0)
\end{tikzcd}$$

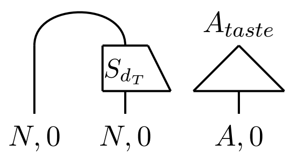
In partial vector notation, every concept $o = (*, \cdots, o)$ in the first span is extended with the unique concept of the form $o = (o, \cdots, o)$ in the second span to give rise to the concept $(o,o) = (*, \cdots, o)$ where distances are measured with the hemi-metric $0 \times d$. It is easy to see that that space of concepts is isomorphic to the space containing $o = (*, \cdots, o)$ as concepts with hemi-metric d.

For instance, the salience of the *apple* state $(\{*\},0) \xleftarrow{!} (N_{apple},0) \xhookrightarrow{i} (N,0)$ can be set so that a linear combination of the *colour* and *taste* dimensions is the relevant metric by composing it with the salience morphism $(N,0) \xhookleftarrow{i} (N, w_C d_C \times w_T d_T) \xhookrightarrow{i} (N,0)$.

On the other hand, a state encapsulating a verb concept will be given by the morphism:



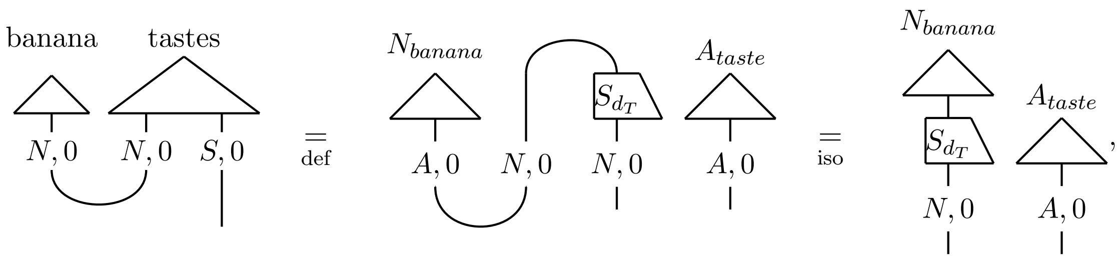
Essentially, S_d is a salience morphism that will adjust the salience of the attached noun, providing a contextualisation; and D, additionally, is a state preparing stimuli pertaining the dynamics of the verb with its corresponding metric interpretation $(\{*\},0) \xleftarrow{!} (D, d_D) \xhookrightarrow{i} (A,0)$. For example, the state associated with the verb *taste* will be given by



,

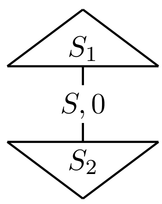
where the salience morphism S_{d_T} sets the metric of the attached noun to the metric $0 \times d_T$, giving only importance to the taste domain, and A_{taste} is providing information about the dynamics of *to taste*.

Having defined how words map into states in the semantic category, the meaning map can be applied to obtain the final concept. Applying it to the sentence *banana tastes*, and using string diagrams calculations, the following results:



which corresponds to the span $(\{*\}, 0) \xleftarrow{!} (N_{banana} \times A_{taste}, (0 \times d_T) \times d_A) \overset{i \times i}{\hookrightarrow} (N \times A, 0)$. Notice that the resulting span summarises all the stimuli embodied by *banana tastes*, i.e. visual, taste and action stimuli, and gives the right salience to each of the domains. Similarly, the resulting span for *apple shines* is $(\{*\}, 0) \xleftarrow{!} (N_{apple} \times A_{shine}, (d_C \times 0) \times d_A) \overset{i \times i}{\hookrightarrow} (N \times A, 0)$, where a change of salience morphism into the metric $d_C \times 0$, giving only importance to the colour domain, would be used instead.

Finally, the inner product provides a way to compare how similar the two sentences are. Let $(\{*\}, 0) \xleftarrow{!} (S_1, d_1) \overset{i}{\hookrightarrow} (S, 0)$ be the span corresponding to *banana tastes* and $(\{*\}, 0) \xleftarrow{!} (S_2, d_2) \overset{i}{\hookrightarrow} (S, 0)$ the span corresponding to *apple shines*, then their inner product



is given by composing $(\{*\}, 0) \xleftarrow{!} (S_1, d_1) \overset{i}{\hookrightarrow} (S, 0)$ with $(S, 0) \overset{i}{\hookleftarrow} (S_2, d_2) \xrightarrow{!} (\{*\}, 0)$, which results in $(\{*\}, 0) \xleftarrow{!} (S_1 \times_S S_2, d_1 \times d_2) \xrightarrow{!} (\{*\}, 0)$, with

$$S_1 \times_S S_2 = \{(s_1, s_2) \in S_1 \times S_2 : i(s_1) = s_1 = s_2 = i(s_2)\}.$$

This resulting span will turn out to be equivalent to a span containing shared stimuli. As in the case of the salience morphism composition, there exists a 2-isomorphism

$(S_1 \times_S S_2, d_1 \times d_2) \xrightarrow{\cong} (S_1 \cap S_2, \max\{d_1, d_2\}) : (s, s) \mapsto s$ that makes the following diagram commute

$$\begin{tikzcd}
 & (S_1 \cap S_2, \max\{d_1, d_2\}) \arrow[d, "\cong"] \arrow[ddl, "!"'] \arrow[ddr, "!"] & \\
 & (S_1 \times_S S_2, d_1 \times d_2) \arrow[dl, "!"] \arrow[dr, "!"'] & \\
(\{*\}, 0) & & (\{*\}, 0)
\end{tikzcd}.$$

This equivalent span $(\{*\}, 0) \xleftarrow{!} (S_1 \cap S_2, \max\{d_1, d_2\}) \xrightarrow{!} (\{*\}, 0)$ is the conceptual space of shared stimuli, with a product metric that takes into account the two original interpretations. In the example of this section, the inner product of *banana tastes* and *apple shines* $(S_1 \cap S_2, \max\{d_1, d_2\})$ is given by

$$((N_{banana} \times A_{tastes}) \cap (N_{apple} \times A_{shines}), \max\{(0 \times d_T) \times d_A, (d_C \times 0) \times d_A)\}).$$

Using properties of the cartesian product and the definition of product metric, it is easy to show that the previous conceptual space is equivalent to

$$((N_{banana} \cap N_{apple}) \times (A_{taste} \cap A_{shine}), (d_C \times d_T) \times d_A),$$

which corresponds to the space of shared stimuli between *banana tastes* and *apple shines* and where the salience of the domains has been adjusted to take into account both the colour and taste dimensions with equal importance.

Chapter 4

A model for Structured Query Language

This chapter will explore a completely different intuition for spans of sets and metric spaces that connects them with *algebra of bags*, the underpinning of Structured Query Language.

4.1 Preliminaries

4.1.1 Locally distributive categories

Some of the results in this chapter will take place in the setting of locally distributive categories, whose definition is given in terms of *slice categories* [4].

Definition 17 (Slice category). *The slice category* $\mathbf{C}/C$ *of a category* $\mathbf{C}$ *over an object* $C \in \mathbf{Ob}(\mathbf{C})$ *is the category that has arrows* $X \xrightarrow{x} C \in \mathbf{Arr}(\mathbf{C})$ *with codomain* C *as objects and arrows* $f : X \to X' \in \mathbf{Arr}(\mathbf{C})$ *such that*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & X' \arrow[dl, "x'"] \\ & C & \end{tikzcd}$$

as morphisms from $X \xrightarrow{x} C$ *to* $X' \xrightarrow{x'} C$.

Slice categories are interesting because products in them correspond to pullbacks in the base category and coproducts are preserved. The following are well-known results in the literature on slice categories:

Lemma 5 (Product in slice categories). *Given a category* $\mathbf{C}$ *with pullbacks, in the slice category* $\mathbf{C}/C$ *the product of* $X \xrightarrow{x} C$ *with* $Y \xrightarrow{y} C$ *is given by* $X \times_C Y \xrightarrow{p} C$*,* *where $X \times_C Y$ is the pullback along the pair x, y and p is such that*

$$\begin{tikzcd} X \arrow[dr, "x"'] & X \times_C Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "p"] & Y \arrow[dl, "y"] \\ & C & \end{tikzcd} ,$$

with projections given by the pullback projections $\pi_X : X \times_C Y \to X$, $\pi_Y : X \times_C Y \to Y$. In addition, given arrows $f : (Z \xrightarrow{z} C) \to (X \xrightarrow{x} C)$ and $g : (Z \xrightarrow{z} C) \to (Y \xrightarrow{y} C)$ in $\mathbf{C}/C$ (where $x \circ f = z$ and $y \circ f = z$), the unique arrow for the UMP is given by the UMP of pullbacks in $\mathbf{C}$ $\langle f, g\rangle : Z \to X \times_C Y$:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ X \arrow[dr, "x"'] & X \times_C Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \arrow[dl, "y"] \\ & C & \end{tikzcd} .$$

This is the unique arrow that satisfies $\pi_X \circ \langle f, g\rangle = f$ and $\pi_Y \circ \langle f, g\rangle = g$, where π_X and π_Y are the pullback projections previously defined.

Lemma 6 (Coproduct in slice categories)**.** *Given a category $\mathbf{C}$ with coproducts, in the slice category $\mathbf{C}/C$ the coproduct of $X \xrightarrow{x} C$ with $Y \xrightarrow{y} C$ is given by $X + Y \xrightarrow{[x,y]} C$, with injections given by the coproduct injections $i_1 : X \to X + Y$ and $i_2 : Y \to X + Y$:*

$$\begin{tikzcd} X \arrow[r, "i_1"'] \arrow[dr, "x"'] & X + Y \arrow[d, "{[x,y]}"] & Y \arrow[l, "i_2"] \arrow[dl, "y"] \\ & C & \end{tikzcd} .$$

Given arrows $f : (X \xrightarrow{x} C) \to (Z \xrightarrow{z} C)$ and $g : (Y \xrightarrow{y} C) \to (Z \xrightarrow{z} C)$ in $\mathbf{C}/C$ (where $z \circ f = x$ and $z \circ f = y$), the unique arrow for the UMP is given by the UMP of coproducts in $\mathbf{C}$ $[f, g] : X + Y \to Z$,

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "i_1"'] & X + Y \arrow[u, "{[f,g]}"] & Y \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd} .$$

This is the unique arrow that satisfies $[f, g] \circ i_1 = f$ and $[f, g] \circ i_2 = g$, where i_1 and i_2 are the coproduct injections previously defined.

Finally, locally distributive categories are just categories that exhibit distributivity in a *local* way, which is given in terms of slice categories. Refer to [8] for further information on distributive categories.

Definition 18 (Distributive category)**.** *A category* $\mathbf{C}$ *with products and coproducts is said to be distributive if for all objects* $X, Y, Z \in \mathbf{Ob}(\mathbf{C})$*, the canonical distributive morphism*

$$[i_1 \times id_Z, i_2 \times id_Z] : X \times Z + Y \times Z \longrightarrow (X + Y) \times Z$$

is an isomorphism.

Definition 19 (Locally distributive category)**.** *A locally distributive category* $\mathbf{C}$ *is a category whose slice categories* $\mathbf{C}/C$ *are distributive.*

4.1.2 Tables and operations

For the unfamiliar reader, this section will introduce the most fundamental notions in Structured Query Language (SQL) [19], which revolve around the concept of tables and operations on them.

