Completeness of the ZW and ZX calculi

Kang Feng Ng

Linacre College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2018

Acknowledgements

Firstly, I would like to thank Bob Coecke, my supervisor, for adopting me into his large family of students. His multiple efforts in advocating the use of diagrammatic calculus is what brought me to this field. Bob is very generous in letting me explore the field freely, and providing guidance and direction when I'm lost. He even found a job for me after this DPhil, which I am eternally grateful for. Outside of academics, he is also very nice to organise events like Christmas party and many other events. I am very fortunate to have a great supervisor like him.

Another important person I would like to thank is Quanlong Wang. He brought to my attention about the ZX calculus, which then quickly developed to the content of this thesis. You could say that, without him, this thesis is not possible. Under his tutelage, I really learnt a lot about this diagrammatic approach. Outside of work, his family is also very welcoming. I would like to also thank his wife and daughters for making me feel like home in a foreign land. I really appreciate their hospitality, and even showing care and support when I'm not feeling well.

I would also like to thank my friends and colleagues; Nori for being a night owl together with me (and teaching me quite a bit of the Japanese culture, and category theory too); Maaike, Hector and Joe for being such a good officemates and putting up with my messiness; Amar for his patience in teaching me the ZW calculus in simple words, and also writing and editing the couple of papers we have together; Giovanni for his hard work in proving the completeness of the fermionic ZW calculus, and the discussion we had for the completeness of the even fragment; Robin for the discussion of his work on the diagrammatic axiomatisation of resource calculus. There are many more people to thank, like the people I have met during numerous conferences and meetings — Ross Duncan, Aleks Kissinger, Simon Perdrix, Dominic Horsman, Renaud Vilmart, and many more — who taught me so much during our short encounters, and also my personal friends (who I shall not name) who supported me through this journey.

Thanks to the administrative staff of the college and department for their help whenever needed, be it for applying for conference funding or transfer and confirmation processes.

Last but not least, big thanks to my family and friends back home! We are always in touch, always supportive, and always updating me of the current affairs. I never felt disconnected whenever I go back. I love you people.

The pictures in this thesis are drawn with the aid of the software TikZit [1].

Contents

Chapter 1

Introduction

Abstraction is an important part of our lives. We use abstraction all the time; we learn to recognise ideas and concepts rather than the actual events. For example, when playing a jigsaw puzzle, we often can extrapolate the missing picture and tell whether a piece is correctly placed rather than trying out all possible ways.

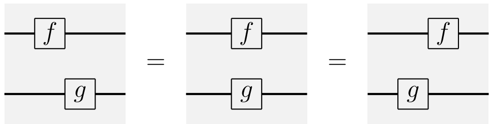
Computer scientists are the master of abstraction. There are many layers of abstractions from the hardware all the way to modern programming languages. Each layer hides the unnecessary details of the previous layer, and expresses the important aspect in a fashion that is easier for us to understand; that is, we don't have to worry about the stuff going on in the previous layers and code in confidence using high level languages, only focusing on the important parts. This is absolutely needed for any progress in the field; as computer programmes get increasingly complex, there is a need for more and more abstract layers to reason about the complicated processes in the programme. It will be unimaginable to think about writing even a simple phone app by manipulating the current and voltage in each part of the circuit.

In the realm of quantum computing, this is not the case; abstraction is still at a pretty low level. One layer above the hardware is the mathematics governing all quantum systems, the mathematics of Hilbert space written by von Neumann in the 1930s, and we are very much still "programming" with it. This is somewhat like an assembly code, and understanding these quantum operations can be horrendous. Take for example, swapping two quantum bits (qubits) is the calculation

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

However, conceptually we know that swapping twice is equals to doing nothing. Perhaps, this is the reason why it took decades to design the first quantum algorithm.

There is a level above this, the quantum circuits, which uses diagrams that have a strong connection to the implementable operations. At this level, things become more intuitive. For example, two operations placed side-by-side is understood that it doesn't matter which one is done first:



,

where the diagrams are read left to right. Perhaps, with diagrammatic abstractions, our minds are able to grapple with the quantum processes better, using similar part of our brains to reason jigsaw puzzles.

The quantum circuit formalism is good as it abstracts quantum processes while keeping in mind what are the allowed processes in laboratories. However, because of this, it is kind of restricting. It will be desirable to have a higher level language which provides further abstraction. This is where ZX and ZW calculus comes in. These two calculi are diagrammatic abstraction of quantum operations, similar to quantum circuits, but is more liberal than the circuit formalism.

Abstraction may be good, but it is, after all, not the actual thing; there are limitations to how much the abstraction can reason about the systems. This is the concept of completeness. If an abstraction can capture all of the system's dynamics, then we say that it is complete. The circuit formalism and the ZX calculus are known to be incomplete, and it greatly limits their ability to reason quantum operations. This thesis is about completeness, but we won't explore the completeness of the quantum circuit formalism; we will present a ZX calculus that is complete.

The ZW calculus is complete, but it is hard to make connections to the implementable quantum operations. We could modify the calculus to obtain a calculus that is closely related to what is called fermionic circuits, but then this calculus is not known to be complete. In this thesis, we will complete it. As a bonus, we will show an even more restricted version, that may have connections to the understanding of the universe, which is complete. This may be an useful abstraction for the study of the fundamental properties of fermions.

The gist of the thesis is not about the complete ZW and ZX calculi; it is the formula to produce a complete diagrammatic calculus. The thesis is organised as follows.

Chapter 2 lays down the foundation for the thesis. It explains the groundwork needed while avoiding over elaborating its mathematics. First, we will covered the intuition of *string diagrams*, explaining with examples. Concepts such as *soundness*, *universality*, and *completeness* are loosely explained, which is mostly good enough for understanding the remaining of the thesis. Then, a simple mathematical definition, using category theory, is given to justify string diagrams and all the concepts introduced. We may use the categorical jargon for the rest of the thesis, but the intuition from previous section should suffice the reader to have a good idea what is going on. We will next provide a quick recap to *finite-dimensional quantum theory*, stripping down to only the required knowledge for the thesis. Finally, we will introduce the common elements to a family of diagrammatic calculus, called the *Z* calculus* which includes ZX, ZW and ZH calculi. The common elements is essentially saying that the Z* calculus are string diagrams.

Chapter 3 will be on the ZW calculus. We will introduce the calculus by first stating the motivation behind inventing the calculus, namely to aid in solving the *entanglement problem*. Then we list the algebraic properties that the ZW calculus exhibits. These properties together gives an complete axiomatisation for the qubit theory, as proved by Hadzihasanovic in his Ph.D. thesis [31] and paper [33]. We will explain the idea of the proof, which is via a *normal form* argument, and give a recipe which can be used for other diagrammatic calculi. We will show how to make slight modifications to the calculus to suit different scenarios, in particular, the one for the *Clifford+T fragment* of qubit theory.

Hadzihasanovic also noted in his Ph.D. thesis that a certain fragment of the ZW calculus, called the *pure fragment*, is suited to model a particular type of quantum computing, called *fermionic quantum computing*. This fragment was not complete. We will complete this fragment using the recipe given previously; this part is based on our paper [32] which is a joint work with Hadzihasanovic and de Felice.

We will restrict the calculus further to the *even fragment*, and using the same recipe, complete the calculus. The even fragment has a couple of interesting properties: it is a calculus with two symmetric braiding, which on its own is interesting as it is closely related to virtual knot theory; second, it may be interpreted as a theory of the fermionic universe where particles are simulated by the curvature of space-time. This work is not published anywhere at the time of writing this thesis.

Chapter 4 is about the ZX calculus. The first part will introduce the calculus' properties and its connection to quantum circuits. Then, we will quickly summarise the ideas used to complete the Clifford+T fragment of the ZX calculus by Jeandel, Perdrix and Vilmart [36], which is to translate the ZX calculus to ZW, and back. Improving on their work, we came up with an algorithm to perform the translation.

The result is completing various fragments of the ZX calculus, more notably the general qubits quantum computing which is a joint work with Wang in our paper [33]. In fact, we will show that any fragments containing the Clifford+T fragment is automatically complete, a work unannounced yet.

Chapter 5 states several future work. They are mainly unfinished work, and many of them are low hanging fruits. We will introduce the ideas of having a ZX calculus that could perform both quantum and classical computations. This may have great applications in the hybrid system of quantum and classical computers. We also state a generalisation of the ZX and ZW calculus to qudit systems. The completeness is in principle doable according to our recipes, but the challenge is to identify the axioms of the calculus that categorise the calculi. Finally, we gave another direction of generalising the ZW calculus, which is to have matrix elements over a semiring. The completeness should not be hard, but requires some effort to check.

Finally, we end off with some concluding words in the last chapter.

For convenience of the reader, the appendices contain a summary of all the axioms for the different versions of the ZW and ZX calculi.

Chapter 2

String diagrams

Overview:

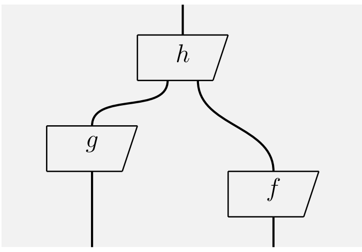
- Section 2.1 — We informally introduce the concept of graphical reasoning with strings diagrams.
- Section 2.2 — We then formalise the concepts using category theory, showing how the axioms are expressed in diagrammatic language. The concepts include: strict symmetric monoidal category, compact close structure, dagger. We also introduce the definition of soundness, universality, and completeness.
- Section 2.3 — This section starts with a brief history of the Z* calculi. Then, we give a quick recap of the finite dimensional quantum theory which is just enough to understand this thesis. Next, we will summarises the few algebraic structures we may encounter in the next couple of chapters, mainly to be familiar with the jargon. Finally, we give a definition of the common ground of the Z* calculi, which is the category of self-dual, compact close PROP.

2.1 Working knowledge of string diagrams

Process theory is a kind of diagrammatic representation which is used to study processes abstractly. The philosophy of such theories is to make analysing and designing processes as intuitive as possible. In this section, we will look at some examples of process theories to get a feeling of what it is.

2.1.1 Process theory

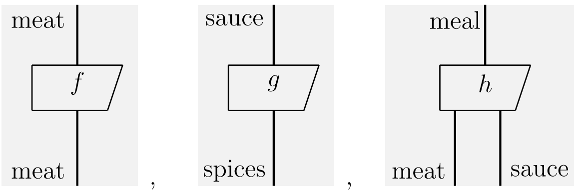
This is an example of a diagram in some process theory:



.

f, g, h are abstract processes, drawn as boxes with a bulge on the right-hand corner, and they are wired together as in the picture. The diagram should be read from bottom to top, and stuff would enter from the bottom wires and processed stuff would exit from the top wire. These processes can be mathematical processes like functions and relations, physical processes like laboratory equipments, or even cooking processes like described in the book [17]. The processes can only take in certain types of information, and they should be wired in a sensible manner.

Taking a cooking example, f could be chopping some meat, g could be preparing your favourite sauce, and h could be cooking in the oven. The types on these processes are

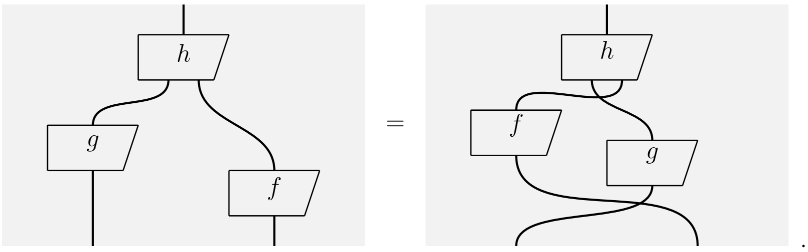


And wiring them in a sensible manner means that the types should match when connecting the wires, that is we shouldn't connect f and g wires together. Usually, we will keep the types implicit and not show them on the digram.

The above example should give us some intuition on what a process theory is. Our example tells us a general way of preparing a meal, ignoring and the choice of meat, spices, and preparation. In summary, a process theory consist of boxes representing abstract processes, and wires connecting these processes in a reasonable manner. The focus of process theories is how the (simple) processes are wired together, giving rise to new (complex) processes.

2.1.2 Twisting wires and moving boxes

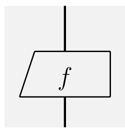
A nice feature of process theory is that we are able to move freely the processes up and down along a wire, and even twist the wires. For example,



We have twisted the wires and slide the processes f and g along them. It should be obvious to the eye that these two process diagrams are the same by untangling the boxes. Operationally, taking the cooking example again, it is saying that it doesn't matter if we chop the chicken or prepare the sauce first or do them simultaneously, and it also doesn't matter if we prepare the chicken on the left or right to the sauce, as long as they are fed into the oven in the correct order.

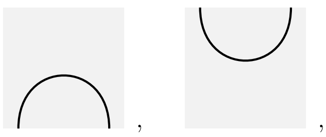
2.1.3 Bending wires and rotating boxes

Not all deformations of the diagrams are allowed. For example, if we rotate the chopping meat process,



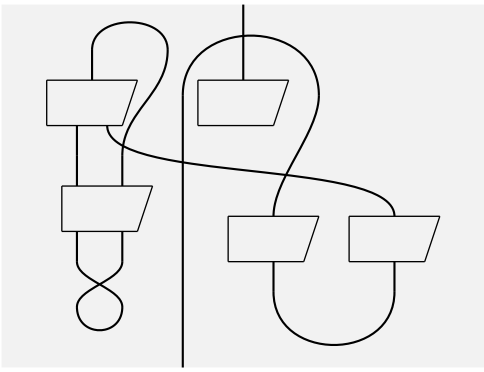
,

we will get a process which 'unchops' the meat, which is absurd! We also cannot bend an input to an output and vice versa, as the bending operations

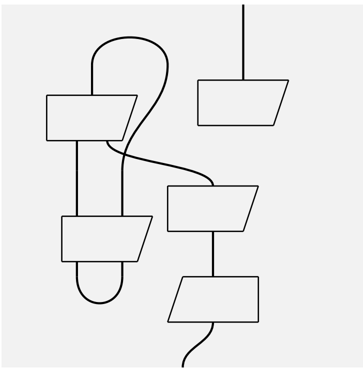


simply don't have a physical meaning in the theory of cooking processes. However, the processes we are interested in this thesis – quantum processes – can allow such moves.

A process theory where it is legal to rotate the boxes and move the inputs to outputs and vice versa is called a *string diagram*. An example would be as such:

.

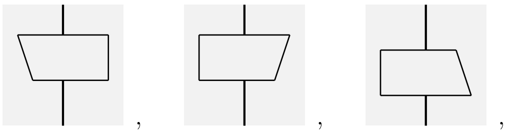
We can untangle the wires to give a slightly neater diagram:

.

The punchline of a string diagram is that we are allowed to deform the wires, rotate the boxes, move them freely on the surface of a plane, as long as the input and output wires are in the proper order. This is the background of all the diagrams in the subsequent parts of the thesis.

2.1.4 Reflecting boxes

There is one more diagrammatic transformation which is to mirror image the boxes:

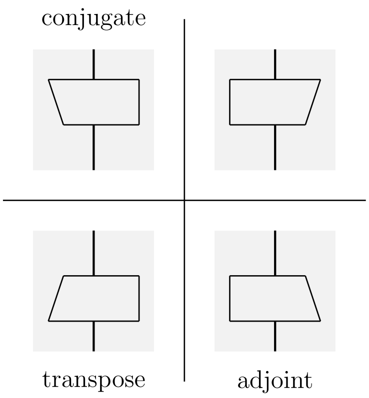


where the middle is the original box, the left is reflecting horizontally and is called the *conjugate*, and the right is reflecting vertically and is called the *adjoint*. The conjugate and adjoint is related by a rotation.

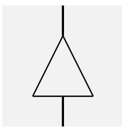
A string diagram which we can do a mirror image is said to be equipped with a *dagger*. This move is not the focus of the thesis, but we briefly mention it as it is a nice thing to have in mind.

2.1.5 Terminology and shapes of boxes

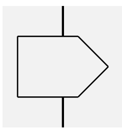
We will state some terminology of the diagrammatic moves we have introduced so far. This is summarised in the picture below:



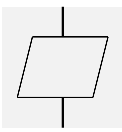
Since the aim of a pictorial theory is to make them easy to grasp, the boxes should be given meaningful shapes to show certain symmetries of the processes. For example, a process with no symmetries should be given an irregular shape like the one we have seen so far, a process that is invariant under conjugation may be shaped like

 ,

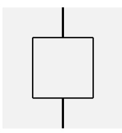
a process that is invariant under adjunction may be shaped like

 ,

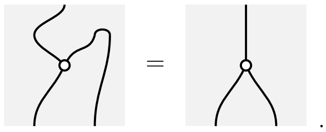
a process that is invariant under transposition may be shaped like

 ,

a process that is invariant under all these transformations may be shaped like

 ,

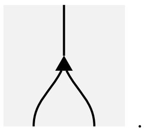
and a process where the inputs and outputs are interchangeable may be shaped like



2.1.6 Diagrammatic calculus

So far, we have encountered some diagrammatic equations, that is equations stating when two diagrams are equal. They are usually pertaining to the deformation of the wires and sliding the boxes along them. While this is already a powerful tool to analyse and design processes, we can make it more expressive by adding some diagrammatic rules regarding when some of the boxes are equal. We will end this section with a simple demonstration of the use of diagrammatic rules with a process theory of basic arithmetic (addition, subtraction, multiplication, and division) with the natural numbers $\mathbb{N}$.

Example 1. *Addition* is the simplest process in basic arithmetic, and we will denote it as the diagram



It takes in a pair of numbers $m, n \in \mathbb{N}$, and gives a number $m + n$. In mathematical terms, it is a *relation*

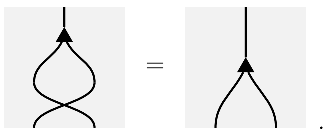
$$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{+} \{\texttt{TRUE}, \texttt{FALSE}\}$$
$$((m, n), p) \mapsto \begin{cases} \texttt{TRUE} & \text{if } p = m + n, \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

Remark 2. A more familiar, equivalent definition for relations is

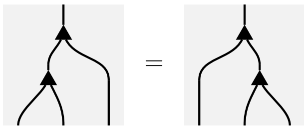
$$+ \subseteq (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$$
$$+ = \{((m, n), m + n) | m, n \in \mathbb{N}\}.$$

Associating the diagrams to some mathematics is called *interpreting* the diagrams. In this case, the addition diagram has interpretation as the + relation.

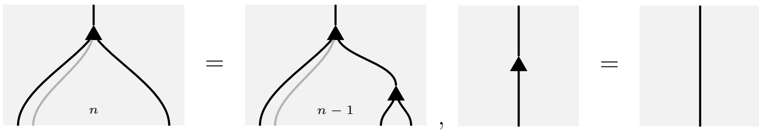
We know that addition is commutative, and it is depicted with a diagrammatic equation as



The addition is also associative, and we will show it as



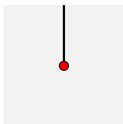
Because of this, we may define a family of diagrams for multiple additions as such:



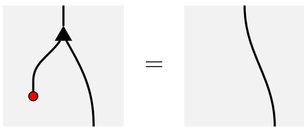
for $n \geq 1$.

The lighter shade of wires represents repeated pattern. This convention is explained more clearly in Section 2.3.3. As for now, don't be too worried about the convention as this example is meant to be informal.

Furthermore, there exist a additive identity 0, drawn as



which satisfies the equation



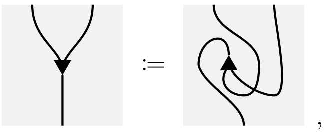
Note that the $|\,0\,\rangle$ can be defined using the addition as such:

When interpreted as a relation, the additive identity 0 is a relation between the singleton set $\{*\}$ and $\mathbb{N}$, where

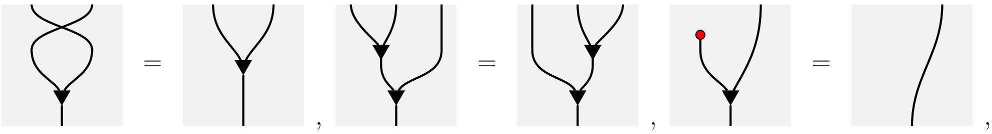
$$(*, 0) \mapsto \texttt{TRUE}$$
$$(*, n) \mapsto \texttt{FALSE}, n > 0.$$

The natural numbers form a commutative monoid under addition, and the diagrams introduced is a typical representation for this structure.

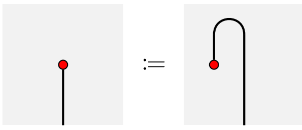
The *subtraction* process, or more technically correct – the *coaddition*, is commonly thought as the reverse of addition. In diagrammatic language, that is just reading the diagram backwards. In other words, it means that we just have to transpose all the diagrams: the subtraction process may be defined as such



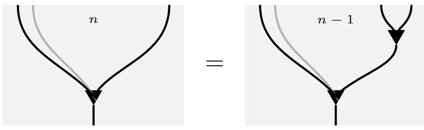
and the rules are



where

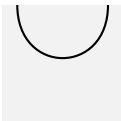


is the counit of the subtraction process. Likewise, we can define a family of diagrams for multiple subtractions as such:

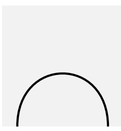


for $n \geq 1$.

The bendy wires responsible for rotating the diagrams can be interpreted as such:



$$\mapsto \quad (*, (j,k)) \mapsto \begin{cases} \texttt{TRUE} & \text{if } j = k \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$



$$\mapsto \quad ((j,k), *) \mapsto \begin{cases} \texttt{TRUE} & \text{if } j = k \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

Interpreting these subtraction diagrams can be a bit tricky. The input wire is the minued, either of the output is the subtrahend and the other one is the difference. We could bend the subtrahend wire down to make the diagram looks like the familiar subtraction equation, but that is not necessary.

Written as relations, the subtraction process is

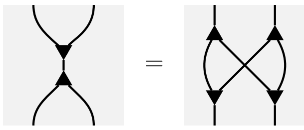
$$(n,(j,k)) \mapsto \begin{cases} \texttt{TRUE} & \text{for } k = n - j, \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

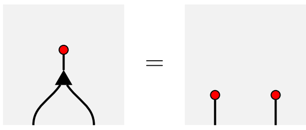
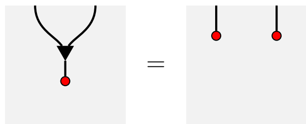
It should be fairly obvious that if we insist that one of the output in the subtraction is 0, then the remaining diagram is not doing anything, that is, $n - 0 = n$. Hence, the counit of the subtraction is the relation

$$(n,*) \mapsto \begin{cases} \texttt{TRUE} & \text{for } n = 0, \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

The structure of the subtraction process is known as the commutative comonoid.

The addition and subtraction processes also interact in a nice way. They interact via the diagram


,


,

,

which forms a bialgebra, and


,

which is called specialness property.

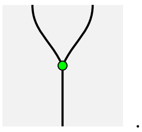
The first equation in the bialgebra equations may be a bit hard to apprehend. The left-hand side, which is easier to understand, is first summing then splitting up the sum. To visualise algebraically, it is $m + n = s = j + k$ for all j, k that sums up to s. The right-hand side can be understood as splitting up in all possible ways, then sum them up. Hence, the output will range from $(0, s)$ to $(s, 0)$.

The specialness property is pretty easy to understand as if add the difference and subtrahend of a subtraction process, we get our original minued. That means we essentially did nothing.

The next thing to introduce is *multiplication*. Multiplication of natural numbers can be thought as iterative addition of the same number. For instance, m times of n is

$$\underbrace{n + \cdots + n}_{m}.$$

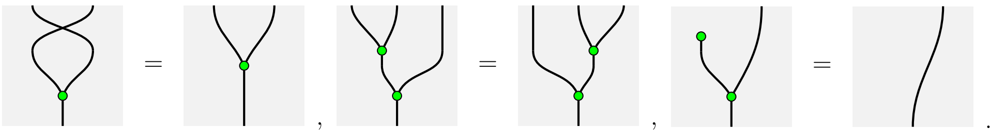
For this, we will include a *copy* process, and we will denote it as the diagram



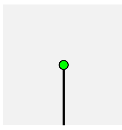
This process takes in a number $n \in \mathbb{N}$ and gives a pair of the same number, that is

$$(n, (j, k)) \mapsto \begin{cases} \texttt{TRUE} & \text{for } j = n = k, \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

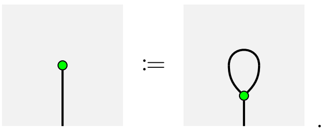
It is straightforward to check that this copy process satisfies the commutative comonoid conditions:



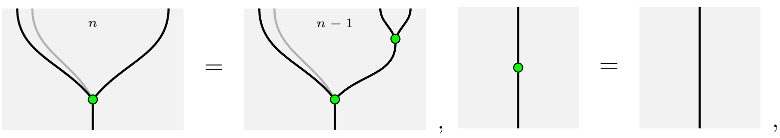
where the counit of the copy process



maps any number to `TRUE`. The counit can also be defined using the copy process like this:

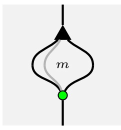


Like before, we can define a multiple copying process



for $n \geq 1$.

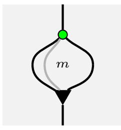
The multiplication by m can then be formed by



.

This multiplication diagram is sometimes known as the multiplicity gate in quantum computing.

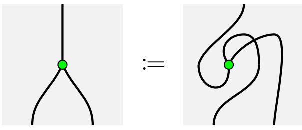
Similar to the subtraction, the division process can be obtained by transposing the multiplication diagram:



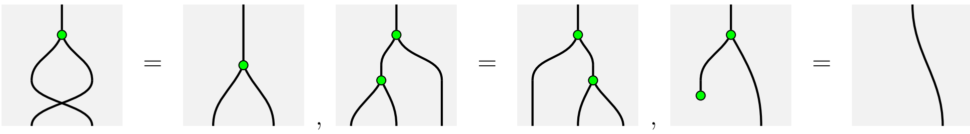
.

For this, we need to define a *cocopy*, also known as the comparing process for obvious reasons.

The cocopy process is defined to be


,

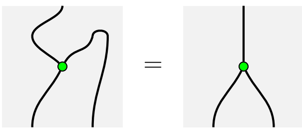
and it satisfies the commutative monoid conditions


,

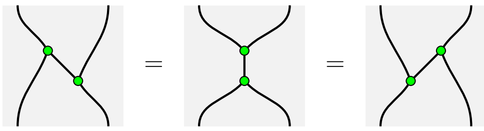
where the unit

can be interpreted as the “all possible number” process.

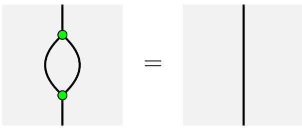
We have chose to draw the copy process with a circle because of this rule:


.

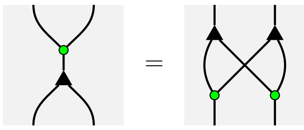
Due to this rule, the copy and compare processes satisfy the Frobenius conditions

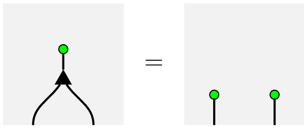
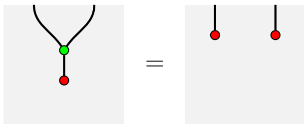

,

and also the specialness property

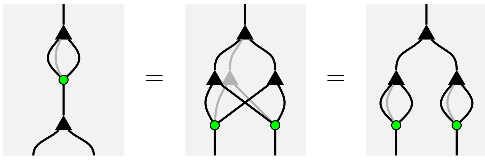


We can also put in some rules for the interaction between the addition and the copy processes. They interact via a bialgebra:



 , 

With all these rules, we can show some nice identities of arithmetic diagrammatically. For instance, $k \times (m+n) = k \times m + k \times n$ can be shown as such:



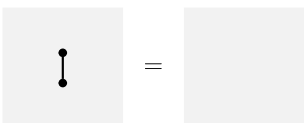
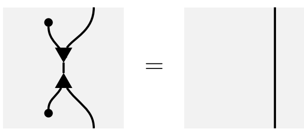
So far, we have just focused on the arithmetic processes. Sometimes it is nice to be able to feed in a number (other than 0 which is the (co)unit) and count with it. Since the natural number is generated by a single element 1, we only have to introduce a new symbol for it:

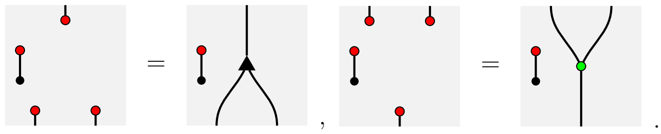
$$\mapsto (*, n) \mapsto \begin{cases} \texttt{TRUE} & \text{for } n = 1, \\ \texttt{FALSE} & \text{otherwise,} \end{cases}$$

and its transpose

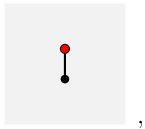
$$\mapsto (n, *) \mapsto \begin{cases} \texttt{TRUE} & \text{for } n = 1, \\ \texttt{FALSE} & \text{otherwise.} \end{cases}$$

Then, we give a few rules on how 1 interacts with all the processes we have defined so far:

 ,  ,  ,

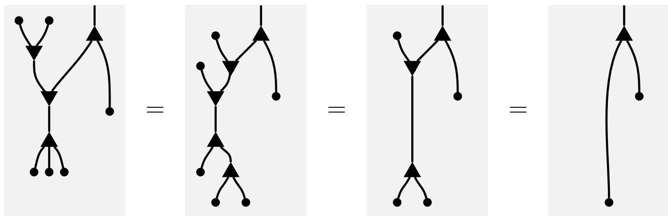


The last two diagram equations reflects the properties of the `FALSE` outcome: once we see a `FALSE`

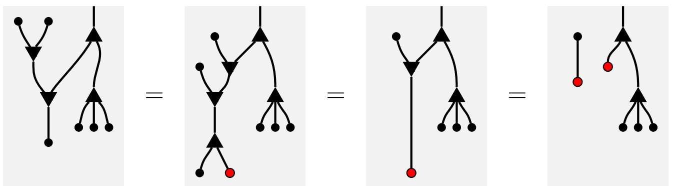


all diagram falls apart and all calculation is meaningless.

We will demonstrate calculating with these rules with a baby example. Say we want to do $(3-2)+1$, the diagrammatic calculation for this is

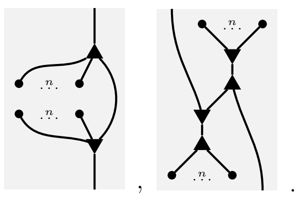


which then can be read as the number 2. A fun observation is that $(3-2)+1$ may seem the same as $(1-2)+3$, a diagrammatic calculation shows otherwise:



where we see a `FALSE` outcome. The diagram shows this because $(1-2)$ is not in the natural numbers, hence it is impossible to do this calculation.

Some other fun operations are the *greater than and equals to* $\geq$ and the *less than and equals to* $\leq$. The following operators allow numbers greater than or less than n to pass through respectively:

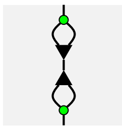


2.1.7 Soundness, universality, completeness

We have defined all four basic arithmetic operations using diagrams. By how we constructed these diagrams, if two diagrams are equal under these diagrammatic rules, then the underlying arithmetic equation is equal. We call this diagrammatic calculus *sound* with respect to the arithmetic of natural numbers.

Furthermore, we can represent all arithmetic equations using diagrams. This fact is called *universality*, that is, this diagrammatic calculus is universal with respect to arithmetic of natural numbers.

However, we aren't sure whether we can diagrammatically proof all truths in arithmetic. For instance, $m \times 2 \div 2$ should give m, but the diagrammatic equation



=

,

may not be provable with our existing rules. If all truths are provable, then we call this calculus *complete* with respect to arithmetic of natural numbers, otherwise we call it *incomplete*. Completeness of a diagrammatic calculus is important because if it is incomplete, then it indicates that we are not understanding the essentials of the arithmetic operations. The completeness problem is usually the one plaguing a diagrammatic calculus because there is usually no clear strategy of proving it or completing the calculus. The completeness problem is the main focus of the thesis.

2.2 Formalising string diagrams

In the previous section, we introduced how diagrammatic calculus is used in practice; it involves twisting and bending of wires, and sliding of boxes along the wires. This graphical representation of processes may seem like a back-of-the envelope calculation, but they actually are mathematically consistent. In this section, we will go through the essential mathematics to make these diagrammatic calculus rigorous. The numbering of the subsections in this section corresponds to the numbering of the previous section where the concepts were introduced.

Remark 3. Everything in this section can be found in a survey paper by Selinger [49]. The paper explores the mathematics with much more rigour than what is presented in this thesis, so interested readers should definitely head over. Fortunately, we don't have to go to such depth.

2.2.1 Basic category theory

We are going to use category theory to formalise process theories. We are going to define what a category is, and how process theories are related to categories.

Definition 4. A category $\mathcal{C}$ consists of:

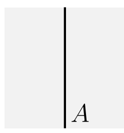
- a collection $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- for each pair A, B of objects, a collection $\mathcal{C}(A, B)$ of *arrows* from A to B;
- for each object A, there exists an *identity* arrow $\mathrm{id}_A \in \mathcal{C}(A, A)$;
- a binary operation $\circ$ assigning a pair $(g, f) \in \mathcal{C}(B, C) \times \mathcal{C}(A, B)$ of arrows to a composite arrow $g \circ f \in \mathcal{C}(A, C)$, such that:

  (i) the arrows compose associatively, that is, $h \circ (g \circ f) = (h \circ g) \circ f$ for arrows $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(C, D)$;

  (ii) the identity arrows satisfy the following property: $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$ for arrows $f \in \mathcal{C}(A, B)$ and identity arrows $\mathrm{id}_A, \mathrm{id}_B$ for objects A, B respectively.

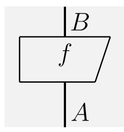
An arrow $f \in \mathcal{C}(A, B)$ is also called a *morphism* and sometimes written as $f : A \to B$ or $A \xrightarrow{f} B$.

We can express the definition of category theory pictorially: The objects are represented by the labels on the wires



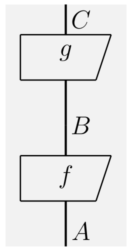
,

and the arrows are boxes



,

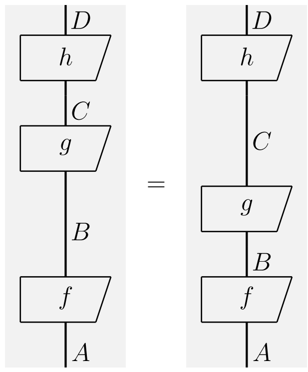
The identity arrows are just the plain wires (no boxes). Composition of arrows are connecting the wires appropriately, like this:



.

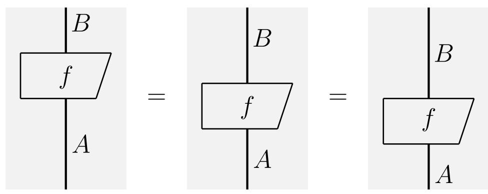
The arrows must satisfy the following properties:

(i) associativity of composition is drawn as such:



,

(ii) the identity arrows satisfy the diagram equation



.

Very often, we will keep the labelling on the wires implicit.

The pictorial form makes the definition of a category really intuitive. In fact, it looks almost like process theories. An interpretation of a process diagram assigns each diagram to a underlying theory in a well-behaved manner. In categorical terms, this assignment is called a *functor*.

Definition 5. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns

- for each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$,

- for each arrow $f \in \mathcal{C}(A, B)$ an arrow $F(f) \in \mathcal{D}(F(A), F(B))$ such that:

- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for every $A \in \mathrm{Ob}(\mathcal{C})$,
- $F(g \circ f) = F(g) \circ F(f)$ for all composable arrows $f, g \in \mathcal{C}$.

A functor F is called *contravariant functor* if it satisfy all the conditions except that $F(g \circ f) = F(f) \circ F(g)$.

An assignment is called *functorial* if it forms a functor. Some people may recognise a functor as a homomorphism between categories.

There is something more to process theories; we need to define what does it mean to put the processes side-by-side, and also twisting of wires. We will define these in the next subsection.

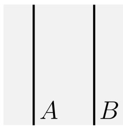
2.2.2 Strict symmetric monoidal category

A *strict monoidal category* is often used as a foundation for the mathematics for a process theory. This is fairly obvious after we introduce its definition.

Definition 6. A *strict monoidal category* is a category $\mathcal{C}$ equipped with a *monoidal product* $(- \otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that:

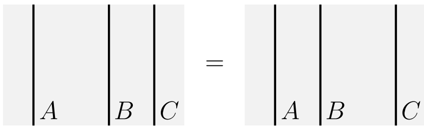
- when acts on objects:
  - the objects $A, B \in \mathrm{Ob}(\mathcal{C})$ are mapped to $A \otimes B \in \mathrm{Ob}(\mathcal{C})$,
  - the mapping is associative $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$,
  - there exist an object called the *monoidal unit* I such that $I \otimes A = A = A \otimes I$ for any object $A \in \mathrm{Ob}(\mathcal{C})$,
- when acts on arrows:
  - the arrows $f \in \mathcal{C}(A, B), g \in \mathcal{C}(A', B')$ are mapped to an arrow $f \otimes g \in \mathcal{C}(A \otimes A', B \otimes B')$,
  - the mapping is associative $f \otimes (g \otimes h) = (f \otimes g) \otimes h$ for arrows f, g, h,
  - $f \otimes e = f = e \otimes f$ for an identity arrow e of the monoidal unit,
  - acts nicely with the usual composition $(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$.

Expressing the definition pictorially: The monoidal product on objects is represented by juxtaposition of the wire labels (together with the wires)



,

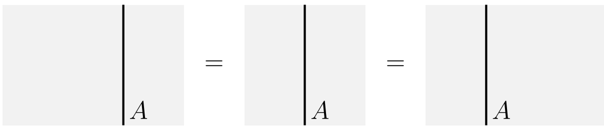
and its associativity is drawn as



,

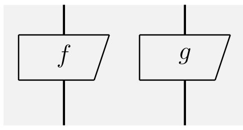
while the monoidal unit is just an 'empty wire'

satisfying



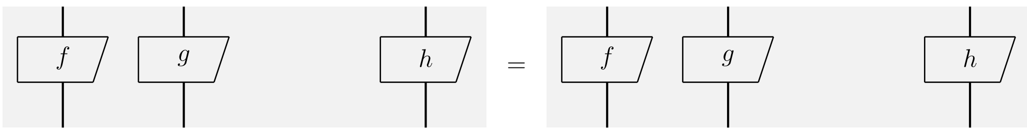
.

The monoidal product on arrows is represented by juxtaposition of the boxes



,

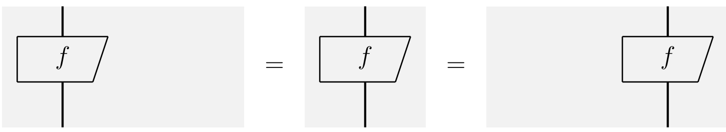
where the associativity is drawn as



,

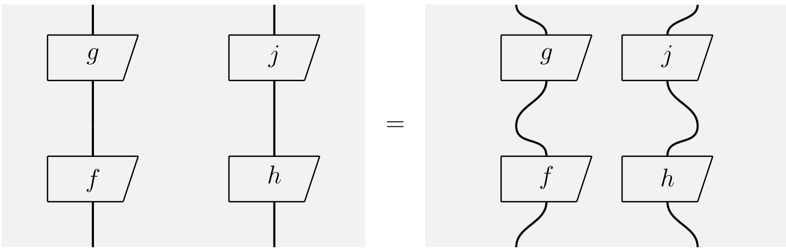
the identity arrow of the monoidal unit is just an 'empty box'

satisfying



,

and it acts nicely with the usual composition



.