Definition 20 (Table)**.** *A* table S *in SQL is a data entity that consists in a set of data tuples or* rows*. Each tuple component is a* column*. A* schema $S(c_1, \cdots, c_n)$ *defines which are the columns of the table.*

A first example of a SQL table is given in Table 4.1. This table gathers three pieces of information regarding students, namely, *id*, *name* and *supervisor*, and records it into several rows. Its schema is S(ID,Name,Supervisor).

ID	Name	Supervisor
1	Alice	Daniel
2	Beth	Eduard
3	Carl	Daniel

Table 4.1: Table of *students*

Each column in SQL has a *data type* that declares the type of information that can contain, i.e. the range of values or *domain* that it allows. In the previous example, the columns *Name* and *Supervisor* are type *string of characters*, or simply *String*, and contains text. On the other hand, the column *ID* is type *Integer* and contains integer numbers. The set of all possible columns is $\mathbb{C}$, the set of all possible data types is $\mathbb{T}$ and the set of all possible values is $\mathbb{V}$. There are functions linking each column with its data type $type : \mathbb{C} \to \mathbb{T}$, each data type with its domain $dom : \mathbb{T} \to \mathcal{P}(\mathbb{V})$ and, transitively, each column with its domain $val : \mathbb{C} \to \mathcal{P}(\mathbb{V})$.

Furthermore, tables can be manipulated by means of operators. The core ones are *projection*, *selection*, *rename*, *cartesian product*, *natural join*, *union* and *difference*.

Definition 21 (Projection). *A projection $\Pi_{c_1,\cdots,c_k}(S)$ of a table S is a new table where all the tuples have been restricted to the columns $c_1, \cdots, c_k$.*

For instance, projecting the previous table S into the columns *Name* and *Supervisor* $\Pi_{\text{Name,Supervisor}}(S)$ results in Table 4.2.

Name	Supervisor
Alice	Daniel
Beth	Eduard
Carl	Daniel

Table 4.2: Projection of the table *students*

Definition 22 (Natural join). *Let S,T be tables, their natural join $S \bowtie T$ is a table consisting in all possible combinations of rows in S and T that agree on their shared column names.*

Consider the table of *supervisors* T(Supervisor, Department) given by Table 4.3. The natural join $S \bowtie T$ of *students* with *supervisors* is the collection of combined rows that agree on the column *Supervisor* (See Table 4.4)

Supervisor	Department
Daniel	Maths
Eduard	Biology

Table 4.3: *Supervisors* table

ID	Name	Supervisor	Department
1	Alice	Daniel	Maths
2	Beth	Eduard	Biology
3	Carl	Daniel	Maths

Table 4.4: Natural join of *students* with *supervisors*

Definition 23 (Rename). *Let S be a table with schema $S(c_1, \cdots, c_n)$, then a* rename *$\rho_{d_1,\cdots,d_n}(S)$ is a table with the same rows that differs from the original one in the column names; its schema is renamed to $S(d_1, \cdots, d_n)$.*

The Table 4.3 T(Supervisor,Department) can be renamed to a different schema T(Professor,Department) applying the *rename* operation $\rho_{\text{Professor,Department}}(T)$.

Definition 24 (Selection)**.** *Given a table S, a* selection $\sigma_\psi(S)$ *is a new table consisting of all tuples of S that satisfy the condition ψ on a set of columns.*

The selection $\sigma_{\text{ID}\geq 2 \wedge \text{Supervisor}=\text{Daniel}}(S)$ on Table 4.1 filters out all the rows with *ID* greater or equal than two and *Supervisor* name *Daniel*, which results in Table 4.5.

ID	Name	Supervisor
3	Carl	Daniel

Table 4.5: Filtering of *students*

Definition 25 (Cartesian product)**.** *The cartesian product of two tables S,T is a new table $S \times T$ that consists in all possible combinations of rows in S with rows in T.*

The cartesian product of *students* (See Table 4.1) with *supervisors* (See Table 4.3) results in Table 4.6.

ID	Name	Supervisor	Supervisor	Department
1	Alice	Daniel	Daniel	Maths
1	Alice	Daniel	Eduard	Biology
2	Beth	Eduard	Daniel	Maths
2	Beth	Eduard	Eduard	Biology
3	Carl	Daniel	Daniel	Maths
3	Carl	Daniel	Eduard	Biology

Table 4.6: Cartesian product of *students* with *supervisors*

Definition 26 (Union of tables)**.** *The union of two tables with the same schema S, T is a new table $S \sqcup T$ consisting of all the rows in S together with the rows in T. In particular, an element appears in the union as many times as it appears in both S and T jointly.*

Define the table of *teaching assistants* as in 4.7, then the union of *students* with *teaching assistants*, which have the same schema, results in Table 4.8 of *supervised students*. Note that *Beth* occurs two times in the union, since she is a regular student and a teaching assistant at the same time.

Definition 27 (Difference of tables)**.** *Given two tables S, T with the same schema, their difference $S \setminus T$ is a sub-collection of rows in S where each row appears as many times as it does in S minus the number of times it appears in T or no occurrences if the number of instances is greater in T*

ID	Name	Supervisor
2	Beth	Eduard
4	Becky	Emma

Table 4.7: Table of teaching assistants

ID	Name	Supervisor
1	Alice	Daniel
2	Beth	Eduard
3	Carl	Daniel
2	Beth	Eduard
4	Becky	Emma

Table 4.8: Supervised students

Subtracting Table 4.9 from *supervised students* (Table 4.8) results in Table 4.10. Notice that there is still an occurrence of *Beth* in the difference, since there were two copies in *supervised students*.

ID	Name	Supervisor
1	Alice	Daniel
2	Beth	Eduard

Table 4.9: Substracted table

4.2 A model for Structured Query Language

4.2.1 Interpretation

Spans can be used to model tables and operations on them. Let $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ be a span in **Set**, then $S = \{r_i\}_{i \in I}$ is some collection of rows, i.e. an instance of a table, and the two legs of the span are the projections $\Pi_X(S) \equiv s_X : r_i \mapsto r_i|_X$ and $\Pi_Y(S) \equiv s_Y : r_i \mapsto r_i|_Y$ into columns X and Y of the table respectively. Notice that row repetitions are modelled by simply taking several instances of the same row in S and regarding them as different elements. The fundamental difference between *algebra of bags* and *relation algebra* is that the former allows for row repetitions and this makes this interpretation closer to algebra of bags.

Take as an example Table 4.1, with schema S(ID,Name,Supervisor) and its projections $\Pi_{\text{ID}}(S)$ and $\Pi_{\text{Name,Supervisor}}(S)$. They can be modelled into a span $\mathbf{ID} \xleftarrow{s_X} S \xrightarrow{s_Y} \mathbf{Name} \times \mathbf{Supervisor}$ by letting S be the collection of rows

$$S = \{(1, \text{Alice,Daniel}), (2, \text{Beth,Eduard}), (3, \text{Carl,Daniel})\}$$

ID	Name	Supervisor
3	Carl	Daniel
2	Beth	Eduard
4	Becky	Emma

Table 4.10: Difference of tables

and

$$\begin{aligned} s_X :& S \to \mathbf{ID} : (t_{ID}, t_{Name}, t_{Supervisor}) \mapsto t_{ID} \\ s_Y :& S \to \mathbf{Name} \times \mathbf{Supervisor} : (t_{ID}, t_{Name}, t_{Supervisor}) \mapsto (t_{Name}, t_{Supervisor}) \end{aligned}$$

the two projections.

More precisely, column names are not witnessed by this abstract framework; sets are rather data types whose elements constitute its possible range of values or domain. Recall that, in set theory, two sets are equal if they contain the exact same elements, hence, two columns with the same domain are indeed the same object. In the previous example, $\mathbf{ID} = \mathbb{Z}$, the data type of integers, and $\mathbf{Name} = \mathbf{Supervisor} = \Sigma^*$, the set of finite sequences of elements over an alphabet Σ. Bearing this in mind, $\mathsf{Span}(\mathbf{Set})$ can be seen as the category of *data types* as objects and *tables and projections* as arrows.

Furthermore, it is possible to extend this setting and instead move to the wider framework of spans of (hemi-,quasi-) metric spaces. As tables are collections of tuples, they are tightly connected with elements in some product space of columns. For example, the table *students* is a generalised relation in the product space $S \subseteq \mathbf{ID} \times \mathbf{Name} \times \mathbf{Supervisor} = \mathbb{Z} \times \Sigma^* \times \Sigma^*$. A product metric or, more abstractly, any particular metric can be given to this space of rows S, making it into a metric space. Data types are given a metric too, e.g. the usual metric for integers $\mathbb{Z}$ or any *string metric*, such as the widely used in information theory Levenshtein metric, for Σ^*.

In addition, the weak contraction conditions in the span of (hemi-,quasi-) metric spaces $(X, d_X) \xleftarrow{s_X} (S, d_S) \xrightarrow{s_Y} (Y, d_Y)$,

$$d_S(s, s') \geq d_X(s_X(s), s_X(s'), d_Y(s_Y(s), s_Y(s'),$$

express that the metric on the space of tuples S must take into account the metric in the columns, in the sense that the distance between two tuples must be greater than the distance between its components. In the example of *students*, the metric $d_{\mathbb{Z}} \times d_{\Sigma^*} \times d_{\Sigma^*}$, where $d_{\mathbb{Z}}$, d_{Σ^*} are some data type metrics, would make a suitable metric for this spaces of tuples.

The choice of category for this chapter will be hemi-metric spaces **HMet**, as they allow row repetitions and, therefore, they will model algebra of bags. Although this will be discussed in more detail at the end of the chapter, intuitively, if there were two instances s,s' of the same row, then their distance would be zero $d(s,s') = 0$, as they represent the same row; and the T_0 requirement would collapse them into a single one.

To simplify the notation, the metric on objects will not be written and instead they will be assumed to implicitly carry a suitable metric; when needed, it will be referred to as d_X for an object X. Finally, a span $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ will be interpreted as a table where some columns are type X and some others type Y. The first column will be referred as the *domain column* and the second one as the *codomain column*.

4.2.2 Natural join, rename and selection

It turns out that span composition under this interpretation models natural join of tables. The composite of a table $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ with a table $Y \xleftarrow{s_Y} S \xrightarrow{s_Z} Z$ is another table $X \xleftarrow{s_X \circ \pi_S} S \times_Y T \xrightarrow{s_Z \circ \pi_T} Z$ where the rows

$$S \times_Y T = \{(s,t) \in S \times T : s_Y(s) = s_Y(t)\}$$

are combinations of rows of S with rows of T that agree on the columns of type Y. Additionally, the resulting span carries the projections into the column of type X in S and into the column of type Z in T. The metric $d_S \times d_T$ on $S \times_Y T$ creates a product metric that accounts for both the metric on tuples of S and the metric on tuples of T.

Notice that there is an implicit renaming in the natural join of two spans. The coinciding legs of the two spans establish the agreement condition, thus, implicitly renaming their associated columns to the same name. They only require that the data type of the joint be the same. Consequently, *rename* is an operation categorically encoded in arrow composition in this framework.