The pictorial presentation of the definition makes many of the conditions really easy to grasp. Because of the pictures, the composition of arrows is sometimes called *sequential composition* while the monoidal product is called *parallel composition*. In this thesis, as we are dealing with mostly diagrams, we will use the word 'plugging' for the sequential composition, and the word 'juxtaposition' for parallel composition.

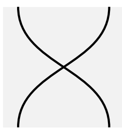
The ability to twist the wires is an added feature on a strict monoidal category. This is called the *symmetric* property.

Definition 7. A strict *symmetric* monoidal category $\mathcal{C}$ is a strict monoidal category where for each pair of objects $A, B \in \mathcal{C}$, there is a *swap* arrow $\sigma_{A,B} : A \otimes B \to B \otimes A$. The swap arrows satisfy the following properties:

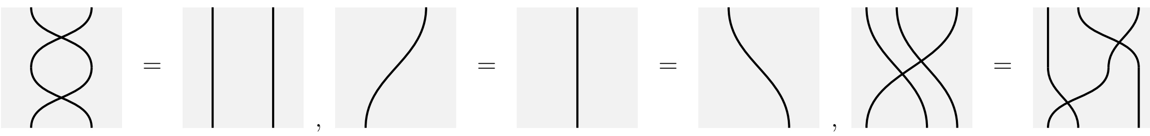
- $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$,
- $\sigma_{A,I} = \mathrm{id}_A = \sigma_{I,A}$ for $A \in \mathrm{Ob}(\mathcal{C})$ and the monoidal unit I,
- $\sigma_{A,B\otimes C} = (\mathrm{id}_B \otimes \sigma_{A,C}) \circ (\sigma_{A,B} \otimes \mathrm{id}_C)$ for $A, B, C \in \mathrm{Ob}(\mathcal{C})$,
- for arrows $f : A \to B$ and $g : A' \to B'$, $\sigma_{B,B'} \circ (f \otimes g) = (g \otimes f) \circ \sigma_{A,A'}$.

The last bullet point is sometimes call the *naturality* condition.

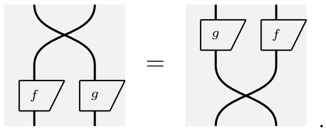
The symmetric property can be easily summarised with some simple diagrams: A swap arrow is represented by



where it satisfies the following four diagrams corresponding to the four bullet points respectively:



,



Definition 8. A functor F between two strict monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is called a *strict monoidal functor* if it respects their monoidal structures. That is,

- $F(A \otimes B) = F(A) \otimes F(B)$ for $A, B \in \mathrm{Ob}(\mathcal{C})$,
- $F(I_{\mathcal{C}}) = I_{\mathcal{D}}$ where $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$ are the monoidal units for $\mathcal{C}$ and $\mathcal{D}$ respectively,
- $F(f \otimes g) = F(f) \otimes F(g)$ for arrows $f, g \in \mathcal{C}$.

We have already made all the necessary definitions to make process theories mathematically rigorous. A process theory is basically a strict symmetric monoidal category. An interpretation is usually a strict monoidal functor (an unstrict version is not too much different, at least the idea is the same). We are now free to use process diagrams to analyse problems and forget about the laborious mathematics. Our next goal is to make string diagrams mathematically rigorous.

Remark 9. The strict symmetric monoidal category can be relaxed to braided monoidal category where many of the equalities are replaced with natural isomorphisms. However, process theories do not feature those as it is unnatural to draw these natural isomorphisms. Furthermore, a monoidal category is monoidally equivalent to a strict one by Mac-Lane's coherence theorem, so we will not bother defining an unstrict, braided version.

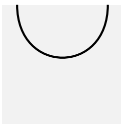
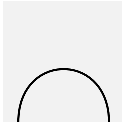
2.2.3 Compact closed

The *compact closed* structure is responsible for the bending of wires in a string diagram.

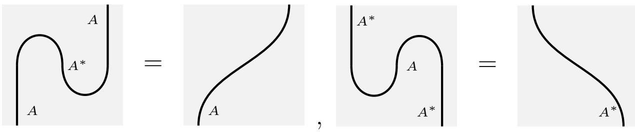
Definition 10. Suppose $\mathcal{C}$ is a strict symmetric monoidal category. It is called a *compact closed category* if for all $A \in \mathrm{Ob}(\mathcal{C})$, there is a *dual* object $A^* \in \mathrm{Ob}(\mathcal{C})$ and a pair of arrows $\eta_A : I \to A^* \otimes A$ and $\varepsilon_A : A \otimes A^* \to I$ such that

$$(\varepsilon_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \eta_A) = \mathrm{id}_A,$$
$$(\mathrm{id}_{A^*} \otimes \varepsilon_A) \circ (\eta_A \otimes \mathrm{id}_{A^*}) = \mathrm{id}_{A^*}.$$

A pictorial presentation of this compact closed structure is sometimes called the snake or yanking equations because of the following diagrams: The η_A and ε_A are drawn as

 ,  ,

respectively. They are sometimes called the cup and cap respectively. They satisfy the snake or yanking equations

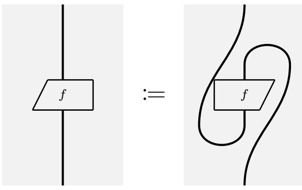


.

We can use the compact structure to *transpose* an arrow. That is, for any arrow $f : A \to B$, its transpose $f^T : B^* \to A^*$ is defined as

$$f^* := (\mathrm{id}_{A^*} \otimes \varepsilon_B) \circ (\mathrm{id}_{A^*} \otimes f \otimes \mathrm{id}_{B^*}) \circ (\eta_A \otimes \mathrm{id}_{B^*}).$$

In pictorial form,

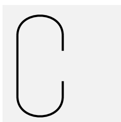


,

where the irregular shaped box will tell us whether it is f or f^T.

From the pictorial definitions, it's not hard to see that we have a string diagram. We are pretty much done with all the hard mathematics. Although not needed for this thesis, we will define another structure on a compact closed category just because we will be dealing with quantum processes later – the *dagger*.

Remark 11. Using the compact closed structure, we can define a partial trace operation

 .

In this case, we need $A^{**} = A$. We can even weaken the compact closed structure to what is called a *traced monoidal category* where we only have the trace operations and not able to freely bend wires.

Example 12. Here is a cute example of a compact structure: it lives in the category of relations. The objects are iterative monoidal product of the natural numbers $\mathbb{N}$, the arrows are just the identities, swaps, and cups and caps. The identities and swaps are defined in the usual way. The cups and caps are defined and generated by the relations:

$$\eta_{\mathbb{N}} : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \{\texttt{TRUE}, \texttt{FALSE}\},$$
$$\varepsilon_{\mathbb{N}} : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \{\texttt{TRUE}, \texttt{FALSE}\}$$

The definition is cumbersome to state, but an simple example will illustrate it clearly.

For example, for $\eta_{\mathbb{N}}$, these triplets are mapped to `TRUE`:

$$((5, 3), 53),$$
$$((12, 34), 1324),$$
$$((213, 10), 201130),$$
$$((10, 213), 21103);$$

The first pair of numbers are encoded in the third by first taking the right-most digit of the second number, then append the right-most digit of the first number on the left, then append the second right-most digit of the second number on the left, and so on. If there are no more digits, then append 0's.

$\varepsilon_{\mathbb{N}}$ is defined in a similar way which decodes the $\eta_{\mathbb{N}}$. One can check that this forms a compact structure.

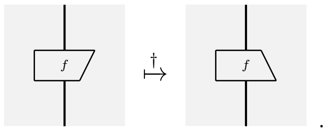
2.2.4 Dagger

The dagger is mainly inspired from quantum mechanics. It was introduced by Abramsky and Coecke to give a categorical description of quantum systems [2]. As the name suggests, it is to describe the adjoint operation in quantum mechanics.

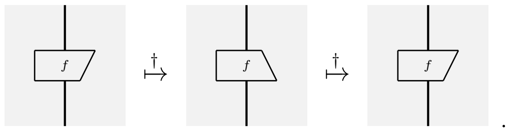
Definition 13. Suppose $\mathcal{C}$ is a category. A *dagger* $(-)^\dagger : \mathcal{C} \to \mathcal{C}$ is a contravariant functor which maps

- $A^\dagger = A$ for $A \in \mathrm{Ob}(\mathcal{C})$,
- $A \xrightarrow{f} B \mapsto B \xrightarrow{f^\dagger} A$ for arrow f, satisfying
  - $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ for arrows f, g,
  - $(f^\dagger)^\dagger = f$ for arrow f.

Pictorially, the dagger is reflecting the diagram along the horizontal axis:



The conditions is simply the following moves:



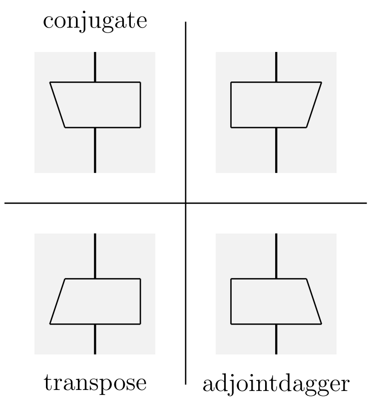
The move for $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ is so trivial in diagrams that drawing it out looks really silly...

The pictorial presentation is very fitting because the adjoint in quantum mechanics is sometimes seen as a 'time reversal', and the dagger can be seen as reading the diagram backwards which is like reversing time.

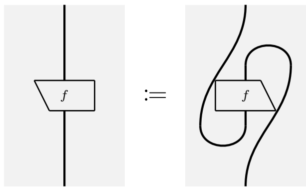
Remark 14. Because of its motivation from quantum mechanics adjoint, dagger and adjoint is often used interchangeably.

2.2.5 Terminology

The terminology is pretty much the same as the one introduced in subsection 2.1.5:



But, we have yet to define what is a conjugate. It can be defined as a combination of transpose and adjoint (in any order):



where we will write equationally as $f^* : A^* \to B^*$.

A fun fact: if we look at the diagrams, it is clear these moves form a group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.2.6 Diagrammatic calculus

Diagrammatic calculus, as the name suggests, is a method for using diagrams to perform calculations. So far, we have presented and justified the use of string diagrams for compact closed categories. However, such presentation is not yet a diagrammatic calculus because it lacks the ability to calculate. All we have now are boxes and all we can do is to move them around. We still have to specify the rules of the calculations, that is, the axioms of the calculus.

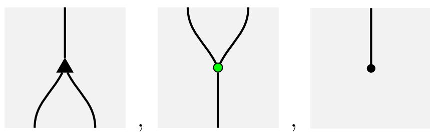
To define a diagrammatic calculus with an underlying string diagram, we

1. state what are the objects in the string diagram $\mathcal{S}$ which will be just labels on the strings;

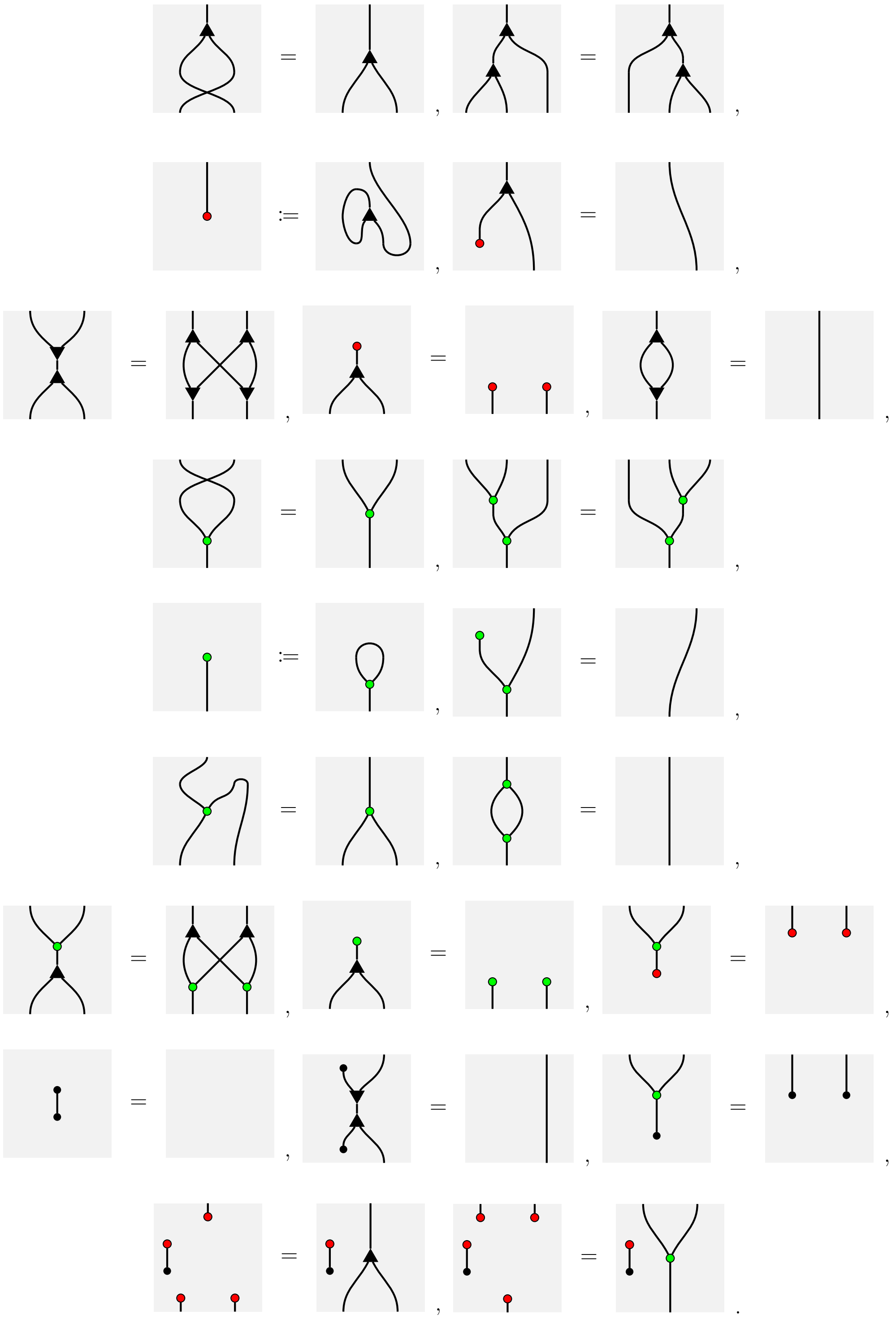
2. specify a *signature*, that is, a collection of boxes $T := \{f_i : A_i \to B_i\}_{i \in I}$ that will be used to generate all other boxes. We now obtain a free string diagram $\mathcal{S}[T]$ on T, that is, the boxes are free plugging and juxtaposition of f_i, modulo the axioms of string diagrams. This is called the *language* of the calculus;

3. state a collection of equations between diagrams E in $S[T]$. Then, quotient $S[T]$ by the smallest equivalence relation including E, compatible with string diagram axioms to obtain the diagrammatic calculus $S[T/E]$. E is the axioms of the calculus which tells us how to calculate with diagrams.

We will now dissect Example 1 to show how this works. Arithmetic process theory is a string diagram where the objects are iterative monoidal product of the natural numbers $\mathbb{N}$. The signature contains the boxes



The axioms of the calculus are



Remark 15. We can, in fact, start from a pictorial representation of strict monoidal category, then specify a monoidal signature which contains the swap maps, cups and caps. The equation between diagrams will then have the axioms for the swap maps, and the snake/yanking equations.

2.2.7 Soundness, universality, completeness

After defining a diagrammatic calculus $\mathcal{D}$, we want to see how fitting this calculus is in modelling other compact closed categories $\mathcal{C}$. We have a map $i : \mathcal{D} \to \mathcal{C}$ which interprets the diagrams in $\mathcal{D}$ as some arrows in $\mathcal{C}$. This interpretation should be *sound*, meaning that diagram equalities in $\mathcal{D}$ should be mapped to arrow equalities in $\mathcal{C}$. In other words, a sound interpretation shouldn't break the axioms of $\mathcal{C}$. This can be formalised as the following definition.

Definition 16 (Soundness)**.** Suppose $\mathcal{D}$ is a diagrammatic calculus and $\mathcal{C}$ is a symmetric monoidal category. An interpretation map $i : \mathcal{D} \to \mathcal{C}$ is called *sound* if i is a symmetric monoidal functor.

After obtaining a sound interpretation, which is very important because we don't want to be using the diagrams to make false reasoning on the category that it is modelling, we want to see if everything can be modelled. That means, we want to see if all the arrows in $\mathcal{C}$ has a corresponding diagram in $\mathcal{D}$. This concept is called *universality*, which has the following formal definition.

Definition 17 (Universality)**.** Given an interpretation functor $i : \mathcal{D} \to \mathcal{C}$, the diagrammatic calculus $\mathcal{D}$ is called *universal* with respect to the category $\mathcal{C}$ if i is surjective on objects and arrows, that is, the mapping of the objects $\mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$ is surjective, and the function induced from the functor $\mathcal{D}(A, B) \to \mathcal{C}(i(A), i(B))$ is surjective for all $A, B \in \mathrm{Ob}(\mathcal{D})$.

Remark 18. Sometimes, we may want to relax the definition for universality to be *essentially surjective* instead of surjective on objects (the condition on arrows remains unchanged). Essentially surjective means that each object in $\mathcal{C}$ just has to be isomorphic to an object that has a preimage in $\mathcal{D}$.

Finally, we may want to ask if everything provable in $\mathcal{C}$ can be proved using the diagrams in $\mathcal{D}$. Phrasing in a more logic-theoretic language, are all truths in $\mathcal{C}$ provable in the model $\mathcal{D}$, that is, is $\mathcal{D}$ *complete* with respect to $\mathcal{C}$. We can formulate this as the following definition.

Definition 19 (Completeness)**.** Given an interpretation functor $i : \mathcal{D} \to \mathcal{C}$, the diagrammatic calculus $\mathcal{D}$ is called *complete* with respect to the category $\mathcal{C}$ if i is a monoidal equivalence, that is, i is a monoidal functor, the function induced from the functor $\mathcal{D}(A, B) \to \mathcal{C}(i(A), i(B))$ is bijective for all $A, B \in \mathrm{Ob}(\mathcal{D})$, and every $X \in \mathrm{Ob}(\mathcal{C})$ is isomorphic to an $i(A) \in \mathrm{Ob}(\mathcal{C})$ for some $A \in \mathrm{Ob}(\mathcal{D})$.

Remark 20. Some authors would define completeness as an isomorphism of categories, which is stronger than equivalence.

2.3 Brief history of Z* calculi for quantum theory

There is a family of diagrammatic calculi used to model pure-state qubit quantum computing, the *ZX*, *ZW* and *ZH calculi*. They have, sometimes, been affectionately known as the *Z* series*. The main highlight of the thesis is on two of them – the ZX and ZW calculi.

The Z* series stem from the research on categorising finite-dimensional quantum theory [2], which this area of research is now known as categorical quantum mechanics. Categorical quantum mechanics studies the structures of quantum theory as an abstract process theory, in particular the structures relevant to quantum computing. The model is often a dagger compact closed category [17, 50]. Diagrammatic calculus is commonly employed as a calculation tool, and as a heuristic for determining algebraic structures that fit naturally in the framework [19]. Compared to matrix calculus, which has been compared to "programming with bit strings" [18], diagrammatic calculi are a higher-level language, allowing one to focus on the connections between gates and the flow of information. Complete axiomatisations of fragments of quantum theory would provide quantum programmers with the possibility of understanding the behaviour of a circuit entirely within this language, without resorting to linear algebra.

Within this framework, Coecke and Duncan proposed an axiomatisation of complementary quantum observables in terms of a pair of special Frobenius algebras whose monoid forms a bialgebra with the comonoid of the other [13, 9]. These structures, with the addition of phases [15], seemed to capture enough interesting aspects of pure-state quantum theory, such as *non-locality* [14]. This partial axiomatisation is the start of a family of diagrammatic calculi called the *ZX calculi*. A reasonable question to ask is if one could take this as a basis for a complete axiomatisation of quantum theory.

The development of ZX calculi is mainly geared towards qubit quantum computing, and many different versions were written down for different fragments of qubits systems. More notably, Backens has provided a ZX calculus for the stabiliser fragment, and one for single-qubit processes in the approximately universal Clifford+T fragment, both being complete for their respective fragments [3, 4]. Completeness for the whole of pure-state qubit theory has remained an open problem [22].

Observing that the components of the ZX calculus seemed ill-suited to analysing finer properties of entangled qubits, such as their separation in SLOCC classes [25], Coecke and Kissinger proposed a variant where one Frobenius algebra is replaced with one satisfying an “anti-special” equation [16]. In [30], Hadzihasanovic refined and extended this theory into a calculus modelled on ZX calculi, the *ZW calculus*, and proved its completeness for maps of qubits that have only integer coefficients. In [36], Jeandel, Perdrix, and Vilmart used a non-trivial translation of the ZW calculus into the ZX calculus to obtain a complete axiomatisation of the entire Clifford+T fragment.

Soon, there is an avalanche of completeness result for the ZX and ZW calculi:

- ZW calculus for the more general pure-state qubit theory [33],
- ZX calculus for the entire pure-state qubit theory [33],
- ZW calculus for fermionic circuits [32],
- ZX calculus for a ‘finer’ Clifford+T fragment [37, 35],
- ZX calculus for Toffoli-Hadamard and other fragments [51].

Remark 21. There is another diagrammatic calculus that is closely related to the Z* series – the *Y calculus* [34].

In this thesis, we will look at some of the new completeness result, in particular the ones on the second and third bullet points.

At the same time while all these completeness result were produced, a new variant of the ZX calculus was developed by Backens and Kissinger – the *ZH calculus* [5]. It aims to provide a language that is useful for representing hypergraph states. Like in the ZW calculus, one of the Frobenius algebra in ZX calculus is replaced with something that is a generalised Hadamard gate in qubit quantum theory; an arbitrary arity Hadamard node. This Hadamard node represents the hyperedges. If the arity is 2, then it reduces back to the normal graph state. This language is also complete for the entire pure-state qubit theory. We will not explore this calculus in this thesis.

2.3.1 Quick recap of finite-dimensional quantum theory

We will quickly look at what is a finite-dimensional quantum theory. We are not going to explain the physics, nor the in-depth mathematics. We are just going to touch on the surface of the subject, just enough for this thesis.

Finite-dimensional quantum theory is basically *finite-dimensional Hilbert spaces*. A n-dimensional Hilbert space is a *n-dimensional complex vector space* V equipped with an *inner product* $\langle - | - \rangle : V \times V \to \mathbb{C}$. We deliberately didn't define what an inner product is because its precise definition is not important for our work. We will define it along the way. And in later part, when we say 'Hilbert space', we meant it to be finite-dimensional, otherwise stated.

Suppose H is a n-dimensional Hilbert space. A vector in H is called a *state*. Since it is a vector space, we can pick a basis set where all other vectors are written as linear combination of states from the basis set. A *computational basis* is a set of n orthonormal vectors, written as the kets

$$| 0 \rangle, | 1 \rangle, \ldots, | n-1 \rangle.$$

These computational basis states are orthonormal, meaning that their inner product is

$$\langle m | n \rangle = \delta_{mn},$$

where $\delta_{mn} = 1$ if $m = n$, and 0 otherwise, for $m, n = 0, \ldots n-1$. An inner product between two vectors, say

$$| v \rangle = \sum_{j=0}^{n-1} a_j | j \rangle,$$
$$| w \rangle = \sum_{k=0}^{n-1} b_k | k \rangle,$$

where $a_j, b_k \in \mathbb{C}$, is evaluated as

$$\langle w | v \rangle = \sum_{j,k=0}^{n-1} a_j b_k^* \langle k | j \rangle = \sum_{l=0}^{n-1} a_l b_l^*,$$

where the asterisk on the complex numbers indicates its complex conjugation. Hence, the computational basis states are also written as the column vectors

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We will use these two presentation interchangeably.

We can define a dual space H^*. It is a space of linear functionals that map $H \to \mathbb{C}$ induced from the inner product. A vector in H^* is called an *effect.* There is a computational basis induced from H, written as the bras

$$\langle\, 0 \,|, \langle\, 1 \,|, \ldots, \langle\, n-1 \,|.$$

They act linearly on the vectors in H as characterised by

$$\langle\, m \,| : |\, n \,\rangle \mapsto \langle\, m \,|n \,\rangle.$$

Hence, these bras are sometimes written as the row vectors

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \ldots & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix},$$

and the mapping will then be the usual matrix product between the row vector and column vector.

We will now state what is a homomorphism between Hilbert spaces. A homomorphism between Hilbert spaces is a linear map between them. Let $\phi : H_m \to H_n$ be a linear map between a Hilbert space of dimension m to n. Then, ϕ is completely described by how the computational basis in H_m map to in H_n, that is, we only have to specify the maps

$$|\, j \,\rangle_m \mapsto |\, v_j \,\rangle_n$$

for each $j = 0, \ldots, m-1$, where the $|\, j \,\rangle_m$ are the computational basis in H_m, and $|\, v_j \,\rangle_n$ are the vectors in H_n. In matrix form, this is just a m by n matrix where the columns are the matrix representation of $|\, v \,\rangle_n$

$$\begin{bmatrix} \vdots & & \vdots \\ |\, v_0 \,\rangle_n & \ldots & |\, v_{m-1} \,\rangle_n \\ \vdots & & \vdots \end{bmatrix},$$

and the mapping is by matrix multiplication of the state on the right. In fact, we can view the dual space of a Hilbert space as a set of linear maps to the 1-dimensional Hilbert space $\mathbb{C}$.

Next in row is introducing something called a *tensor product.* Suppose H_m and H_n are m and n-dimensional Hilbert spaces respectively, the tensor product of them, written as $H_m \otimes H_n$, is a Hilbert space of dimension $m \times n$. The computational basis of $H_m \otimes H_n$ is the set of $|\, j \,\rangle \otimes |\, k \,\rangle$, abbreviated as $|\, jk \,\rangle$, for $j = 0, \ldots, m-1$ and

$k = 0, \dots, n-1$. The inner product and dual space are just a extension in the most natural way: suppose

$$
\begin{aligned}
|\, v\,\rangle &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} a_{jk} |\, jk\,\rangle, \\
|\, w\,\rangle &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} b_{jk} |\, jk\,\rangle
\end{aligned}
$$

are vectors in $H_m \otimes H_n$, then their inner product is

$$\langle\, w\,|v\,\rangle = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} a_{jk} b_{jk}^{*}.$$

The dual space is just a set of functional $\langle\, j\ k\,|$ for $j = 0, \dots, m-1, k = 0, \dots, n-1$ where

$$\langle\, j\ k\,|j'\ k'\,\rangle = \langle\, j\,|j'\,\rangle\langle\, k\,|k'\,\rangle.$$

We can also act partially on this combined Hilbert space as such: suppose

$$
\begin{aligned}
&|\, w\,\rangle_n \in H_n, \\
&|\, v\,\rangle_{mn} = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} a_{jk} |\, jk\,\rangle \in H_m \otimes H_n,
\end{aligned}
$$

their inner product is define as

$${}_n\langle\, w\,|v\,\rangle_{mn} = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} a_{jk} |\, j\,\rangle_m\ {}_n\langle\, w\,|k\,\rangle_m.$$

A well-known fact about finite-dimensional Hilbert spaces is that if their dimensions are the same, then there exist an isomorphism between them. This means that they are essentially the same thing. We may insist that they are the same with not much drawback. Then we will write a n-dimensional Hilbert space as $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$, a n direct sum of $\mathbb{C}$.

We can now define what is finite-dimensional quantum theory. It is the category of finite-dimensional Hilbert space **FHilb**. The objects are n-dimensional Hilbert spaces for $n \in \mathbb{N}$, and the arrows are linear maps. A state in a n-dimensional Hilbert space is a represented by a linear map

$$
\begin{aligned}
\mathbb{C} &\to H_n \coloneqq \mathbb{C} \otimes \cdots \otimes \mathbb{C} \\
1 &\mapsto \sum_{j=0}^{n-1} c_j |\, j\,\rangle,
\end{aligned}
$$

where $c_j \in \mathbb{C}$. Normally, we will just write the state as

$$\sum_{j=0}^{n-1} c_j |\, j\,\rangle.$$

This category is a dagger compact closed category. The monoidal product is the *tensor product* and the monoidal unit is $\mathbb{C}$, the symmetric map can be defined on the Hilbert spaces H_m and H_n

$$\sigma_{mn} : |\, j\,\rangle_m \otimes |\, k\,\rangle_n \mapsto |\, k\,\rangle_n \otimes |\, j\,\rangle_m,$$

compact structure on the Hilbert space H is the (generalised) *bell state* and *bell effect*

$$\text{Bell state: } 1 \mapsto \sum_j |\, jj\,\rangle,$$
$$\text{Bell effect: } |\, jj\,\rangle \mapsto 1,$$

and the dagger is the functor

$$H_m \xrightarrow{f} H_n \overset{\dagger}{\mapsto} H_m \xleftarrow{f^\dagger} H_n.$$

In particular, a state is mapped to its dual space:

$$\mathbb{C} \to H_n \overset{\dagger}{\mapsto} \mathbb{C} \leftarrow H_n.$$

We will write the linear map in a dual space as a bra

$$\langle\, v\,| : H_n \to \mathbb{C}.$$

We are mostly interested in the category of qubit theory **Qubit**. It is the full monoidal subcategory of **FHilb** whose objects are tensor product of a finite copies of 2-dimensional Hilbert space $\mathbb{C} \oplus \mathbb{C}$. We can generalise **Qubit** to $\mathbf{Qudit}_n$, the full monoidal subcategory of **FHilb** whose objects are tensor product of a finite copies of n-dimensional Hilbert space $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$.

In this thesis, we can safely forget about the dagger structure. We can generalise **Qubit** and $\mathbf{Qudit}_n$ by generalising the objects to tensor product of a finite copies of $R \oplus \cdots \oplus R$ for some commutative ring R. Such categories are called $R\mathbf{bit}$ and $R\mathbf{dit}_n$ respectively. They work exactly the same, each state is written as

$$r_0|\, 0\,\rangle + r_1|\, 1\,\rangle + \cdots + r_{m-1}|\, m-1\,\rangle$$

for $r_0, \ldots, r_{m-1} \in R$, and the maps are still linear maps defined in the same way. These objects are also known as *R-modules*, and the linear maps are also known as *R-module homomorphisms.*

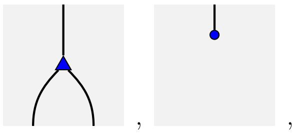
Remark 22. Forgetting the dagger structure, **FHilb** is just $\mathbf{FVect}_{\mathbb{C}}$, the category of finite-dimensional vector space over the field of complex numbers.

2.3.2 Some algebraic structures

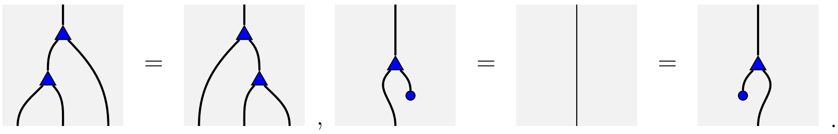
We have encountered many terms like "monoid", "commutativity", "bialgebra", and "Frobenius algebra". We are going to look at some of them in this subsection in diagrammatic form. This subsection is more like a taxonomy of algebraic structures than explaining them.

Many of the algebraic structures we will introduce have a corresponding coalgebraic structure. A *coalgebra* is an algebra with all the arrows reversed. In diagrammatic terms, a coalgebra is basically an algebra rotated (or flipped) upside down.

A *monoid* is a pair of arrows

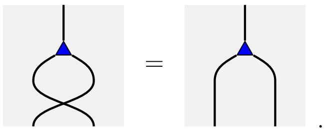


satisfying the following axioms:



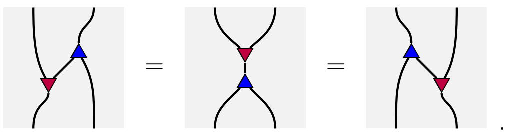
The unary blue vertex is called the *unit*.

A monoid is *commutative* if it satisfies the following swapping axiom:



A *Frobenius algebra* is a pair of monoid and comonoid

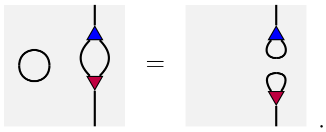
satisfying the Frobenius law:



A Frobenius algebra is commutative if its associated (co)monoid is (co)commutative.

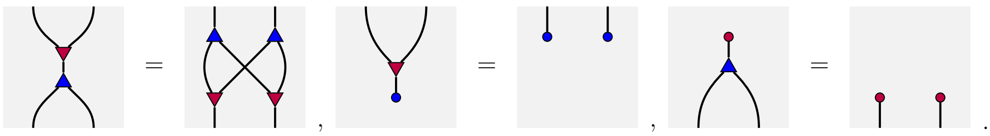
A Frobenius algebra is *special* if the loops close as such:

A Frobenius algebra is *anti-special* if the loops are broken as such:



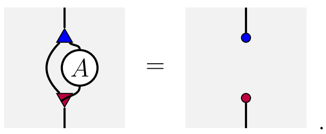
A *bialgebra* is a pair of monoid and comonoid

satisfying the following bialgebra axioms:



We have a choice of the braiding. In this case, we have depicted the swap map.

A *Hopf algebra* is a bialgebra which satisfies the additional Hopf axiom:



The vertex A is called the antipode.

2.3.3 Preliminary to Z* calculi

As the title suggests, we are going to introduce some important categories to the Z* series.

The ambient category in which the Z* series live is the free *self-dual*, *compact closed PROP*. We will define these terms in turn.

Definition 23. A *products category* (*PRO*) is a strict monoidal category whose objects are finitely iterated monoidal products of a single object X. A *products and permutations category* (*PROP*) is a PRO that is also symmetric.

Remark 24. Since the objects of a PRO and PROP are finitely iterated monoidal products of a single object, the objects are sometimes define as the set of natural numbers instead.

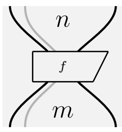
Definition 25. A *self-dual, compact closed* PROP is a PROP with two arrows, $\varepsilon : X \otimes X \to 1$ and $\eta : 1 \to X \otimes X$, satisfying

$$\varepsilon \circ \sigma = \varepsilon, \quad \sigma \circ \eta = \eta,$$

$$(\varepsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ (\eta \otimes \mathrm{id}),$$

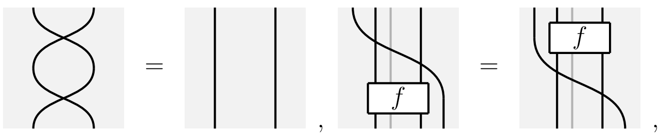
where $\mathrm{id} : X \to X$ is the identity arrow on X, and $\sigma : X \otimes X \to X \otimes X$ is the swap arrow on $X \otimes X$.

Using the diagrammatic language we have developed so far for strict monoidal categories, we will give a quick run through of the conventions we will expect in a PROP diagram. An arrow in a PROP $f : X^{\otimes m} \to X^{\otimes n}$ is depicted diagrammatically as a box labelled f with m inputs and n outputs:

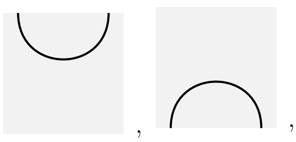


.

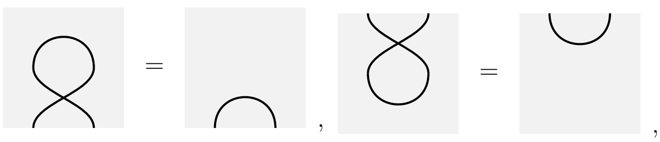
Composing arrows is "plugging" the wires of the diagrams together and monoidal product of arrows is "juxtaposition" of diagrams. The monoidal unit I and its identity arrow $e : 1 \to 1$ are just an empty diagram, the identity morphism $\mathrm{id} : X \to X$ is a straight wire, and the *swap* arrow $\sigma : X \otimes X \to X \otimes X$ is two intersecting wires satisfying



for all morphisms f. The ε and η are depicted as



respectively, satisfying the diagrammatic equations



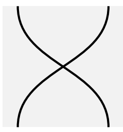
In short, it is a string diagram where we don't have to be concern with the labels on the wires because they all have the same label.

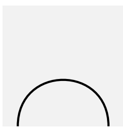
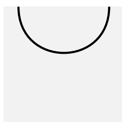
We will draw a diagram with lines of various shades; a darker shade means that that portion of the diagram is the focus of the discussion while a lighter shade means otherwise. A lighter shade lines can also indicate a repeated pattern. Occasionally, we may zoom in on the diagram while not showing the rest of it. It should be clear from the context what the lighter shade of lines mean or that we have zoomed in on the diagram while the rest remains the same.

If one recalls about $\mathbf{Qubit}, \mathbf{Qudit}_n, R\mathbf{bit}, R\mathbf{dit}_n$, it shouldn't be hard to realise that they are self-dual, compact closed PROPs. Example 1 on arithmetic processes is also a self-dual, compact closed PROP.

The ZX, ZW and ZH calculi are self-dual, compact closed PROPs, with a signature T, quotient by an equivalence relation E on arrows respecting the number of inputs and outputs, that is, they are string diagrams with a single generating object X, and the arrows are freely generated by the set T quotient by the axioms E. These graphical calculi are used to model **Qubit** theory, or even $\mathbf{Qudit}_n$ theories, and, after fixing a computational basis, we can set up an interpretation map $i_{Z*} : Z* \to \mathbf{Qudit}_n$. The interpretation on the structures of the categories are as follows:

$e :=$ $\mapsto 1,$ $\qquad$ $\mathrm{id} :=$ $\mapsto \sum_{j=0}^{n-1} |\, j \,\rangle\langle\, j \,|,$

$\sigma :=$  $\mapsto \sum_{j,k=0}^{n-1} |\, kj \,\rangle\langle\, jk \,|,$

$\varepsilon :=$  $\mapsto \sum_{j=0}^{n-1} \langle\, jj \,|,$ $\qquad$ $\eta :=$  $\mapsto \sum_{j=0}^{n-1} |\, jj \,\rangle.$

In the next couple of chapters, we will look at the properties of the ZW and ZX calculi with respect to **Qubit**, and the present a version that is complete. The final chapter will be all other, mainly unfinished, generalisations of the calculi, which includes generalising for $\mathbf{Qudit}_n$.

Chapter 3

The ZW calculi

Overview:

- Section 3.1 — We introduce the motivation of the ZW calculus, entanglement classification. We then give a quick run through of its algebraic properties.

- Section 3.2 — We then show the proof of the completeness of the calculus with respect to the category R**bit**. All these so far are already noted by Hadzihasanovic in [31, 33]. Our contribution is noting that the Clifford+T fragment of qubit quantum theory is also complete. This will be used in the next chapter regarding the completeness of the ZX calculus.

- Section 3.3 — This section is the main original contribution in this chapter. We noted the technique used in the previous section to prove the completeness of a calculus, and applying it to a fragment of the ZW calculus, called the fermionic ZW calculus; we will first show its universality, then add axioms to make the calculus complete. So far, this has been reported in the paper [32], a joint work between Hadzihasanovic, de Felice, and Ng.

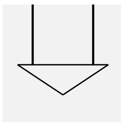
  This technique can be applied for all universal diagrammatic calculus with a clear normal form, and we will demonstrate it by apply the technique again for a fragment of the fermionic ZW calculus, called the even ZW calculus. The even ZW calculus is a calculus which has two different symmetric braidings, which may be of interest to mathematicians. Furthermore, it may have a interpretation of emulating particles with space-time curvature, which physicists might want to look into. This is not reported anywhere yet.

3.1 Basic properties of the ZW calculus

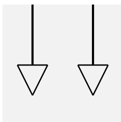
In this section, we shall highlight some of the motivations and properties of the ZW calculus explored by Coecke and Kissinger in [16] and Hadzihasanovic in his DPhil thesis [31]. This is also somewhat summarised in [30, 33].

3.1.1 Entanglement classification

Entanglement of quantum systems means that they are related in some ways. An entangled state can only be drawn as

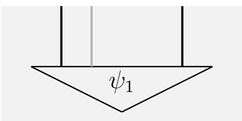

,

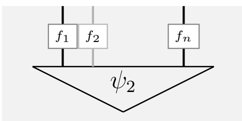
and can never be drawn separately like

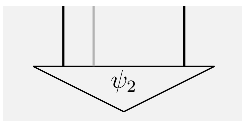

.

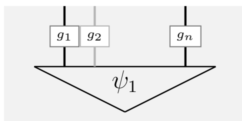
Entangled states can be thought like the two systems are connected informationally, that is, the information is able to flow from one system through the wires to another system. An unentangled state is when the wires are broken. This graphical definition makes it easy to grasps the concept of entanglement, and it seems to place diagrammatic calculus in good stead to analyse entanglement problems. One such problem is to classify them.

One way of classification of entangled states is by stochastic local operations and classical communication (SLOCC) equivalence classes. This loosely means that if two states $|\,\psi_1\,\rangle$ and $|\,\psi_2\,\rangle$ are SLOCC equivalent, then there are some local quantum operations f_j, g_k such that



=


,


=


,

with a nonzero probability of success.

The predecessor of ZW calculus, sometimes called the GHZ-W calculus, was built on ZX calculus for the entanglement classification problem. It features, as the building blocks, the representatives of the only two SLOCC equivalence classes of (non-trivial) three qubit entanglement [25], one is the GHZ state

$$|\, GHZ \,\rangle = \frac{1}{\sqrt{2}}(|\, 0\ 0\ 0 \,\rangle + |\, 1\ 1\ 1 \,\rangle),$$

and the other one the W state

$$|\, W \,\rangle = \frac{1}{\sqrt{3}}(|\, 0\ 0\ 1 \,\rangle + |\, 0\ 1\ 0 \,\rangle + |\, 1\ 0\ 0 \,\rangle).$$

Coecke and Kissinger found that the two different classifications correspond to different Frobenius algebras, the GHZ state corresponds to a special commutative Frobenius algebra and the W state corresponds to an anti-special commutative Frobenius algebra. Hence, the problem is equivalent to the classification of Frobenius algebras. Furthermore, as the GHZ-W calculus (together with arbitrary single qubit states) is universal, Coecke and Kissinger believed that this calculus is useful to analyse multi-partite entanglements.

We are going to state a painfully simplified heuristic to show the two Frobenius algebras.

We are going to represent a GHZ state, up to normalisation, as

 $:=$ $\mapsto |\, 000 \,\rangle + |\, 111 \,\rangle,$

and a W state, up to normalisation, as

 $:=$ $\mapsto |\, 001 \,\rangle + |\, 010 \,\rangle + |\, 100 \,\rangle.$

Their two qubit versions are

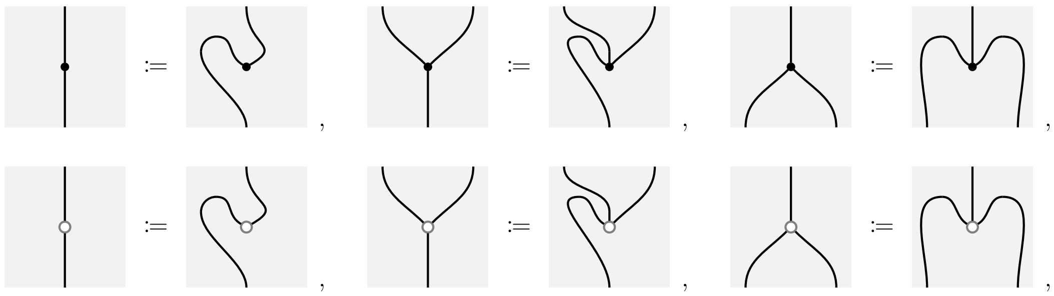
 $:=$ $\mapsto |\, 00 \,\rangle + |\, 11 \,\rangle,$

 $:=$ $\mapsto |\, 01 \,\rangle + |\, 10 \,\rangle,$

respectively.

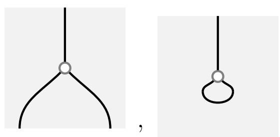
Remark 26. The two qubit entanglements represented by the binary black and white vertices are of the same entanglement class. They are called an EPR state (modulo normalisation) or a Bell state.

We can define a different partition of inputs and outputs on these state by fixing a particular partial transposition of the wires. One such definition can be



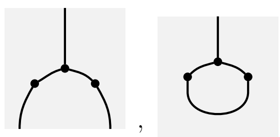
By observing the interpretations, the binary and ternary black and white vertices are symmetric with respect to swapping the wires. Hence, it doesn't matter how we define the partial transpositions.

The ternary white vertex forms a special commutative Frobenius algebra without much effort: the pair

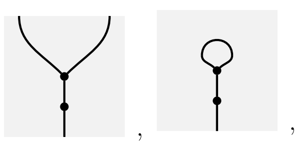


is a commutative monoid, and it forms a special Frobenius algebra with its comonoid obtained by transposition.

We can construct an anti-special commutative Frobenius algebra that is SLOCC equivalent to the ternary black vertex: the pair



is a commutative monoid, and the pair



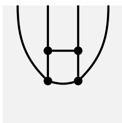
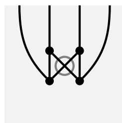
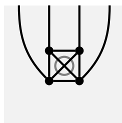
is a commutative comonoid, and the monoid-comonoid forms an anti-special commutative Frobenius algebra.

The classification of $n \geq 4$ entangled qubits is still an active area of research. Hadzihasanovic noted that while we can analyse the problem with two Frobenius algebras, it might be possible to just use the black vertices and study how they are connected. His argument came from the following observations:

- the entangled states are connected diagrams, assuming that the diagrams are simplified to give the maximum number of disconnected components,
- for the classification of bipartite states, the binary black vertex is a representative for the Bell state,
- for the classification of tripartite states, the W-like states are equivalent to the ternary black vertex, and the GHZ-like states are equivalent to

   ,

- for the classification of $n = 4$ qubits states, the following are some examples of states in distinct SLOCC super-classes [44]:

   ,  ,  ,  ,  ,

  where a quaternary black vertex is

   .

  The gray circle indicates another kind of "braiding" which we will talk about in the later subsection.

With all these diagrams, he suggests that maybe we can study these classifications from a topological viewpoint. More specifically, the classification has something to do with topological inequivalence of the networks.

Nothing much has been done beyond this point.

3.1.2 Algebraic properties

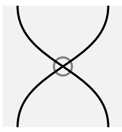
We have seen that Coecke and Kissinger suggested that the main defining structure of the ZW calculus is having a special and anti-special commutative Frobenius algebras. However, Hadzihasanovic thinks otherwise; he suggests that the anti-special commutative Frobenius algebra should be replaced with a Hopf algebra. The Hopf algebra is inspired by studying the *fermionic line* in the theory of quantum groups [45]. We will see how to use ZW calculus to model fermionic quantum theory in section 3.3. With this change, he quickly obtained a completeness result, the completeness of ZW calculus for $\mathbb{Z}$**bit** [30].

Hadzihasanovic split the ZW calculus into a couple fragments: the pure fragment, the vanilla version, and the one enriched by a commutative ring R. The pure fragment is responsible for the Hopf algebra structure, the vanilla ZW calculus adds a special commutative Frobenius algebra in the picture, and then we add a ring structure on top of everything.

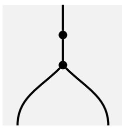
We will introduce the axioms of the calculus as we explore the algebraic properties. These axioms will be labelled with an upper case alphabet.

Pure fragment

The pure fragment of the ZW calculus can be further broken down into an *even fragment* and the rest. The even fragment consists of the generators



$$\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, jk \,\rangle\langle\, kj \,|, \tag{3.1}$$

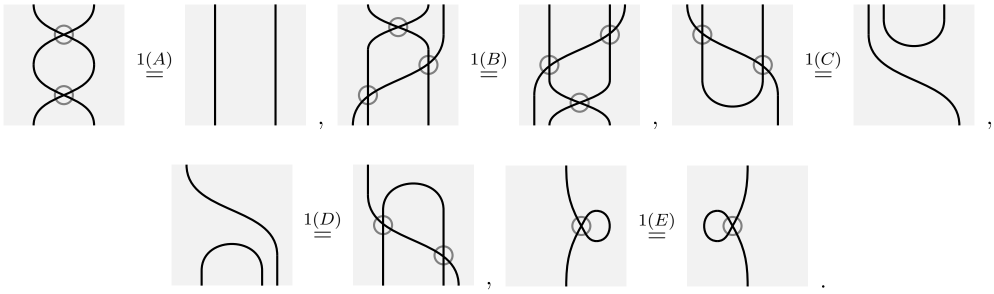


$$\mapsto |\, 0 \,\rangle\langle\, 00 \,| + |\, 1 \,\rangle\langle\, 01 \,| + |\, 1 \,\rangle\langle\, 10 \,|. \tag{3.2}$$

These generators have the characteristic that the sum of the numbers in each bra-ket of the summands is even. For example, the bra-kets for the summands in Equation 3.1 are $|\, jk \,\rangle\langle\, kj \,|$. The sum of the numbers in the bra-kets are $j+k+k+j = 2(j+k)$, which is an even number. Maps with such characteristic is called *even maps*.

The identity map, swap map, cup and cap are also even maps. Hence, any composition of the generators, by plugging or juxtaposition, will preserve the even parity. For this reason, this fragment is called even fragment.

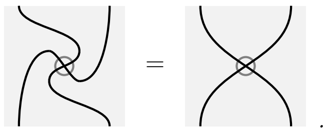
The first generator in Equation 3.1 is called the *crossing*. From the artwork, one can tell that it is some sort of braiding, and from the interpretation, it is a $\mathbb{Z}_2$-grading. Hence, it satisfies the following axioms:



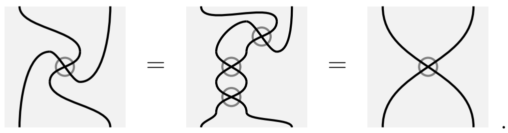
(3.3)

These axioms kind of tell us that we are building a calculus with two symmetric braidings (which is the subject of virtual knot theory [40]), although these axioms are probably not enough. However, there are still some interesting lemmas regarding the crossing.

Lemma 27. *The crossing is invariant under quarter rotation. That is,*

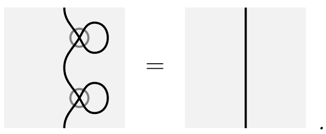


Proof.



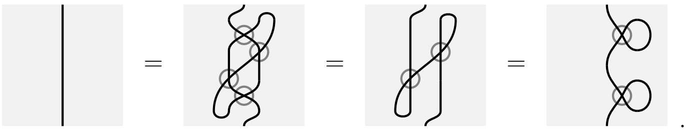
□

Lemma 28. *The self crossing is involutive. That is,*



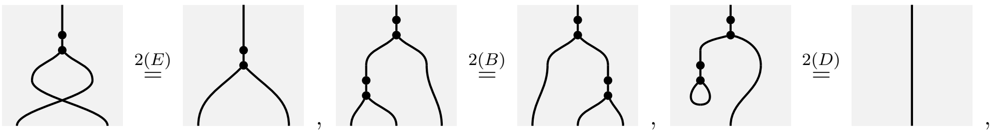
Proof. This follows from $1(E)$, combined with the equation

which is a consequence of the fermionic swap axioms by the Whitney trick [39, p. 484]. More explicitly,



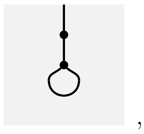
□

Next, we have the generator in Equation 3.2. This generator satisfies the following commutative monoid axioms



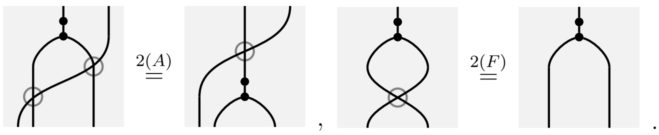
(3.4)

where the unit is



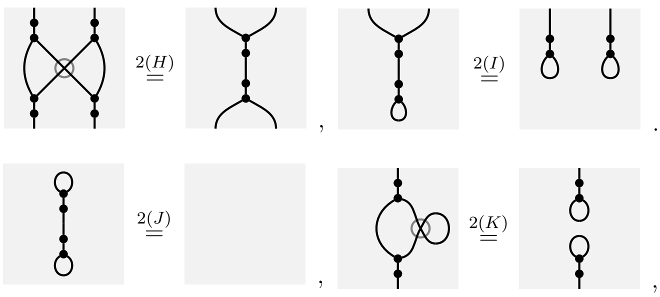
(3.5)

and it is natural and commutative with respect to the crossing:



(3.6)

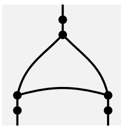
It is the monoid in the Hopf algebra from the fermionic line in the theory of quantum groups. The comonoid in the fermionic line is the canonical transposition of the monoid, hence we have the following axioms:



(3.7)

where the braiding in the bialgebra is the crossing, and the antipode in the Hopf algebra is the self crossing.

The monoid can be seen as a partial function that "adds" numbers, but adding to 2 or greater is not allowed, as noted by Kissinger. This is closely related to the beam splitter formalism which we will talk about in section 3.3. We can also construct an operation that adds mod 2:

 $\mapsto |\,0\,\rangle\langle\,0\;0\,| + |\,0\,\rangle\langle\,1\;1\,| + |\,1\,\rangle\langle\,0\;1\,| + |\,1\,\rangle\langle\,1\;0\,|.$

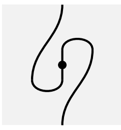
Next we will introduce the odd maps, the maps where if you add up the numbers in each bra-ket of the summands, you will always get an odd number. This is simply obtained by introducing to the even fragment a binary black vertex

 $\mapsto |\,0\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,0\,|,$ (3.8)

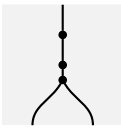
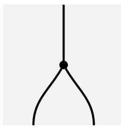
as a parity reversing operation, or in other words, it is a bit flip operator.

Remark 29. The bit flip operation is an odd state. We have many choices of odd states to add to the even fragment, like $|\,1\,\rangle$, but using the bit flip seems most natural.

The bit flip is a self transpose and involutive operation:

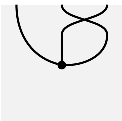
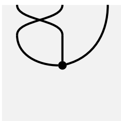
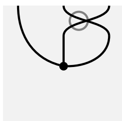
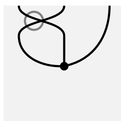
 $=$  , $\overset{2(C)}{=}$. (3.9)

From how the monoid in 3.4 is drawn, it is suggestive that

 $=$  , (3.10)

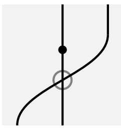
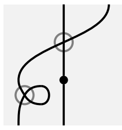
which we get a ternary black vertex. This is exactly why we drew the monoid in that manner in the first place; this is the W state, and from now we will treat this as a generator instead. So, the monoid is constructed by composing the W state with a bit flip on any one of the wire, and partial transposing the appropriate wires.

We can then rewrite thes axioms $2(E, F)$ as such:

 $\overset{2(E)}{=}$  $\overset{2(E)}{=}$  ,  $\overset{2(F)}{=}$  $\overset{2(F)}{=}$  . (3.11)

The parity of the diagram can be determined by counting the number of black vertices; even number of black vertices is a even state, and odd number of black vertices is a odd state.

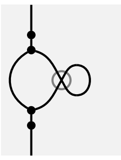
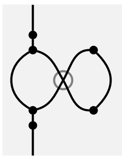
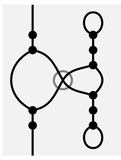
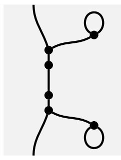
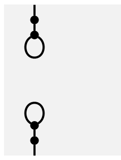
In the even fragment, the crossing can be treated as a second braiding. However, in the entire pure fragment, the odd states is not natural with the crossing. It induces a self crossing when passing through a crossing, hence we have the axiom:

 $\overset{2(G)}{=}$  . (3.12)

With all these axioms, the axioms for the Hopf algebra $2(K)$ can now be derived.

Lemma 30. *The axiom $2(K)$ is derivable.*

Proof.

 $\overset{2(C)}{=}$  $\overset{2(D)}{=}$  $\overset{2(H)}{=}$  $\underset{2(J)}{\overset{2(I)}{=}}$  . (3.13)

□

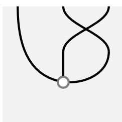
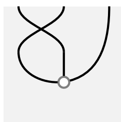
This concludes the portion on the pure fragment of ZW calculus.

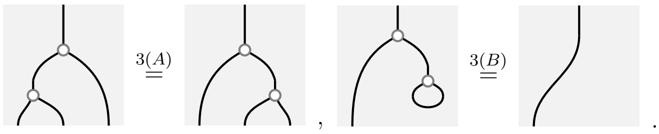
Vanilla ZW calculus

The vanilla ZW calculus has the addition of a ternary white vertex as generator:

 $\mapsto |\,0\ 0\ 0\,\rangle + |\,1\ 1\ 1\,\rangle.$ (3.14)

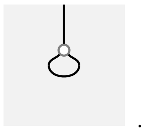
This generator is a mixed parity generator, that is, it has both odd and even components. This is the GHZ state (up to normalisation), and it satisfies the following commutative monoid axioms:

 $\overset{3(C)}{=}$  $\overset{3(C)}{=}$  ,



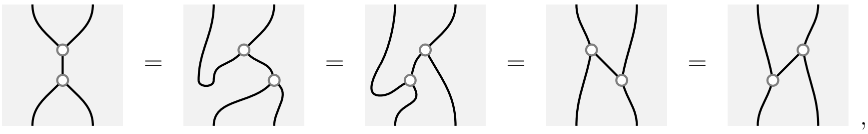
(3.15)

The monoid is obtained by transposing any of the two output wires, and the unit is



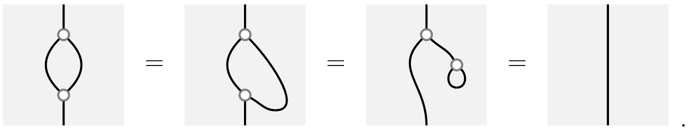
(3.16)

With these axioms, we get a Frobenius algebra for free:



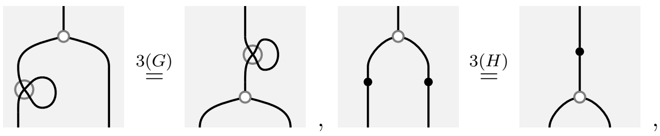
(3.17)

and this Frobenius algebra is special:



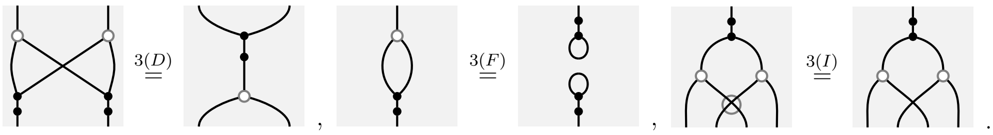
(3.18)

This monoid has a few nice properties. The self crossing is a module homomorphism, and the bit flip is a monoid homomorphism with respect to it:



(3.19)

and it interacts with the black (co)monoid in Equation 3.4:



(3.20)

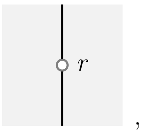
The the first two axioms in 3.20 seems like a Hopf algebra, but they only have the form of a Hopf algebra and are not even true bialgebra. The last axiom is telling us that when plugging the black and white ternary vertices together, the swap and crossing are interchangeable.

It is not hard to notice that the calculus is about the connectivity of the vertices, and not having strict distinction of inputs and outputs. Furthermore, the black and white vertices are all symmetric, hence we obtain a diagram that resembles a network where only the topology matters. This is a meta-rule, which is often known as the *only topology matters* rule.

We are done with all the axioms of the vanilla ZW calculus. With all these axioms introduced so far, we are ready to show that the vanilla ZW calculus is complete for the category $\mathbb{Z}$**bit** [30]. However, we will enrich the calculus with a ring structure.

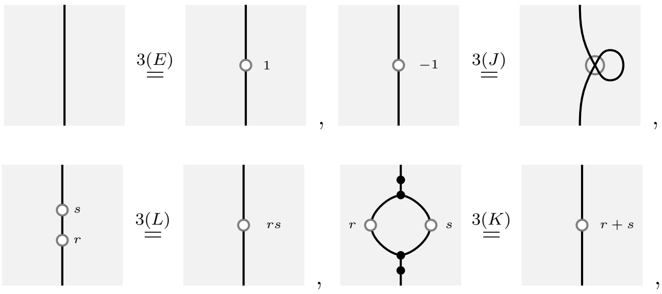
Full ZW calculus

Suppose r is an element of a commutative ring R, we are going to add a r labelled binary white vertex:



$$, \quad \mapsto |\,0\,\rangle\langle\,0\,| + r|\,1\,\rangle\langle\,1\,|. \tag{3.21}$$

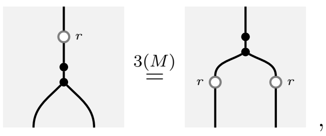
We can define the ring operations in ZW calculus as:



(3.22)

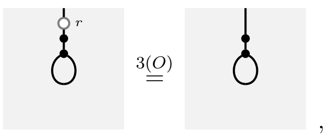
where the composition and convolution by black vertices correspond to multiplication and addition of the ring elements.

The binary white vertex interacts nicely with the black and white monoids: it is a monoid homomorphism for the black monoid



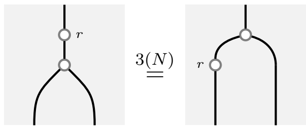
(3.23)

is eliminated when plugged to the unit for the black monoid



(3.24)

and it is a module homomorphism for the white monoid



(3.25)

We are pretty much done with the introduction to the axioms of ZW calculus. We obtain a diagrammatic calculus which is in the form of an undirected graph, that is, only topology matters, and the axioms are mostly motivated by having nice algebraic properties, which is what Hadzihasanovic envisioned. He also proved that this set of axioms is complete for R**bit**, for some commutative ring R. The completeness result is briefly summarised in the next section.

3.2 Completeness of the ZW calculus

In [30], Hadzihasanovic proved that the vanilla ZW calculus is complete for $\mathbb{Z}$**bit**, and in [31, 33], Hadzihasanovic extended the ZW calculus and proved the completeness for R**bit** for any commutative ring R. We shall recap the proof in this section.

The most important message of this chapter is the idea of the proof. The proof follows a normal form argument, and is repeated in the subsequent few sections for different versions of the ZW calculi. The completeness proof is in three stages:

1. First, we will associate to any state $v : R \to (R \oplus R)^{\otimes n}$ of R**bit** a diagram $g(v) : 0 \to n$ in ZW_R such that $i_{zw}(g(v)) = v$. Because both categories are compact closed, and the dualities of R**bit** are in the image of i_{zw}, this assignment can be extended to any map of R**bit**, proving universality of our interpretation. We will say that a diagram is in *normal form* if it is of the form $g(v)$ for some v.

2. Then, we will show that any composite of diagrams in normal form can be rewritten in normal form using the equations in E_{zw}, proving that g determines a monoidal functor from R**bit** to ZW_R.

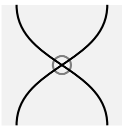
3. Finally, we will show that all the generators of ZW_R can be rewritten in normal form using the equations in E_{zw}, proving that g and i_{zw} are two sides of a monoidal equivalence between ZW_R and R**bit**.

The most important stage is stage 2. That is because we know the initial and final diagrams of the rewriting process. While rewriting a composite of normal form diagrams in a single normal form, we are most likely going to encounter some equalities which we have no idea how to proceed. Then that will be an axiom for the calculus. This procedure has also been employed for the recent ZH calculus [5].

The section is separated into two parts for clarity. The first part is the preparation phase, where we state the axioms and the derive rules that may be used in the completeness proof. Then is the execution phase, where we demonstrate the completeness proof. The completeness proof of the other versions of ZW calculi also follows the same format, but we won't state them as "preparation" and "execution" explicitly.

3.2.1 The preparation

A quick summary of ZW calculus: The ZW calculus is a self dual, compact closed PROP with the following generators T_{zw} and interpretation i_{zw}:

 $\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, j\ k \,\rangle\langle\, k\ j \,|,$

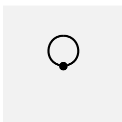
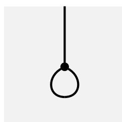
 $\mapsto |\,0\ 0\ 1\,\rangle + |\,0\ 1\ 0\,\rangle + |\,1\ 0\ 0\,\rangle,$  $\mapsto |\,0\ 1\,\rangle + |\,1\ 0\,\rangle,$

 $\mapsto |\,0\ 0\ 0\,\rangle + |\,1\ 1\ 1\,\rangle,$ 

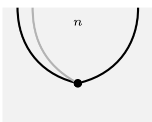
$\mapsto |\,00\,\rangle + r|\,11\,\rangle.$

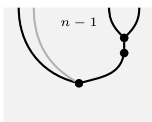
Because of the monoid structure of the black and white vertices, we can define the arbitrary arity vertices as such:

1. *Black vertices.* The nullary and unary black vertices are defined as follows:

 $:=$  $\mapsto 0,$ $:=$  $\mapsto |\,1\,\rangle.$

We already have binary and ternary black vertices. For $n > 3$, the n-ary black vertex is defined inductively by

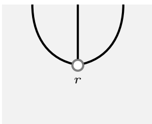


$:=$ 

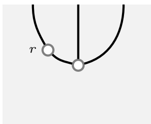
$\mapsto \sum_{k=1}^{n} |\, \underbrace{0\ldots0}_{k-1} 1 \underbrace{0\ldots0}_{n-k} \rangle.$

Here, and in what follows, lighter wires and vertices indicate the repetition of a pattern for a number of times, which may or may not be specified.

2. *White vertices.* First, define a ternary vertex with parameter $r \in R$ as

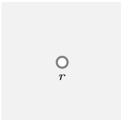


$:=$

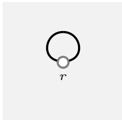


$\mapsto |\,000\,\rangle + r|\,111\,\rangle.$

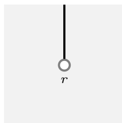
The nullary and unary white vertex with parameter $r \in R$ is then defined by



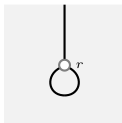
$:=$



$\mapsto 2 + r,$

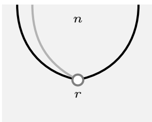


$:=$

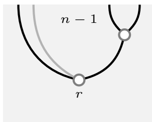


$\mapsto |\,0\,\rangle + r|\,1\,\rangle.$

We already have binary and ternary white vertices with parameters. For $n > 3$, the n-ary white vertex with parameter $r \in R$ is defined inductively by

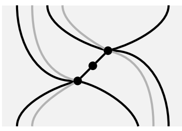
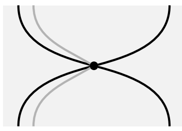
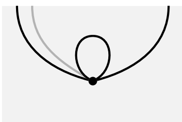
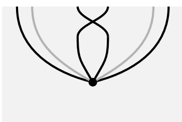
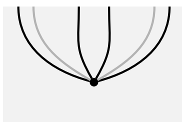
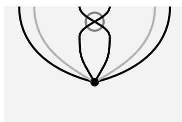


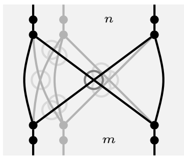
$:=$

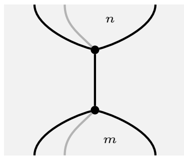


$\mapsto |\,\underbrace{0\ldots0}_{n}\rangle + r|\,\underbrace{1\ldots1}_{n}\rangle.$

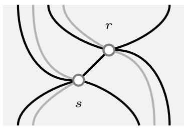
This arbitrary arity vertices presentation is also known as *spiders presentation*. With the spiders, we can write the axioms more compactly with some inductive schemes. Some of them subsume several axioms at once. It is possible to perform diagram rewriting directly with such equation schemes, using for example the theory of pattern graphs developed in [41]. In the following table, the non-condensed axioms are listed on the left, while the condensed version is on the right.

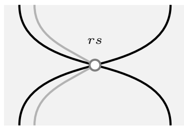
Axioms	Condense axioms
$2(B)$	 $\overset{2(b)}{=}$ 
$2(D)$	 $\overset{2(d)}{=}$ 
$2(E, F)$	 $\overset{2(e)}{=}$  $\overset{2(f)}{=}$ 

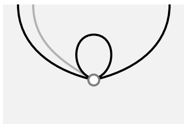
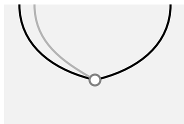
$2(H, I, J)$ 

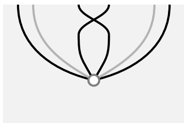
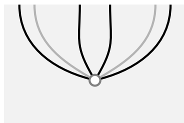
$\overset{2(h)}{=}$ 

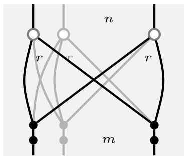
for $m, n \geq 0$

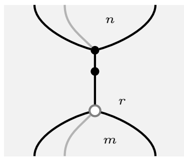
$3(A, L, N)$ 

$\overset{3(a)}{=}$ 

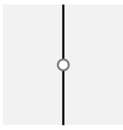
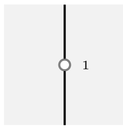
$3(B)$  $\overset{3(b)}{=}$ 

$3(C)$  $\overset{3(c)}{=}$ 

$3(D, M, O)$ 

$\overset{3(d)}{=}$ 

for $m = 0 = n$, or $m > 0, n \geq 0$

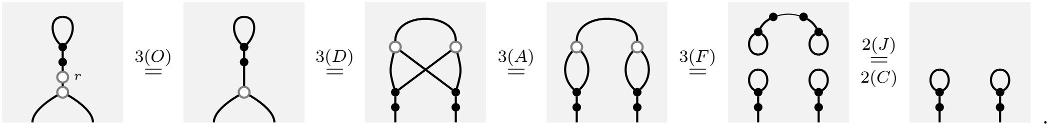
$3(E)$  $\overset{3(e)}{=}$ $\overset{3(e)}{=}$ 

The condense axioms show some nice properties: both black and white vertices are symmetric under permutation of wires (axioms $2(e), 3(c)$), which allows us to write vertices with different numbers of inputs and outputs, transposing some of them with no ambiguity. Most importantly, they satisfy certain "fusion" rules, as shown in axioms $2(b, d), 3(a, b)$.

All these condensed axioms are easy to prove. Perhaps, the axioms that require some justification are $2(h)$ and $3(d)$.

The axiom $2(h)$ can be proved case by case. For $m = n = 0$, this is $2(J)$. The case $n = 0$, $m > 1$ is an inductive generalisation of $2(I)$, and similarly for $n > 0$,$m = 0$. The case $m = n = 1$ follows from $2(C)$. The case $m = n = 2$ is simply $2(H)$. From here on, it is a double induction on n, m using $2(A)$ to slide the black vertices through the crossings.

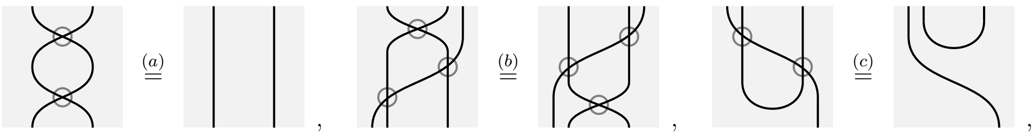
Finally, the axiom $3(d)$ can be proved case by case. For $m = 2$, $n = 0$,

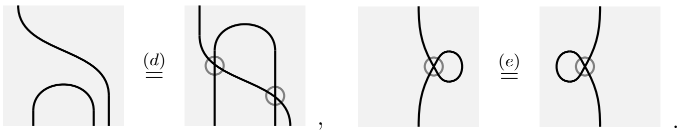


The case $m = n = 0$ then simply follows by connecting the two outputs and apply $2(C, J)$ to get the empty diagram. The case $m = 1$,$n = 0$ is $3(O)$, $m = n = 1$ follows from $2(C)$, $m = 1, n = 2$ is $3(M)$, $m = 2$,$n = 2$ is combining $3(D, M)$. From here on, it is a double induction on n, m.

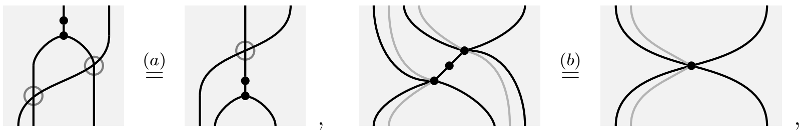
A summary of the axioms E_{zw} in the condense form is as follows: with appropriate partial transposition,

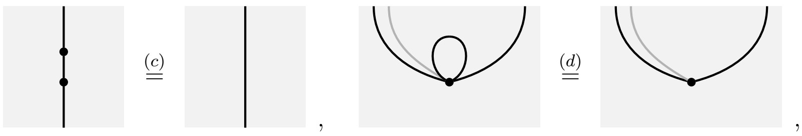
1. The following are the axioms for the crossings:

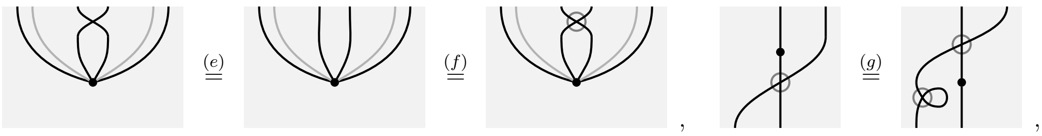


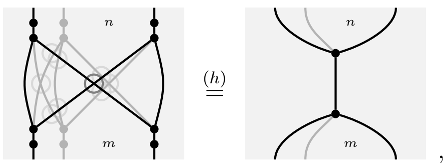


2. The following are the axioms for the black vertices:



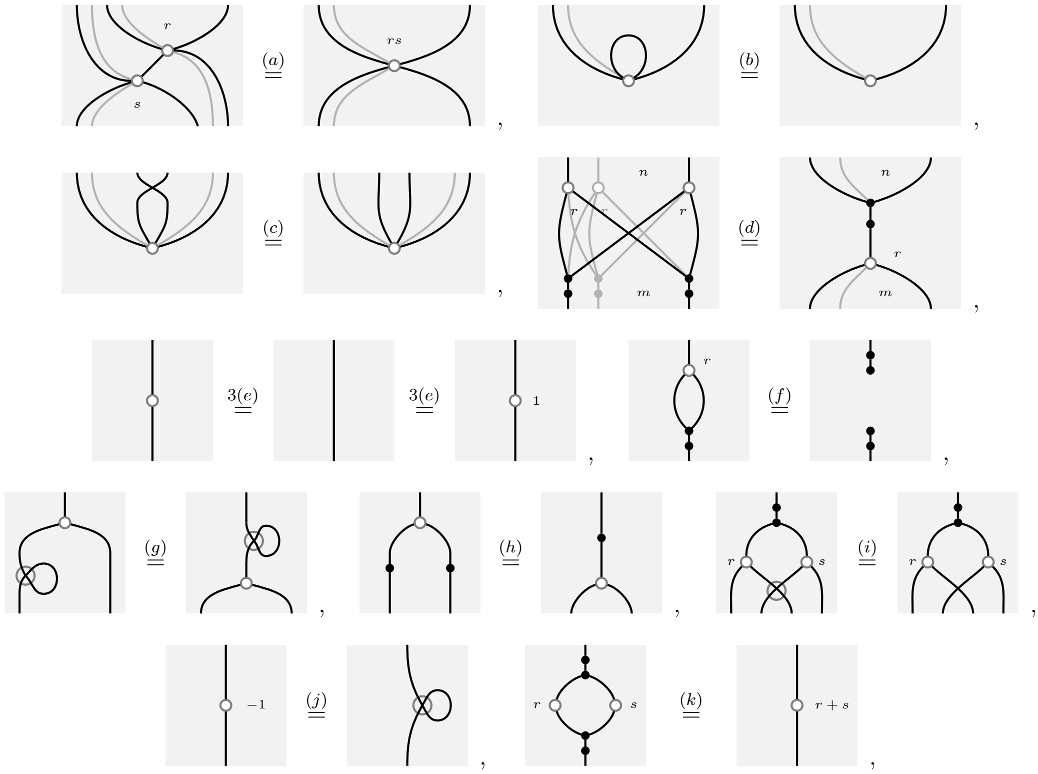






for all $m, n \in \mathbb{N}$.

3. Given a commutative ring R, the following are the axioms for the white vertices:

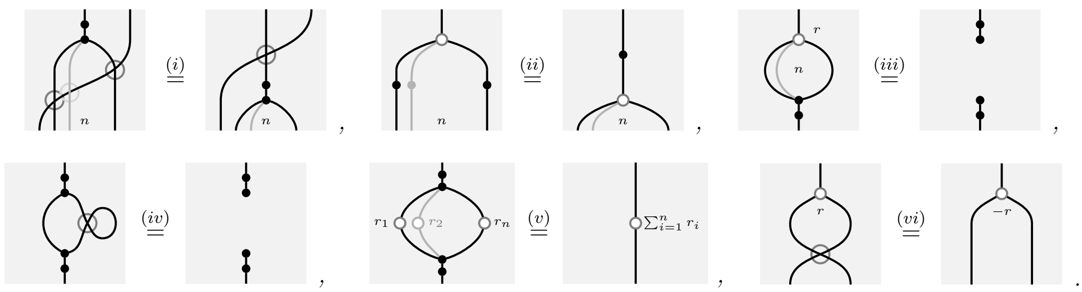


for $r, s \in R$, $m, n \in \mathbb{N}$ such that either $m = n = 0$ or $m > 0$.

The set of axioms E_{zw} is sound with respect to the interpretation i_{zw}, that is, the map i_{zw} is functorial. This can be easily check by verifying the interpretation of each axiom.

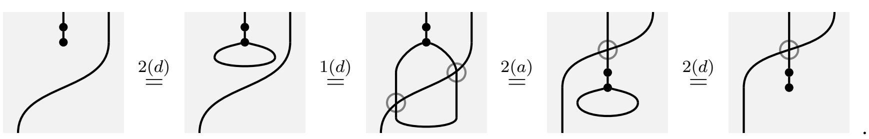
There are some derived rules that are useful for the completeness proof.