Let $\mathbf{Name} \xleftarrow{s_X} S \xrightarrow{s_Y} \mathbf{Supervisor}$ be Table 4.1 and $\mathbf{Supervisor} \xleftarrow{s_Y} T \xrightarrow{s_Z} \mathbf{Department}$ be Table 4.3, with corresponding projections, their composite is the table $\mathbf{Name} \xleftarrow{s_X \circ \pi_S} S \times_{\mathbf{Supervisor}} T \xrightarrow{s_Z \circ \pi_T} \mathbf{Department}$, where

$$\begin{aligned}
&S \times_{\mathbf{Supervisor}} T = \\
&= \{(s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \in S \times T : s_{Supervisor} = t_{Supervisor}\} \\
&s_X \circ \pi_S : (s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \mapsto s_{Name} \\
&s_Z \circ \pi_T : (s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \mapsto t_{Department}.
\end{aligned}$$

In addition, the metric on the new rows will be given by

$$d((s,t),(s',t')) = \max\{d_S(s,s'), d_T(t,t')\},$$

which captures the maximum dissimilarity between the original rows.

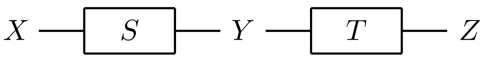
Note that the resulting table $X \xleftarrow{s_X \circ \pi_S} S \times_Y T \xrightarrow{s_Z \circ \pi_T} Z$ is not quite the natural join according to its definition, since the joint columns are repeated rather than being fused into a single one, i.e. $S \times_Y T = \{((u,y),(y,v) \in S \times T\}$. However, this span is equivalent to the natural join $X \xleftarrow{s'_X} S \bowtie T \xrightarrow{s'_Z} Z$, where

$$\begin{aligned} S \bowtie T =& \{(u,y,v) : (u,y) \in S, (y,v) \in T\} \\ s'_X :& (u,y,v) \mapsto s_X(u,y) \\ s'_Z :& (u,y,v) \mapsto s_Z(y,v) \\ d((u,y,v),(u',y',v')) =& \max\{d_S((u,y),(u',y')), d_T((y,v),(y',v'))\}, \end{aligned}$$

since there is a 2-isomorphism $S \bowtie T \xrightarrow{\cong} S \times_Y T : (u,y,v) \mapsto (u,y,y,v)$ that makes the following diagram commute

$$\begin{tikzcd} & S \bowtie T \arrow[ddl, "s'_X"'] \arrow[d, "\cong"] \arrow[ddr, "s'_Z"] & \\ & S \times_Y T \arrow[dl, "s_X \circ \pi_S"] \arrow[dr, "s_Z \circ \pi_T"'] & \\ X & & Z \end{tikzcd} .$$

Lastly, natural join of tables is expressed in the diagrammatic calculus by means of morphism composition



,

which instantiated to the previous example gives the following representation

.

Notice that the orientation of the diagrams has been changed to left-to-right, in order to provide a better resemblance with actual tables, which suggest a horizontal distribution.

Selection, on the other hand, can be modelled using span composition as well. Recall that a selection $\sigma_\psi(S)$ from a table S filters out those rows that satisfy the condition ψ on a particular set of columns, i.e.

$$\sigma_\psi(S) = \{s \in S : \psi(s_X(s))\},$$

where s_X is the projection onto the target columns. In that sense, ψ is a predicate on the data type X of the column, $\psi : X \to \{\textbf{True}, \textbf{False}\}$, that filters out the table. Equivalently, as a predicate is essentially a characteristic function on a set, it can be seen as a subset of the data type $P = \{x \in X : \psi(x)\} \subseteq X$, which can be encoded into a span

$$\mathbf{1} \xleftarrow{!} P = \{x \in X : \psi(x)\} \xhookrightarrow{i} X$$

by taking the inclusion map i into X. P is a metric subspace of X and, therefore, it inherits the metric, i.e. $d_P = d_X|_P$ or simply d_X to simplify the notation.

The composite of this filter span $\mathbf{1} \xleftarrow{!} P \xhookrightarrow{i} X$ with a table $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ results in the span $\mathbf{1} \xleftarrow{!} P \times_X S \xrightarrow{s_Y \circ \pi_S} Y$, where

$$\begin{aligned} P \times_X S &= \{(p, s) \in P \times S : i(p) = p = s_X(s)\} \\ &= \{(p, s) \in X \times S : \psi(p), p = s_X(s)\} \\ &= \{(s_X(s), s) \in X \times S : \psi(s_X(s))\} \end{aligned}$$

with metric $d_X \times d_S$. This is isomorphic to the selection $\mathbf{1} \xleftarrow{!} \sigma_\psi(S) \xrightarrow{s'_Y} Y$, with

$$\begin{aligned} \sigma_\psi(S) &= \{s \in S : \psi(s_X(s))\} \subseteq S \\ s'_Y &: s \mapsto s_Y(s) \end{aligned}$$

and metric d_S inherited from S, since the 2-isomorphism $\sigma_\psi(S) \xrightarrow{\cong} P \times_X S : s \mapsto (s_X(s), s)$ makes the following diagram commute

$$\begin{tikzcd} & \sigma_\psi(S) \arrow[dl, "!"'] \arrow[d, "\cong"] \arrow[dr, "s'_Y"] & \\ & P \times_X S \arrow[dl, "!"] \arrow[dr, "s_Y \circ \pi_S"'] & \\ \mathbf{1} & & Y \end{tikzcd} .$$

That it is a bijection follows easily from the definitions of $\sigma_\psi(S)$ and $P \times_X S$. To see that the metric is preserved, note that

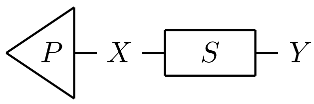
$$\begin{aligned} (d_X \times d_S)((s_X(s), s), (s_X(s'), s')) &\underset{\text{def}}{=} \max\{d_X(s_X(s), s_X(s')), d_S(s, s')\} \\ &= d_S(s, s'), \end{aligned}$$

with the last equality following from the fact that $s_X : S \to X$ is a weak contraction that satisfies $d_X(s_X(s), s_X(s')) \leq d_S(s, s')$.

Take as an instance the selection $\sigma_{\text{ID}\geq 2 \wedge \text{Name}=\text{Daniel}}$ on the *students* Table 4.1. The corresponding predicate is given by $P = \{z \in \mathbb{Z} : z \geq 2\} \times \{\text{Daniel}\} \subseteq \mathbb{Z} \times \Sigma^*$. Composing $\mathbf{1} \xleftarrow{!} \{z \geq 2\} \times \{\text{Daniel}\} \xhookrightarrow{i} \mathbb{Z} \times \Sigma^*$ with $\mathbb{Z} \times \Sigma^* = \mathbf{ID} \times \mathbf{Supervisor} \xleftarrow{s_X} S \xrightarrow{s_Y} \mathbf{Name}$ outputs the span $\mathbf{1} \xleftarrow{!} P \times_{\mathbf{ID}\times\mathbf{Supervisor}} S \xrightarrow{s_Y \circ \pi_S} \mathbf{Name}$, where

$$\begin{aligned} P \times_{\mathbf{ID}\times\mathbf{Supervisor}} S =& \{(p_1, p_2, s_{ID}, s_{Name}, s_{Supervisor}) \in P \times S : p_1 = s_{ID}, p_2 = s_{Supervisor}\} \\ =& \{(3, \text{Daniel}, 3, \text{Carl}, \text{Daniel})\} \\ s_Y \circ \pi_S :& (3, \text{Daniel}, 3, \text{Carl}, \text{Daniel}) \mapsto \text{Daniel} \end{aligned}$$

Finally, using string diagrams, selection can be depicted as



,

where P is the filter morphism and S the table to select from. In the particular case of the previous example, the following representation is obtained:

.

4.2.3 Cartesian product

The cartesian product of tables can be modelled in this setting by means of the tensor product. Given two tables $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ and $X' \xleftarrow{s_{X'}} S' \xrightarrow{s_{Y'}} Y'$, their tensor product $X \times X' \xleftarrow{s_X \times s_{X'}} S \times S' \xrightarrow{s_Y \times s_{Y'}} Y \times Y'$ is a new table where the rows

$$S \times S' = \{(s, s') \in S \times S'\}$$

are all possible combinations of rows in S with rows in S'. Moreover, the final span carries the projections into the original columns and, as in the case of the natural join, $S \times S'$ is given a product metric $d_S \times d_{S'}$ that accounts for the metrics in both S and S'.

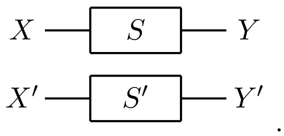
For instance, consider the tables for *students* $\mathbf{ID} \xleftarrow{s_X} S \xrightarrow{s_Y} \mathbf{Name}$ and *supervisors* $\mathbf{Supervisor} \xleftarrow{s_{X'}} T \xrightarrow{s_{Y'}} \mathbf{Department}$. Their tensor product is given by $\mathbf{ID} \times \mathbf{Supervisor} \xleftarrow{s_X \times s_{X'}} S \times T \xrightarrow{s_Y \times s_{Y'}} \mathbf{Name} \times \mathbf{Department}$, where

$$\begin{aligned} & S \times T = \{(s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \in S \times T\} \\ & s_X \times s_{X'} : (s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \mapsto (s_{ID}, t_{Supervisor}) \\ & s_Y \times s_{Y'} : (s_{ID}, s_{Name}, s_{Supervisor}, t_{Supervisor}, t_{Department}) \mapsto (s_{Name}, t_{Department}). \end{aligned}$$

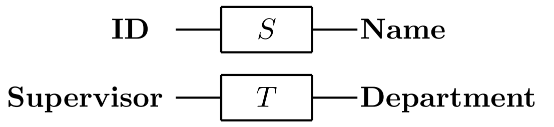
Moreover, the metric measures the maximum dissimilarity between the original rows

$$(d_S \times d_{S'})((s_1, s'_1), (s_2, s'_2)) = \max\{d_S(s_1, s_2), d_{S'}(s'_1, s'_2)\}.$$

In the graphical calculus, cartesian product of two tables is depicted as two morphisms in parallel:



Instantiated for the previous example leads to the following diagram



4.2.4 Union of tables

Union of tables are naturally modelled using coproducts, as in this setting they correspond to disjoint unions. Recall that a row appears in the union of two tables as many times as it does in both tables jointly. For that reason, the disjoint union operation will provide a successful tool to capture row repetitions. In this section, a more general notion of union will be investigated; two tables will not necessarily have the same schema but agree on the projected columns types.

Given two tables $X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ and $X \xleftarrow{s_X} T \xrightarrow{s_Y} Y$, their union is defined as the span $X \xleftarrow{[s_X, s_X]} S + T \xrightarrow{[s_Y, s_Y]} Y$, where the rows

$$S + T = \{(s, 1) : x \in S\} \cup \{(t, 2) : x \in T\}$$

are rows in S together with rows in T and the projections take each row to its corresponding original projection. The metric on $S + T$

$$d((x, i), (x', i')) = \begin{cases} d_S(x, x') & \text{if } i = i' = 1 \\ d_T(x, x') & \text{if } i = i' = 2 \\ \infty & \text{otherwise} \end{cases}$$

preserves the distances in the original tables and adds no metric information between rows in different tables.

Notice that there is not enough context to provide metric information for rows in different tables. The next section will explore a new tool, that of *universes*, to define such context. Interestingly, union of spans, defined as above, defines an enrichment over commutative monoids.