Proposition 31. *The following are derived rules:*



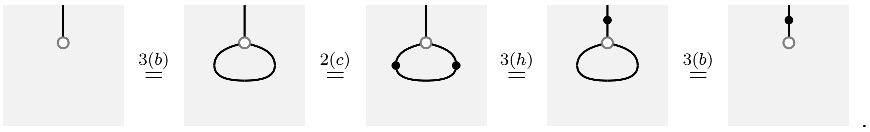
for all $n \in \mathbb{N}$ in rules (i, ii, v), $n \geq 2$ in rule (iii), and $r, r_i \in R$ in rule (v, vi).

Proof. Rule (i): The case for $n = 0$ is as follows:



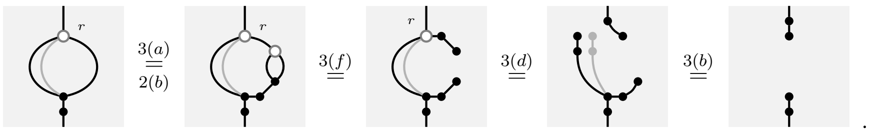
The case for $n = 1$ is just using axiom $2(c)$, and $n = 2$ is axiom $2(a)$. The cases for $n \geq 2$ are induction starting from axiom $2(a)$ and using the axiom $2(b)$.

Rule (ii): The case for $n = 0$ is as follows:



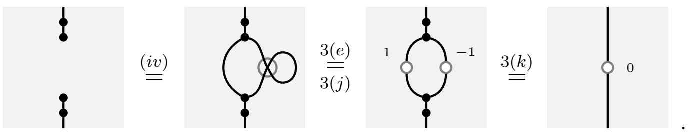
The case for $n = 1$ is just using axiom $3(e)$, and $n = 2$ is axiom $3(h)$. The cases for $n \geq 2$ are induction starting from axiom $3(h)$ and using the axiom $3(a)$.

Rule (iii): The case $n = 2$ is axiom $3(f)$. The cases $n > 2$ follows by:



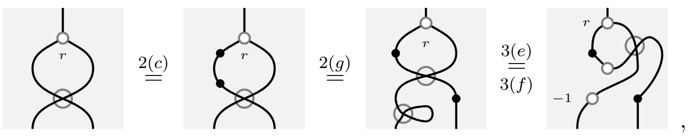
Rule (iv): This is proved in Lemma 30.

Rule (v): The case for $n = 0$ is as follows:

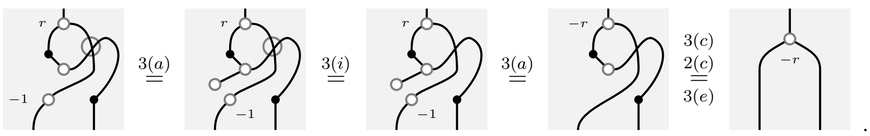


The case for $n = 1$ is simply applying axiom $2(c)$, and the case $n = 2$ is axiom $3(k)$. For cases for $n \geq 2$ are induction starting from axiom $3(k)$ and using the axiom $2(b)$.

Rule (vi): Start with:



where we have transposed some wires in the last step. Then,



□

3.2.2 The execution

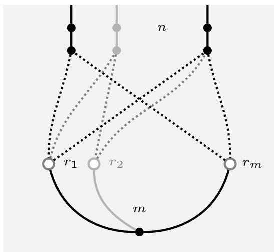
Theorem 32 (Universality of the ZW calculus)**.** *The interpretation functor i_{zw} : $ZW_R \to R\mathbf{bit}$ is full.*

Proof. Due to the presence of self-duality (known as the Choi-Jamiołkowski isomorphism in quantum information theory), every morphism in $R\mathbf{bit}$ can be written as a partial transpose of a state. Hence it suffices to prove that for every state in $R\mathbf{bit}$ there exists a corresponding ZW diagram.

Write an arbitrary n-partite state in $R\mathbf{bit}$ as

$$\sum_{i=1}^{m} r_i |\, b_{i_1} \dots b_{i_n} \,\rangle,$$

where $r_i \neq 0$ and no two kets in the sum are the same. We claim that it is the image of the diagram



(3.26)

,

where the i-th white vertex has one connection to the j-th output if and only if $b_{i_j} = 1$, for $i = 1, \dots m$, $j = 1, \dots n$. The claim can be proved by a direct calculation: the black vertex creates the state

$$\sum_{k=1}^{n} |\, \underbrace{0 \dots 0}_{k-1} 1 \underbrace{0 \dots 0}_{n-k} \rangle,$$

where the i^{th} summand will be transformed to $r_i |\, b_{i_1} \dots b_{i_n} \,\rangle$ by first going through the i^{th} white vertex to give

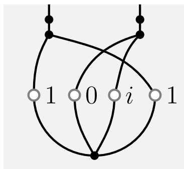
$$r_i |\, 0 \dots 0 \underbrace{1 \dots 1}_{p} 0 \dots 0 \,\rangle$$

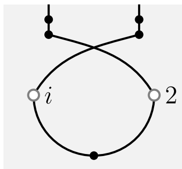
where there are p number of 1's in $b_{i_1} \dots b_{i_n}$. Then the 1's are swapped according and added using the black vertices to get the desired result. $\square$

The proof of universality showed that any state is the image of the diagram in (3.26). We will call this diagram the *normal form*. From how the normal form is constructed, it is clear that it is unique up to a permutation of the white vertices. It is possible to give an ordering to the white vertices, but due to the symmetry of the black vertex with respect to swap it does not matter what ordering we take.

Allowing diagram (3.26) to have two white vertices with the same connections, and r_i to be 0 for some i, we obtain what we call a diagram in *pre-normal form.*

Example 33. The following diagrams are in pre-normal form, but only the second one is in normal form:



,


.

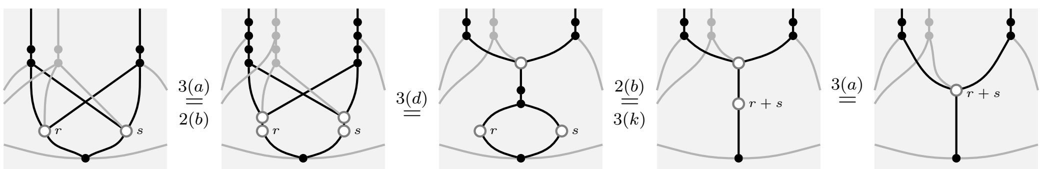
Both are interpreted as the state $1 \mapsto i|\,01\,\rangle + 2|\,10\,\rangle$.

The pre-normal form can be reduced to a normal form, as we will now show. Then, the structure of our completeness proof is as follows:

- any composite of two diagrams in normal form can be rewritten to pre-normal form:
  - the juxtaposition of diagrams in normal form can be rewritten to pre-normal form;
  - the plugging of an output of a diagram in normal form into another (self-plugging) can be rewritten to pre-normal form;
  - an arbitrary composition of two diagrams can be factored as a juxtaposition, followed by a self-plugging;
- all generators can be rewritten to normal form.

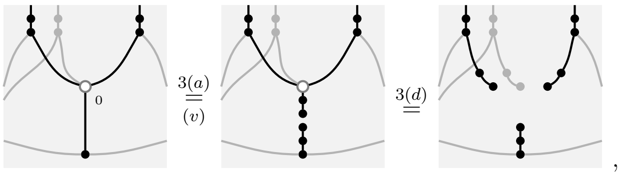
Lemma 34. *A ZW diagram in pre-normal form can be rewritten in normal form.*

Proof. Suppose a diagram is in pre-normal form. If two white vertices have the same connections, then we can "sum" them by



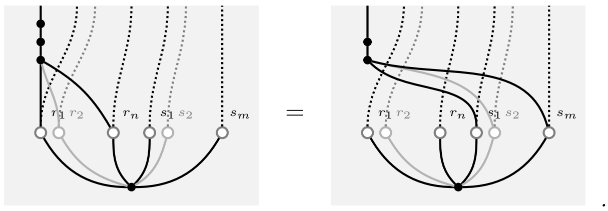
.

If $r_i = 0$ for some i, then we can eliminate that white vertex by



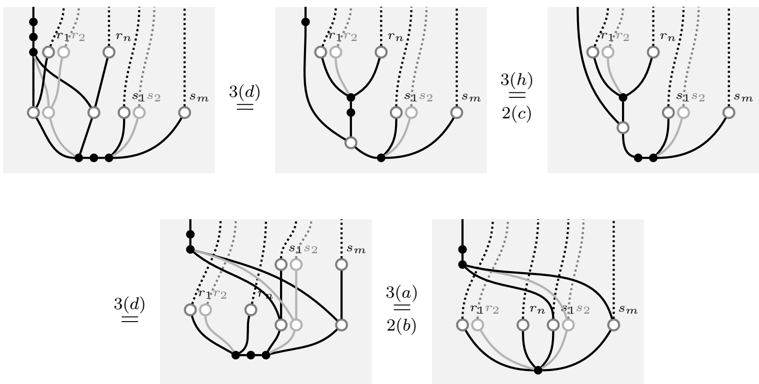
and use axiom $2(b)$ to simplify the diagram. □

Lemma 35 (Negation)**.** *Given a diagram in pre-normal form, the diagram obtained by composing an output with a binary black vertex can be rewritten in pre-normal form, which complements the connections of that output to the white vertices; that is, locally,*



Remark 36. In the picture, the dotted wires can stand for a multitude of wires.

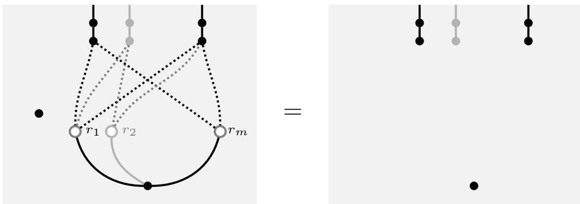
Proof. First isolate the first n white vertices on the left hand diagram using axioms $3(a)$ and $2(b)$, then proceed as follows:



□

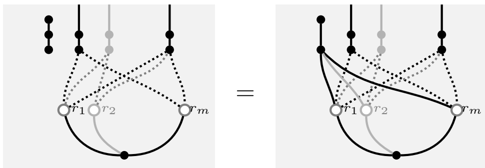
A nullary black vertex is interpreted as the scaler 0; the following lemma shows that it acts as an "absorbing element" for diagrams in pre-normal form, that is, it kills off everything.

Lemma 37 (Absorption)**.** *For all diagrams in pre-normal form, a nullary black vertex eliminates all the white vertices; that is,*



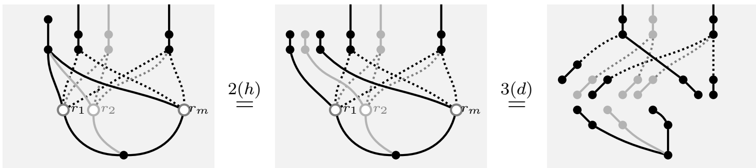
.

Proof. Use axiom 2(b) to expand the nullary black vertex and apply the negation lemma:



.

Then apply axioms 2(h) and 3(d), which eliminate all the white vertices:

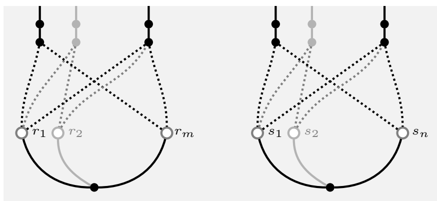


.

The final diagram can be simplified using axiom 2(b) which merges the black vertices to get the desired result. □

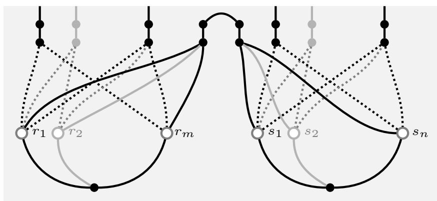
Lemma 38 (Juxtaposition)**.** *The juxtaposition of two diagrams in pre-normal form can be rewritten to pre-normal form.*

Proof. Consider the following juxtaposition of diagrams,



.

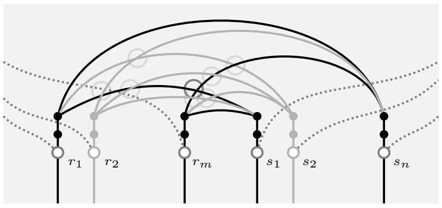
Using the axiom 2(h), we can produce a pair of connected black vertices, and using the negation lemma we can connect the pair of black vertices to the diagrams:



.

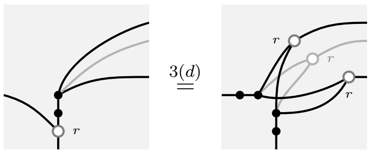
The only case in which this still leaves two disconnected diagrams is when one of the diagrams has no white vertices, that is, it is a nullary black vertex with some outputs like in right-hand side of the the negation lemma. In this case, we can use the nullary black vertex to "absorb" the other diagram, which produces a diagram in pre-normal form.

So assuming both our diagrams have more than or equals to one outputs. Applying axiom $2(h)$ again on the pair of black vertices (after eliminating the middle two black vertices using the $2(c)$ axiom) gives the following diagram (zooming in to the relevant part of the diagram):



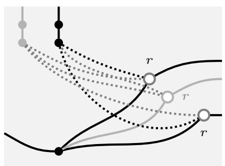
.

We can then push all the white vertices through the top black vertices using axiom $3(d)$, for instance,



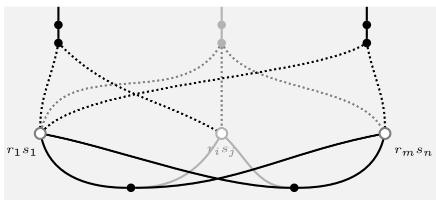
.

We would want to merge the white vertices, but they cannot pass through the crossing. However, using $2(b)$ in the final diagram, the lower black vertex merges with the bottom black vertex and the higher black vertex merges with some outputs, for instance



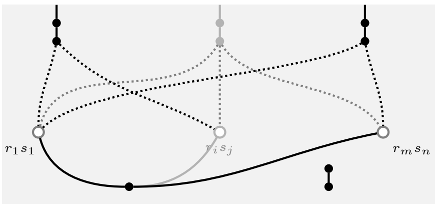
.

The white vertices are all connected to either of the two bottom black vertices and now we can eliminate all the crossings using $3(i)$. After merging white vertices pairwise, there is a white vertex with label $r_i s_j$ for each $i = 1, \ldots, m$, $j = 1, \ldots, n$



.

Finally, one of the bottom black vertices can be eliminated with the negation lemma

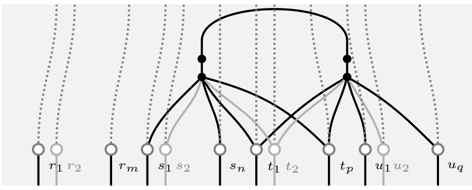


,

and the floating pair of black vertices is eliminated using $2(h)$. $\square$

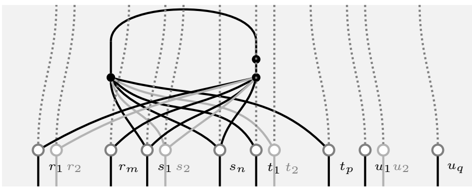
Lemma 39 (Trace)**.** *A self-plugging on a diagram in pre-normal form can be rewritten in pre-normal form.*

Proof. The proof involves some tactful use of the negation lemma. Suppose that we have the following self-plugging diagram, only zooming in to the white vertices and the pair of self-plugged outputs,



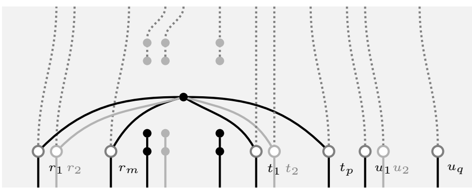
,

where the r white vertices are not connected to either of the outputs, the s white vertices are connected to the left output only, the u white vertices are connected to the right output only, and the t white vertices are connected to both outputs. From the interpretation in R**bit**, we expect the r and t labelled white vertices to survive while the s and u labelled white vertices are eliminated. This can be shown diagrammatically by first performing a negation to obtain the following diagram:



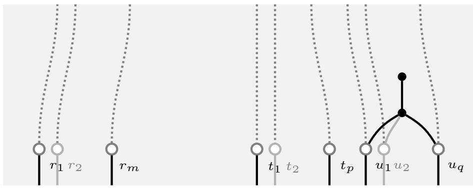
.

We can merge the black vertices using the axiom $2(b)$ and the s labelled white vertices have two connections with the black vertex. This means that we can apply $3(f)$ to eliminate all the s labelled white vertices:



.

We can apply negation lemma again to obtain



.

Finally, we can eliminate all the u labelled white vertices as in the absorption lemma, which completes the proof. □

With the juxtaposition and trace lemma, we have proved the following theorem:

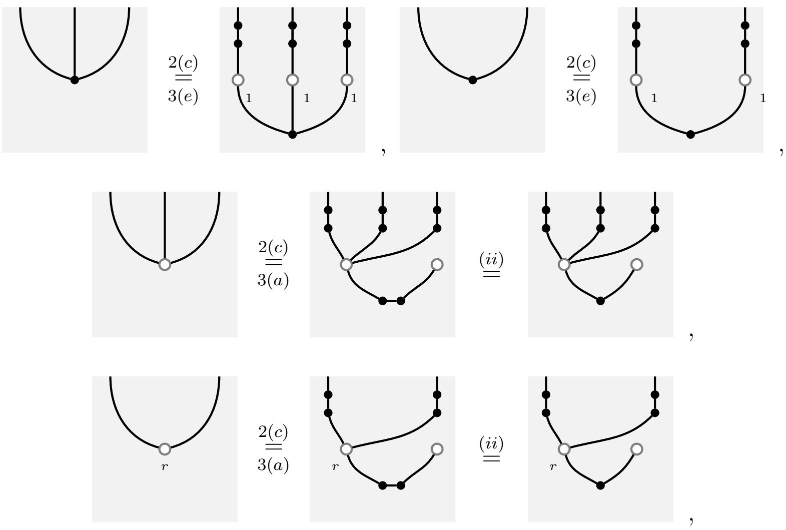
Theorem 40 (Functoriality)**.** *Any composition of two ZW diagrams in normal form can be rewritten in pre-normal form.*

Theorem 40 shows that the assignment $R\mathbf{bit} \to ZW_R$ defined by the normal form is functorial: every morphism in $R\mathbf{bit}$ is mapped to a ZW normal form with some transposing of outputs, and composition of morphisms in $R\mathbf{bit}$ (tensoring and vertical composition) is composition of normal forms in ZW (juxtaposition and plugging), which can be rewritten in normal form. This functor is a right inverse to the interpretation $ZW_R \to R\mathbf{bit}$. It remains to show that it is a two-sided inverse.

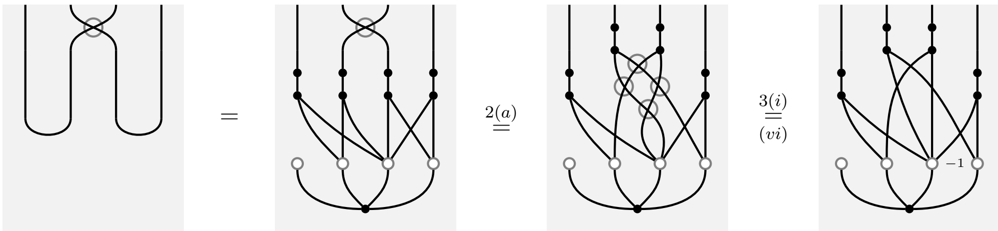
Theorem 41 (Completeness of the ZW calculus)**.** *The interpretation $ZW_R \to R\mathbf{bit}$ is an isomorphism of PROPs.*

Proof. It suffices to show that every generator of ZW can be rewritten to normal form.

The binary and ternary black and white vertices are rewritten in normal form as follows:



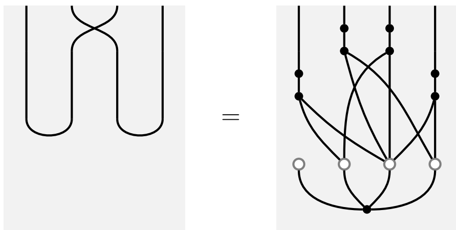
We rewrite the crossing as follows:



,

where in the first step we have rewritten the juxtaposition of two self-duality arrows in normal form, as made possible by the juxtaposition lemma.

Finally, the case for swap is similar to the case for crossing, except that there are no crossings in the third diagram. Hence the normal form is simply:



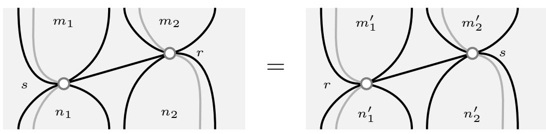
.

□

Just some remarks of the proof. We relied heavily on the bialgebra equations $2(h), 3(d)$ which allows us to move the monoid and comonoid pass each other. For this reason, it is believed that the Hopf algebra structure of the black vertices is a more crucial defining feature of the pure fragment of the ZW calculus than the anti-special commutative Frobenius algebra.

3.2.2.1 Some interesting ZW calculi for different rings

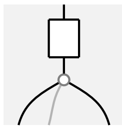
We have now proved the completeness of the ZW calculus for R**bit**. The calculus features an n-ary R-labelled white vertex for each $r \in R$. Given a presentation of R, we could in fact have just a label for each generator of R. The calculus remains largely the same; we just have to add an axiom for each relation the generators satisfy, and also a slight modification to some of the axioms involving the ring operations on the white vertices. For instance the axiom $3(k)$ can be slightly modified to suit the generators, and $3(a)$ can be replaced with



,

where r, s are some generators, $m_1, m_2, n_1, n_2, m'_1, m'_2, n'_1, n'_2$ are natural numbers, and $m_1 + m_2 = m'_1 + m'_2$, $n_1 + n_2 = n'_1 + n'_2$.

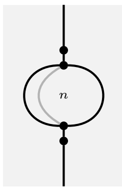
The completeness proof also remains largely the same since any composition of diagrams in pre-normal form can be rewritten in pre-normal form. We can also modify the normal form by replacing the R-labelled white vertices with

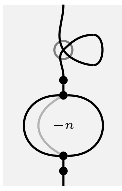
 ,

where the box is some fixed expression for the ring elements as sums and products of generators, expressed by convolution with black vertices and vertical composition.

We will look at some different commutative rings as examples.

If R is the free commutative ring on one generator $\mathbb{Z}$, we recover the result of [30]. We can express $n \in \mathbb{Z}$ as

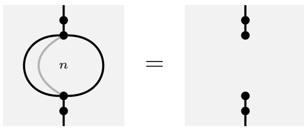


,


,

depending on whether n is positive or negative.

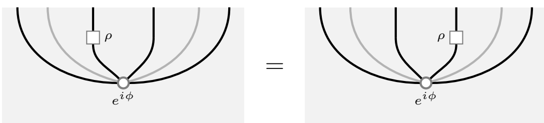
If $R := \mathbb{Z}_n$, then we just need to add one axiom for the relation $n = 0$,



.

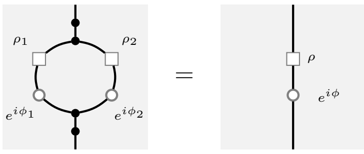
In particular, for $\mathbb{Z}_2$**bit**, also called modal quantum theory in [47], the crossing is equal to the swap and many of the axioms regarding the crossing become redundant.

If we take $R := \mathbb{C}$, the ring of complex numbers, we obtain completeness for **Qubit**. It might be convenient to separate the phase part and the length part and express each element in $\mathbb{C}$ as $\rho e^{i\phi}$ for ρ the positive reals and $\phi \in [0, 2\pi)$. Then all we need to modify is to have an n-ary white vertex for $e^{i\phi}$, a binary white vertex for ρ which we require is a (co)module homomorphism with respect to the n-ary white vertices



,

and change the axiom $3(k)$ to



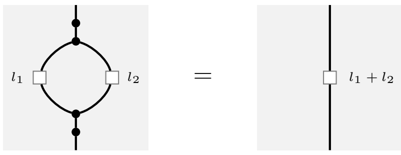
.

where $\rho e^{i\phi} = \rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2}$. We will call this example the $ZW_{\mathbb{C}}$ calculus.

As a final example, we let $R := \mathbb{Z}\left[\frac{1}{2}, e^{i\pi/4}\right]$. It is possible to have an n-ary white vertex with label $e^{i\frac{\pi}{4}}$ and a binary white vertex with label $\frac{1}{2}$, and some axioms for the generators. However, for convenience, we can use the same convention as in the complex case: an n-ary white vertex with labels $e^{i\phi}$ for $\phi = k\frac{\pi}{4}, k = 0, 1, \ldots, 7$, and a binary one with labels $0 < l \in \mathbb{Z}\left[\frac{1}{2}\right]$. Then an expression for an arbitrary ring element is

$$\sum_{k=0}^{7} l_k e^{ki\frac{\pi}{4}}$$

for $0 < l_k \in \mathbb{Z}\left[\frac{1}{2}\right]$, $k = 0, 1, \ldots, 7$. The axiom $3(k)$ is now



,

for $0 < l_1, l_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$. The $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** corresponds to the Clifford + T fragment of **Qubit** as proved in [36]. We will call this example the $\mathrm{ZW}_{\frac{\pi}{4}}$ calculus.

The last two examples, where $R = \mathbb{C}$ and $R = \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$, are of great importance to the completeness of the ZX calculus for **Qubit** and the Clifford+T fragment, respectively. The proof for the completeness results is via a direct translation from ZX to ZW calculus, which we will detail in the next chapter.

Remark 42. For readers unfamiliar with the notation $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$, it is defined as follows: given a commutative ring R and a symbol X,

$$R[X] := \{r_n X^n + r_{n-1} X^{n-1} + \cdots + r_1 X + r_0 | n \geq 0, r_j \in R\}.$$

It is an easy undergraduate exercise to check that $R[X]$ is a commutative ring. Given another symbol Y,

$$R[X, Y] := R[X][Y].$$

3.3 ZW calculus for fermionic circuits

Hadzihasanovic suggested some generalisations of the ZW calculus. One of them is to explore the limits of the core of ZW calculus, the pure fragment. This core has the property of only representing maps that have a definite parity with respect to the computational basis: the subspaces spanned by basis states with an even or odd number of 1's are either preserved, or interchanged by a map. This happens to be compatible with an interpretation of the basis states of a single qubit as the empty and occupied states of a *local fermionic mode*, the unit of information of the *fermionic quantum computing* (FQC) model.

Fermionic quantum computing is computationally equivalent to qubit computing [10], although it presents very different properties with regard to entanglement and non-locality [28, 20]. This connection suggested that an independent fermionic version of the ZW calculus could be developed, combining the best of both worlds with respect to FQC rather than qubit computing: the superior structural properties of the ZW calculus, including an intuitive normalisation procedure for diagrams, together with the superior hands-on features of the ZX calculus.

In this section, we will lay down the axioms for this ZW calculus for FQC, which we will call *fermionic ZW calculus*, and introduce how the physical gates can be constructed in the language of the fermionic ZW calculus. This section is based on the work done in the paper [32].

3.3.1 Preliminary for fermionic quantum computing

The basic systems in FQC are *local fermionic modes* (LFMs), physical sites that are either empty or occupied by a single spinless fermionic particle [10]. We indicate the empty and occupied states of a LFM as $|\,0\,\rangle$ and $|\,1\,\rangle$, respectively, in bra-ket notation.

Much like the computational basis states of a qubit, we can see these as an orthonormal basis for the two-dimensional complex Hilbert space B. States of a composite system of n LFMs then correspond to states of the n-fold tensor product $B^{\otimes n}$. However, not all physical states or operations on qubits are accessible as physical states or operations on LFMs. The Hilbert space of a system of n LFMs splits as $H_0 \oplus H_1$, where H_0 is spanned by states where an *even* number of LFMs is occupied, and H_1 by states where an *odd* number of LFMs is occupied. Then, any physical operation $f : H_0 \oplus H_1 \to K_0 \oplus K_1$ must either preserve, or invert the parity, that is, either map H_0 to K_0 and H_1 to K_1, or map H_0 to K_1 and H_1 to K_0. This is called the *parity superselection rule*; see [8, 20] for a discussion.

Remark 43. Note that Bravyi and Kitaev only consider parity-preserving operations in [10], although odd states are allowed.

These operations assemble into a category, as follows.

Definition 44. A $\mathbb{Z}_2$-*graded Hilbert space* is a complex Hilbert space H decomposed as a direct sum $H_0 \oplus H_1$.

A *pure map* $f : H \to K$ of $\mathbb{Z}_2$-graded Hilbert spaces is a bounded linear map $f : H \to K$ such that $f(H_0) \subseteq K_0$ and $f(H_1) \subseteq K_1$ (*even* map), or $f(H_0) \subseteq K_1$ and $f(H_1) \subseteq K_0$ (*odd* map).

Given two $\mathbb{Z}_2$-graded Hilbert spaces H, K, the tensor product $H \otimes K$ can be decomposed as $(H \otimes K)_0 := (H_0 \otimes K_0) \oplus (H_1 \otimes K_1)$, and $(H \otimes K)_1 := (H_0 \otimes K_1) \oplus (H_1 \otimes K_0)$. Then, the tensor product (as maps of Hilbert spaces) of a pair of pure maps $f : H \to K$, $f' : H' \to K'$ is a pure map $f \otimes f' : H \otimes H' \to K \otimes K'$ of $\mathbb{Z}_2$-graded Hilbert spaces. The $\mathbb{Z}_2$-graded Hilbert space $\mathbb{C} \oplus 0$ acts as a unit for the tensor product.

We write $\mathbf{Hilb}^{\mathbb{Z}_2}$ for the symmetric monoidal category of $\mathbb{Z}_2$-graded Hilbert spaces and pure maps, with the tensor product as monoidal product.

Remark 45. The zero maps $0 : H \to K$ are the only pure maps between $\mathbb{Z}_2$-graded Hilbert spaces that are both even and odd.

Definition 46. We write **LFM** for the full monoidal subcategory of $\mathbf{Hilb}^{\mathbb{Z}_2}$ whose objects are n-fold tensor products of $B := \mathbb{C} \oplus \mathbb{C}$, for all $n \in \mathbb{N}$.

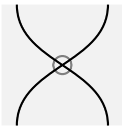
Here, B_0 is the span of $|\, 0\,\rangle$, and B_1 as the span of $|\, 1\,\rangle$. As usual, we write $|\, b_1 \ldots b_n\,\rangle$ for the basis state $|\, b_1\,\rangle \otimes \ldots \otimes |\, b_n\,\rangle$ of $B^{\otimes n}$, where $b_i \in \{0, 1\}$, for $i = 1, \ldots, n$.

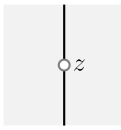
Remark 47. The category **LFM** admits, in fact, the structure of a dagger compact closed category in the sense of [50]: each object $B^{\otimes n}$ is self-dual, and the dagger of a pure map $f : B^{\otimes n} \to B^{\otimes k}$ is its adjoint $f^\dagger : B^{\otimes k} \to B^{\otimes n}$.

3.3.2 The model

The pure fragment of the ZW calculus is perfect for modelling the FQC as its operations are pure maps. We will add a labelled binary white vertex to the pure fragment, and this is call this the fermionic ZW calculus.

The following set T_{fzw} (written with their interpretations i_{fzw} in **LFM**) is the generators of the fermionic ZW calculus:

 $\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, jk \,\rangle\langle\, kj \,|$,



$\mapsto |\, 0 \,\rangle\langle\, 0 \,| + z|\, 1 \,\rangle\langle\, 1 \,|$,

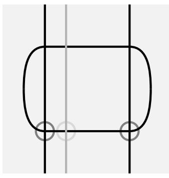
 $\mapsto |\, 001 \,\rangle + |\, 010 \,\rangle + |\, 100 \,\rangle$,

 $\mapsto |\, 01 \,\rangle + |\, 10 \,\rangle$.

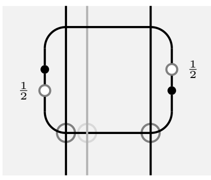
for $z \in \mathbb{C}$. The crossing is seen as a statement of the antisymmetry nature of fermions, where exchange two identical fermions result in a π phase rotation. For this, we will now call it the *fermionic swap*. The usual swap will be called the *structural swap*, or simply just the swap.

The difference between the fermionic ZW calculus from the general one is that this one doesn't have a ternary white vertex because it is of mixed parity. This caused some difficulty in defining a normal form like the one before in the general ZW calculus, and because of this, it is unclear whether it is universal with respect to **LFM**.

However, it doesn't rule out the fact that we may construct a quaternary white vertex. With a quaternary white vertex, we may define arbitrary even arity white vertices, and we can define a normal form like the one before. This became the key challenge, and soon, we found a solution by studying a family of diagrams:

 .

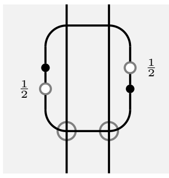
This family is called the *even projectors*. They only allow even states to pass and kill the odd states, hence the name. It also alters the state by multiplying it with the scalar 2, but we can simply fix that by



.

We just have to keep in mind that the decoration on this map doesn't affect its property of projecting onto the even subspace; it's there to fix the scalar.

Consider the following even projector:

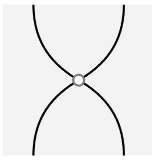


.

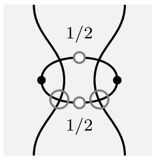
This is interpreted as

$$|\,00\,\rangle \mapsto |\,00\,\rangle, \qquad |\,11\,\rangle \mapsto |\,11\,\rangle,$$
$$|\,01\,\rangle, |\,10\,\rangle \mapsto 0,$$

which is exactly the map for the quaternary white vertex! We will adopt the white vertex notation and define a shorthand for it:



$:=$



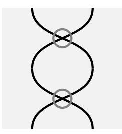
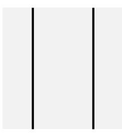
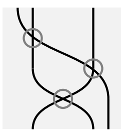
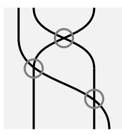
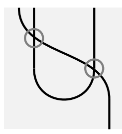
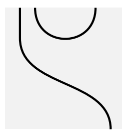
.

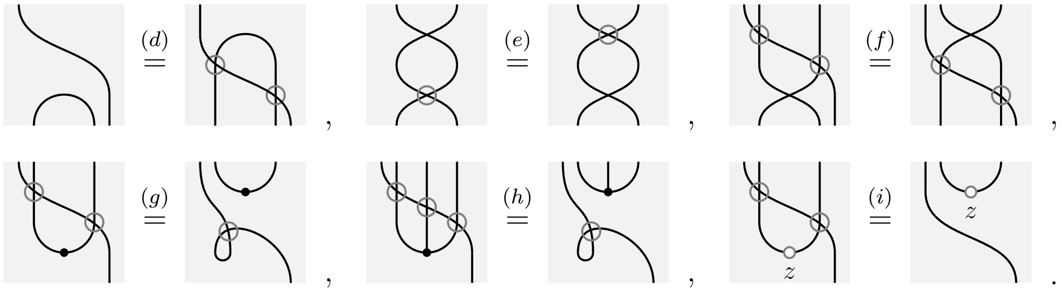
With this, the rest is mainly modifying the original ZW axioms and a routine proof of its completeness. While modifying the axioms, we had some revelation about the white vertices. It seems like many of the algebraic structures are now a consequence rather than an axiom (see Proposition 49).

3.3.3 The axioms

Like before, we divide the set E_{fzw} of axioms into three groups, based on the generators to which they mainly pertain. We have rearranged some of the rules when compared to the original ZW calculus. We did that because it seems more natural to group them in this manner. Just to avoid confusion, only refer to these axioms numbering for the fermionic ZW calculus and not the original ones.

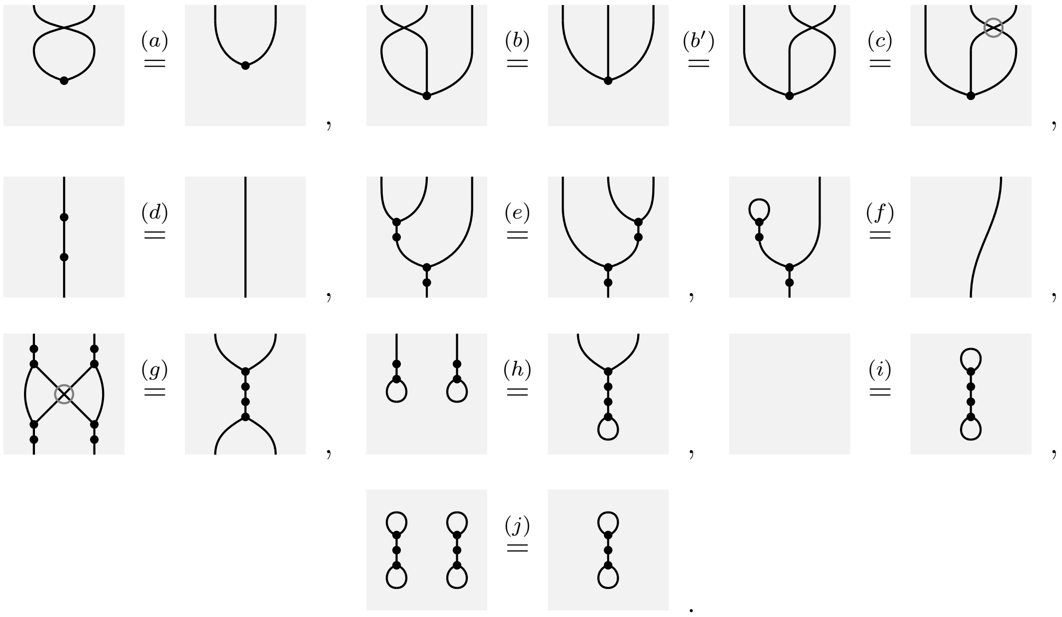
1. The following are the axioms for the fermionic swap:

 $\overset{(a)}{=}$  ,  $\overset{(b)}{=}$  ,  $\overset{(c)}{=}$  ,



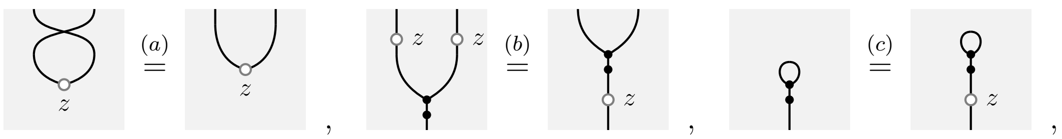
There are a few new additions to the axioms of the crossing. The axioms on the interplay between the structural and fermionic swaps imply that only the number of fermionic swaps between two wires matters, and not their direction. And we have an axiom saying that the binary white vertex is natural with the fermionic swap.