Lemma 7. *Given a category* $\mathbf{C}$ *with finite coproducts and pullbacks, the operation that assigns two spans* $\mathbf{S} : X \xleftarrow{s_X^S} S \xrightarrow{s_Y^S} Y$, $\mathbf{T} : X \xleftarrow{s_X^T} T \xrightarrow{s_Y^T} Y$ *in* $\mathsf{Span}(\mathbf{C})$ *to the span*

$$\mathbf{S}+\mathbf{T} : X \xleftarrow{[s_X^S, s_X^T]} S+T \xrightarrow{[s_Y^S, s_Y^T]} Y$$

defines a commutative monoid structure over the hom-set.

Proof. Commutativity, associativity and units follow from properties of coproducts:

(**Commutativity**) Commutativity requires that

$$\mathbf{S}+\mathbf{T} = \begin{tikzcd} & S+T \arrow[dl, "{[s_X^S, s_X^T]}"'] \arrow[dr, "{[s_Y^S, s_Y^T]}"] & \\ X & & Y \end{tikzcd} \cong \begin{tikzcd} & T+S \arrow[dl, "{[s_X^T, s_X^S]}"'] \arrow[dr, "{[s_Y^T, s_Y^S]}"] & \\ X & & Y \end{tikzcd} = \mathbf{T}+\mathbf{S},$$

which can be reduced to commutativity of coproducts. Coproducts are commutative by means of the natural isomorphism $[i_2, i_1] : S+T \overset{\cong}{\to} T+S$, which is a 2-isomorphism for the two spans above that makes the following diagram commute:

$$\begin{tikzcd} & S+T \arrow[d, "{[i_2,i_1]}"] \arrow[ddl, "{[s_X^S, s_X^T]}"'] \arrow[ddr, "{[s_Y^S, s_Y^T]}"] & \\ & T+S \arrow[dl, "{[s_X^T, s_X^S]}"] \arrow[dr, "{[s_Y^T, s_Y^S]}"'] & \\ X & & Y \end{tikzcd} .$$

The left triangle is shown to commute using properties of the coproduct

$$[s_X^T, s_X^S] \circ [i_2, i_1] = [[s_X^T, s_X^S] \circ i_2, [s_X^T, s_X^S] \circ i_1] = [s_X^S, s_X^T]$$

and commutativity of the right triangle is proved analogously.

(**Associativity**) For associativity the following must hold:

$$\mathbf{S}\text{+}(\mathbf{T}\text{+}\mathbf{U}) = \begin{tikzcd} & S+(T+U) \arrow[dl, "{[s_X^S,[s_X^T,s_X^U]]}"'] \arrow[dr, "{[s_Y^S,[s_Y^T,s_Y^U]]}"] & \\ X & & Y \end{tikzcd} \cong \begin{tikzcd} & (S+T)+U \arrow[dl, "{[[s_X^S,s_X^T],s_X^U]}"'] \arrow[dr, "{[[s_Y^S,s_Y^T],s_Y^U]}"] & \\ X & & Y \end{tikzcd} = (\mathbf{S}\text{+}\mathbf{T})\text{+}\mathbf{U}.$$

Coproducts are associative with the natural isomorphism $[i_1 \circ i_1, [i_1 \circ i_2, i_2]] : S+(T+U) \overset{\cong}{\to} (S+T)+U$, which is a 2-isomorphism for the two spans above that makes the following diagram commutes:

$$\begin{tikzcd} & S+(T+U) \arrow[d, "{[i_1\circ i_1,[i_1\circ i_2,i_2]]}"] \arrow[ddl, "{[s_X^S,[s_X^T,s_X^U]]}"'] \arrow[ddr, "{[s_Y^S,[s_Y^T,s_Y^U]]}"] & \\ & (S+T)+U \arrow[dl, "{[[s_X^S,s_X^T],s_X^U]}"] \arrow[dr, "{[[s_Y^S,s_Y^T],s_Y^U]}"'] & \\ X & & Y \end{tikzcd} .$$

The left triangle again commutes due to properties of the coproduct

$$
\begin{aligned}
&[[s^S_X, s^T_X], s^U_X] \circ [i_1 \circ i_1, [i_1 \circ i_2, i_2]] \\
=&[[[s^S_X, s^T_X], s^U_X] \circ i_1 \circ i_1, [[[s^S_X, s^T_X], s^U_X] \circ i_1 \circ i_2, [[s^S_X, s^T_X], s^U_X] \circ i_2]] \\
=&[s^S_X, [s^T_X, s^U_X]].
\end{aligned}
$$

and an analogous proof holds for the right side of the diagram.

(**Units**) Define the unit $u_{X,Y}$ as

$$
u_{X,Y} \underset{\text{def}}{\equiv}
\begin{tikzcd}
 & \mathbf{0} \arrow[dl, "i_X"'] \arrow[dr, "i_Y"] & \\
X & & Y
\end{tikzcd}
$$

where $\mathbf{0}$ is the initial object and i_X,i_Y are the unique arrows from the initial object to X and Y. Without loss of generality, the unit span for a particular hom-set will be refereed as $\mathbf{0}$ in the following. The unit law requires that

$$
\mathbf{S} + \mathbf{0} =
\begin{tikzcd}
 & S + \mathbf{0} \arrow[dl, "{[s^S_X, i_X]}"'] \arrow[dr, "{[s^S_Y, i_Y]}"] & \\
X & & Y
\end{tikzcd}
\cong
\begin{tikzcd}
 & S \arrow[dl, "s^S_X"'] \arrow[dr, "s^S_Y"] & \\
X & & Y
\end{tikzcd}
= \mathbf{S}.
$$

Using the natural isomorphism $[\mathrm{id}_S, i_S] : S + \mathbf{0} \overset{\cong}{\to} S$, the diagram

$$
\begin{tikzcd}
 & S + \mathbf{0} \arrow[ddl, "{[s^S_X, i_X]}"'] \arrow[d, "{[\mathrm{id}_S, i_S]}"] \arrow[ddr, "{[s^S_Y, i_Y]}"] & \\
 & S \arrow[dl, "s^S_X"] \arrow[dr, "s^S_Y"'] & \\
X & & Y
\end{tikzcd}
$$

commutes and the spans above are equivalent. To see that it commutes, take the left triangle, as an instance, and notice that

$$
s^S_X \circ [\mathrm{id}_S, i_S] = [s^S_X \circ \mathrm{id}_S, s^S_X \circ i_S] = [s^S_X, i_X].
$$

Similarly, the same is shown for the remaining triangle. $\square$

Before showing that there is indeed an enrichment over commutative monoids, observe that addition of spans interacts well with the dagger functor.

Proposition 6. *The dagger functor* $\dagger$ *preserves superposition of spans, i.e.*

$$
\dagger(\mathbf{S} + \mathbf{T}) = \dagger(\mathbf{T}) + \dagger(\mathbf{S})
$$

Proof.

$$\dagger(\mathbf{S}+\mathbf{T}) = \dagger\left(\begin{tikzcd} & S+T \arrow[dl, "{[s_X^S, s_X^T]}"'] \arrow[dr, "{[s_Y^S, s_Y^T]}"] & \\ X & & Y \end{tikzcd}\right) = \begin{tikzcd} & S+T \arrow[dl, "{[s_Y^S, s_Y^T]}"'] \arrow[dr, "{[s_X^S, s_X^T]}"] & \\ Y & & X \end{tikzcd}$$

$$= \begin{tikzcd} & S \arrow[dl, "s_Y^S"'] \arrow[dr, "s_X^S"] & \\ Y & & X \end{tikzcd} + \begin{tikzcd} & T \arrow[dl, "s_Y^T"'] \arrow[dr, "s_X^T"] & \\ Y & & X \end{tikzcd} = \dagger(\mathbf{T}) + \dagger(\mathbf{S})$$

□

Proposition 7. *The dagger functor $\dagger$ preserves units, i.e.*

$$\dagger(\mathbf{0}) = \mathbf{0}$$

Proof.

$$\dagger(\mathbf{0}) = \dagger\left(\begin{tikzcd} & \mathbf{0} \arrow[dl, "i_X"'] \arrow[dr, "i_Y"] & \\ X & & Y \end{tikzcd}\right) = \begin{tikzcd} & \mathbf{0} \arrow[dl, "i_Y"'] \arrow[dr, "i_X"] & \\ Y & & X \end{tikzcd} = \mathbf{0}$$

□

Theorem 2. *Given spans of metric spaces $\mathbf{S} : X \xleftarrow{s_X^S} S \xrightarrow{s_Y^S} Y$, $\mathbf{T} : X \xleftarrow{s_X^T} T \xrightarrow{s_Y^T} Y$, the following defines an enrichment over commutative monoids:*

$$\mathbf{S}+\mathbf{T} : X \xleftarrow{[s_X^S, s_X^T]} S+T \xrightarrow{[s_Y^S, s_Y^T]} Y$$

Proof. The commutative monoid structure follows directly from Lemma 7. The enrichment follows from the following:

(**Addition is preserved by composition**) In this case, the following must hold:

$$(\mathbf{S}+\mathbf{T}); \mathbf{U} \equiv \mathbf{S}; \mathbf{U} + \mathbf{T}; \mathbf{U}$$
$$\mathbf{U}; (\mathbf{S}+\mathbf{T}) \equiv \mathbf{U}; \mathbf{S} + \mathbf{U}; \mathbf{T};$$

It suffices to prove that the first equality holds, since the second one can be derived simply relying on the dagger functor, as it will be shown later. The left hand side of this first equality amounts to

$$(\mathbf{S}+\mathbf{T}); \mathbf{U} = \left(\begin{tikzcd} & S+T \arrow[dl, "{[s_X^S, s_X^T]}"'] \arrow[dr, "{[s_Y^S, s_Y^T]}"] & \\ X & & Y \end{tikzcd}\right); \left(\begin{tikzcd} & U \arrow[dl, "s_Y^U"'] \arrow[dr, "s_Z^U"] & \\ Y & & Z \end{tikzcd}\right)$$

$$= \begin{tikzcd} & (S+T) \times_Y U \arrow[dl, "{[s_X^S, s_X^T] \circ \pi_{S+T}}"'] \arrow[dr, "s_Z^U \circ \pi_U"] & \\ X & & Z \end{tikzcd},$$

with metric $(d_S + d_T) \times d_U$ on $(S+T) \times_Y U$ and where

$$\begin{aligned}(S+T)\times_Y U =&\{((x,L),u) : (x,L) \in (S+T), u \in U, [s^S_Y, s^T_Y](x,L) = s^U_Y(u)\}\\ =&\{((s,S),u) : s \in S, u \in U, [s^S_Y, s^T_Y](s,S) = s^S_Y(s) = s^U_Y(u)\}\\ &\cup\{((t,T),u) : t \in T, u \in U, [s^S_Y, s^T_Y](t,T) = s^T_Y(t) = s^U_Y(u)\}\end{aligned}$$