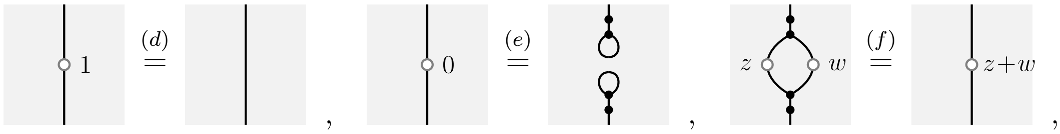
2. The following are the axioms for the black vertices:

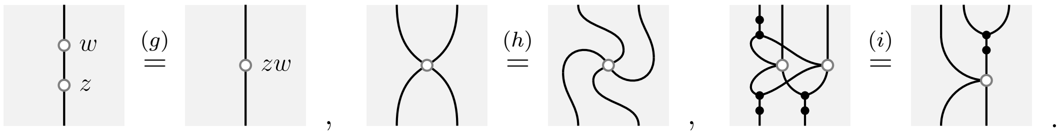


These axioms are pretty much the same as the original ones. The only notable one is the final axiom, which says that 0 times 0 is 0; it serves to ensure that there is a unique zero map, rather than an "even" and an "odd" zero.

3. The following are the axioms for the white vertices:



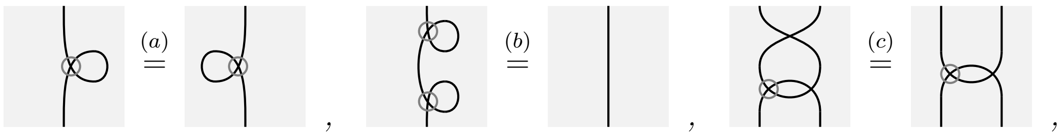


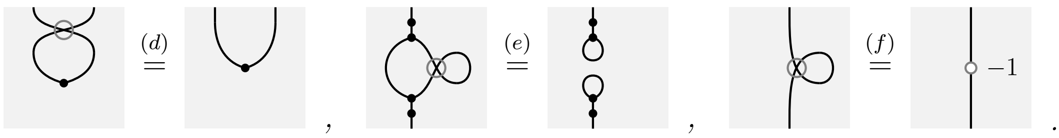


The axioms for the binary white vertex is the same. We have some axioms for the quaternary white vertex as expected. There is a modified bialgebra-like equation between the black and white vertices to account for the number of wires on the white vertex. Furthermore, the projector is symmetric under cyclic permutation of its wires, and it determines a kind of mixed action/coaction of the algebra on itself.

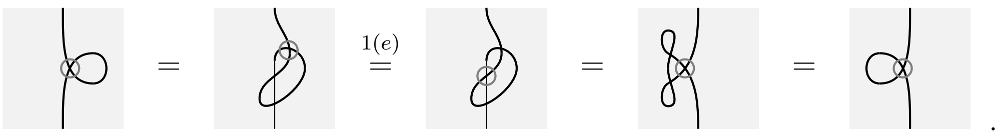
We state some useful derived equations.

Proposition 48. *The following are derived rules concerning the black vertices:*



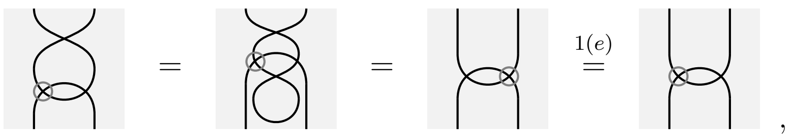


Proof. Equation (a) comes from the following manipulation:

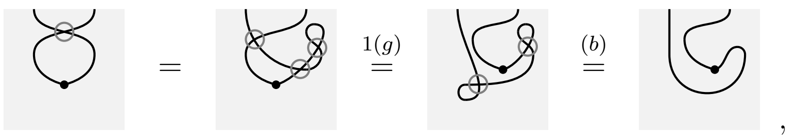


Equation (b) is proved in Lemma 28.

Equation (c) is proved by the following argument:



whereas (d) comes from

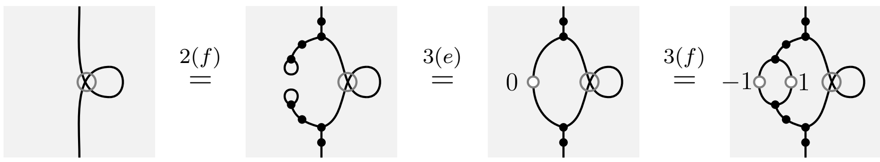


finally using the symmetry or the black vertex under the structural swap.

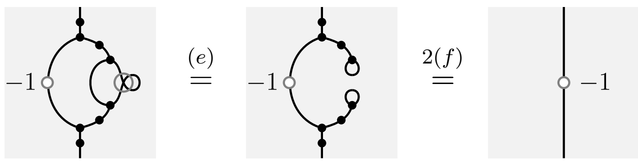
Equation (e) is proved by the following argument:

where we tacitly used axiom $2(d)$ to introduce or eliminate pairs of binary black vertices in several occasions.

Finally, for equation (f), start by considering that



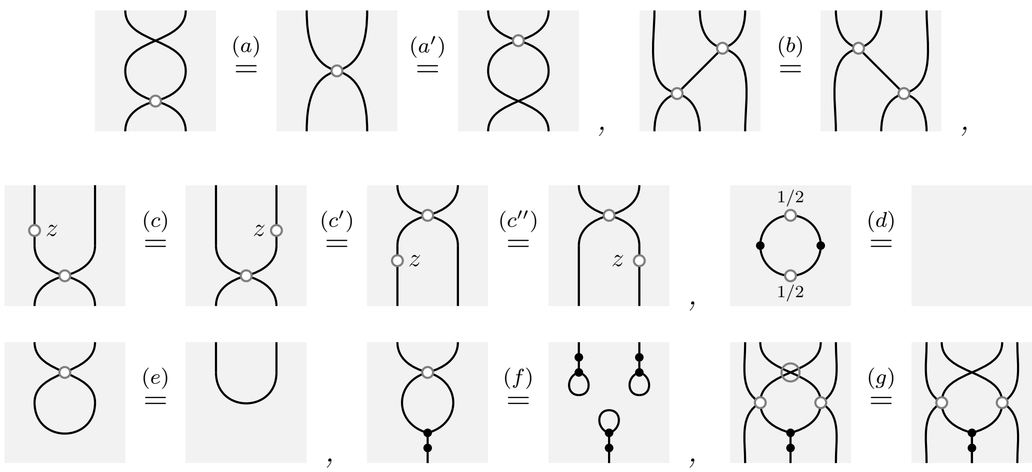
by axioms $2(e)$ and $3(d)$, this is equal to



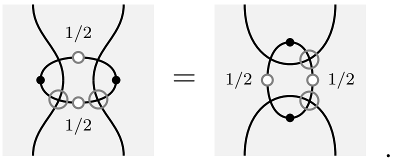
This completes the proof. □

The following Proposition shows that all the least natural-looking axioms about white vertices in the original ZW calculus become provable equations for (quaternary, rather than ternary) white vertices in the fermionic ZW calculus.

Proposition 49. *The following are derived rules concerning the white vertices:*



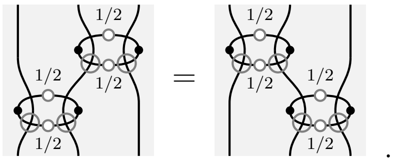
Proof. Substituting the definition of the projector, axiom $3(h)$ becomes the following equation:



(3.27)

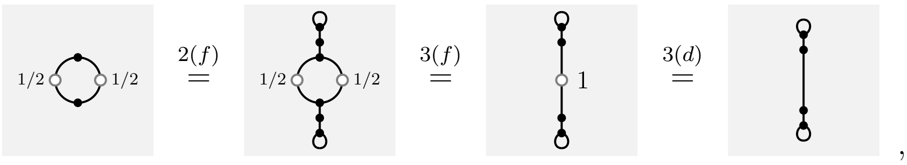
Equations (a) and (a') are then immediate consequences of Proposition 48.(c) and its transposes, applied to the right-hand side of (3.27).

Equation (b) is also immediate from the definition: because swaps slide through fermionic swaps and vice versa, we can slide one "circle" past another to get

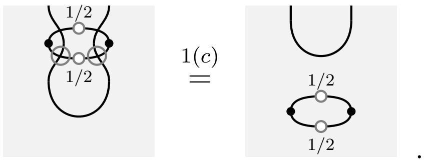


For equations (c), (c'), and (c''), we use either of the forms in equation (3.27) and slide the binary white vertex through a fermionic swap using axiom $1(i)$, to move it to a different wire.

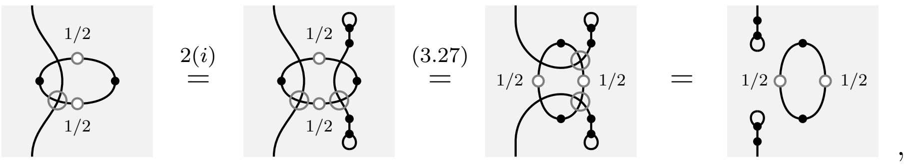
Equation (d) comes from



finally applying axiom $2(i)$. Then, equation (e) follows from it by

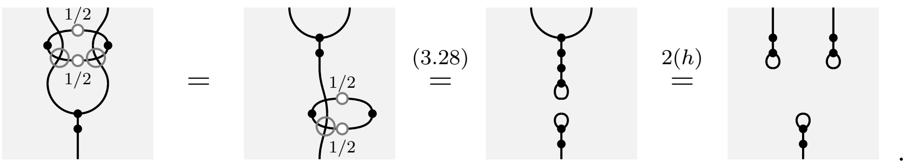


In order to prove equation (f), consider first that

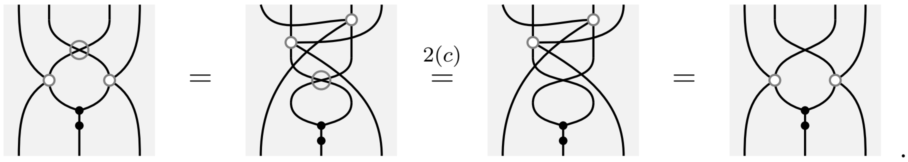


(3.28)

and we can eliminate the circle by equation (d). Then,



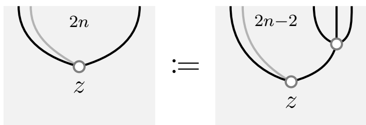
Finally, for equation (g), observe that the projector can slide past fermionic swaps naturally using axiom $1(f)$. Therefore,



This concludes the proof. $\square$

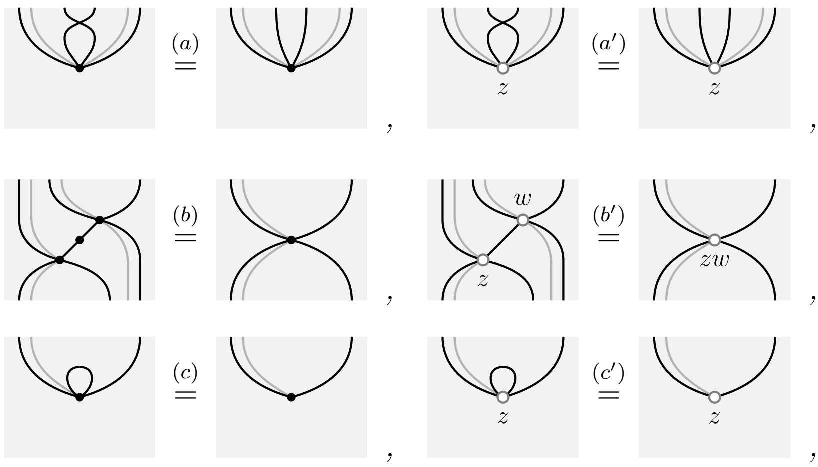
Together with its invariance under cyclic permutation of wires, the first two equations justify the arbitrary transposition of inputs and outputs of the quaternary white vertex.

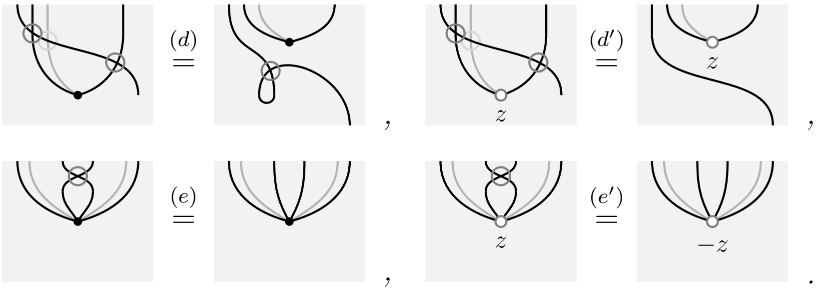
Like before, it is convenient to introduce a condensed notation, a spider presentation including black vertices with n wires and white vertices with $2n$ wires for all $n \in \mathbb{N}$, and derive inductive equation schemes to use directly in proofs. The definition is exactly the same as the original ZW calculus, with a slight modification for the white vertices as there are now even number of wires:



We state some basic properties of black and white vertices in the Proposition 50. We have already seen most of them in the original ZW calculus. A more noteworthy one is that all black vertices correspond to odd maps, while white vertices correspond to even maps, as reflected in their sliding through fermionic swaps (d, d').

Proposition 50. *The following equations hold in FZW for black and white vertices of any arity:*

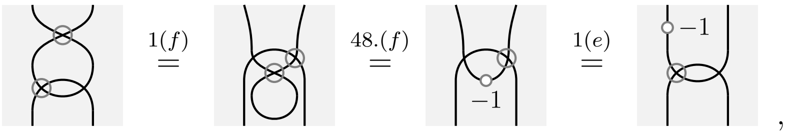




Proof. All the equations are proved by induction on the arity of the vertices involved. They are pretty easy to proof. The only ones that require some attention are (d', e').

Equation (d') is a consequence of axiom $1(i)$ together with the definition of the quaternary white vertex.

For equation (e'), let $2n > 1$ be the arity of the white vertex. The case $n = 1$ is a consequence of Proposition 48.(f), and $n = 2$ follows from the following argument:

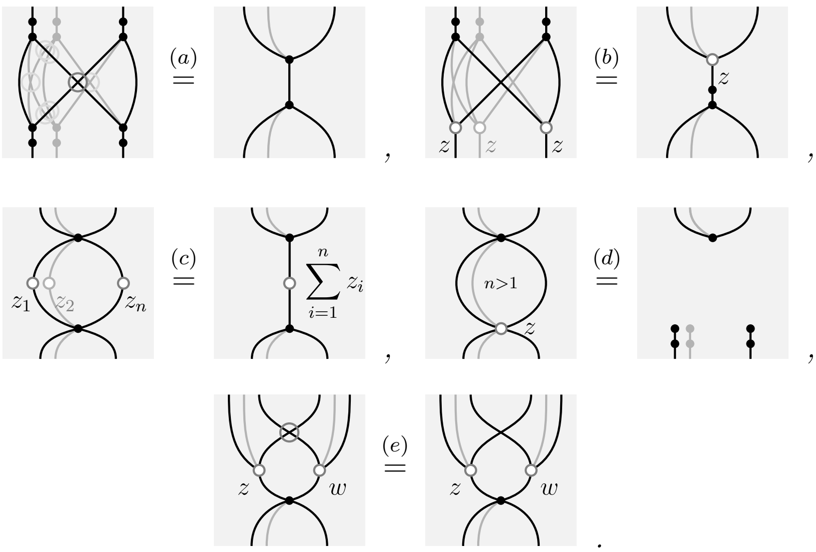


applied to the definition of the quaternary white vertex, as in the right-hand side of (3.27). All other cases follow from this one, by symmetry.

□

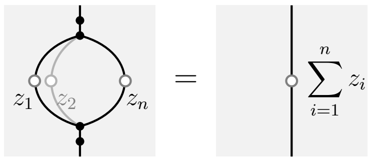
Several other equations, both axioms and derived, admit inductive generalisations; we list them in the following Proposition.

Proposition 51. *The following equations hold in FZW for black and white vertices of any compatible arities:*



Proof. Equation (a) has already been proved when discussing the axiom $2(h)$ in the original ZW calculus. Equation (b) is proved in a similar fashion as the discussion of axiom $3(d)$ in the original ZW calculus.

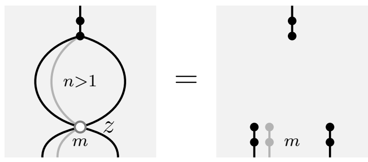
For equation (c), by Proposition 50.(b) it suffices to prove



,

which is proved in Proposition 31.

Similarly, for equation (d), it suffices, by Proposition 50.(b) and (b'), to prove



which is proved in Proposition 31.

Finally, equation (e) is an immediate generalisation of Proposition 49.(g), using Proposition 50.(b) and (b'). □

Remark 52. Unlike in the original ZW calculus, we have listed the these condensed equations as propositions instead of axioms. There is nothing mysterious about this as it is simply because there is no need to restate all the axioms again.

In the next section, we will use these equations to prove that our axioms are complete for **LFM**.

3.3.4 Completeness of the fermionic ZW calculus

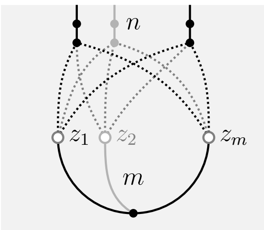
The proof for the completeness follows a very similar procedure as the original ZW calculus. It is via a normal form argument, and we will define a normal form in the following theorem on the universality. In fact, many of the steps are the same, but we will state them anyway to make them easier to follow.

Theorem 53 (Universality)**.** *The interpretation functor* $i_{fzw} : FZW \to$ **LFM** *is full.*

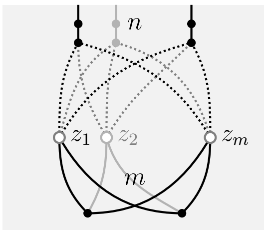
Proof. Write an arbitrary state $v : \mathbb{C} \to B^{\otimes n}$ in the form

$$\sum_{i=1}^{m} z_i |\, b_{i1} \dots b_{in} \,\rangle,$$

where $z_i \neq 0$ for all i, and no pair of n-tuples $(b_{i1}, \ldots, b_{in})$ is equal. Then, we claim that the it is the image of either of the diagrams



if v is odd,



if v is even, (3.29)

where, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, the dotted wire connecting the i-th white vertex to the j-th output is present if and only if $b_{ij} = 1$. The definition is only ambiguous if $v = 0$, in which case we arbitrarily pick one of the two forms.

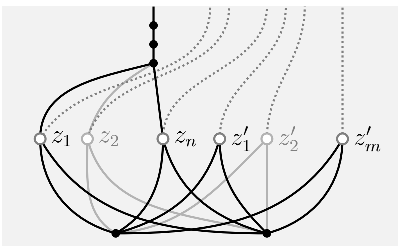
Because for all summands of an odd (respectively, even) state v, we have $b_{ij} = 1$ for an odd (respectively, even) number of bits, the white vertices in the diagram have an odd (respectively, even) number of outputs. The two distinct diagrams for odd and even states ensure that only white vertices with an even arity appear.

The claim can be proved by a direct calculation, which suffices to prove the statement. □

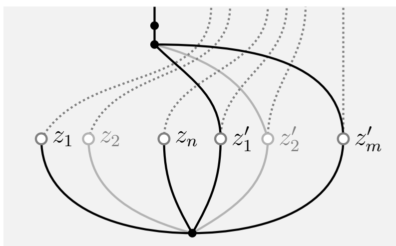
We will define Equation 3.29 as the *normal form* for a state in **LFM**. Like before, this is unique up to a permutation of the white vertices.

We can define a *pre-normal form* like before: two or more white vertices may be connected to the exact same outputs; z_i may be 0 for some i. We will mostly work with diagrams in pre-normal form, as they are more flexible, but we will later prove that any diagram in pre-normal form can be rewritten in normal form. We will speak of *even* and *odd* diagrams for diagrams in pre-normal form corresponding to even and odd states, respectively.

Lemma 54 (Negation)**.** *The composition of one output of a diagram in pre-normal form with a binary black vertex can be rewritten in pre-normal form, and that has the effect of "complementing" the connections of the output to white vertices: that is, locally,*



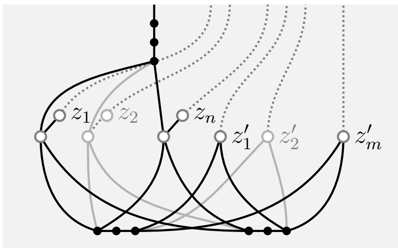
$=$



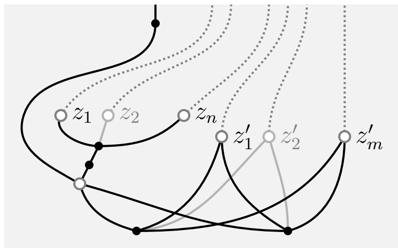
.

Remark 55. The version where the original diagram is odd, rather than even, is obtained by composing again both sides with a binary black vertex and using axiom 2(d).

Proof. Using the "fusion rules" Proposition 50.(b) and (b'), we rewrite the left-hand side as

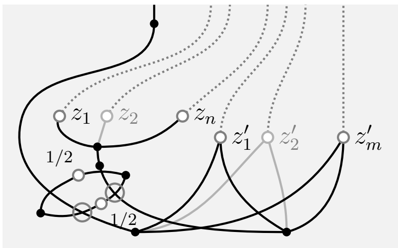


$\overset{51.(b)}{=}$

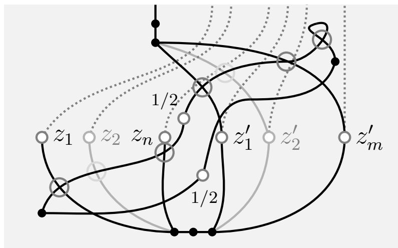


.

By definition of the quaternary white vertex, this is equal to

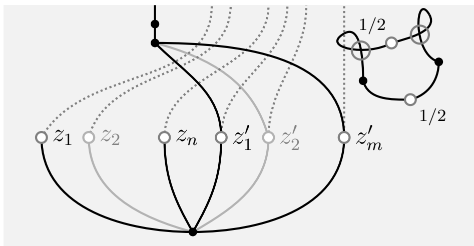


$\overset{50.(d)}{=}$



,

where we made implicit use of some symmetry properties of vertices. Now, fusing black vertices, and using Proposition 50.(d) and (d') to move the closed loop to the outside of the main diagram, we see that this is equal to

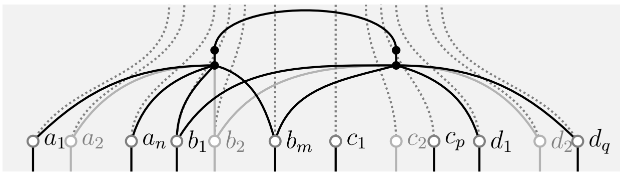


,

and we can conclude by Proposition 48.(b) and 49.(d). □

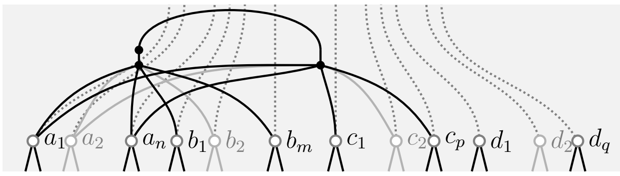
Lemma 56 (Trace). *The plugging of two outputs of a diagram in pre-normal form into each other can be rewritten in pre-normal form.*

Proof. We consider the case of an odd diagram; the even case is completely analogous. Focus on the two relevant outputs, and subdivide the white vertices into four groups, based on their being connected to both outputs, to one of them, or neither of them:



.

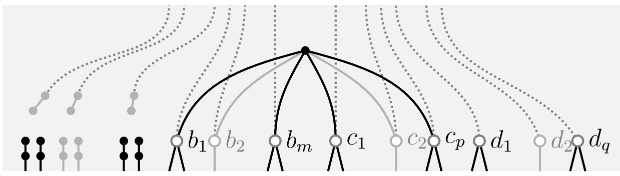
Again, the dotted wires can stand for a multitude of wires. Using the negation lemma on the rightmost output, this becomes



,

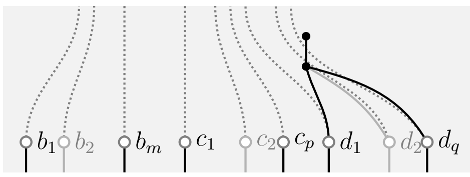
where there are now two black vertices (not pictured) at the bottom, one leading to all the left-hand inputs, and one leading to all the right-hand inputs of the white vertices.

After fusing the pictured black vertices, the leftmost n white vertices have two wires connecting them to the same black vertex. This means that Proposition 51.(d) is applicable, leaving us with



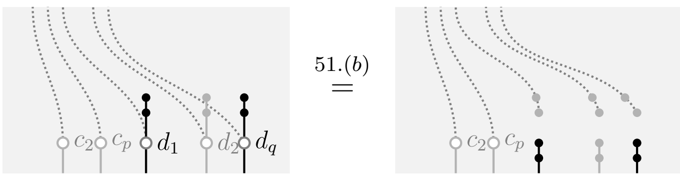
.

The leftmost black vertices can be fused with any black vertices they are connected to, or eliminated with axiom $2(i)$ if there is none, which rids us of the leftmost n white vertices. Now, we can apply the negation lemma again, to find that the remaining portion of the diagram is equal to



,

which, focussing on the rightmost part, is equal, by Proposition 51.(a), to

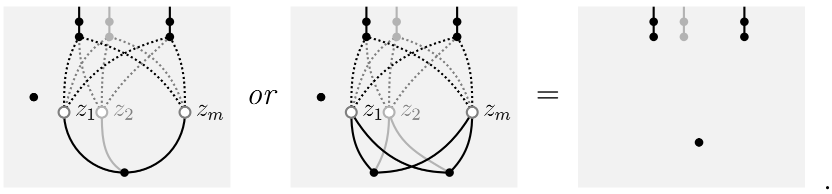


.

This rids us of the rightmost q white vertices, and leaves us with a diagram in pre-normal form. □

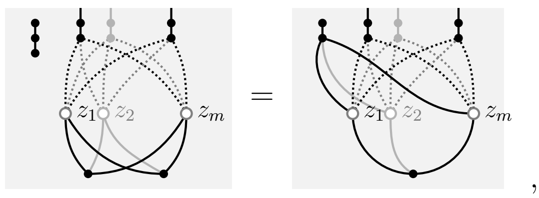
The nullary black vertex is interpreted as the scalar 0; the following lemma shows that it acts as an "absorbing element" for diagrams in pre-normal form.

Lemma 57 (Absorption)**.** *For all diagrams in pre-normal form, a nullary black vertex eliminates all the white vertices; that is,*



(3.30)

Proof. Suppose the diagram is even. Expanding the nullary black vertex, we can treat it as an additional output of the diagram, with no connections to the white vertices, composed with a unary black vertex. Applying the negation lemma,



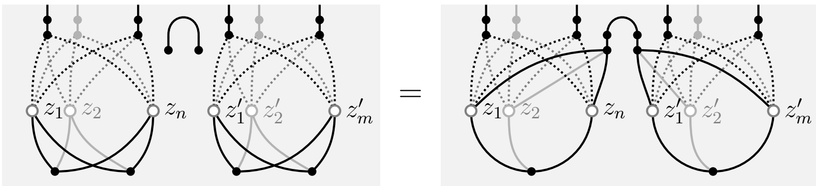
where the new output is connected to all the white vertices. From here, we can proceed as in the last part of the proof of the trace lemma in order to eliminate all the white vertices.

Now, suppose the diagram is odd. If it has at least one output wire, we can freely introduce two binary black vertices on it; applying the negation lemma once, we obtain a negated even diagram, to which the first part of the proof can be applied. Another application of the negation lemma, followed by axiom $2(j)$, produces the desired equation. If the diagram has no outputs, it necessarily consists of a single nullary black vertex, and the statement follows immediately from axiom $2(j)$. □

Lemma 58 (Functoriality)**.** *Any composition of two diagrams in pre-normal form can be rewritten in pre-normal form.*

Proof. We can factorise any composition of diagrams in pre-normal form as a tensor product followed by a sequence of "self-pluggings"; thus, by the trace lemma, it suffices to prove that a tensor product — diagrammatically, the juxtaposition of two diagrams in pre-normal form — can be rewritten in pre-normal form.

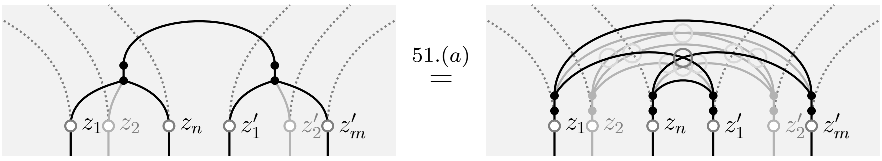
Suppose first that the two diagrams are both even. Then, we can create a pair of unary black vertices connected by a wire by axiom $2(i)$, and treat them as additional outputs, one for each diagram. Applying the negation lemma on both sides, we obtain



,

which is the plugging of two outputs connected to all the white vertices of their respective diagrams. The only case in which this still leaves the two diagrams disconnected is when one of the diagrams has no white vertices, that is, it looks like the right-hand side of equation (3.30). In this case, by the absorption lemma, we can use its isolated black vertex to "absorb" the other diagram, which produces a diagram in pre-normal form.

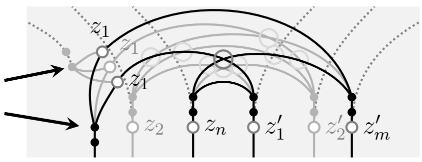
So, suppose that $n, m > 0$. Focussing on the two outputs,



(3.31)

;

then, we can use Proposition 51.(b) on each of the white vertices: for example, on the leftmost one, it leads to

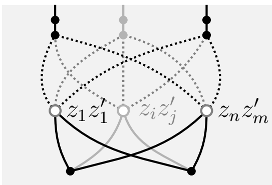


.

Each of the outgoing wires leads to a black vertex, so we can push the vertices indicated by arrows to the outside, and fuse them. Repeating this operation, we can push all black vertices to the outside, which leaves us with a tangle of $n \cdot m$ wires connecting white vertices, one for each pair (z_i, z'_j), where $i = 1, \ldots, n$, and $j = 1, \ldots, m$.

This tangle is made of fermionic swaps; however, each of the white vertices has at least one wire connecting it to a black vertex, which means that Proposition 51.(e) is applicable: this allows us to turn all the fermionic swaps into structural swaps.

Finally, we can fuse all pairs of white vertices connected by a wire. After some rearranging, this leaves us with the pre-normal form diagram



,

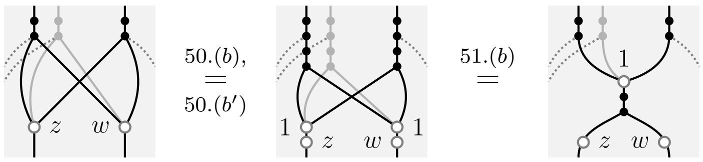
where the white vertex with parameter $z_iz'_j$ is connected both to the outputs to which the original vertex with parameter z_i was connected, and to the outputs to which the original vertex with parameter z'_j was connected, for $i = 1, \ldots, n$, and $j = 1, \ldots, m$.

Now, suppose one diagram is odd, or they both are odd. If the odd diagrams have at least one output wire, we can introduce a pair of black vertices on it, and apply the negation lemma to produce negated even diagrams. We can then apply the first part of the proof to obtain a diagram in pre-normal form negated once or twice, then apply the negation lemma again to conclude. If one of the odd diagrams has no outputs, it necessarily consists of a single nullary black vertex, and we can conclude with an application of the absorption lemma. □

Lemma 59. *Any diagram in pre-normal form can be rewritten in normal form.*

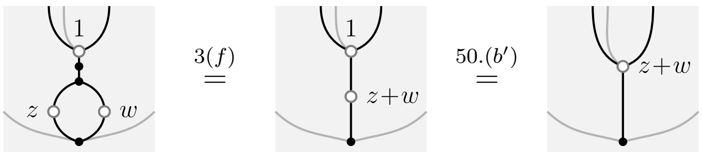
Proof. First of all, by the symmetry property of vertices, we can always reshuffle the white vertices of a diagram in pre-normal form to make them follow our preferred ordering of n-tuples of bits.

Suppose the diagram is odd, and two white vertices are connected to the same outputs. The relevant portion of the diagram looks like



.

The two input wires both lead to the bottom black vertex; zooming in on that, we find

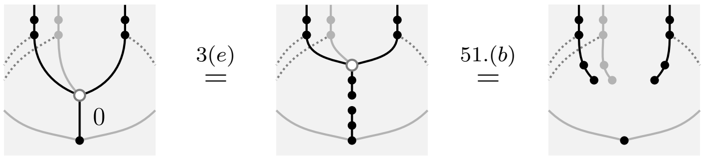


(3.32)

,

which has a single white vertex, connected to the same outputs, replacing the two initial ones.

Now, suppose that there is a white vertex with parameter 0. The relevant part of the diagram is

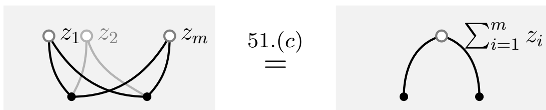


,

where we can fuse black vertices, which simply eliminates the white vertex.

If the diagram is even, and has at least one output wire, we can introduce a pair of binary black vertices, apply the negation lemma once to produce a negated odd diagram, reduce that to normal form, and apply the negation lemma again; it is easy to see that negation turns diagrams in normal form into diagrams in normal form, *modulo* a reshuffling of white vertices.

If the diagram has no output wires, then it is of the form



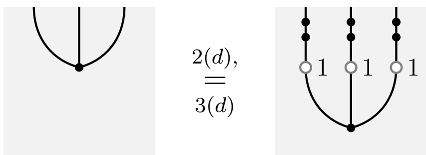
,

where the right-hand side is in normal form. This concludes the proof. $\square$

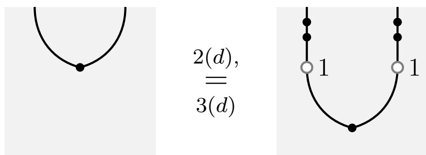
Theorem 60 (Completeness)**.** *The interpretation functor* $i_{fzw} : FZW \to \mathbf{LFM}$ *is a monoidal equivalence.*

Proof. It suffices to show that all the generators can be rewritten in normal form.

For the ternary and binary black vertices:

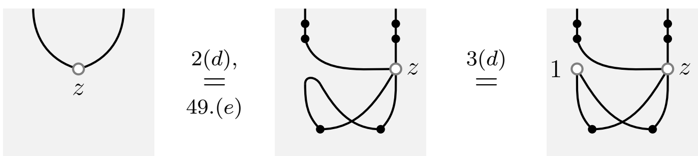


,



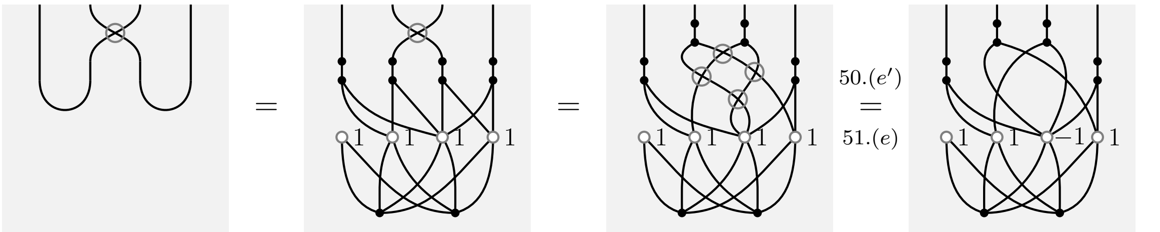
.

For the binary white vertex with parameter $z \in \mathbb{C}$:



.

For the fermionic swap, we use the fact that we know how to rewrite the tensor product of two dualities in normal form:

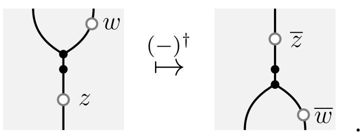


The case of the structural swap is similar, and easier. This concludes the proof. □

Remark 61. The only properties of complex numbers that were used in the proof are that they form a commutative ring, and that they contain an element z such that $z + z = 1$ (namely, $\frac{1}{2}$). Thus, we can replace $\mathbb{C}$ with any commutative ring R that has the latter property (for example, $\mathbb{Z}_{2n+1}$, for each $n \in \mathbb{N}$), and obtain a similar completeness result for "LFMs with coefficients in R".

As with the original ZW calculus, instead of introducing binary white vertices with arbitrary parameters $r \in R$, we can introduce one binary white vertex for each element of a family of generators of R, together with one axiom for each relation that they satisfy. For example, in the complex case, it may be convenient to have separate phase gates, that is, white vertices with parameter $e^{i\vartheta}$, for $\vartheta \in [0, 2\pi)$, and "resistor" gates, with real parameter $r > 0$.

Remark 62. The fermionic ZW calculus doesn't have a dagger structure. We can equip it with one by defining the dagger to be the vertical reflection of the diagram, with parameters $z \in \mathbb{C}$ of white vertices turned into their complex conjugates $\overline{z}$. For example,

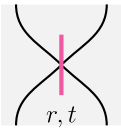


Then, we will obtain an equivalence of dagger compact closed categories between the fermionic ZW calculus and the category **LFM**.

3.3.5 Connection to fermionic circuits

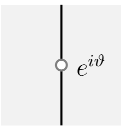
Operationally, we are interested in representing circuits built from the following *logical* components, shown here in diagrammatic form, next to their interpretation as maps in **LFM**.

1. The *beam splitter* with parameters $r, t \in \mathbb{C}$, such that $|r|^2 + |t|^2 = 1$:



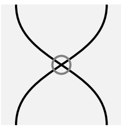
$$|\,00\,\rangle \mapsto |\,00\,\rangle, \qquad |\,10\,\rangle \mapsto r|\,10\,\rangle + t|\,01\,\rangle,$$
$$|\,01\,\rangle \mapsto -\bar{t}|\,10\,\rangle + \bar{r}|\,01\,\rangle, \qquad |\,11\,\rangle \mapsto |\,11\,\rangle.$$

2. The *phase* gate with parameter $\vartheta \in [0, 2\pi)$:



$$|\,0\,\rangle \mapsto |\,0\,\rangle, \qquad |\,1\,\rangle \mapsto e^{i\vartheta}|\,1\,\rangle.$$

3. The *fermionic swap* gate:



$$|\,00\,\rangle \mapsto |\,00\,\rangle, \qquad |\,10\,\rangle \mapsto |\,01\,\rangle,$$
$$|\,01\,\rangle \mapsto |\,10\,\rangle, \qquad |\,11\,\rangle \mapsto -|\,11\,\rangle.$$

4. *Empty state* and *occupied state* preparation:

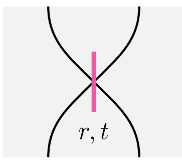
$$1 \mapsto |\,0\,\rangle,$$

$$1 \mapsto |\,1\,\rangle.$$

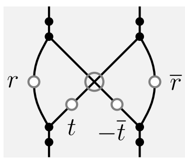
All of these are isometries, which makes them, at least in principle, physically implementable gates; see for example [38] for the description of an electron beam splitter.

Apart from the fermionic swap gate, which exploits the antisymmetry of fermionic particles under exchange, these operations are structurally the same as those used in implementations of linear optical quantum computing (LOQC), such as the Knill-Laflamme-Milburn scheme [43], which employ photons, that is, bosonic particles as resources. The two models seem closely related; given the way that the fermionic swap ties the other components together, and that the impossibility for two particles to occupy the same mode — a constraint for the bosons in LOQC — is simply a consequence of Pauli exclusion for fermions, it seems likely to us that the logical features of the optical model are a consequence of the features of the fermionic model, rather than the other way around.