Notice the choice of $L = S, T$ to index and identify the original sets in the disjoint union; it will simplify calculations in the remaining. On the other hand, the right hand side equates to

$$\mathbf{S};\mathbf{U}+\mathbf{T};\mathbf{U} = \left(\begin{tikzcd} & S\times_Y U \arrow[dl, "s^S_X\circ\pi_S"'] \arrow[dr, "s^U_Z\circ\pi_U"] & \\ X & & Z\end{tikzcd}\right) + \left(\begin{tikzcd} & T\times_Y U \arrow[dl, "s^T_X\circ\pi_T"'] \arrow[dr, "s^U_Z\circ\pi_U"] & \\ X & & Z\end{tikzcd}\right)$$

$$= \begin{tikzcd} & S\times_Y U + T\times_Y U \arrow[dl, "{[s^S_X\circ\pi_S, s^T_X\circ\pi_T]}"'] \arrow[dr, "{[s^U_Z\circ\pi_U, s^U_Z\circ\pi_U]}"] & \\ X & & Z\end{tikzcd},$$

with metric $d_S \times d_U + d_T \times d_U$ on $S\times_Y U + T\times_Y U$ and where

$$\begin{aligned}S\times_Y U + T\times_Y U = &\{(s,u) : s \in S, u \in U : s^S_Y(s) = s^U_Y(u)\}\\ &+\{(t,u) : t \in T, u \in U : s^T_Y(t) = s^U_Y(u)\}\\ &=\{((s,u),S) : s \in S, u \in U, s^S_Y(s) = s^U_Y(u)\}\\ &\quad\cup\{((t,u),T) : t \in T, u \in U, s^T_Y(t) = s^U_Y(u)\}.\end{aligned}$$

These two sets are isomorphic by means of the isomorphism

$$m : (S+T)\times_Y U \to S\times_Y U + T\times_Y U : ((x,L),u) \mapsto ((x,u),L),$$

which is a bijection due to the almost identical definitions of domain and codomain and also preserves distances:

$$\begin{aligned}&(d_S \times d_U + d_T \times d_U)\ (m((x,L),u), m((x',L'),u'))\\ =&(d_S \times d_U + d_T \times d_U)\ (((x,u),L),((x',u'),L'))\\ =&\begin{cases}(d_L \times d_U)((x,u),(x',u')) & L = L'\\ \infty & L \neq L'\end{cases}\\ =&\begin{cases}\max\{d_L(x,x'), d_U(u,u')\} & L = L'\\ \infty & L \neq L'\end{cases},\end{aligned}$$

on the one hand, which is equal to

$$
\begin{aligned}
&(d_S + d_T) \times d_U \ (((x, L), u), ((x', L'), u')) \\
=& \max\{d_S + d_T((x, L), (x', L')), d_U(u, u')\} \\
=& \begin{cases} \max\{d_L(x, x'), d_U(u, u')\} & L = L' \\ \infty & L \neq L' \end{cases}.
\end{aligned}
$$

It is moreover a 2-isomorphism that makes the diagram

$$
\begin{tikzcd}
& (S+T) \times_Y U \arrow[d, "m"] \arrow[ddl, "{[s_X^S, s_X^T] \circ \pi_{S+T}}"'] \arrow[ddr, "s_Z^U \circ \pi_U"] & \\
& S \times_Y U + T \times_Y U \arrow[dl, "{[s_X^S \circ \pi_S, s_X^T \circ \pi_T]}"] \arrow[dr, "{[s_Z^U \circ \pi_U, s_Z^U \circ \pi_U]}"'] & \\
X & & Z
\end{tikzcd}
$$

commute and, therefore, the inital spans are indeed isomorphic. Commutation of the left triangle reduces to showing that

$$
\begin{aligned}
&[s_X^S \circ \pi_S, s_X^T \circ \pi_T] \circ m \ ((x, L), u) \\
=&[s_X^S \circ \pi_S, s_X^T \circ \pi_T] \ ((x, u), L) \\
=&s_X^L \circ \pi_L \ (x, u) \\
=&s_X^L \ (x) \\
=&[s_X^S, s_X^T] \ (x, L) \\
=&[s_X^S, s_X^T] \circ \pi_{S+T} \ ((x, L), u),
\end{aligned}
$$

whereas for the right triangle it reduces to showing that

$$
\begin{aligned}
&[s_Z^U \circ \pi_U, s_Z^U \circ \pi_U] \circ m \ ((x, L), u) \\
=&[s_Z^U \circ \pi_U, s_Z^U \circ \pi_U] \ ((x, u), L) \\
=&s_Z^U \circ \pi_U \ (x, u) \\
=&s_Z^U \ (u) \\
=&s_Z^U(u) \circ \pi_U \ ((x, L), u).
\end{aligned}
$$

As far as the second equivalence $\mathbf{U}; (\mathbf{S} + \mathbf{T}) \equiv \mathbf{U}; \mathbf{S} + \mathbf{U}; \mathbf{T}$ is concerned, recall by Proposition 6 that the dagger functor preserves superposition of spans. Applying the dagger functor to the right-composition equation proved before gives a dualised version that corresponds to the left-composition equation

$$
\begin{aligned}
&\dagger\left((\mathbf{S}+\mathbf{T});\mathbf{U}\right) \equiv \dagger(\mathbf{S};\mathbf{U}+\mathbf{T};\mathbf{U}) \\
\iff &\dagger(\mathbf{U});\dagger(\mathbf{S}+\mathbf{T}) \equiv \dagger(\mathbf{S};\mathbf{U})+\dagger(\mathbf{T};\mathbf{U}) \\
\iff &\dagger(\mathbf{U});(\dagger(\mathbf{S})+\dagger(\mathbf{T})) \equiv \dagger(\mathbf{U});\dagger(\mathbf{S})+\dagger(\mathbf{U});\dagger(\mathbf{T}).
\end{aligned}
$$

(**Units are compatible with composition**) In this case, it must hold that

$$\mathbf{S};\mathbf{0} \equiv \mathbf{0} \equiv \mathbf{0};\mathbf{S}$$

The left hand side equates to

$$
\mathbf{S};\mathbf{0} = \left(\begin{tikzcd} & S \arrow[dl, "s_X^S"'] \arrow[dr, "s_Y^S"] & \\ X & & Y \end{tikzcd}\right);\left(\begin{tikzcd} & \mathbf{0} \arrow[dl, "i_Y"'] \arrow[dr, "i_Z"] & \\ Y & & Z \end{tikzcd}\right) = \begin{tikzcd} & S\times_Y \mathbf{0} \arrow[dl, "s_X^S\circ\pi_S"'] \arrow[dr, "i_Z\circ\pi_{\mathbf{0}}"] & \\ X & & Z \end{tikzcd}
$$

where

$$S\times_Y \mathbf{0} = S\times_Y \emptyset = \{(s,t)\in S\times\emptyset : s_Y^S(s) = i_Y(t)\} = \emptyset = \mathbf{0}.$$

As a result, $s_X^S\circ\pi_S$ and $i_Z\circ s_{\mathbf{0}}$ must be the unique arrows from the initial object to X and Z respectively so

$$
\mathbf{S};\mathbf{0} = \begin{tikzcd} & S\times_Y \mathbf{0} \arrow[dl, "s_X^S\circ\pi_S"'] \arrow[dr, "i_Z\circ\pi_{\mathbf{0}}"] & \\ X & & Z \end{tikzcd} = \begin{tikzcd} & \mathbf{0} \arrow[dl, "i_X"'] \arrow[dr, "i_Z"] & \\ X & & Z \end{tikzcd} = \mathbf{0}
$$

and the first equation holds. The second equation is proven analogously or, similar to the previous case, by means of the dagger functor. □

In fact, this theorem is an instance of a much more general fact in span categories. Notice how the previous proof only relies in properties of coproducts (commutativity, associativity and units) and distributivity of pullbacks with respect to coproducts (composition preserves addition and units are compatible with composition). The following theorem extends the previous result to a wider range of categories.

Theorem 3 (Enrichment in commutative monoids for span categories)**.** *Let* $\mathbf{C}$ *be a category with finite coproducts and pullbacks which is moreover locally distributive, then* $\mathsf{Span}(\mathbf{C})$ *is enriched over over commutative monoids, where the monoid structure over the hom-set is given by the operation that assigns two spans* $\mathbf{S}: X \xleftarrow{s_X^S} S \xrightarrow{s_Y^S} Y$*,* $\mathbf{T}: X \xleftarrow{s_X^T} T \xrightarrow{s_Y^T} Y$ *to the span*

$$\mathbf{S}+\mathbf{T} : X \xleftarrow{[s_X^S, s_X^T]} S+T \xrightarrow{[s_Y^S, s_Y^T]} Y$$

Proof. The commutative monoid structure is entailed by Lemma 7. The enrichment follows from the subsequent results.

(**Addition is preserved by composition**) The following must hold:

$$(\mathbf{S}+\mathbf{T});\mathbf{U} \equiv \mathbf{S};\mathbf{U}+\mathbf{T};\mathbf{U}$$
$$\mathbf{U};(\mathbf{S}+\mathbf{T}) \equiv \mathbf{U};\mathbf{S}+\mathbf{U};\mathbf{T}$$

As in the previous theorem, the second equivalence can be proven resorting to the dagger functor. The left hand side and the right hand side of the first one equate to:

$$(\mathbf{S}+\mathbf{T});\mathbf{U} = \left(\begin{tikzcd} & S+T \arrow[dl, "{[s^S_X, s^T_X]}"'] \arrow[dr, "{[s^S_Y, s^T_Y]}"] & \\ X & & Y \end{tikzcd}\right);\left(\begin{tikzcd} & U \arrow[dl, "s^U_Y"'] \arrow[dr, "s^U_Z"] & \\ Y & & Z \end{tikzcd}\right)$$

$$= \begin{tikzcd} & (S+T)\times_Y U \arrow[dl, "{[s^T_X, s^S_X]\circ\pi_{S+T}}"'] \arrow[dr, "s^U_Z\circ\pi_U"] & \\ X & & Z \end{tikzcd},$$

$$\mathbf{S};\mathbf{U}+\mathbf{T};\mathbf{U} = \left(\begin{tikzcd} & S\times_Y U \arrow[dl, "s^S_X\circ\pi_S"'] \arrow[dr, "s^U_Z\circ\pi_U"] & \\ X & & Z \end{tikzcd}\right)+\left(\begin{tikzcd} & T\times_Y U \arrow[dl, "s^T_X\circ\pi_T"'] \arrow[dr, "s^U_Z\circ\pi_U"] & \\ X & & Z \end{tikzcd}\right)$$

$$= \begin{tikzcd} & S\times_Y U+T\times_Y U \arrow[dl, "{[s^S_X\circ\pi_S, s^T_X\circ\pi_T]}"'] \arrow[dr, "{[s^U_Z\circ\pi_U, s^U_Z\circ\pi_U]}"] & \\ X & & Z \end{tikzcd},$$

where $S \xleftarrow{\pi_S} S\times_Y U \xrightarrow{\pi_U} U$, $T \xleftarrow{\pi_T} S\times_Y U \xrightarrow{\pi_U} U$, $S+T \xleftarrow{\pi_{S+T}} (S+T)\times_Y U \xrightarrow{\pi_U} U$ are the pullbacks along the pairs of morphism $S \xrightarrow{s^S_Y} Y \xleftarrow{s^U_Y} U$, $T \xrightarrow{s^T_Y} Y \xleftarrow{s^U_Y} U$ and $S+T \xrightarrow{[s^S_Y, s^T_Y]} Y \xleftarrow{s^U_Y} U$ respectively.

Since $\mathbf{C}$ is locally distributive, then $\mathbf{C}/Y$ is distributive, which means that the canonical distributive morphism $[i_1\times\mathrm{id}_U, i_2\times\mathrm{id}_U] : S\times U+T\times U \to (S+T)\times U$ in $\mathbf{C}/Y$ is an isomorphism. Taking $S = S \overset{s^S_Y}{\to} Y$, $T = T \overset{s^T_Y}{\to} Y$ and $U = U \overset{s^U_Y}{\to} Y$, this morphism becomes the isomorphism

$$m = [i_1\times\mathrm{id}_U, i_2\times\mathrm{id}_U] : S\times_Y U+T\times_Y U \to (S+T)\times_Y U$$

in $\mathbf{C}$. Remember that both $i_1\times\mathrm{id}_U = \langle i_1\circ\pi_S, \mathrm{id}_U\circ\pi_U\rangle : S\times_Y U \to (S+T)\times_Y U$ and $i_2\times\mathrm{id}_U = \langle i_2\circ\pi_T, \mathrm{id}_U\circ\pi_U\rangle : T\times_Y U \to (S+T)\times_Y U$ come from the UMP of pullbacks in the product of objects in the slice category and not from the categorical product in $\mathbf{C}$. This morphism is a 2-isomorphism that makes the diagram

$$
\begin{tikzcd}
 & S \times_Y U + T \times_Y U \arrow[ddl, "{[s_X^S \circ \pi_S, s_X^T \circ \pi_T]}"'] \arrow[d, "m"] \arrow[ddr, "{[s_Z^U \circ \pi_U, s_Z^U \circ \pi_U]}"] & \\
 & (S+T) \times_Y U \arrow[dl, "{[s_X^S, s_X^T] \circ \pi_{S+T}}"] \arrow[dr, "s_Z^U \circ \pi_U"'] & \\
X & & Z
\end{tikzcd}
$$

commute, since

$$
\begin{aligned}
&[s_X^S, s_X^T] \circ \pi_{S+T} \circ m \\
=&[s_X^S, s_X^T] \circ \pi_{S+T} \circ [i_1 \times \mathrm{id}_U, i_2 \times \mathrm{id}_U] && \text{Def. of } m \\
=&[s_X^S, s_X^T] \circ [\pi_{S+T} \circ (i_1 \times \mathrm{id}_U), \pi_{S+T} \circ (i_2 \times \mathrm{id}_U)] \\
=&[s_X^S, s_X^T] \circ [\pi_{S+T} \circ \langle i_1 \circ \pi_S, \mathrm{id}_U \circ \pi_U \rangle, \pi_{S+T} \circ \langle i_2 \circ \pi_T, \mathrm{id}_U \circ \pi_U \rangle] \\
=&[s_X^S, s_X^T] \circ [i_1 \circ \pi_S, i_2 \circ \pi_T] && \text{Pullback projections} \\
=&[[s_X^S, s_X^T] \circ i_1 \circ \pi_S, [s_X^S, s_X^T] \circ i_2 \circ \pi_T] \\
=&[s_X^S \circ \pi_S, s_X^T \circ \pi_T]
\end{aligned}
$$

$$
\begin{aligned}
&s_Z^U \circ \pi_U \circ m \\
=&s_Z^U \circ \pi_U \circ [i_1 \times \mathrm{id}_U, i_2 \times \mathrm{id}_U] && \text{Def. of m} \\
=&s_Z^U \circ [\pi_U \circ \langle i_1 \circ \pi_S, \mathrm{id}_U \circ \pi_U \rangle, \pi_U \circ \langle i_2 \circ \pi_T, \mathrm{id}_U \circ \pi_U \rangle] \\
=&s_Z^U \circ [\mathrm{id}_U \circ \pi_U, \mathrm{id}_U \circ \pi_U] && \text{Pullback projections} \\
=&[s_Z^U \circ \pi_U, s_Z^U \circ \pi_U]
\end{aligned}
$$

As a result, the two spans are equivalent. Finally, the second equivalence $\mathbf{U}; (\mathbf{S}+\mathbf{T}) \equiv \mathbf{U}; \mathbf{S} + \mathbf{U}; \mathbf{T}$ can be again obtained as a dualised version of the first one using that the dagger functor preserves addition of spans (Proposition 6):

$$
\begin{aligned}
& \dagger\, ((\mathbf{S}+\mathbf{T}); \mathbf{U}) \equiv \dagger(\mathbf{S}; \mathbf{U} + \mathbf{T}; \mathbf{U}) \\
\iff & \dagger\, (\mathbf{U}); \dagger(\mathbf{S}+\mathbf{T}) \equiv \dagger(\mathbf{S}; \mathbf{U}) + \dagger(\mathbf{T}; \mathbf{U}) \\
\iff & \dagger\, (\mathbf{U}); (\dagger(\mathbf{S}) + \dagger(\mathbf{T})) \equiv \dagger(\mathbf{U}); \dagger(\mathbf{S}) + \dagger(\mathbf{U}); \dagger(\mathbf{T}).
\end{aligned}
$$

(**Units are compatible with composition**) The following must hold:

$$\mathbf{S}; \mathbf{0} \equiv \mathbf{0} \equiv \mathbf{0}; \mathbf{S}$$

Focusing first on the first equivalence, both the left hand side and the right hand side equate to:

$$\mathbf{S};\mathbf{0} = \begin{tikzcd} & S \times_Y \mathbf{0} \arrow[dl, "\pi_X^S \circ \pi_S"'] \arrow[dr, "i_Z \circ \pi_{\mathbf{0}}"] & \\ X & & Z \end{tikzcd} \qquad \mathbf{0} = \begin{tikzcd} & \mathbf{0} \arrow[dl, "i_X"'] \arrow[dr, "i_Z"] & \\ X & & Z \end{tikzcd},$$

where $S \xleftarrow{\pi_S} S \times_Y \mathbf{0} \xrightarrow{\pi_{\mathbf{0}}} \mathbf{0}$ is the pullback along the pair of morphisms $S \xrightarrow{s_Y} Y \xleftarrow{i_Y} \mathbf{0}$. Since $\mathbf{C}$ is locally distributive, the slice category $\mathbf{C}/Y$ is distributive. In addition, in a distributive category, the projection $\pi_{\mathbf{0}} : S \times \mathbf{0} \to \mathbf{0}$ is an isomorphism [8, Prop. 3.2]. Taking $S = S \overset{s_Y}{\to} Y$ and $\mathbf{0} = \mathbf{0} \overset{i_Y}{\to} Y$, the projection becomes the pullback projection

$$\pi_{\mathbf{0}} : S \times_Y \mathbf{0} \to \mathbf{0}$$

in $\mathbf{C}$. This means that $S \times_Y \mathbf{0}$ is an initial object in $\mathbf{C}$ and, therefore, the following diagram commutes:

$$\begin{tikzcd} & S \times_Y \mathbf{0} \arrow[ddl, "\pi_X \circ \pi_S"'] \arrow[d, "\pi_{\mathbf{0}}"] \arrow[ddr, "\pi_Z \circ \pi_{\mathbf{0}}"] & \\ & \mathbf{0} \arrow[dl, "i_X"] \arrow[dr, "i_Z"'] & \\ X & & Z \end{tikzcd}$$

Finally, the second equivalence $\mathbf{0};\mathbf{S} \equiv \mathbf{0}$ can be obtained as a dualised version of the first one by means of the dagger functor, using that it preserves units (Proposition 7):

$$\begin{aligned} & \dagger\,(\mathbf{S};\mathbf{0}) \equiv \dagger(\mathbf{0}) \\ \iff & \dagger\,(\mathbf{0});\dagger(\mathbf{S}) \equiv \dagger(\mathbf{0}) \\ \iff & \mathbf{0};\dagger(\mathbf{S}) \equiv \mathbf{0} \end{aligned}$$

$\square$

A major example of locally distributive categories with finite coproducts and pullbacks are *lextensive* categories [8, Cor. 4.9]. **Set** and, in general, any *topos* are instances of this type of categories [11, Sec. 3.1].

Finally, addition of morphism, and in this particular case union of tables, are depicted graphically using summations

$$X \;—\; \boxed{S+T} \;—\; Y \quad .$$

Moreover, since they interact well with composition, they can be taken out of the morphisms and decorate globally string diagrams. As a result, union of an arbitrary collection of tables can be represented by

$$\sum_{i\in I} \; X \;\text{—}\boxed{S_i}\text{—}\; Y \quad .$$

4.2.5 Schemas

Some set-theoretic operations such as set difference and set complement rely strongly on a set universe that is taken as reference. In order to model the difference of tables operation, it will be necessary to introduce a notion of *universes* for spans. As a first stage in the following sections, it will be conceptually easier to drop the metric and work with spans of sets instead; once the basic ideas are developed, introduction of metric spaces again will be briefly explored.

In this particular interpretation, it is of interest to model the concept of universe of a table, which is tightly connected with its schema. The schema of a table essentially declares the range of values that it is allowed to contain. For example, the Table 4.1 of *students* has schema S(Name, ID, Supervisor), so it can only contain tuples formed by a string, an integer and another string, i.e. $S \subseteq \Sigma^* \times \mathbb{Z} \times \Sigma^*$ and the universe of S is $\mathcal{U} = \textbf{Name} \times \textbf{ID} \times \textbf{Supervisor} = \Sigma^* \times \mathbb{Z} \times \Sigma^*$.

Given a table S, there is a map $i_S : S \to \mathcal{U}$ that maps each row with its corresponding row in the universe of rows. Notice that repeated rows are accounted by just simply sending several elements $t_1, \cdots, t_n \in S$ to the same row $u \in \mathcal{U}$ in the universe. When applied to the particular setting of spans, projections into columns of elements in S must agree with projections of the corresponding elements in $\mathcal{U}$, since they represent the same row, i.e.

$$\begin{tikzcd}
 & S \arrow[d, "i_S"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\
 & \mathcal{U} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\
X & & Y
\end{tikzcd}$$

This essentially means that i_S is a 2-morphism that links a span $\mathbf{S} : X \xleftarrow{s_X} S \xrightarrow{s_Y} Y$ with its universe span $\boldsymbol{\mathcal{U}} : X \xleftarrow{s_X} \mathcal{U} \xrightarrow{s_Y} Y$. Therefore, schemas are encoded by means of 2-morphisms $i_S : \mathbf{S} \Longrightarrow \boldsymbol{\mathcal{U}}$ in this framework.