The beam splitter with parameters r, t can be decomposed as follows:



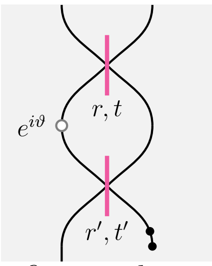
$:=$



.

We will end this part with a simple example, Mach-Zehnder interferometer.

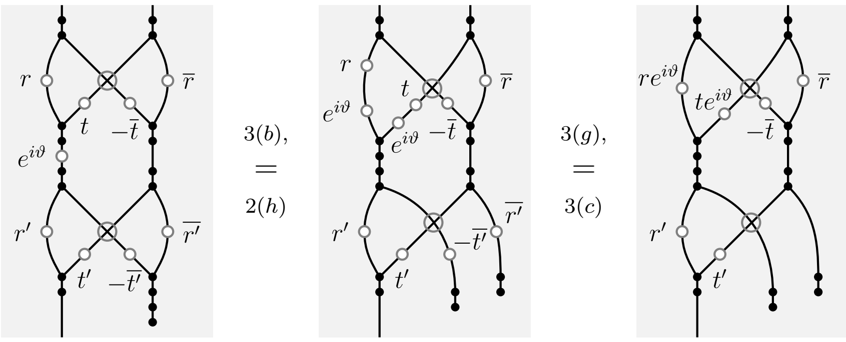
Example 63 (The Mach-Zehnder interferometer). The Mach-Zehnder interferometer is a classic quantum optical setup (see for example [48]), which, despite its simplicity, can demonstrate interesting features of quantum mechanics, as in the Elitzur-Vaidman bomb tester experiment [26]. The theoretical setup can be straightforwardly imported into FQC, with the same statistics as long as single-particle experiments are concerned; an electronic analogue of the Mach-Zehnder interferometer has also been realised in practice [38].



With the graphical notation introduced, the experimental setup is represented by the diagram on the left, where $r, t, r', t' \in \mathbb{C}$ and $\vartheta \in [0, 2\pi)$ are parameters subject to $|r|^2 + |t|^2 = |r'|^2 + |t'|^2 = 1$. In practice, it would also include "mirrors", or beam splitters with $|r| = 1$, which we omit in the picture, instead taking the liberty of bending wires at will.

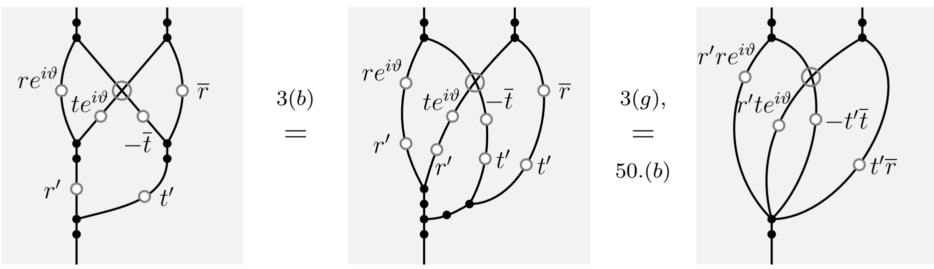
As a first application of the fermionic ZW calculus, we show how this circuit diagram can be simplified in just a few steps using our axioms, in such a way that its statistics become immediately readable from the diagram.

In our language, the diagram becomes



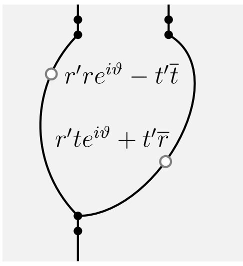
,

which, sliding the leftmost empty state past the fermionic swap, and using axiom $2(f)$ twice, becomes



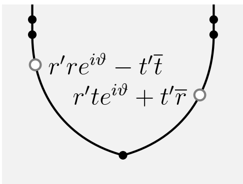
.

Finally, using the fermionic swap symmetry of black vertices (Proposition 50.(e)), together with Proposition 51.(c), this simplifies to



.

If we input one particle, after fusing the bottom black vertices, we obtain a diagram in normal form, whose interpretation in **LFM** we can readily deduce:



$$\mapsto (r're^{i\vartheta} - t'\overline{t}) \,|\, 10 \,\rangle + (r'te^{i\vartheta} + t'\overline{r}) \,|\, 01 \,\rangle.$$

So, the probability of detecting the particle at the left-hand output is $|r're^{i\vartheta} - t'\overline{t}|^2$, and the probability of detecting the particle at the right-hand output is $|r'te^{i\vartheta} + t'\overline{r}|^2$. If the beam splitters are symmetric, that is, $r = r' = \frac{1}{\sqrt{2}}$, and $t = t' = \frac{i}{\sqrt{2}}$, the probability amplitudes become

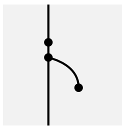
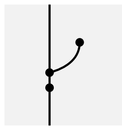
$$\frac{1}{2}(e^{i\vartheta} - 1) = e^{i(\frac{\vartheta+\pi}{2})}\sin\vartheta, \qquad \frac{i}{2}(e^{i\vartheta} + 1) = e^{i(\frac{\vartheta+\pi}{2})}\cos\vartheta,$$

leading to probabilities $\sin^2\vartheta$ of detecting the particle at the left-hand output, and $\cos^2\vartheta$ of detecting it at the right-hand output.

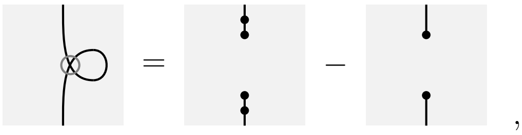
Arguably, given that this particular example involves at most binary gates, a matrix calculation would not have been considerably harder. On the other hand, the result appears here as the outcome of a short sequence of intuitive, algebraically motivated local steps, rather than the unexplained product of a large matrix multiplication. Moreover, since the juxtaposition of two diagrams only adds up their sizes, while the tensor product of two matrices multiplies them, we expect that the diagrammatic calculus should outperform matrix calculus, as the number of systems involved grows.

3.3.6 Understanding the fermionic world with ZW calculus

It is customary to describe the fermionic behaviour of a multi-particle system in terms of a pair of operators $a^\dagger$ (creation) and a (annihilation) that satisfy the anti-commutation relation $aa^\dagger = 1 - a^\dagger a$; see for example [52, Chapter 27]. In our language, these operators can be defined as

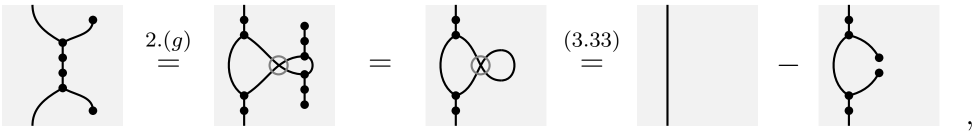
$a^\dagger :=$  , $a :=$  .

We can see the anti-commutation relation as subsumed by the axioms in the following way: pulling back the linear structure of **LFM** to FZW through the equivalence, we have



(3.33)

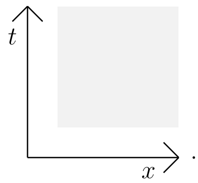
from which we obtain



which can be read as the equation $aa^\dagger = 1 - a^\dagger a$.

We are going to be bold and take the black (co)monoid as a fundamental operation of the fermionic world instead of the creation and annihilation operation. The creation operation is injecting a particle into the monoid, and the annihilation is expelling a particle from the comonoid. Injecting or expelling a vacuum state (empty state) amounts to doing nothing.

We can go further and start imposing some sort of space-time structure on the diagrams. The horizontal axis may be read as the space axis, and the vertical axis is the time axis:

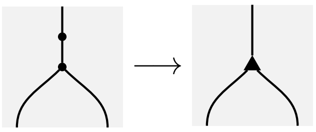


The curvy wires are structures on space-time, and the self-duality maps can be seen as moving back in time. An entangled state can be seen as connecting parts of space-time in a inseparable manner.

With a space-time that allows for moving backwards in time, we get many interesting phenomena, especially with feedback loops. The self crossing is one such example, which acts like a π phase rotation. The projectors we have seen in the previous subsection is also a consequence of feedback loops.

All these are not new ideas (perhaps the projector construction is a new idea?). But we are going to be even more bold and think about what if we remove particle injection. Can the fermionic universe work without a particle to initiate all the “fun” operations?

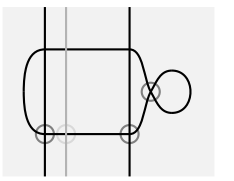
This particle-less, fermionic universe has as generators the fermionic swap, binary $z \in \mathbb{Z}$ labelled white vertices, and a black monoid. We are going to draw the black monoid in this fashion to signify that there is no longer a particle injection:



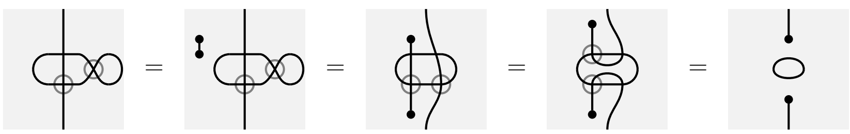
Remark 64. In principle, we could have white vertices with just complex phases, since phase shift is physical. But having labels for the all the complex numbers will be more convenient for discussion.

Remark 65. With the absence of a particle injection, the ZW calculus has now two full fledge symmetric braidings. All the operations are natural with the structural swap and the fermionic swap.

We have seen that the quaternary even projectors can emulate white spiders. The odd projectors, which is the following diagram (modulo scalar)

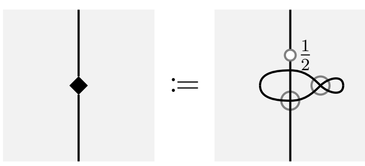


also can emulate something; the binary one actually emulates a pair of particles!



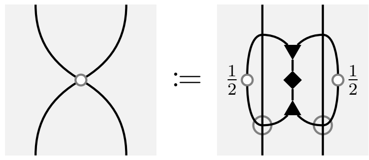
The curvature of space-time can actually emulate particles. The particles that we perceive may be a manifestation of space-time interacting with the fermionic swaps.

We will introduce a short-hand for this binary odd projector:



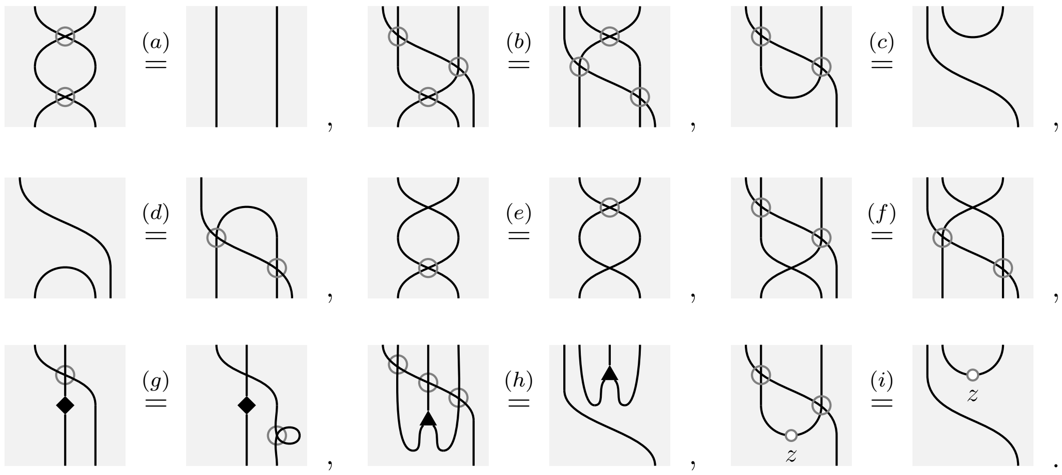
$\mapsto |1\rangle\langle 1|.$

We also need a new short-hand for the quaternary white vertex:

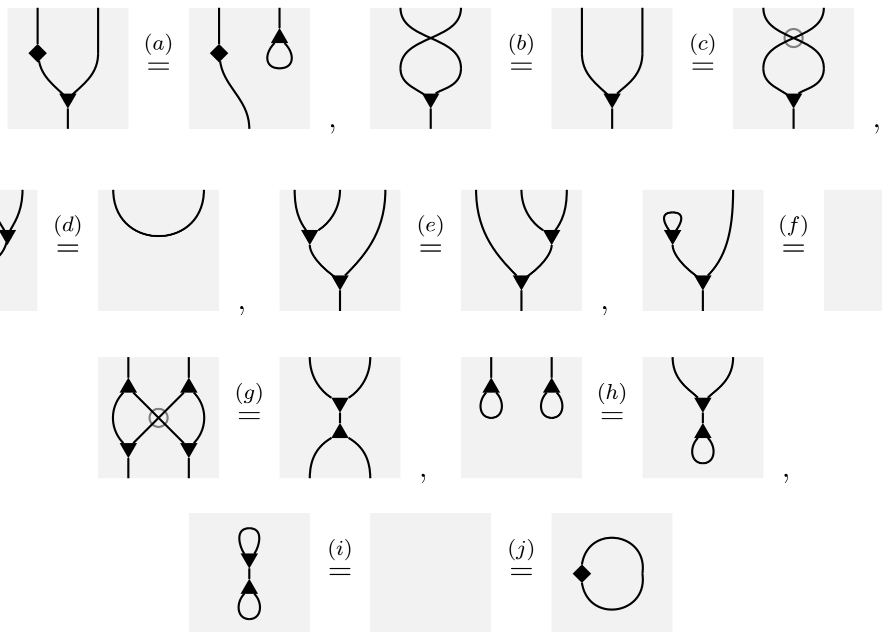


In this scenario, we only have the even maps. We will build a ZW calculus, we will call it *even ZW calculus (EZW)*, which is complete for the subcategory of only even maps, which we will call the category of *even LFM* (**ELFM**).

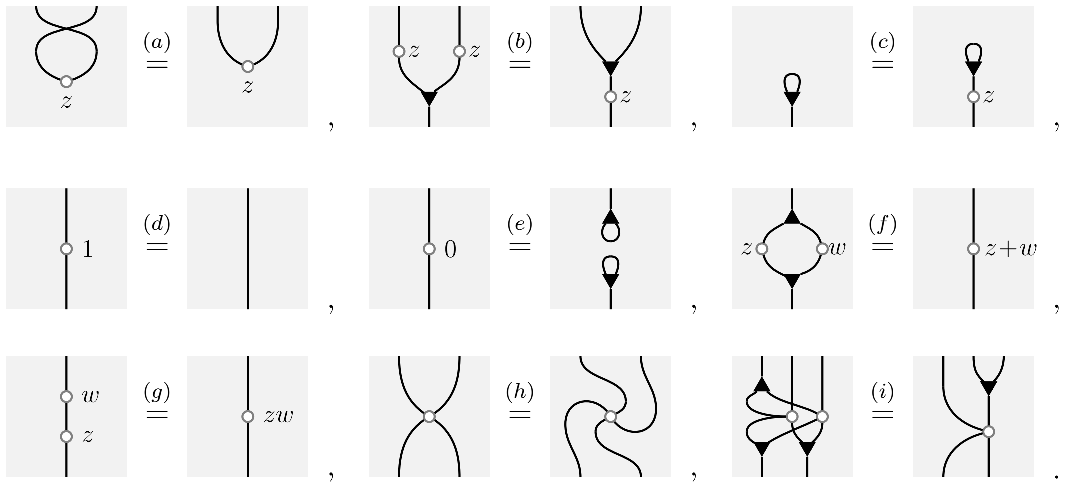
1. The following are the axioms for the fermionic swap:



2. The following are the axioms for the black vertices:



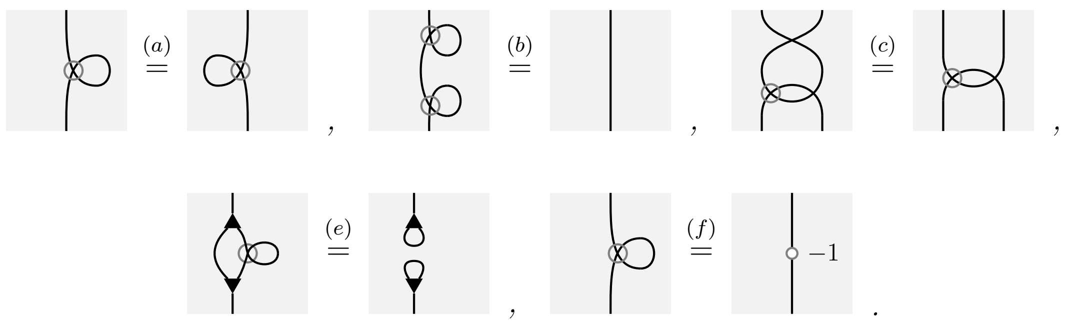
3. The following are the axioms for the white vertices:



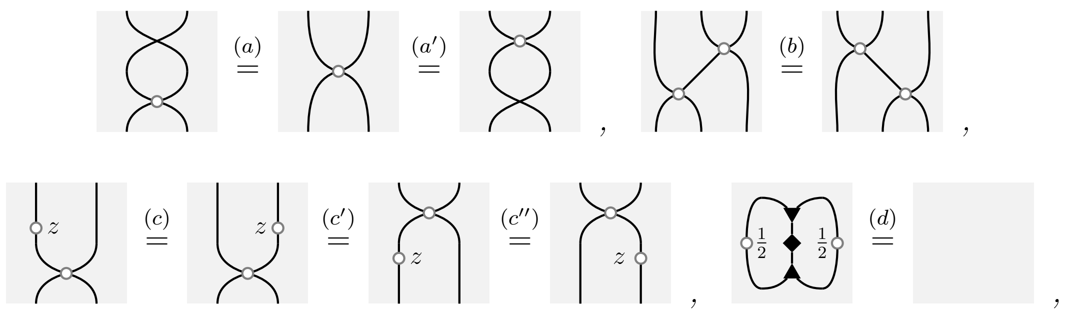
The axioms for the even ZW calculus are more or less the same as the fermionic ZW calculus; we have replaced all the axioms that has odd states with a corresponding even ones. Notable ones are axioms $2(a)$ which is another way of representing an annihilation operator, and axiom $2(j)$ is another empty rule.

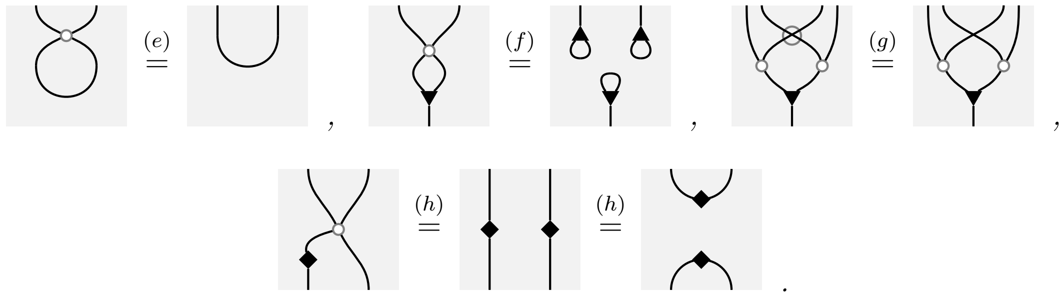
Most of the derived equations are the same as the fermionic ZW calculus. The proofs are also more or less the same. I will state the derived equations without proof.

Proposition 66. *The following are derived rules concerning the black vertices:*



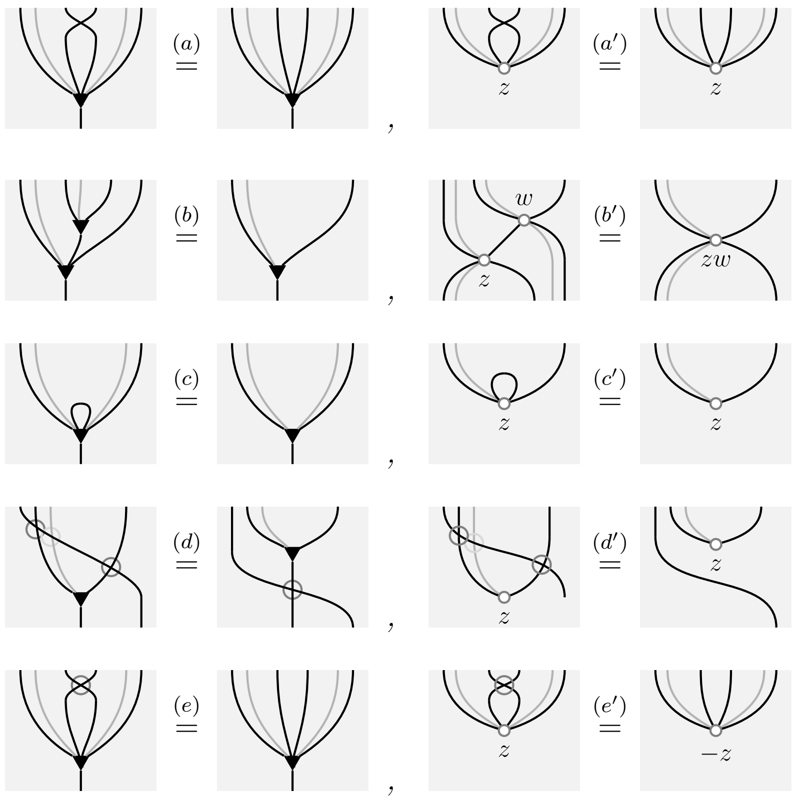
Proposition 67. *The following are derived rules concerning the white vertices:*



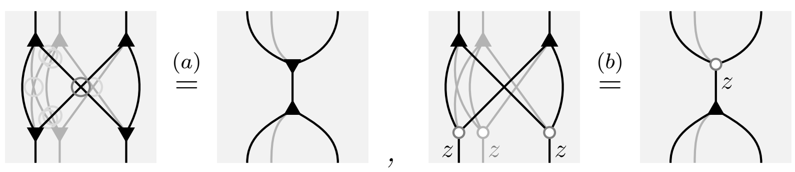


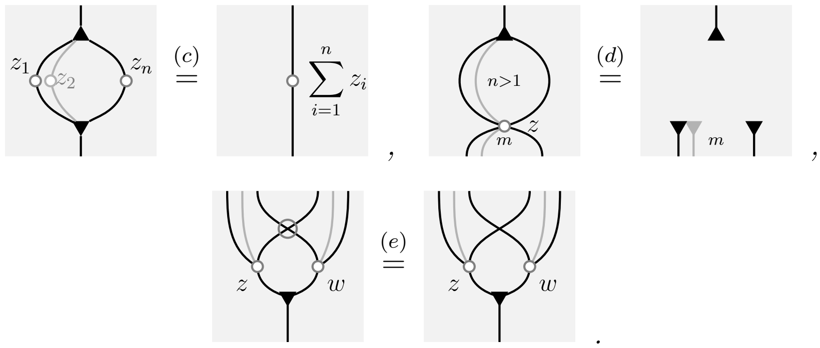
We will define some short-hand notation for the black monoid:

Proposition 68. *The following equations hold in EZW for black and white vertices of any arity:*



Proposition 69. *The following equations hold in EZW for black and white vertices of any compatible arities:*





The proof for the completeness follows a very similar route. There are some slight modifications, but they aren't major. Perhaps, the only big difference is that we don't have the negation lemma, but that can be circumvented.

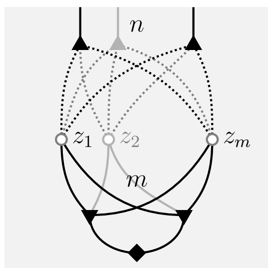
We will kick off by proving that is it universal with respect to the even ZW calculus.

Theorem 70 (Universality)**.** *The interpretation functor functor i_{fzw} : $EZW \to$* **ELFM** *is full.*

Proof. Write an arbitrary even state $v : \mathbb{C} \to B^{\otimes n}$ in the form

$$\sum_{i=1}^{m} z_i |\, b_{i1} \dots b_{in} \,\rangle,$$

where $z_i \neq 0$ for all i, no pair of n-tuples $(b_{i1}, \dots, b_{in})$ is equal, and $\sum_{j=1}^{n} b_{ij}$ is even. Then, we claim that the it is the image the diagram



(3.34) ,

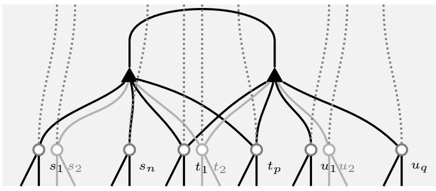
where, for $i = 1, \dots, m$ and $j = 1, \dots, n$, the dotted wire connecting the i-th white vertex to the j-th output is present if and only if $b_{ij} = 1$.

The claim can be proved by a direct calculation. □

We will define Equation 3.34 as the normal form for a state in **ELFM**, and the pre-normal form is defined as before.

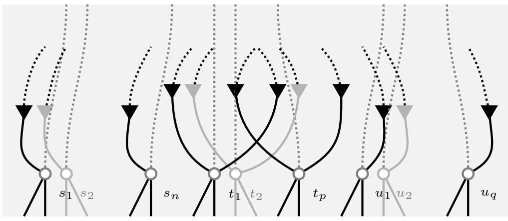
Lemma 71 (Trace)**.** *The plugging of two outputs of a diagram in pre-normal form into each other can be rewritten in pre-normal form.*

Proof. Focus on the two relevant outputs, and subdivide the white vertices into three groups, based on their being connected to both outputs, or to one of them:



.

Again, the dotted wires can stand for a multitude of wires. Using Proposition 69.(a), this becomes



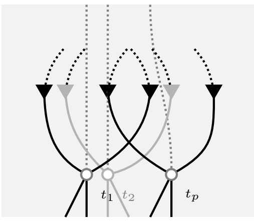
,

where the white vertices labelled with s and u are connected to a black comonoid each, and the ones labelled with t are connected to two black comonoids. The comonoids are separated to two halves, and the the dark dotted lines represents the connections between the comonoids (not pictured), forming a bipartite graph between the left half and right half of the comonoids. The intersections between the comonoids are all fermionic swaps.

Apply the Proposition 69.(b) to the s and u white vertices. Each s vertex produces $p + q$ white vertices, and each u vertex produces $n + p$ white vertices. Out of these white vertices, only p of them survive for each label because they will be paired with another white vertex (a s labelled will be paired with a u labelled one), which then fuse together and has a double connection to the bottom black vertices, and Proposition 69.(d) is applicable to kill them off.

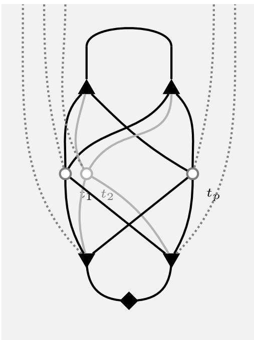
We will refer to the diagram we have now as D (although it is not drawn). Now, apply Proposition 69.(b) between the left half of the black comonoids with the t white vertices. Using the same argument as above, that will eliminate all the u white vertices. Then apply Proposition 69.(b) to reverse what was done to recover back the diagram D, except that now we don't have the u white vertices. Repeat the same procedure to the right half of the black comonoids to eliminate the s white vertices.

We will be left with the following diagram:



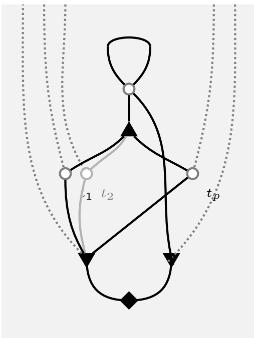
.

We can now apply Proposition 69.(a) to the black comonoids to get



.

Apply Proposition 69.(b) to get

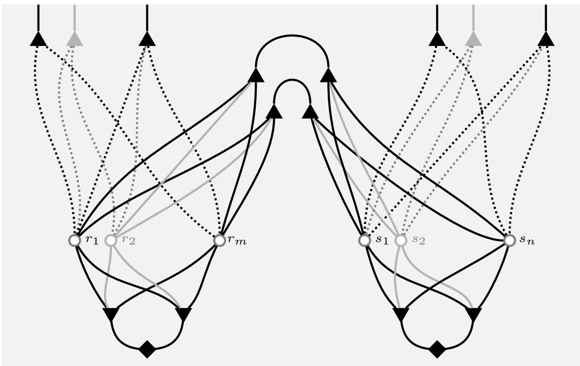


.

Finally, apply Proposition 68.(c', b) and rearrange the diagram. This leaves us with a diagram in pre-normal form. $\square$

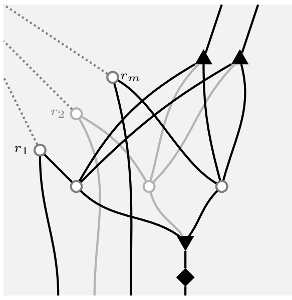
Lemma 72 (Functoriality)**.** *Any composition of two diagrams in pre-normal form can be rewritten in pre-normal form.*

Proof. The only obstacle in this proof is to connect the two states together without the use of negation lemma like before. Consider the following state:



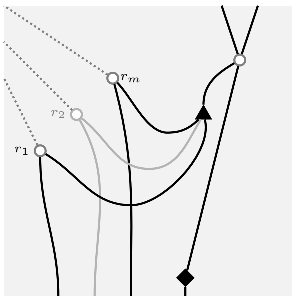
,

where there are a pair of black monoid connected to all the white vertices on each side of the state. We will zoom in on this part of the diagram. The other part is analogous:



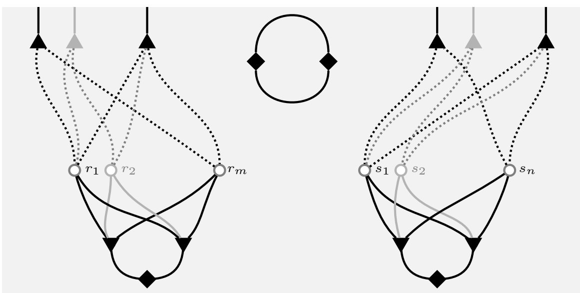
,

We can apply Proposition 69.(b) to obtain



,

Finally, apply Proposition 67.(h), we obtain disconnected states

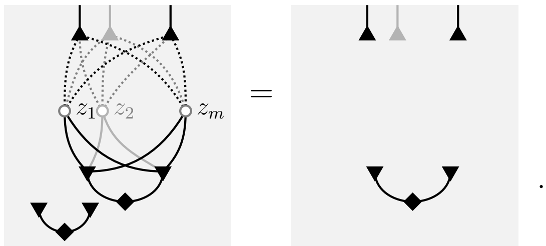


,

where we can remove the scalar easily by applying Proposition 67.(h) and axiom 2(j).

We have shown how to connect the two states, the rest of the procedures are pretty much the same as the previous functoriality lemma. Finally, we obtain a single state with a pair of outputs plugged to each other. This can be eliminated by applying the trace lemma, which completes the proof. □

Lemma 73 (Absorption). *For all diagrams in pre-normal form, a null diagram eliminates all the white vertices; that is,*



(3.35)

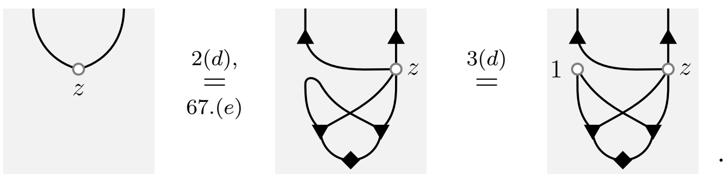
Proof. Take the bottom black diamond vertex and do the same trick as the functoriality proof to connect to all the white vertices. Then, the diagram can be simplified using Proposition 69.(a, b). With some further simplification, we obtain what we require. □

Lemma 74. *Any diagram in pre-normal form can be rewritten in normal form.*

Proof. This proof is the same as Lemma 59. □

Theorem 75 (Completeness). *The interpretation functor* $i_{ezw} : EZW \to \mathbf{LFM}$ *is a monoidal equivalence.*

Proof. We start writing the binary white vertex with parameter $z \in \mathbb{Z}$:



For the black monoid, fermionic swap, and structural swap we use the fact that we know how to rewrite the tensor product of two dualities in normal form, then plug them on top.

This concludes the proof. □

Chapter 4

The ZX calculi

Overview:

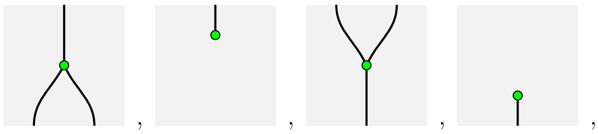
- Section 4.1 — We introduce the core of the ZX calculus, which is interacting complementary observables as a pair of Frobenius algebras interacting via Hopf algebra. We then show a heuristic on determining the scalars that may appear in diagram equalities. Finally, we introduce the progress of the calculus, till the point just before the stuff in the next section.
- Section 4.2 — This section summaries the results of the completeness of the Clifford+T ZX calculus by Jeandel, Perdrix and Vilmart. The important message is their technique to complete the calculus.
- Section 4.3 — This section is the main original contribution of this chapter. We improve on the work by Jeandel et al., and give a procedure to complete any universal diagrammatic calculus without a clear normal form, by translating to a already complete calculus. The technique is essentially making the two calculi monoidally equivalent, and hence complete. This is based on the published work [33] by Ng and Wang.
- Section 4.4 — We apply our technique developed in the previous section to the Clifford+T fragment. We also notice that with this new improved technique, we proved a more general result, which is that the calculus is complete for R**bit**, for all commutative ring R containing a ring $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$. This result is not yet reported, and it generalises the result in [37] where they state the completeness for the rings $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{2^n}}]$ for $n \geq 2$.

4.1 Basic properties of the ZX calculus

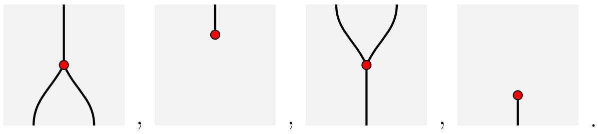
In this section, we shall give an operational overview of the ZX calculus. We will mainly be dealing with qubits systems in the discussions. Readers should refer to the book [17] for a more complete introduction of the calculus. There are also many references in the book which readers can further dig into.

4.1.1 ZX calculus and quantum circuits

The ZX calculus also lives in a self-dual, compact closed PROP. The crux of the ZX calculus is a pair of special commutative Frobenius algebra, draw as such:



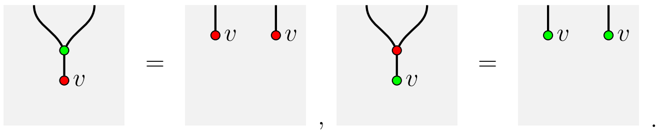
(4.1)



(4.2)

where the comonoids is the canonical transposition of the monoids.

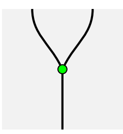
Given a Frobenius algebra, it uniquely determines an orthonormal basis, and vice versa. The elements of the basis v are the copyable states, that is:

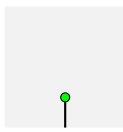


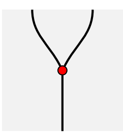
In qubits systems, we will interpret the green Frobenius algebra as the copier for the computational basis $\{|0\rangle, |1\rangle\}$, which is commonly called the Z-basis. The red one is then the copier for the complementary basis

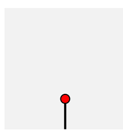
$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\
|-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),
\end{aligned}
$$

called the X-basis. Hence, we can set up an interpretation map i_{zx} as such:

 $\mapsto |00\rangle\langle 0| + |11\rangle\langle 1|$,

 $\mapsto \langle 0| + \langle 1|$,

 $\mapsto |++\rangle\langle +| + |--\rangle\langle -|$,

 $\mapsto \sqrt{2}\langle 0|$.

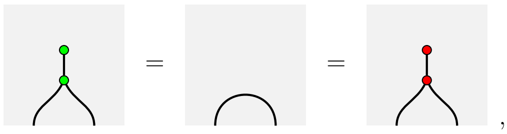
It is clear from this interpretation that

Hence we can take this as a definition for the (co)unit. The (co)monoids are obviously commutative, forms a Frobenius algebra, and the specialness comes almost for free.

Remark 76. The copyable states are sometimes called *classical states*. They are classical with respect to their respective copiers.

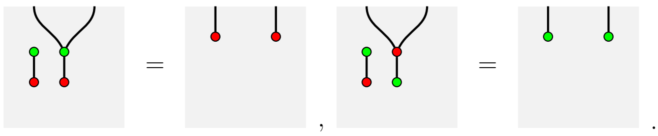
Remark 77. Because of the names of the bases, this calculus is called the ZX calculus. The ZW calculus retains the Z Frobenius algebra, which is a GHZ state, and defines a W state in the Z basis.

Under this interpretation i_{zx}, we see that

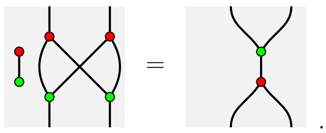


which means that we can bend one of the output wires of the comonoid down to obtain a monoid. Hence, what matters in the drawing is the connectivity of the vertices.

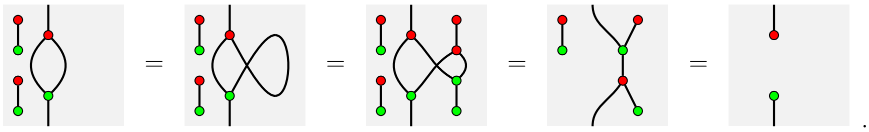
The green unit is proportional to the $|+\rangle$ state, and the red unit is proportional to the $|0\rangle$ state. Hence, the green copier copies the red unit, and the red copier copies the green unit (up to scalar):



When viewed in the Z basis, the green comonoid is the copier and the red monoid is the adder modulo 2 (up to scalar). Together, they form a bialgebra



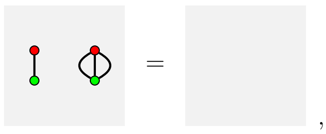
We call the two Frobenius algebras *strongly complementary* if they satisfy the bialgebra equation. In fact, we can prove that it forms a Hopf algebra with the identity wire as the antipode:



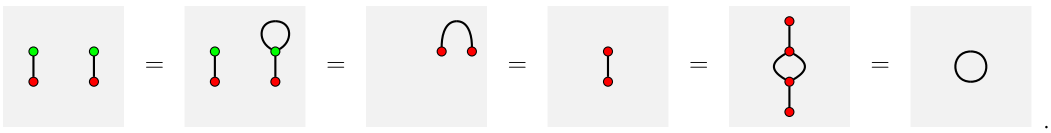
(4.3)

The Hopf algebra equation is a statement showing the principle of complementarity in quantum physics.

We see that there are a few scalars in the diagrammatic equations. They are there to balance the $\left(\sqrt{2}\right)^m$ for $m \in \mathbb{Z}$ that originated from the red vertices; the unary red vertex contributes a $\sqrt{2}$ while a ternary red vertex contributes a $\frac{1}{\sqrt{2}}$. So, an informal way of checking whether the scalar balances is to count the number of wires from the red vertices. A unary red vertex counts as -1, while the ternary one counts as $+1$. Take for example the first and last of the proof of the Hopf equation in 4.3. The first one counts as $-1-1+1$ and the last one counts as -1. We see that their counts balances, so we know that we got the right scalars. Another example is the following:

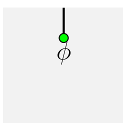


where both sides counts to 0. However, one must be careful when applying this rule because there are cases where it fails:

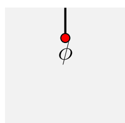


The specialness breaks this rule.