In the example of the table for *students* with projections into *name* and *ID* attributes, the inclusion map into its schema constitutes a suitable 2-morphism:

$$
\begin{tikzcd}
 & S \arrow[d, hook, "i_S"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\
 & \mathbf{Name} \times \mathbf{ID} \times \mathbf{Supervisor} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\
\mathbf{Name} & & \mathbf{ID}.
\end{tikzcd}
$$

Furthermore, the two main operations in this framework, composition (natural join) and tensor product (cartesian product), interact well with universes. The natural join of two table-schema pairs $\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ and $\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}}$ is given by the unique 2-morphism $\mathbf{S};\mathbf{U} \overset{i_S \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}};\boldsymbol{\mathcal{W}}$ that results from the UMP of pullbacks. More concretely,

$$
\left(
\begin{tikzcd}
 & S \arrow[d, "i_S"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\
 & \mathcal{U} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\
X & & Y
\end{tikzcd}
\right)
;
\left(
\begin{tikzcd}
 & U \arrow[d, "i_U"] \arrow[ddl, "s_Y"'] \arrow[ddr, "s_Z"] & \\
 & \mathcal{W} \arrow[dl, "s_Y"] \arrow[dr, "s_Z"'] & \\
Y & & Z
\end{tikzcd}
\right)
=
\begin{tikzcd}
 & S \times_Y U \arrow[d, "i_S \times i_U"] \arrow[ddl, "s_X \circ \pi_S"'] \arrow[ddr, "s_Z \circ \pi_U"] & \\
 & \mathcal{U} \times_Y \mathcal{W} \arrow[dl, "s_X \circ \pi_{\mathcal{U}}"] \arrow[dr, "s_Z \circ \pi_{\mathcal{W}}"'] & \\
X & & Z
\end{tikzcd}
,
$$

where $i_S \times i_U$ is the unique arrow that transforms the pullback $S \times_Y U$ into the pullback $\mathcal{U} \times_Y \mathcal{W}$, according to its UMP:

$$
\begin{tikzcd}
 & S \times_Y U \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_U"] \arrow[dd, dotted, "i_S \times i_U"] & \\
S \arrow[dd, "i_S"'] & & U \arrow[dd, "i_U"] \\
 & \mathcal{U} \times_Y \mathcal{W} \arrow[dl, "\pi_{\mathcal{U}}"'] \arrow[dr, "\pi_{\mathcal{W}}"] & \\
\mathcal{U} \arrow[dr, "s_Y"'] & & \mathcal{W} \arrow[dl, "s_Y"] \\
 & Y &
\end{tikzcd}
$$

Notice that this is not the cartesian product of arrows, since codomain and domain are not products but pullbacks; however, it can be shown that it is an equalised version of the actual cartesian product $i_S \times i_U$ and hence the notation. In **Set**, this arrow is given by $i_S \times i_U : S \times_Y U \to \mathcal{U} \times_Y \mathcal{W} : (s, u) \mapsto (i_S(s), i_U(u))$, which, under the interpretation of tables, sends each combination of rows that agree on shared columns to the corresponding combination in the universe.

On the other hand, the tensor product of two table-schema pairs $\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ and $\mathbf{S}' \overset{i_{S'}}{\Longrightarrow} \boldsymbol{\mathcal{U}}'$ is given by the unique 2-morphism $\mathbf{S} \otimes \mathbf{U} \overset{i_S \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}} \otimes \boldsymbol{\mathcal{W}}$ defined by the cartesian product of arrows. Specifically,

$$\left(\begin{tikzcd} & S \arrow[dl, "s_X"'] \arrow[d, "i_S"] \arrow[dr, "s_Y"] & \\ X & \mathcal{U} \arrow[l, "s_X"] \arrow[r, "s_Y"'] & Y \end{tikzcd}\right) \otimes \left(\begin{tikzcd} & S' \arrow[dl, "s_{X'}"'] \arrow[d, "i_{S'}"] \arrow[dr, "s_{Y'}"] & \\ X' & \mathcal{U}' \arrow[l, "s_{X'}"] \arrow[r, "s_{Y'}"'] & Y' \end{tikzcd}\right) = \begin{tikzcd} & S \times S' \arrow[dl, "s_X \times s_{X'}"'] \arrow[d, "i_S \times i_U"] \arrow[dr, "s_Y \times s_{Y'}"] & \\ X \times X' & \mathcal{U} \times \mathcal{U}' \arrow[l, "s_X \times s_{X'}"] \arrow[r, "s_Y \times s_{Y'}"'] & Y \times Y' \end{tikzcd}$$

In **Set**, this map is given by $i_S \times i_{S'} : S \times S' \to \mathcal{U} \times \mathcal{U}' : (s, s') \mapsto (i_S(s), i_{S'}(s'))$, which, under the interpretation of tables, takes each combinations of rows to its corresponding combination in the universe.

4.2.6 Difference of tables

Once the concept of universe span has been delved into, set theoretic operations, such as difference of tables, can be modelled naturally. However, before introducing the notion of difference of spans, some previous lemmas need to be stated.

Lemma 8. *Let $S \xrightarrow{m} U$, $S' \xrightarrow{n} U$ be maps that satisfy*

$$\forall u \in U \quad m^{-1}(u) \cong n^{-1}(u),$$

then $S \cong S'$ and there exists an isomorphism $i : S \xrightarrow{\cong} S'$ such that $m = n \circ i$.

Proof. Given the assumption, there exists a collection of a isomorphisms $f_u : m^{-1}(u) \xrightarrow{\cong} n^{-1}(u)$. In addition, recall that every map creates a partition of the domain in disjoint sets by simply taking the inverse images of the elements in the codomain; so, in particular, $S = \bigsqcup_{u \in U} m^{-1}(u)$ and $S' = \bigsqcup_{u \in U} n^{-1}(u)$. Moreover, the isomorphisms f_u can be extended to a higher isomorphism $\sum_{u \in \mathcal{U}} f_u : \bigsqcup_{u \in U} m^{-1}(u) \xrightarrow{\sum_{u \in U} f_u} \bigsqcup_{u \in U} n^{-1}(u)$ taking coproducts, which by construction satisfies

$$\begin{tikzcd} S \arrow[r, "\sum_{u \in \mathcal{U}} f_u"] \arrow[d, "m"] & S' \arrow[dl, "n"] \\ U & \end{tikzcd},$$

i.e. $m = n \circ \sum_{u \in \mathcal{U}} f_u$. □

Intuitively, if two collections of rows S, S' with the same universe U have the same collections of instances of a given row, $\forall u \in U \quad m^{-1}(u) \cong n^{-1}(u)$, then they should be isomorphic and that isomorphism should preserve each particular row $m = n \circ i$. The next lemma generalises this idea to table-schema pairs and shows that projections are indeed preserved too.

Lemma 9. *Let* $\mathbf{S} \overset{m}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ *and* $\mathbf{S}' \overset{n}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ *be 2-morphisms that satisfy*

$$\forall u \in \mathcal{U} \quad m^{-1}(u) \cong n^{-1}(u),$$

then $\mathbf{S} \cong \mathbf{S}'$.

Proof. Expanding the 2-morphisms, the following diagrams commute:

$$\begin{tikzcd} & S \arrow[d, "m"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\ & \mathcal{U} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & S' \arrow[d, "n"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\ & \mathcal{U} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\ X & & Y \end{tikzcd}.$$

On the other hand, given the assumption from the statement and applying Lemma 8, there exists an isomorphism $i : S \xrightarrow{\cong} S'$ such that $m = n \circ i$. This morphism is moreover a 2-isomorphism that makes the diagram

$$\begin{tikzcd} & S \arrow[d, "i"] \arrow[ddl, "s_X"'] \arrow[ddr, "s_Y"] & \\ & S' \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\ X & & Y \end{tikzcd}$$

commute, since the left triangle reduces to

$$\begin{aligned} s_X \circ i &= s_X \circ n \circ i && \text{From } \mathbf{S}' \overset{n}{\Longrightarrow} \boldsymbol{\mathcal{U}} \\ &= s_X \circ m && \text{Lemma 8} \\ &= s_X && \text{From } \mathbf{S} \overset{m}{\Longrightarrow} \boldsymbol{\mathcal{U}}, \end{aligned}$$

and the right triangle is proven analogously. As a result, the two spans are isomorphic $\mathbf{S} \cong \mathbf{S}'$. □

Difference of spans can now be defined as follows:

Definition 28 (Difference)**.** *Let* $\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}$, $\mathbf{T} \overset{i_T}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ *be 2-morphisms, a difference is a 2-morphism*

$$\mathbf{D} \overset{i_D}{\Longrightarrow} \boldsymbol{\mathcal{U}} \underset{def}{\equiv} \begin{tikzcd} & D \arrow[d, "i_D"] \arrow[ddl, "s_X \circ i_D"'] \arrow[ddr, "s_Y \circ i_D"] & \\ & \mathcal{U} \arrow[dl, "s_X"] \arrow[dr, "s_Y"'] & \\ X & & Y \end{tikzcd}$$

such that $i_D : D \to \mathcal{U}$ *satisfies*

$$\forall u \in \mathcal{U} \quad card(i_D^{-1}(u)) = \max(card(i_S^{-1}(u)) - card(i_T^{-1}(u)), 0).$$

The above condition, inspired on [3], basically states that the number of instances of a row u in the difference D is equal to the number of instances in S minus the number of instances in T, or 0 if there are more in T. Notice that there are several choices for the difference, however, it is unique up to isomorphism.

Proposition 8. *Difference is defined uniquely up to isomorphism.*

Proof. Given 2-morphisms $\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ and $\mathbf{T} \overset{i_T}{\Longrightarrow} \boldsymbol{\mathcal{U}}$, let $\mathbf{D} \overset{i_D}{\Longrightarrow} \boldsymbol{\mathcal{U}}$, $\mathbf{D}' \overset{i_{D'}}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ be two distinct choices of differences. In particular, they both satisfy

$$\begin{aligned} \forall u \in \mathcal{U} \quad card(i_D^{-1}(u)) &= \max(card(i_S^{-1}(u)) - card(i_T^{-1}(u)), 0) \\ \forall u \in \mathcal{U} \quad card(i_{D'}^{-1}(u)) &= \max(card(i_S^{-1}(u)) - card(i_T^{-1}(u)), 0), \end{aligned}$$

and, therefore,

$$\forall u \in \mathcal{U} \quad card(i_D^{-1}(u)) = card(i_{D'}^{-1}(u)).$$

Since the sets have the same cardinality, they are isomorphic, i.e. $\forall u \in \mathcal{U} \quad i_D^{-1}(u) \cong i_{D'}^{-1}(u)$, and Lemma 9 gives $\mathbf{D} \cong \mathbf{D}'$. $\square$

As a result, a choice of 2-morphism together with a representative of the difference span class will be assumed and denoted as

$$\mathbf{S} \setminus \mathbf{T} \overset{\delta(i_S, i_T)}{\Longrightarrow} \boldsymbol{\mathcal{U}},$$

where $\delta(i_S, i_T)$ satisfies

$$\forall u \in \mathcal{U} : card(\delta(i_S, i_T)^{-1}(u)) = \max(card(i_S^{-1}(u)) - card(i_T^{-1}(u)), 0).$$

In algebra of bags, difference of tables is compatible with natural join and so is span difference with span composition.

Lemma 10. *The map* $i_S \times i_V : S \times_Y V \to \mathcal{U} \times_Y \mathcal{W} : (s, v) \mapsto (i_S(s), i_V(v))$ *satisfies*

$$(i_S \times i_V)^{-1}(u, w) = i_S^{-1}(u) \times i_V^{-1}(w).$$

Proof.

$$\begin{aligned}
(s, v) \in (i_S \times i_V)^{-1}(u, w) &\iff (i_S \times i_V)(s, v) = (u, w) && \text{Def. of inverse image} \\
&\iff (i_S(s), i_V(v)) = (u, w) && \text{Def. of } i_S \times i_V \\
&\iff i_S(s) = u \wedge i_V(v) = w && \\
&\iff s \in i_S^{-1}(u) \wedge v \in i_V^{-1}(u) && \text{Def. of inverse image} \\
&\iff (s, v) \in i_S^{-1}(u) \times i_V^{-1}(u) && \text{Def. of } \times
\end{aligned}$$

□

Theorem 4 (Span difference is compatible with composition)**.** *Given 2-morphisms* $\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}$, $\mathbf{T} \overset{i_T}{\Longrightarrow} \boldsymbol{\mathcal{U}}$ *and* $\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}}$, *then the following holds:*

$$\begin{aligned}
(\mathbf{S} \setminus \mathbf{T}); \mathbf{U} &\equiv \mathbf{S}; \mathbf{U} \setminus \mathbf{T}; \mathbf{U} \\
\mathbf{U}; (\mathbf{S} \setminus \mathbf{T}) &\equiv \mathbf{U}; \mathbf{S} \setminus \mathbf{U}; \mathbf{T}
\end{aligned}$$

Proof. The right-composition equation will be proven first. Using operations from the algebra of universes, the left hand side reduces to

$$\begin{aligned}
&((\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}) \setminus (\mathbf{T} \overset{i_T}{\Longrightarrow} \boldsymbol{\mathcal{U}})); (\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}}) \\
=&(\mathbf{S} \setminus \mathbf{T} \overset{\delta(i_S, i_T)}{\Longrightarrow} \boldsymbol{\mathcal{U}}); (\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}}) \\
=&(\mathbf{S} \setminus \mathbf{T}); \mathbf{U} \overset{\delta(i_S, i_T) \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}},
\end{aligned}$$

which corresponds to the following map between spans:

$$\begin{tikzcd}
 & (S \setminus T) \times_Y U \arrow[d, "\delta(i_S, i_T) \times i_U"] \arrow[ddl, "s_X \circ \delta(i_S, i_T) \circ \pi_{S \setminus T}"'] \arrow[ddr, "s_Z \circ \pi_U"] & \\
 & \mathcal{U} \times_Y \mathcal{W} \arrow[dl, "s_X \circ \pi_{\mathcal{U}}"] \arrow[dr, "s_Z \circ \pi_{\mathcal{W}}"'] & \\
X & & Z
\end{tikzcd}.$$

On the other hand, the final form for the right hand side is

$$\begin{aligned}
&((\mathbf{S} \overset{i_S}{\Longrightarrow} \boldsymbol{\mathcal{U}}); (\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}})) \setminus ((\mathbf{T} \overset{i_T}{\Longrightarrow} \boldsymbol{\mathcal{U}}); (\mathbf{U} \overset{i_U}{\Longrightarrow} \boldsymbol{\mathcal{W}})) \\
=&(\mathbf{S}; \mathbf{U} \overset{i_S \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}}) \setminus (\mathbf{T}; \mathbf{U} \overset{i_T \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}}) \\
=&\mathbf{S}; \mathbf{U} \setminus \mathbf{T}; \mathbf{U} \overset{\delta(i_S \times i_U, i_T \times i_U)}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}},
\end{aligned}$$

which corresponds to the following map between spans:

$$\begin{tikzcd}
& (S \times_Y U) \setminus (T \times_Y U) \arrow[d, "\delta(i_S \times i_U, i_T \times i_U)"] \arrow[ddl, "s_X \circ \pi_{\mathcal{U}} \circ \delta(i_S \times i_U, i_T \times i_U)"'] \arrow[ddr, "s_Z \circ \pi_{\mathcal{W}} \circ \delta(i_S \times i_U, i_T \times i_U)"] & \\
& \mathcal{U} \times_Y \mathcal{W} \arrow[dl, "s_X \circ \pi_{\mathcal{U}}"] \arrow[dr, "s_Z \circ \pi_{\mathcal{W}}"'] & \\
X & & Z
\end{tikzcd}.$$

Moreover, $(\mathbf{S} \setminus \mathbf{T}); \mathbf{U} \overset{\delta(i_S, i_T) \times i_U}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}}$ satisfies

$$\begin{aligned}
&\forall (u, w) \in \mathcal{U} \times_Y \mathcal{W} \quad card((\delta(i_S, i_T) \times i_U)^{-1}(u, w)) && \\
=& card((\delta(i_S, i_T)^{-1}(u) \times i_U^{-1}(w)) && \text{Lemma 10} \\
=& card((\delta(i_S, i_T)^{-1}(u)) card(i_U^{-1}(w)) && \text{Cardinal of the product} \\
=& \max(card(i_S^{-1}(u)) - card(i_T^{-1}(u)), 0) card(i_U^{-1}(w)) && \text{Def. of difference} \\
=& \max(card(i_S^{-1}(u)) card(i_U^{-1}(w)) - card(i_T^{-1}(u)) card(i_U^{-1}(w)), 0) && \text{Distributivity of cardinals}
\end{aligned}$$

and $\mathbf{S}; \mathbf{U} \setminus \mathbf{T}; \mathbf{U} \overset{\delta(i_S \times i_U, i_T \times i_U)}{\Longrightarrow} \boldsymbol{\mathcal{U}}; \boldsymbol{\mathcal{W}}$ meets

$$\begin{aligned}
&\forall (u, w) \in \mathcal{U} \times_Y \mathcal{W} \quad card(\delta(i_S \times i_U, i_T \times i_U)^{-1}(u, w)) && \\
=& \max(card((i_S \times i_U)^{-1}(u, w)) - card((i_T \times i_U)^{-1}(u, w)), 0) && \text{Def. of difference} \\
=& \max(card(i_S^{-1}(u) \times i_U^{-1}(w)) - card(i_T^{-1}(u) \times i_U^{-1}(w)), 0) && \text{Lemma 10} \\
=& \max(card(i_S^{-1}(u)) card(i_U^{-1}(w)) - card(i_T^{-1}(u)) card(i_U^{-1}(w)), 0) && \text{Cardinal of the product}
\end{aligned}$$

As a result,

$$\forall (u, w) \in \mathcal{U} \times_Y \mathcal{W} \quad card((\delta(i_S, i_T) \times i_U)^{-1}(u, w)) = card(\delta(i_S \times i_U, i_T \times i_U)^{-1}(u, w)),$$

so $\forall (u, w) \in \mathcal{U} \times_Y \mathcal{W} \quad (\delta(i_S, i_T) \times i_U)^{-1}(u, w) \cong \delta(i_S \times i_U, i_T \times i_U)^{-1}(u, w)$ and, applying Lemma 9, it follows that $(\mathbf{S} \setminus \mathbf{T}); \mathbf{U} \cong \mathbf{S}; \mathbf{U} \setminus \mathbf{T}; \mathbf{U}$, i.e they belong to the same class of isomorphic spans:

$$(\mathbf{S} \setminus \mathbf{T}); \mathbf{U} \equiv \mathbf{S}; \mathbf{U} \setminus \mathbf{T}; \mathbf{U}.$$

Finally, the left-composition equation is proved similarly. Using algebra of universes, the two sides reduce to $\mathbf{U}; (\mathbf{S} \setminus \mathbf{T}) \overset{i_U \times \delta(i_S, i_T)}{\Longrightarrow} \boldsymbol{\mathcal{W}}; \boldsymbol{\mathcal{U}}$ and $\mathbf{U}; \mathbf{S} \setminus \mathbf{U}; \mathbf{T} \overset{\delta(i_U \times i_S, i_U \times i_T)}{\Longrightarrow} \boldsymbol{\mathcal{W}}; \boldsymbol{\mathcal{U}}$ respectively. Since cardinals distribute over both sides, an analogous proof to the one before shows that $\mathbf{U}; (\mathbf{S} \setminus \mathbf{T}) \cong \mathbf{U}; \mathbf{S} \setminus \mathbf{U}; \mathbf{T}$ and, therefore, they belong to the same class of isomorphic spans:

$$\mathbf{U}; (\mathbf{S} \setminus \mathbf{T}) \equiv \mathbf{U}; \mathbf{S} \setminus \mathbf{U}; \mathbf{T}.$$

$\square$

Once the key ideas for span universes have been grasped, metrics can be introduced again. In this case, it is necessary that the map $i_S : S \longrightarrow \mathcal{U}$, assigning each instance of row with the corresponding row in the universe, also reflect the metric information. For that reason, i_S must be an *isometric embedding*, i.e. a map that preserves distances

$$\forall s, s' \in S \quad d_S(s, s') = d_{\mathcal{U}}(i_S(s), i_S(s')),$$

so any two instances s, s' of rows u, u' inherit the metric from the universe $d_S(s, s') = d_{\mathcal{U}}(u, u')$. Consequently, table-schema pairs $i_S : \mathbf{S} \Rightarrow \mathcal{U}$ are isometric embeddings, which creates two different scenarios. In **Met** the T_0 condition eliminates duplicates; two instances s,s' of the same row u must be the same element

$$d_S(s, s') = d_{\mathcal{U}}(i_S(s), i_S(s')) = d_{\mathcal{U}}(u, u) = 0 \underset{T_0}{\Longrightarrow} s = s',$$

so it describes relational algebra, where row repetitions are not allowed. On the contrary, **HMet** has no such requirement and, therefore, row repetitions are allowed, making it into a model for algebra of bags. Lifting the previous results to the framework of metric spaces is, however, future work.

Chapter 5

Conclusion

5.1 Summary

This work analysed different intuitions and applications for spans of sets and metric spaces. First, John Baez's interpretation for path composition and matrix mechanics was explored. The informal language of paths introduced proved intuitive enough to show how the range of structures for span of sets interact internally.

Afterwards, an interpretation for conceptual spaces was presented: a model for hyper-conceptual spaces. Under this interpretation, spans of metric spaces capture concept - sub-concept relations and extension of separable domains to higher conceptual spaces. They were shown to model successfully cognitive phenomena such as concept correlation and categorical perception. Furthermore, they integrate into the compositional distributional scheme as a semantic category where the morphisms are concepts carrying information about the underlying conceptual space structure, which proves useful to model interaction between linguistic units in the sentence by means of correlation.

Finally, an interpretation for algebra of bags was introduced. Several major operations were given categorical counterparts and integrated into string diagrams taking advantage of the compact closedness in the span construction. In addition, this perspective, and in particular union of tables, led to an enrichment over commutative monoids for spans of locally distributive categories, with potential applications in a wide range of settings. Also, in the case of the category of **Set**, it was shown that difference of tables interacts well with composition, showing consistency with the framework.

5.2 Future work

The following items arise as future work:

1. It is a question whether the graphical path language for the path interpretation could be equipped with more formality.

2. Spans of metric spaces contribute to the compositional distributional scheme with a perspective that focuses on concept correlation. Other constructions over metric spaces may provide alternative perspectives as semantic categories and, therefore, it is worth researching them.

3. Intransitive sentences were semantically modelled using an approach that strongly emphasises interactions between concepts, modelling them using correlation. It would be interesting to study how this approach works for other situations, not necessarily intransitive sentences, in the compositional distributional model for cognition.

4. Explore other applications within Gärdenfors's theory of conceptual spaces that the interpretation presented in this dissertation may explain.

5. Lift the results for spans of sets regarding schemas and difference of tables to the setting of metric spaces.

6. The results for difference of spans rely on set-theoretic notions such as inverse images and cardinalities. It would be interesting to describe it in more categorical terms. For instance, inverse images can be described categorically by means of appropriate pullbacks.

7. Explore the role of Frobenius structures in the interpretation for algebra of bags, as they are likely to model flows of relational information with operations such as duplicating, combining and discarding.

8. Examine the value that this categorical framework and, in particular, string diagrams add to computations in algebra of bags.

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