We can introduce phases with respect to the two bases:

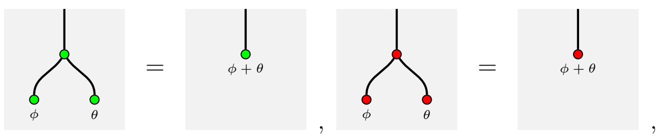


$\mapsto |\,0\,\rangle + e^{i\phi}|\,1\,\rangle,$



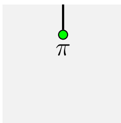
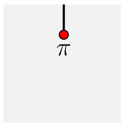
$\mapsto |\,+\,\rangle + e^{i\phi}|\,1\,\rangle.$

Their respective (co)monoids are adder for the phase angles:

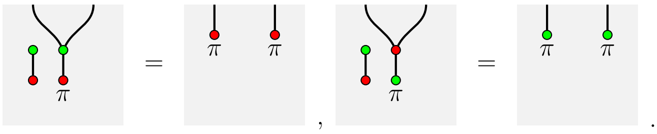


This means that the phases form a $U(1)$ cyclic group.

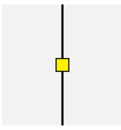
A direct calculation shows that the following phases are copyable states (up to scalar):

 $\mapsto |\,0\,\rangle - |\,1\,\rangle,$  $\mapsto \sqrt{2}\langle\,1\,|,$

that is,



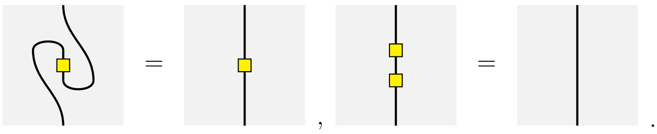
All the equations we have introduced so far are completely symmetric with the exchange of colours. We will add in a colour changing operation, known as the *Hadamard gate* in quantum information, which is changing between the Z and X bases:

 $\mapsto |\,0\,\rangle\langle\,+\,| + |\,1\,\rangle\langle\,-\,|.$

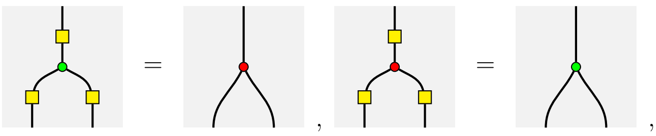
A matrix representation of the Hadamard gate may be more enlightening on its properties:

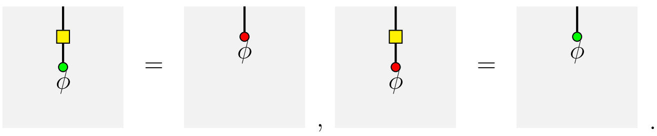
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

The Hadamard gate is a self transpose, and involutive:



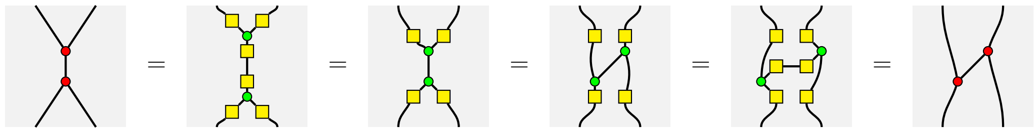
Hence, we have



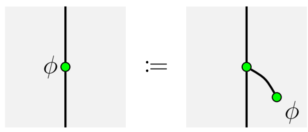


We could then define the ZX calculus entirely using just green vertices and a Hadamard gate, and the red vertices are by definition the green vertices plugged with the

Hadamards. This ensures that the axioms for the green vertices also applies to the red. For example, the Frobenius equation for the red now can be proved as such:

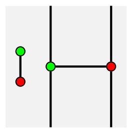

.

The ZX calculus is also related to quantum circuits in a natural way; the basic components of a quantum circuit can be expressed in the ZX language easily. We already have the Hadamard gate, the computational basis, and there is an obvious way to get the phase shifts:



.

We are left with the CNOT gate, which can be constructed as such:


.

The left input is the control input, while the right is the target.

It is well known that the quantum gate set consisting of arbitrary phase shifts, the CNOT gate and the Hadamard gate, together with ancilla bits is universal with respect to **Qubit**. This implies the universality of of the ZX calculus with respect to **Qubit** [13].

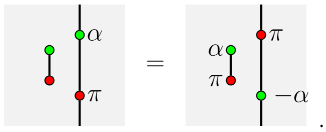
Theorem 78 (Universality of the ZX calculus)**.** *The interpretation functor* i_{zx} : $ZX \to$ **Qubit** *is full.*

Remark 79. We have a natural way to embed quantum circuits into the ZX language. However, the converse is not that obvious. This is an active area of research, to find what is the optimum way to convert a ZX diagram to a circuit.

This is the essence of the ZX calculus. We can define different fragments of the ZX calculus by restricting the angles ϕ: real stabiliser has ϕ a multiple of π, stabiliser has ϕ a multiple of $\frac{\pi}{2}$, and Clifford+T has ϕ a multiple of $\frac{\pi}{4}$.

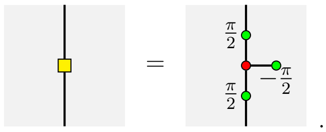
For many years, there are some axioms added to the calculus, but they serve a more operational purpose. We will list a couple of them here:

- K commutation axiom:



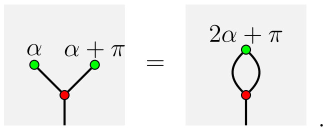
This axiom tells us how a classical point in one colour commutes with any arbitrary angle of another colour.

- Hadamard decomposition [23]:



This axiom tells us that a Hadamard gate can be decomposed into some red and green vertices with phases.

- Supplementarity [46]:



The supplementarity axiom connects angles that are π apart.

Even with these rules, the calculus is still not complete in general [22]. However, some fragments have been completed: stabiliser fragment by Backens [3], real stabiliser fragment by Duncan and Perdrix [24], and single qubit Clifford+T fragment by Backens [4]. The multi-qubit Clifford+T fragment and the general ZX calculus was still not complete.

A ground breaking completeness result came from Jeandel, Perdrix and Vilmart where they added some axioms and proved the completeness of the ZX calculus for the Clifford+T fragment [36]. Their technique is interesting, and is the focus for the next couple of sections.

4.2 Completeness of the Clifford+T ZX calculus

This section is based on the paper [36] which is not by us. However, it contains many results that we may use in the next section. So we will state their results here for easy reference.

4.2.1 The completing procedure

The paper provides a set of axioms for the ZX calculus that is complete for the Clifford+T fragment of the qubit theory. We will write the ZX calculus for Clifford+T as $ZX_{\frac{\pi}{4}}$. Perhaps, their most important result is the proving technique. Their proof involves setting up two functors, $T_1 : ZX_{\frac{\pi}{4}} \to ZW_{\mathbb{Z}[1/2]}$ and $T_2 : ZW_{\mathbb{Z}[1/2]} \to ZX_{\frac{\pi}{4}}$, and requires them to have some properties. We will state that in the procedure below:

1. First, prove that the $ZX_{\frac{\pi}{4}}$ is universal for $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$. With this step, we can encode the ZX calculus into the ZW calculus.
2. Pick a suitable ZW calculus for the encoding. In their case, the commutative ring chosen is $\mathbb{Z}[\frac{1}{2}]$. Set up a translation map $T_1 : ZX_{\frac{\pi}{4}} \to ZW_{\mathbb{Z}[1/2]}$ and make sure that it is functorial. The functor should also have the property that two ZX diagrams D_1 and D_2 have the same interpretation $i_{zx}(D_1) = i_{zx}(D_2)$ if and only if they have the same interpretation after translating to ZW diagrams $i_{zw}(T_1(D_1)) = i_{zw}(T_1(D_2))$. By the completeness of the ZW calculus, we can conclude that $T_1(D_1) = T_1(D_2)$.
3. Then, set up a translation map $T_2 : ZW_{\mathbb{Z}[1/2]} \to ZX_{\frac{\pi}{4}}$ and also make sure this is functorial.
4. Finally, set up a decoding endofunctor $f : ZX_{\frac{\pi}{4}} \to ZX_{\frac{\pi}{4}}$ such that $f \circ T_2 \circ T_1$ is the identity functor.

We will explain how this procedure proves completeness. The idea is to encode the information from ZX to ZW, then encode back to ZX. The encoding processes should preserve the information contained in the diagram. Then, we apply a decoding map to recover the original diagram. More explicitly, suppose we have two ZX diagrams D_1 and D_2 that have the same interpretation under i_{zx}. We send them through the translation functor T_1 such that $i_{zw}(T_1(D_1)) = i_{zw}(T_1(D_2))$. By the completeness of the ZW calculus, we have $T_1(D_1) = T_1(D_2)$. Then send it through the functor $f \circ T_2$, we obtain $D_1 = f \circ T_2 \circ T_1(D_1) = f \circ T_2 \circ T_1(D_2) = D_2$ as desired.

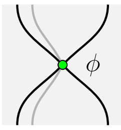
Each encoding step is non-trivial. That is because there is a need to ensure that the encodings are functorial. We are not going to explain how this is done.

The crucial step in completing the ZX calculus is setting up T_2. We want this to be functorial, and that means every axiom in the ZW calculus must be derivable, after translating, in the ZX calculus. This step will be the key step in the subsequent few sections of the chapter when completing other fragments of the ZX calculus.

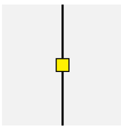
4.2.2 The (old) calculus

We present the ZX calculus as in version 1 of the paper. This is because version 1 is more relevant to our work, and using the latest version introduces unnecessary complications.

The ZX calculus for the Clifford+T fragment, written as $ZX_{\frac{\pi}{4}}$, is a self dual, compact closed PROP with the following generators T_{zxCT} and interpretation i_{zx}:

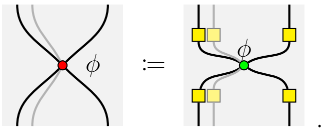


$$\mapsto |\,0\ldots0\,\rangle\langle\,0\ldots0\,| + e^{i\phi}|\,1\ldots1\,\rangle\langle\,1\ldots1\,|,$$



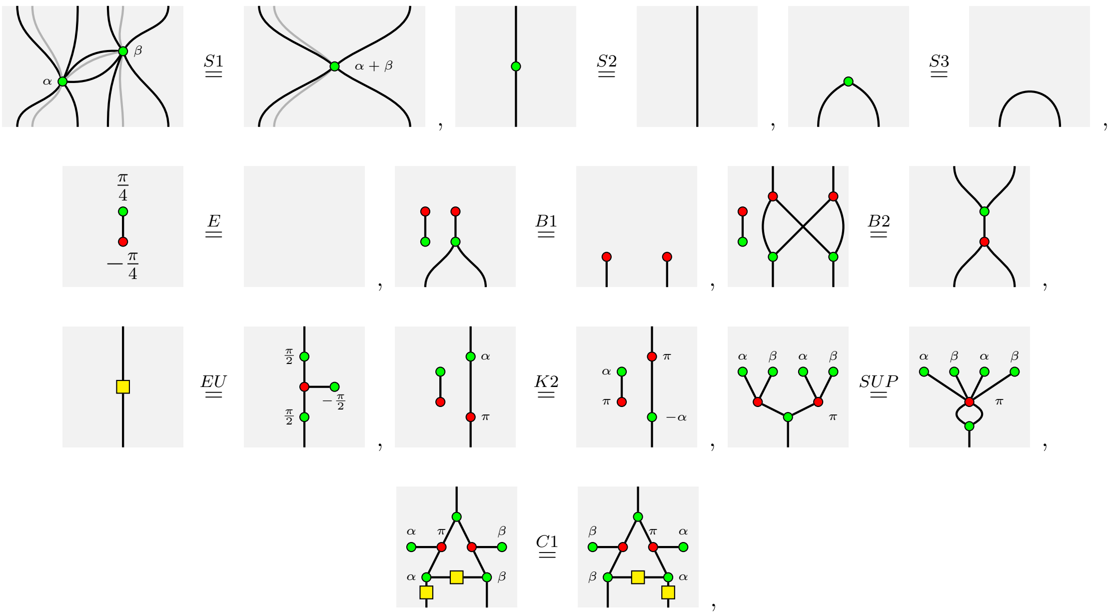
$$\mapsto \frac{1}{\sqrt{2}}(|\,0\,\rangle\langle\,0\,| + |\,0\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,0\,| - |\,1\,\rangle\langle\,1\,|),$$

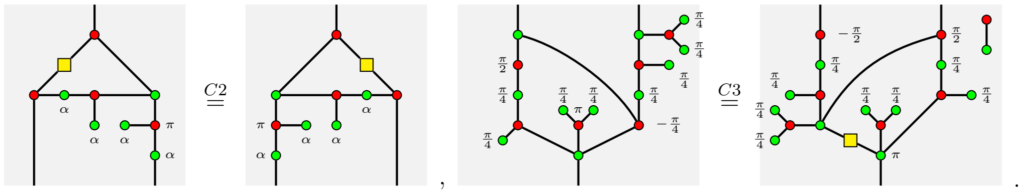
where $\phi = k\frac{\pi}{4}$, $k = 0, 1, \ldots, 7$. Define the red vertex as



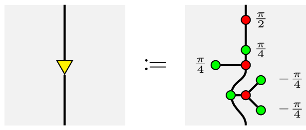
Remark 80. We have used the spider presentation of the green vertices. This is the same as the white vertices in the ZW calculus.

The following are the axioms E_{zxCT} for the calculus:



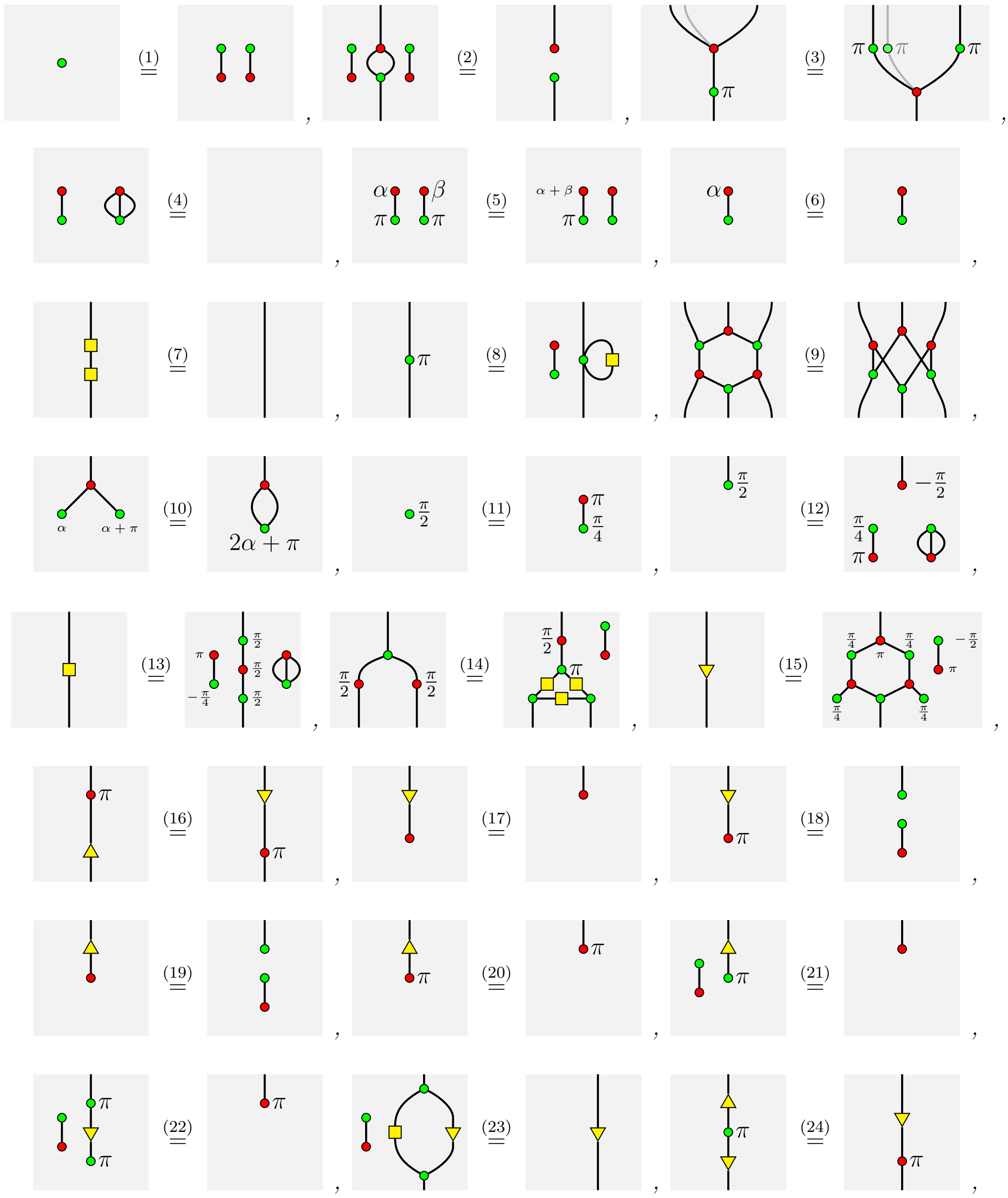


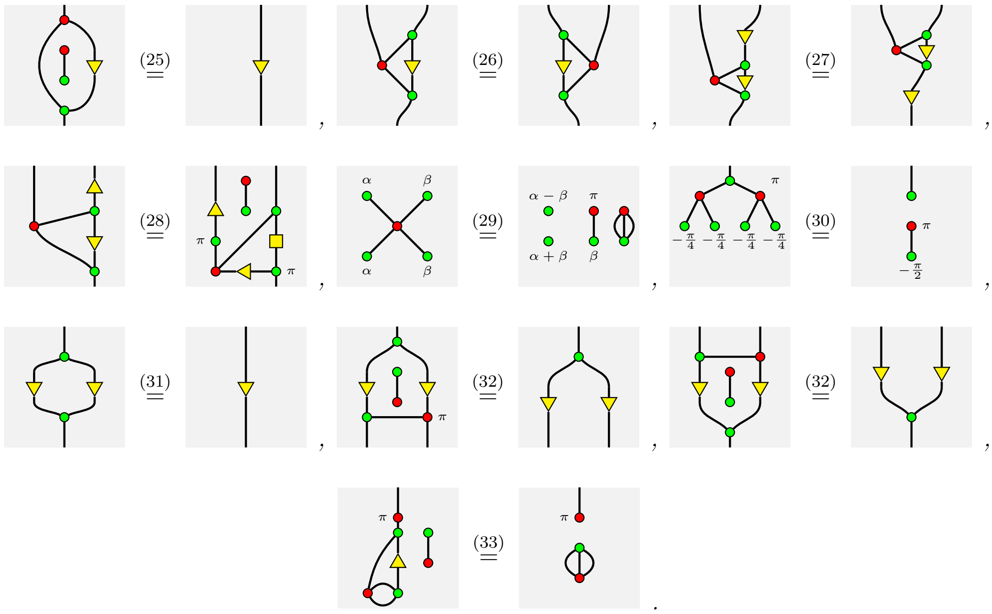
Lemma 81. *Defining the triangle node as such with its interpretation:*



$$\mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

the following are derived rules:





4.2.3 The completeness

Proposition 82. *The interpretation functor $i_{zx} : ZX_{\frac{\pi}{4}} \to \mathbb{Z}[\frac{1}{2}, e^{\pi/4}]$***bit** *is full.*

Their proof made improvements on Giles and Selinger's work to obtain the universality result [29]. We will state another proof which is more direct in the next section.

Proposition 83. *The interpretation functor $i_{zw} : ZW_{\mathbb{Z}[1/2]} \to \mathbb{Z}[1/2]$***bit** *is a monoidal equivalence.*

Theorem 84. *The interpretation functor $i_{zx} : ZX_{\frac{\pi}{4}} \to \mathbb{Z}[\frac{1}{2}, e^{\pi/4}]$***bit** *is a monoidal equivalence.*

Proof. Set up two translation maps, $T_1 : ZX_{\frac{\pi}{4}} \to ZW_{\mathbb{Z}[1/2]}$ and $T_2 : ZW_{\mathbb{Z}[1/2]} \to ZX_{\frac{\pi}{4}}$, which are described in their paper, such that:

- T_1 is functorial. This can be checked by checking all the translated axioms from $ZX_{\frac{\pi}{4}}$ are provable in $ZW_{\mathbb{Z}[1/2]}$;
- $T_1 \circ i_{zx}^{-1}$ and $i_{zx} \circ T_1^{-1}$ are well define;
- T_2 is functorial. This can be checked by checking all the translated axioms from $ZW_{\mathbb{Z}[1/2]}$ are provable in $ZX_{\frac{\pi}{4}}$. Any (seemingly) unprovable axiom is then promoted to an axiom;

- There exist an endofunctor $f : ZX_{\frac{\pi}{4}} \to ZX_{\frac{\pi}{4}}$ such that $f \circ T_2 \circ T_1$ is the identity functor.

This guarantees that for any ZX diagrams D_1 and D_2 such that $i_{zx}(D_1) = i_{zx}(D_2)$, $D_1 = f \circ T_2 \circ T_1(D_1) = f \circ T_2 \circ T_1(D_2) = D_2$, which completes the proof. □

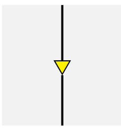
4.3 Completeness of the general ZX calculus

This section is based on the paper [33].

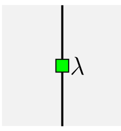
We have seen that the proof for the completeness of the $ZX_{\frac{\pi}{4}}$ is constructive; the axioms are obtained from constructing the translations. Hence, the axioms are dependent on the translation, that is, we get different presentation of the same theory via different translation. In this section, we will give a ZX calculus that is complete for **Qubit**. We made improvements on the technique used in the previous section by choosing a suitable ZW calculus, $ZW_{\mathbb{C}}$, and give a more direct translation, that is, the decoding functor f can be omitted. This immediately shows a stronger result, a monoidal equivalence between ZX and $ZW_{\mathbb{C}}$.

4.3.1 The completing procedure

The aim of the translation is to find out from the perspective of the ZW calculus what are the axioms needed for the ZX calculus to be complete. Therefore, when translating between the two graphical calculi, it is easier if we make them as similar to each other as possible. For this purpose, we will introduce two generators, the triangle and the lambda box, to the ZX calculus:



$$\mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$



$$\mapsto |\,0\,\rangle\langle\,0\,| + \lambda|\,1\,\rangle\langle\,1\,|,$$

for real $\lambda > 0$.

Remark 85. We are allowed to introduce these two generators because of the universality of the calculus. We have to take note not to alter the calculus by introducing generators that are not expressible in the original calculus.

Remark 86. A close examination of the sub-lemmas in Lemma 81 reveals that the triangle vertex plays an important role. It contains the power of the W state, which is a natural consequence since we are translating from the ZW calculus. Because of this, we chose to promote it to a generator instead of a shorthand for some connected red and green vertices. The axioms containing the triangle don't depend on the decomposition of the triangle, but instead take the connected red and green vertices as a whole. That is, different choices of decomposition lead to different axiomatisation.

As we want to complete the ZX calculus, we should translate the calculi (ZX and ZW calculi) with the theory in mind, in this case the category **Qubit**. Hence, we set up a pair of assignments $t_1 : ZX \to ZW_{\mathbb{C}}$ and $t_2 : ZW_{\mathbb{C}} \to ZX$ such that they respect the interpretations in **Qubit**, that is,

$$i_{zw} = i_{zx} \circ t_2,$$
$$i_{zx} = i_{zw} \circ t_1.$$

This implies that

$$i_{zw} = i_{zw} \circ t_1 \circ t_2,$$
$$i_{zx} = i_{zx} \circ t_2 \circ t_1,$$

and we want $t_2 \circ t_1$ and $t_1 \circ t_2$ to be the identity map because they have the same interpretation as before the translations.

Finally, we want t_1 and t_2 to be functorial. The translation t_1 is already functorial because it respect the interpretations and $ZW_{\mathbb{C}}$ is complete, so all we need to do is to make t_2 functorial. This step will tell us what axioms we need. This will give an isomorphism of PROPs, which completes the procedure.

This procedure can be applied to all universal string diagrams S which we want to complete with respect to a theory T:

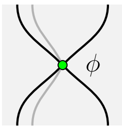
1. Take an already known complete diagrammatic calculus C and set up a pair of translation assignments $t_1 : S \to C$ and $t_2 : C \to S$ such that they respect the interpretation in T. This then requires that the translations to be inverse of each other;

2. Then, make t_2 functorial. In the process, we will identify the axioms from C that we may need to make S complete. This completes the procedure.

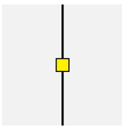
Remark 87. Using this completion method, we could, in principle, build a diagrammatic calculus with a whole bunch of generators and axioms that is relatively easy to complete, then translate this to a calculus that we are interested in.

4.3.2 The calculus

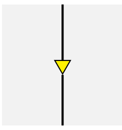
As before, the ZX calculus has the following set of generators T_{zx} with the interpretation i_{zx}:



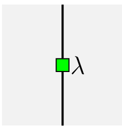
$$\mapsto |0\dots0\rangle\langle0\dots0| + e^{i\phi}|1\dots1\rangle\langle1\dots1|,$$



$$\mapsto \frac{1}{\sqrt{2}}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| - |1\rangle\langle1|),$$

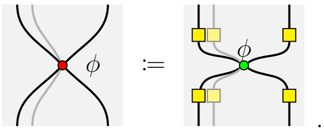


$$\mapsto |0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle1|,$$



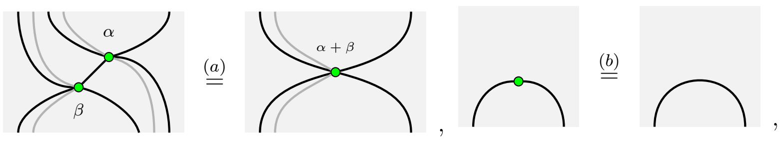
$$\mapsto |0\rangle\langle0| + \lambda|1\rangle\langle1|,$$

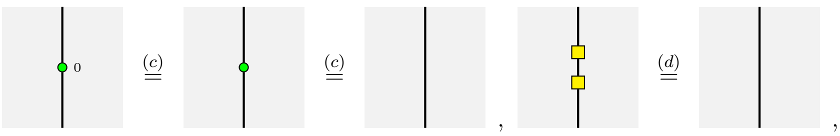
where $\phi \in [0, 2\pi)$, and real $\lambda > 0$. The red vertex is defined in the same way:

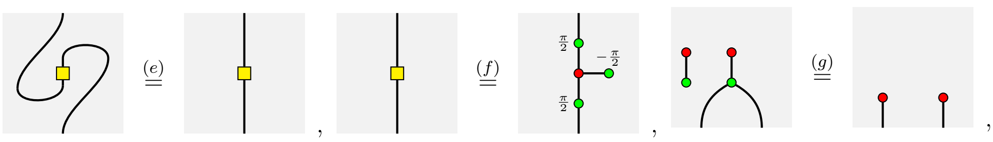


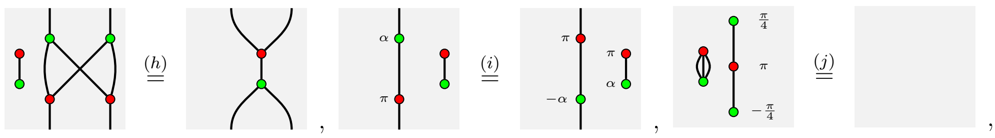
We split the set of axioms E_{zx}, which is sound by a direct calculation, in three groups:

1. The following are the axioms for the traditional generators of the ZX calculus:



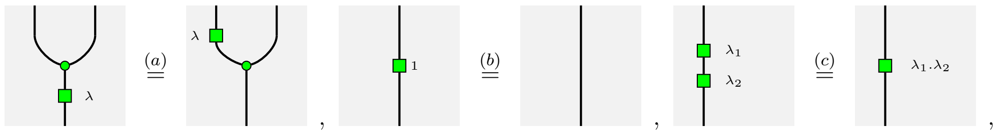


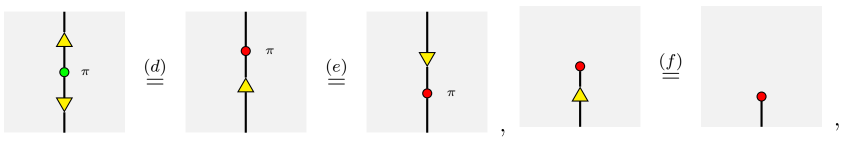


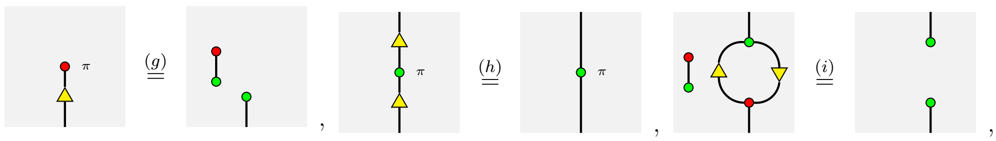


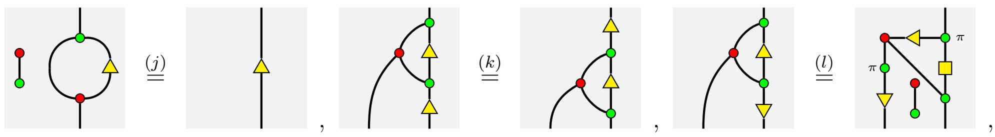
for $\alpha, \beta \in [0, 2\pi)$. It is derivable that the axioms are also true for the interchange of the red and green colours, and for simplicity we will give them the same axiom labels.

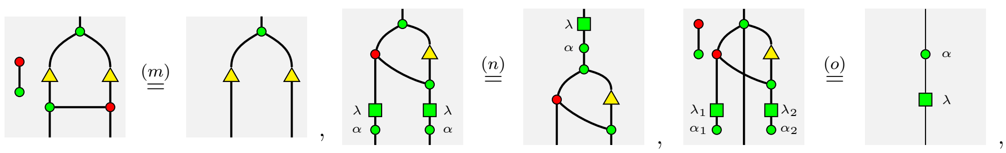
2. The following are the axioms for the extended generators of the ZX calculus:









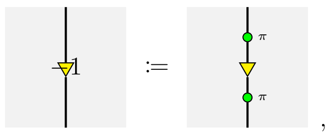


for $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, $\alpha, \alpha_1, \alpha_2 \in [0, 2\pi)$, and in (o), $\lambda e^{i\alpha} = \lambda_1 e^{i\alpha_1} + \lambda_2 e^{i\alpha_2}$.

The axioms in group 1 are the traditional axioms that we know already, and reflects that "only topology matters"; only the connectivity of the vertices are important. There is a new scalar axiom $1(j)$. The axioms in group 2 say that the green box acts in the same way as the green vertex. The rest of group 2 characterises the triangle by axioms with a small number of nodes. Some of the triangle axioms are inspired by [36]. Below is a list of where the axioms came from:

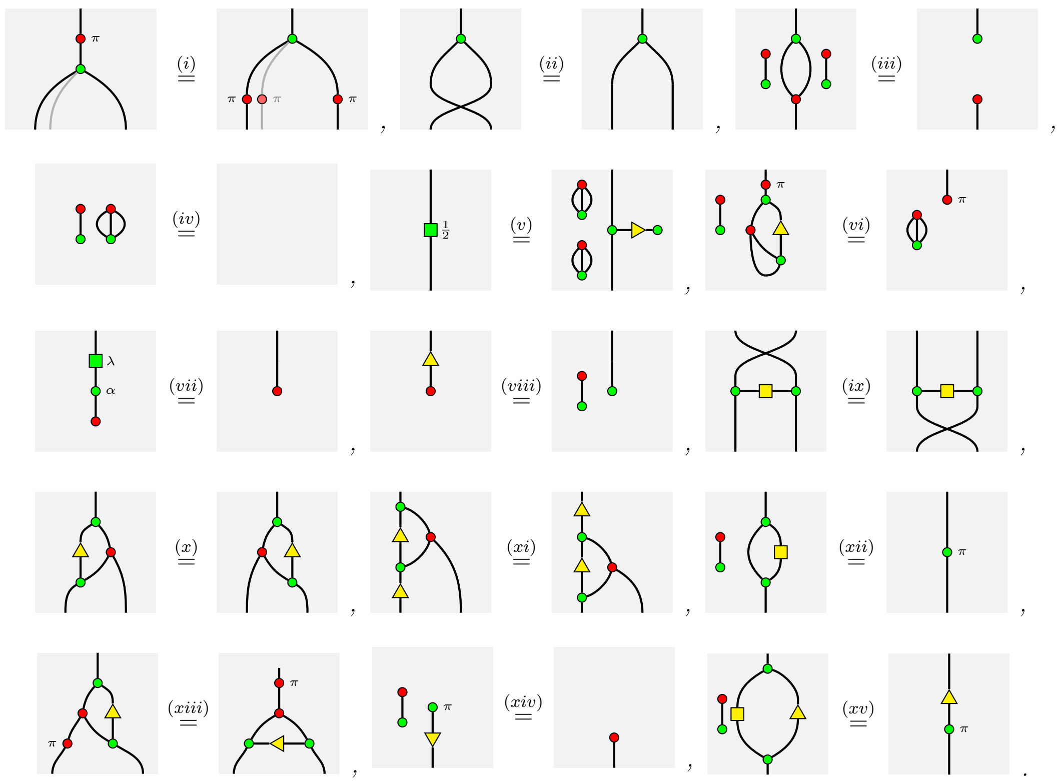
Axiom	Lemma 81
(d)	24
(e)	16
(f)	17
(g)	18
(j)	25
(k)	27
(l)	28
(m)	33

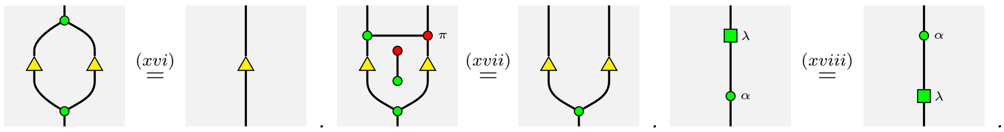
The triangle has an implicit directionality, meaning that not only just topology matters, but there is a difference between input and output. It is possible to mitigate this as suggested by axiom 2(e), but we will not write the axioms using the undirected triangle because there are some nice applications with the directed one. Axiom 2(h) is a statement showing the inverse of the triangle, shown below as a triangle with a -1 in it:



Axiom 2(n, o) are directly from the black monoid in the ZW calculus.

Proposition 88. *The following are derived rules:*





Proof. (i, iii) are proved in [6], (ii) is proved in [7], and $(vi, viii, x)$ are proved in [36]. The theorem references and axioms/rules used are

(i): Lemma A.15 (with inverted colours). The rules used are $1(a, b, c, d, e, f, g, h)$, (iv).

(ii): Lemma 45. The rules used are $1(a, c, d, g, h), (iii)$.

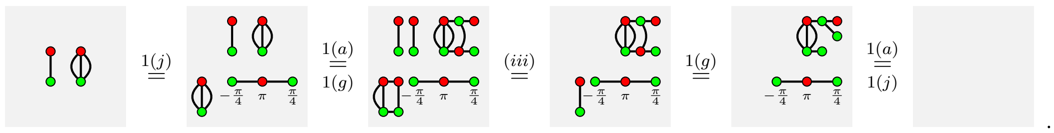
(iii): Lemma A.3. The rules used are $1(a, b, c, d, e, g, h)$.

(vi): Lemma 33. The rules used are $1(g), 2(f), (iii, iv)$.

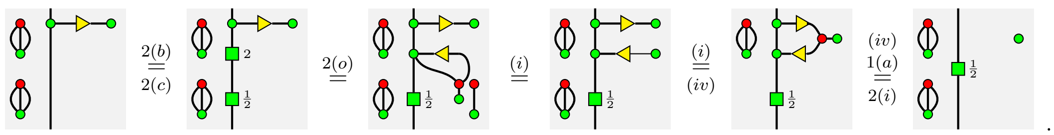
$(viii)$: Lemma 19. The rules used are $1(g), 2(f), (iii, iv)$.

(x): Lemma 26. The rules used are $1(a, h), 2(j)$.

(iv):

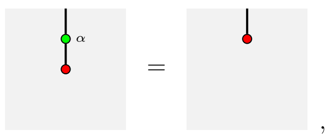


(v):



Finally, the scalars can be eliminated using $(viii)$ with a red unary vertex attached on top, then applying (iv).

(vii): First, we note that the green vertex can be eliminated, that is,



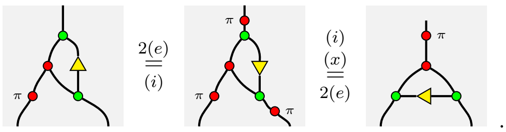
after applying $1(g)$ and (iv), the equation follows immediately from Lemma 3.4 in [6] using $1(a, g, i)$, (i, iv). Hence it suffices to show that it is also true for the green box. Given how the green box can be expressed in terms of the triangle, green and red vertices, it should be clear that using $1(g)$, $2(f)$ and the result stated above, we can eliminate all the vertices to obtain what we need.

(ix): A topological move of the vertices as stated in the discussion of the axioms. Axiom $1(e)$ and (ii) are used.

(xi): A topological move of the vertices in axiom $2(k)$ as stated in the discussion of the axioms. Rule (ii) is used.

(xii): A straightforward application of axiom $1(f)$.

$(xiii)$:



(xiv): The proof is as follows: write the right-hand side using $2(f)$, then create a pair of red π vertices on top and use $2(d)$, and finally use $2(g)$ and $1(a)$ to simplify.

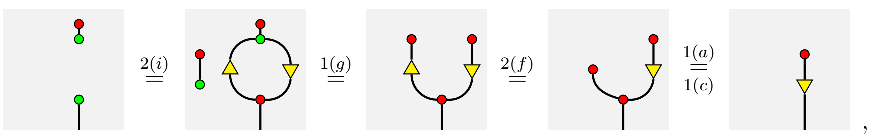
(xv): Starting from $2(l)$, attach a unary red vertex to the left wire. Using $2(f)$ and (vii), the right-hand side is the left-hand equation in (xv) (rotated and a green π vertex on top). So it suffices to prove

This can be proved in a straightforward manner by plugging a unary red π vertex on the left wire of $(xiii)$.

(xvi): Starting from $2(m)$, connect the two bottom wires with a ternary green vertex. Then apply (iii) to the left-hand diagram (with (iv)), and one of the triangles can be removed using $(viii)$. The scalars can then be taken care of using (iv).

$(xvii)$: This can be proved from $2(m)$ by attaching a unary red π vertex to all the wires and moving them around using (i) and $2(e)$. $(xviii)$: This is a simple application of $1(a)$ and $2(a)$. □

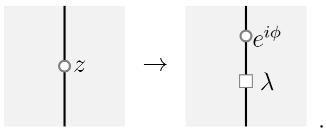
Remark 89. It was later found that the axiom $2(g)$ is derivable:



and apply a red π (with rule (i)) to get the direction of the triangle right. This serves as a reminder that we are not sure whether all the axioms are independent.

4.3.3 The completeness

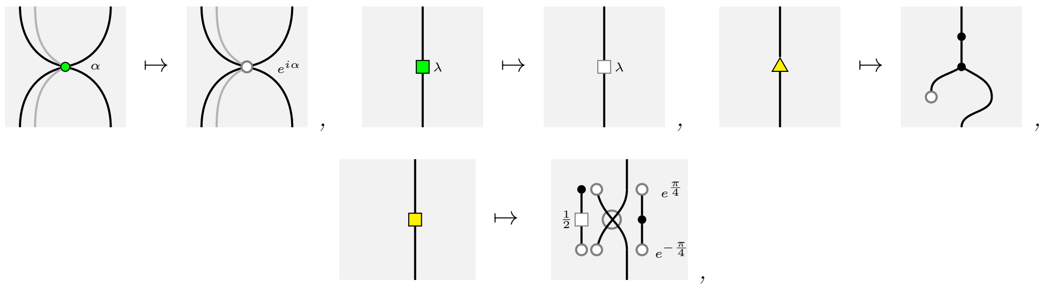
We will start the completeness proof with noting what is it universal to. From Theorem 78, it is universal with respect to **Qubit**. Therefore, we will choose the ZW calculus where $R = \mathbb{C}$. We will modify the ZW calculus, as suggested in the subSection 3.2.2.1, where we separate the white vertices into a phase part and a length part for the elements in $\mathbb{C}$:



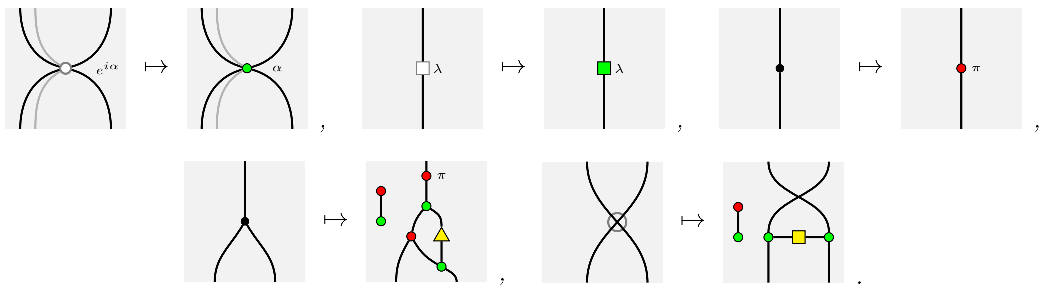
Then, we have to set up a pair of invertible translations that respect their respective interpretations. The translation from ZW calculus to ZX calculus is similar to the one by [36], but the one from ZX calculus to ZW calculus is made simpler.

Lemma 90. *Let t_1, t_2 be the following assignments of diagrams in $ZW_{\mathbb{C}}$ to the generators of ZX, and vice versa:*

1.



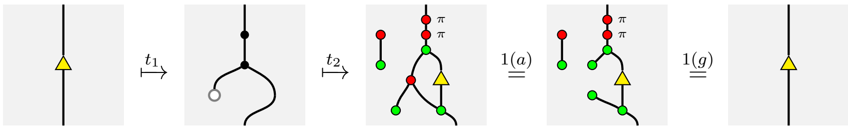
2.



Then t_1 and t_2 respect the interpretations of diagrams in **Qubit**, *and for all generators G of ZX, and G' of $ZW_{\mathbb{C}}$, we have $t_2(t_1(G)) = G$, and $t_1(t_2(G')) = G'$.*

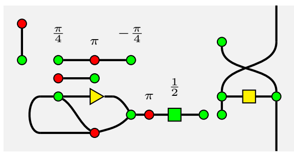
Proof. It is easy to check that the assignments respect the interpretations. Then, $t_1(t_2(G')) = G'$ follows from completeness of ZW and soundness of ZX.

The claim that $t_2(t_1(G)) = G$ for all generators of ZX is trivial to check for the green vertex and green box. Checking the triangle is a simple application of some axioms:



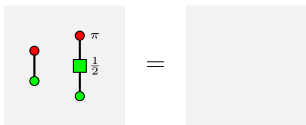
.

For the Hadamard gate, we will get



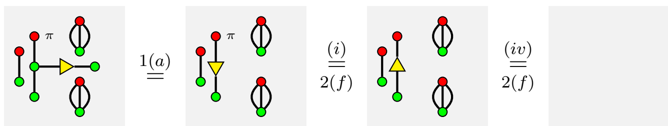
,

and after some simplifications using the derived rule (vi), axiom $1(a)$ and $1(j)$, we are left to show



.

This can be done by applying rule (v) to replace the green box, then

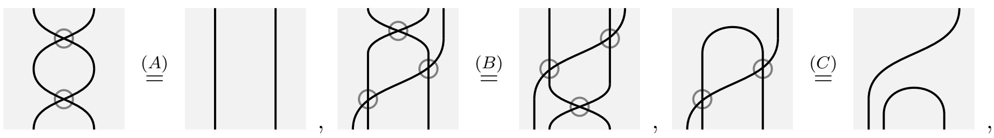


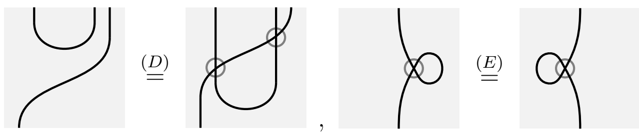
. ∎

Theorem 91 (Completeness of the ZX calculus). *The functor $ZX \to$ **Qubit** is an isomorphism of PROPs.*

Proof. We only need to show that extending t_2, as defined in Lemma 90, to composite diagrams defines a monoidal functor; it will then automatically be an isomorphism. For this, it suffices to check that the translations of all axioms of $ZW_{\mathbb{C}}$ can be derived from the ZX axioms. For convenience, we will list the ZW (expanded) axioms here:

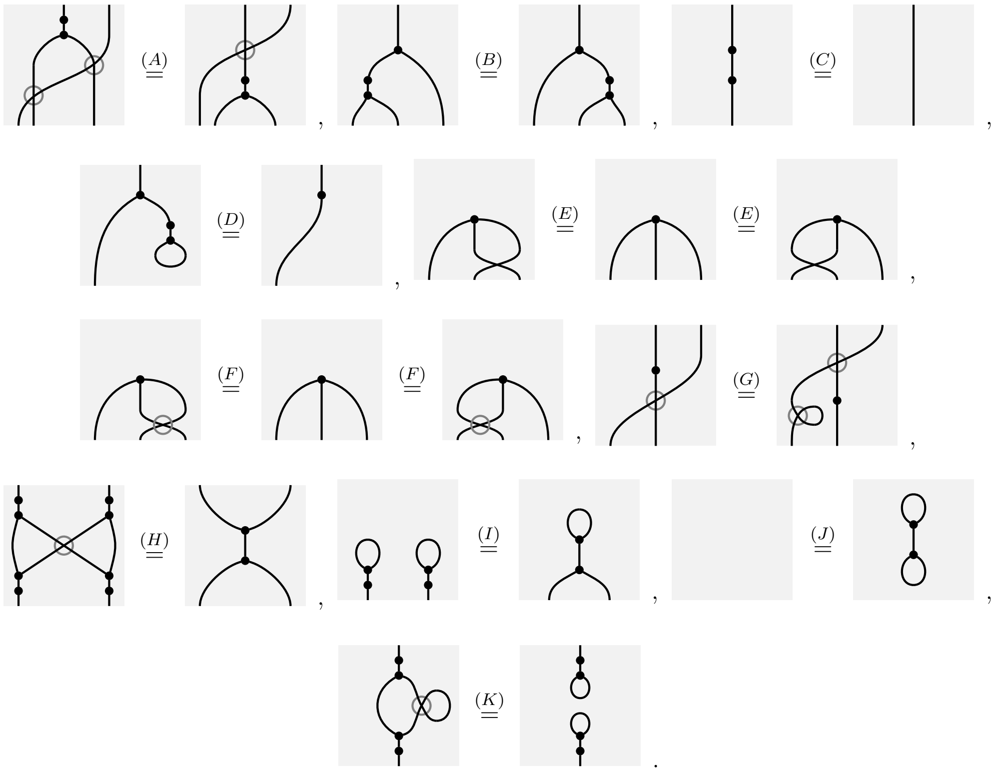
1. The following are the axioms for the crossings:



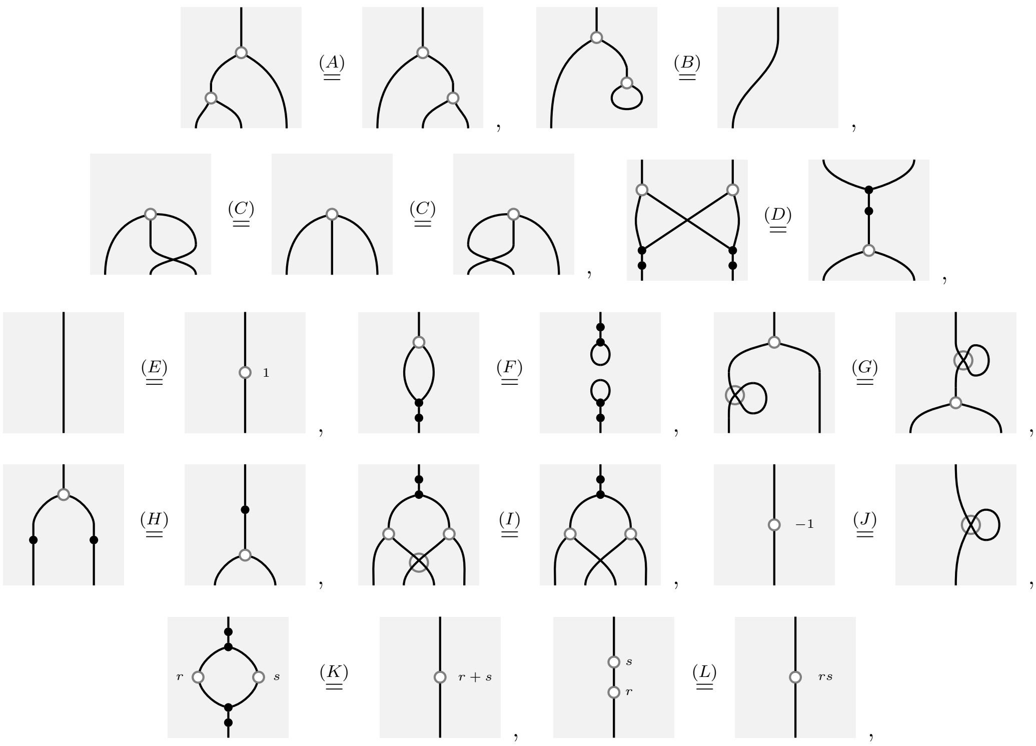


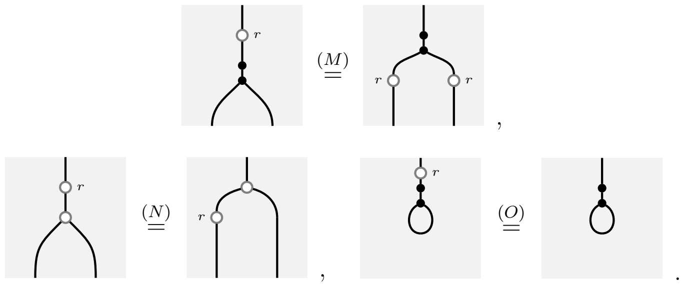
.

2. The following are the axioms for the black vertices:



3. The following are the axioms for the white vertices:

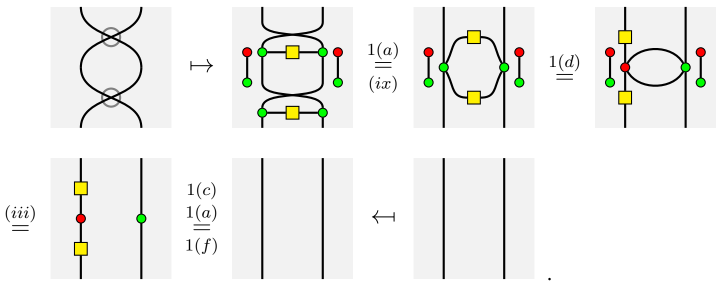




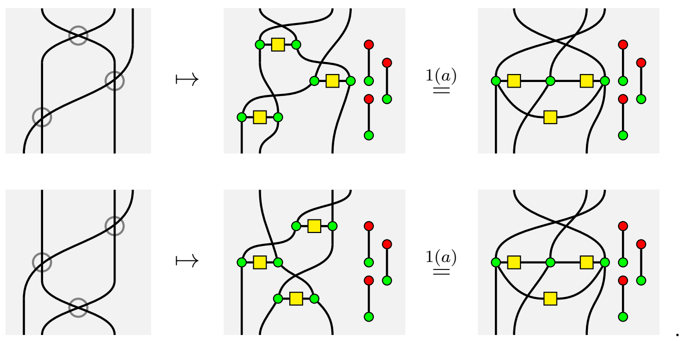
for $r, s \in \mathbb{C}$.

Many of the ZW axioms have been proved to be true in ZX in [36] version 1, and will be proved in exactly the same way with our axioms; we will not repeat those proofs but will simply indicate the theorem number (e.g. Proposition 7 part . . .) and the our axioms needed to prove them.

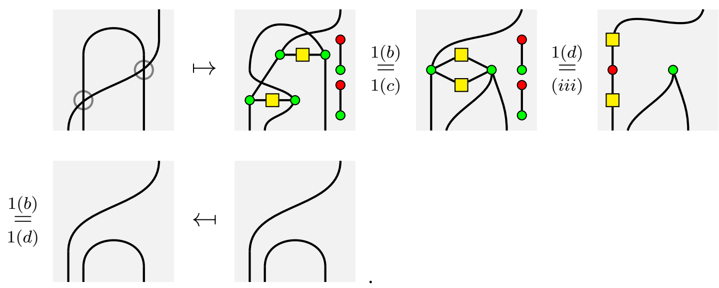
$1(A)$:



$1(B)$:

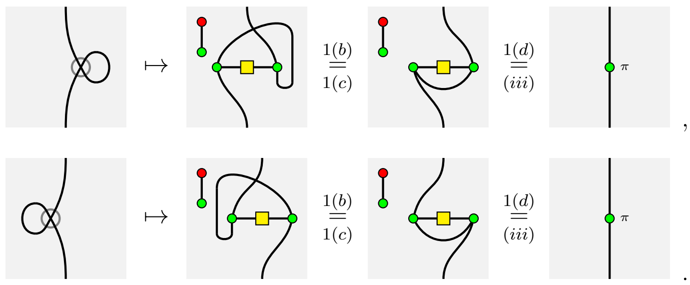


$1(C)$:



1(D): Similar to 1(C).

1(E):



This also proves 3(J).

2(A) Proposition 7 part 7a. The rules used are $1(a, h)$, (ix).

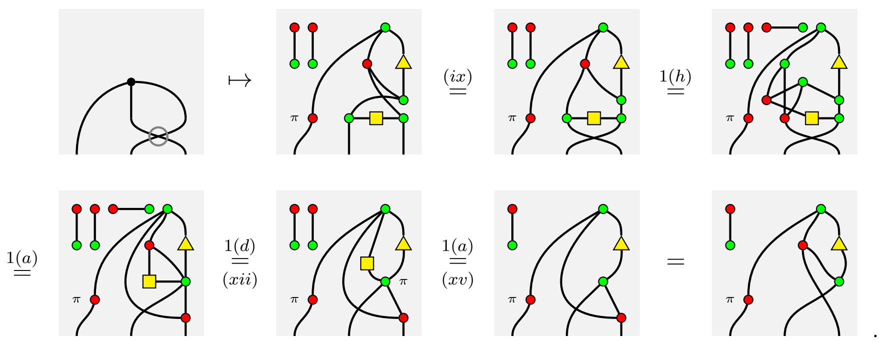
2(B) Proposition 7 part 1a. The rules used are $1(a, h)$, $2(k)$, (x, xi).

2(C) A simple application of $1(a, c)$.

2(D) Proposition 7 part 1b. The rules used are $1(a, c, d, f, j)$, $(iii, viii)$.

2(E) Proposition 7 parts 0b and 0b'. The rules used are $2(e)$, $(i,\ ii, x)$.

2(F)



We then recover case 2(E).

2(G) Proposition 7 part 7b. The rules used are $1(a, d)$, (i).

2(H) Proposition 7 part 5a. This was proved in eight parts, (i-viii). Note that these roman numerals refer to the parts in that paper, and are not the derived rules in this paper. The rules used are:

- i $1(a, d, h)$, (i, xv).
- ii $1(a, h)$, $2(e)$, $(i, x, xvii)$.
- iii $1(a)$, $2(d, k)$, (i, x).

iv $1(a,h)$, $(i,x,xvii)$.

v (i) and the previous parts (i).

vi $1(a,i)$, $2(e,l)$, (i,iii,x) and the previous part (i,ii,iii,v).

vii $1(a,h)$, $2(e,j)$, $(i,x,xvii)$.

viii $1(a,d,h)$, (i,iii,xii)

Finally, the proof combines these parts and the rules $1(a,h)$, $2(e)$, $(i,ii,iii,x,xiii)$.

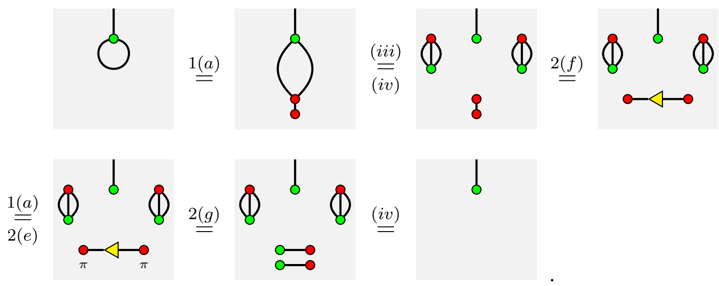
$2(I)$ Proposition 7 part 5c. The rules used are $1(g)$, $2(f)$, (vi).

$2(J)$ Proposition 7 part 5c. The rules used are $1(a,g)$, (iv,vi).

$2(K)$ Proposition 7 part 5d. The rules used are $1(a,g)$, (iii,xiv,xvi) and the result of $3(J)$.

$3(A)$ Follows trivially from $1(a)$.

$3(B)$ The translation is trivial, hence we only have to prove that the green vertices are special:



$3(C)$ Follows directly from (ii).

$3(D)$ Proposition 7 part 6a. The rules used are $1(a,h)$, (xvi).

$3(E)$ This is just $2(b)$.

$3(F)$ Proposition 7 part X. The rules used are $1(a,h)$, (ix,xii,xv).

$3(G)$ This is simply a combination of $3(J)$ and $1(a)$.

$3(H)$ This is (i).

$3(I)$ Proposition 7 part X. The rules used are $1(a,h)$, (ix,xii,xv).

$3(J)$ This is proved in $1(E)$.

3(K) This follows directly from 1(h), 2(j,o).

3(L) This is 1(a) and 2(c).

3(M) This is 2(n).

3(N) This is 1(a) and 2(a).

3(O) This is (vi) combined with (vii). □

4.4 Completeness of its different fragments

This section is based on the last part of the paper [33].

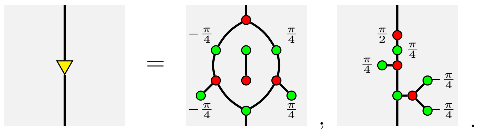
With the new improved completion technique, it will be nice to see how it can be applied to obtain a complete axiomatisation for the Clifford+T fragment different from the one by Jeandel et al as described in Section 4.2; as a result of our design choices, our axiomatisation has a larger number of axioms, which, on the other hand, involve a smaller number of vertices. In fact, our technique suggests a stronger completeness result, that is, completeness with respect to R**bit** for any commutative ring R containing a subring $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$.

The Clifford+T fragment of quantum mechanics is traditionally defined by restricting the Z basis phases to integer multiples of $\frac{\pi}{4}$. Therefore, the generators of the ZX$_{\frac{\pi}{4}}$ calculus are those of the ZX calculus where the labels of the green vertices are restricted to integer multiples of $\frac{\pi}{4}$. However, we would like to have the extended generators – the triangle vertex and green boxes – just like in the general ZX calculus described in the previous section, to aid in the completeness. If we are able to show that the extended generators are expressible in the Clifford+T fragment, then we get the completeness result for free since the axioms can be define to be the same as the general case, and the proof goes through in exactly the same manner. We first need to give a suitable definition for the green box. We are going to define the label on the green box as $0 < \lambda \in \mathbb{Z}[\frac{1}{2}]$.

Remark 92. We do not know, at the moment, whether all our axioms are mutually independent.

Lemma 93. *The triangle and green box are expressible in the Clifford+T ZX calculus.*

Proof. We will explicitly construct the two vertices. A representation of the triangle has been given in [17, 36]:



To represent the green box, we first notice that we are allowed to sum numbers using axiom 2(o), and multiply numbers using axiom 2(c). Hence, it suffice to show that the generators of the ring $\mathbb{Z}[\frac{1}{2}]$ are expressible. The 1 labelled green box is axiom 2(b), and the $\frac{1}{2}$ labelled green box is shown in Proposition 88.(v). $\square$

This automatically gives us the universality of the $ZX_{\frac{\pi}{4}}$ with respect to $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$**bit**, a proof different from the one given by Jeandel et al as described in Section 4.2.

Lemma 94. *The interpretation functor $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$* **bit** *is full.*

Proof. It is clear from the interpretation of the generators that the ZX$_{\frac{\pi}{4}}$ calculus is interpreted in the subcategory $\mathbb{Z}\left[\frac{1}{\sqrt{2}}, e^{i\frac{\pi}{4}}\right]$ **bit** $=$ $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** of **Qubit**. Furthermore, restricting the functor defined in Lemma 90 we obtain a full functor $ZX_{\frac{\pi}{4}} \to ZW_{\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]}$, which then implies that $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** is full. $\square$

With this, we are ready to state a complete ZX calculus for the Clifford+T fragment: The ZX$_{\frac{\pi}{4}}$ calculus features the same axioms as the ZX calculus, with restricted phases $\alpha = k\frac{\pi}{4}$, $k = 0, 1, \ldots 7$, lengths $0 < \lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$; moreover, we will modify the conditions of axiom 2(o) to $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\alpha \equiv \alpha_1 \equiv \alpha_2 (\text{mod } \pi)$.

The reason for the new condition for axiom 2(o) is because the interpreted matrices are over the ring $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$, where each element can be written as $\sum_{j=0}^{7} \lambda_j e^{i\frac{k\pi}{4}}$ for $\lambda_j \in \mathbb{Z}\left[\frac{1}{2}\right]$. Since $-e^{i\frac{k\pi}{4}} = e^{i(\frac{k\pi}{4}+\pi)}$, we can simplify the expression of each element to $\sum_{j=0}^{3} \lambda_j e^{i\frac{k\pi}{4}}$ for $\lambda_j \in \mathbb{Z}\left[\frac{1}{2}\right]$. This explains why we have the mod π condition. This rule is analogous to the addition rule (3(k)) of the ZW calculus for the Clifford+T fragment, only being more compressed.

Theorem 95 (Completeness of the ZX$_{\frac{\pi}{4}}$ calculus)**.** *The interpretation $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$* **bit** *is an isomorphism of PROPs.*

The power of the translation is that we get to see how are the two calculus related. The ZX calculus and ZW calculus are related by the triangle vertex; as long as we can construct a triangle in ZX calculus, we can, in principle, always complete it using the ZW calculus. With the triangle, we essentially get a universality over matrices over a commutative ring because we can start doing addition via axiom 2(o), and multiplication is just plugging them together. In view of this, the stabiliser fragment has no triangle in it. However, any fragment after the Clifford+T fragment is complete, that is, for any commutative ring R containing a subring $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$, the ZX calculus with the same axioms as the general one we presented is complete with respect to R**bit**. This generalises the result in [37], where they state this result for fragments defined by restricting the Z phase angles to integer multiples of $\frac{\pi}{2^n}$ for $n \geq 2$.

Theorem 96. *The ZX calculus is complete with respect to* R**bit** *for any commutative ring containing a subring isomorphic to* $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$.

Chapter 5

Generalisations of the ZW and ZX calculus

Overview: This chapter is about work in progress.

- Section 5.1 — This section talks about the prospect of using the ZX calculus for a hybrid system of quantum and classical computing. In particular, we noted that the classical gates can also be expressed naturally in the ZX language. As a by-product, the Toffoli gate can be expressed elegantly using the triangle vertex as introduced in the previous chapter on ZX calculus.

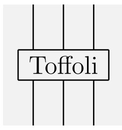
- Section 5.2 — This section is about generalising the ZW and ZX calculus to model higher (finite) dimension quantum mechanics.

- Section 5.3 — This section talks about the possibility to generalise the ZW calculus to model R**bit** where R is a commutative semiring. This may find applications in a wider range of problems, like the axiomatisation of stateless connectors.

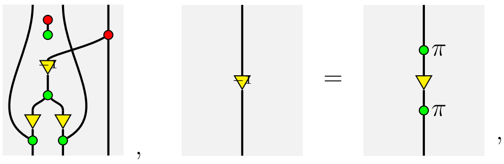
In the previous couple of chapters, we have explored the ZW and ZX calculus that is complete for R**bit** for some suitable commutative ring R. In this chapter, we will look at some on going work on generalising the results. This involves generalising the ZX calculus to model a hybrid system of quantum and classical computing, generalising the ZW and ZX calculi to model $\mathbf{Qudit}_d$ for some fix dimension $d > 2$, and generalising the ZW calculus for R**bit** where R is a commutative semiring.

5.1 ZX calculus for hybrid system

One of the main reason for having the triangle vertex in the ZX calculus is its relation to the Toffoli gate. The Toffoli gate is the control control not gate,



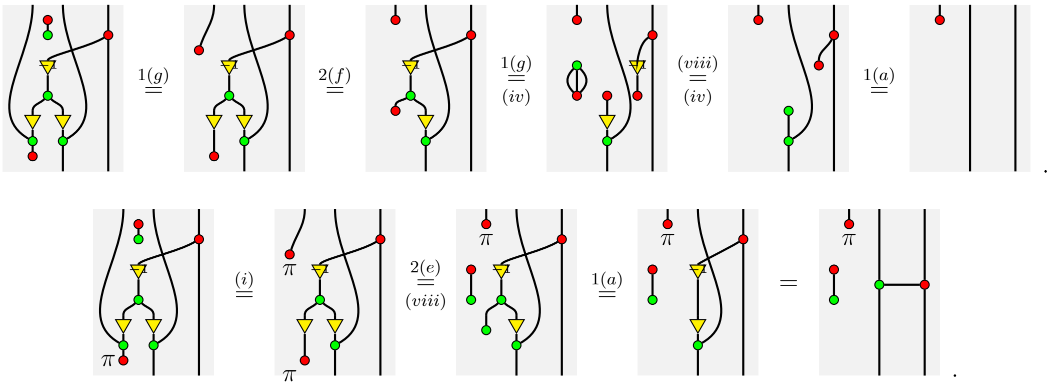
where the left two qubits are controls, and the last qubit is a not gate if either of the left two qubits are $|1\rangle$. The Toffoli gate can be expressed using the triangle vertex as such:



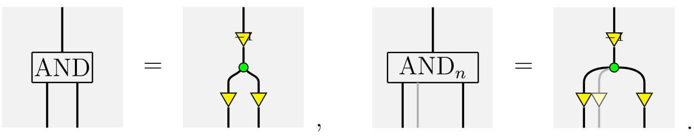
where the triangle with a -1 label is the inverse of the usual triangle.

Remark 97. A similar presentation of the Toffoli gate appeared in the book [17], and it is expressed using an analogous triangle node drawn as a slash box.

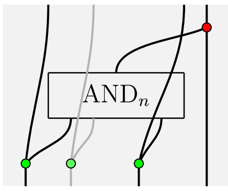
We can prove the correctness of the expression of the Toffoli gate by a direct verification:



It is well-known that the core of the Toffoli gate is the AND gate. Hence, it shouldn't be surprising the three triangle cluster forms a AND gate:

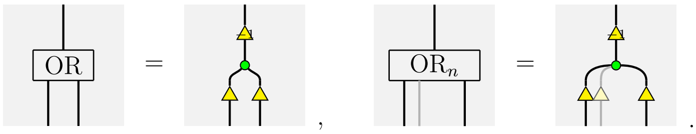


Hence, the generalised Toffoli gate with n controls has the following elegant diagram:



.

Another interesting logic gates, the OR gate, can also be expressed simply using the triangle:



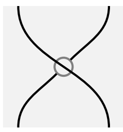
The not gate is simply a binary red vertex with a π phase. Therefore, all the classical gates has a natural expression in the ZX language. In other words, we have a diagrammatic language that incorporates both classical and quantum processes. This language is a natural starting point to design a programming language for a hybrid system of classical and quantum computing.

5.2 Qudits ZW and ZX calculi

In this section, we will only be dealing with matrices over the complex.

5.2.1 Qudits ZW calculus

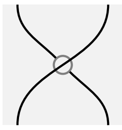
Due to its link to fermionic circuits, Hadzihasanovic sees some potential for the ZW calculus to model behaviours of abelian anyonic oscillators. As with the fermionic systems, the fundamental component of the anyonic calculus is the anyonic swap:



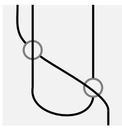
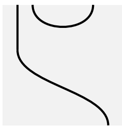
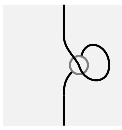
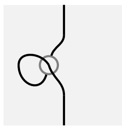
$$\mapsto \sum_{j,k=0}^{d-1} q^{jk} |\, jk \,\rangle\langle\, kj \,|.$$

where $q = e^{i\frac{2\pi}{d}}$ is the primitive dth root of unity. As can be seen from the diagram, the anyonic swap is no longer symmetric. It is now a “braiding” where plugging d of them together gives the identity or structural swap, depending whether d is even or odd.

We can define a dagger of the anyonic swap:

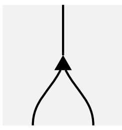
 $\mapsto \sum_{j,k=0}^{d-1} q^{-jk} |\, jk \,\rangle\langle\, kj \,|.$

With these, we obtain a some aesthetically pleasing rules:

 $=$  ,  $=$  .

The first equation tells us that wires that crosses each other with one above and one below passes through each other. The second equation tells us that the self crossing is invariant under conjugation.

Then, we can define the black monoid as

 $\mapsto \sum_{j+k<d} a_{jk} |\, j+k \,\rangle\langle\, jk \,|$

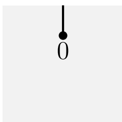
where

$$a_{jk} = \sqrt{\frac{[j+k]_q!}{[j]_q![k]_q!}}$$

and

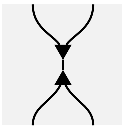
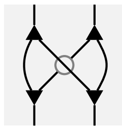
$$[n]_q = \sum_{j=0}^{n-1} q^j, \quad [n]_q! = \prod_{j=1}^{n} [j]_q, \quad [0]_q! = 1, \tag{5.1}$$

and the unit is the vacuum state

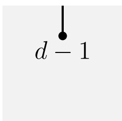


$\mapsto |\, 0 \,\rangle.$

This is inspired by *q-deformed algebra*, which by construction forms a bialgebra with its transpose comonoid and the braiding the is anyonic swap:

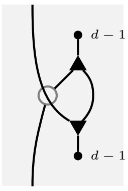
 $=$  .

In fact, it forms a Hopf algebra if we define a state for $|\, d-1 \,\rangle$,



$\mapsto |\, d-1 \,\rangle,$

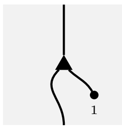
and the antipode is

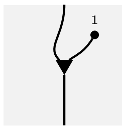


$$\mapsto \sum_{n=0}^{d-1} q^{\frac{n(n-1)}{2}}(-1)^n |\, n\,\rangle\langle\, n\,|,$$

We can make it into a creation $a^\dagger$ and annihilation a operator by introducing a $|\,1\,\rangle$ state:

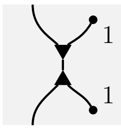
$$\mapsto |\,1\,\rangle,$$

$$a^\dagger =$$ 

, $$a =$$ 

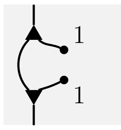
.

Note that the creation and annihilation operators are not dagger of each other; they are transpose of each other. These operators satisfies the q-commutation relation:



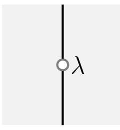
$=$

$+\ q$



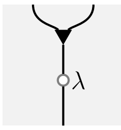
.

Next is to define a binary white vertex as such:

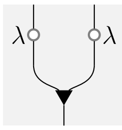


$$\mapsto \sum_{j=0}^{d-1} \lambda^j |\, j\,\rangle\langle\, j\,|,$$

for $\lambda \in \mathbb{C}$. The binary white vertex copies through the black comonoid as such:

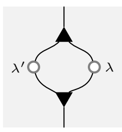


$=$

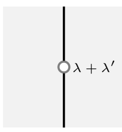


.

However, the sum axiom doesn't hold…, that is,



$\neq$

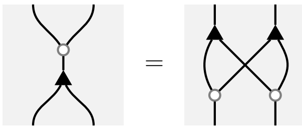


.

Finally, we define a ternary white vertex:

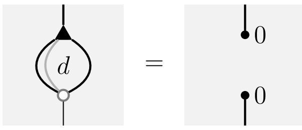
$$\mapsto \sum_{j=0}^{d-1} \sqrt{[j]_q!} |\, j \,\rangle\langle\, jj \,|,$$

where this forms a comonoid with a unary white vertex define as $\sum_{j=0}^{d-1} \frac{1}{\sqrt{[j]_q!}} \langle\, j \,|$. This then satisfies the bialgebra like equation



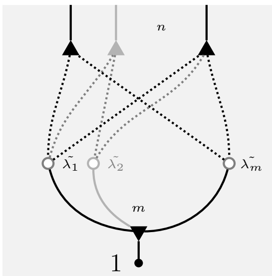
,

and



.

With all these generators, Hadzihasanovic obtained the universality of this qudits ZW calculus for $\mathbf{Qudit}_d$ by defining a normal form:



$$\mapsto \sum_{i=1}^{m} \lambda_i |\, k_{i1}, \ldots, k_{in} \,\rangle,$$

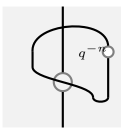
.

where

$$\tilde{\lambda_i} := \lambda_i \prod_{j=1}^{n} \sqrt{\frac{1}{[k_{ij}]_q!}}.$$

This is where Hadzihasanovic stopped.

We first make some slight improvement to the universality result. We don't need the unary black vertex with a label as a generator because any state $|\, n \,\rangle$ can be obtained by a projection like construction:



$$\mapsto d |\, n \,\rangle\langle\, n \,|.$$

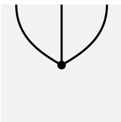
Then, we can, in principle, construct a ZW calculus that is complete for $\mathbf{Qudit}_d$ by having axioms telling us how to do plugging and juxtaposition of normal forms, and telling us how each generator can be written to normal form. The challenge is how do we get meaningful axioms, rather than the very primitive axioms that are no different from matrix calculation. This requires more looking into.

In anyonic quantum mechanics, there are also other choices of defining the black monoid. Hadzihasanovic used the one where the "numbers" are the q-numbers (see Equation 5.1). We can define the "numbers" differently, and one such way is

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

as suggested in [27]. The form of the axioms are more or less the same. It will be nice to see how the two different ways of defining the numbers are related from a diagrammatic perspective. As for now, there are much more to investigate.

We can also study at the calculus qualitatively, that is, ignoring the coefficients. The anyonic swap is essentially a swap, and the black monoid can be decomposed like in the qubit case:



$$\mapsto \sum_{i+j+k<d} f(i,j,k) |\, ijk\, \rangle,$$

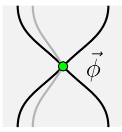
$$\mapsto \sum_{i+j<d} g(i,j) |\, ij\, \rangle.$$

It may be helpful to analyse the qudits ZW calculus with these finer and symmetrical vertices. A lot more work has to be done in this area.

5.2.2 Qudits ZX calculus

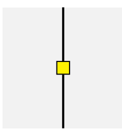
This subsection is a short introduction to the qudits version of ZX calculus, in collaboration with Quanlong Wang.

We could generalise the Z spiders to



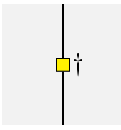
$$\mapsto \sum_{j=0}^{d-1} e^{i\phi_j} |\, j \dots j\, \rangle\langle\, j \dots j\, |,$$

where $\vec{\phi} = (\phi_0, \phi_1, \dots, \phi_{d-1}), \phi_0 = 0, \phi_j \in [0, 2\pi)$. The Hadamard gate is generalised to the Fourier transform



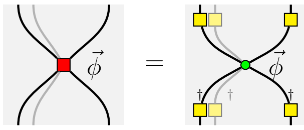
$$\mapsto \sum_{j=0}^{d-1} |\, h_j\, \rangle\langle\, j\, |,$$

where $|\, h_j \,\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} q^{jk} |\, k \,\rangle$ are the Fourier basis. There is a dagger version of the Hadamard



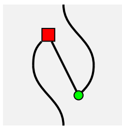
$$\mapsto \sum_{j=0}^{d-1} |\, j \,\rangle\langle\, h_j \,|,$$

We then can define the red vertices:



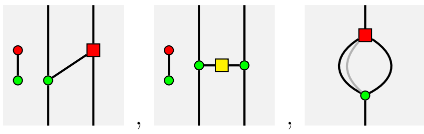
The green vertices play nice with the self dual maps, meaning that the wires can be bent up or down freely. After all, the self dual maps are defined with the Z basis. However, the wires of the red vertices cannot be transposed freely, as suggested by the box shaped vertex. There is an implicit difference between the input and output.

The red and green vertices also forms a Hopf algebra, just like in qubit case, except that the antipode is the *dualiser*:



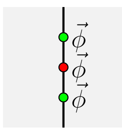
.

There are some interesting quantum gates that can be constructed from these generators:



From left to right, the first one is the CNOT gate, the second is the CZ gate, and the last one is the multiplicity gate. The multiplicity gate is also shown in Example 1, and the number of wires connecting the vertices is modulo d.

The Fourier gate also displays a decomposition into phases: the decomposition is, up to scalar,



,

where the phases $\vec{\phi} = (0, \phi_1, \ldots, \phi_{d-1})$ depends on the parity of the dimension. For even dimensions, there are two solutions:

$$\phi_j = j^2 \frac{\pi}{d} + j\pi,$$

$$\phi_j = j^2\frac{\pi}{d}.$$

For odd dimensions, there is only one solution

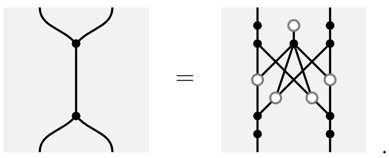
$$\phi_j = j^2\frac{\pi}{d} + j\pi.$$

With the tools at hand, we can use the ZX calculus for qudits graph states. This will be left for future research, and we feel that it is another low hanging fruit. Of course, we can also do a translation from the qudits ZW calculus to get a complete ZX calculus for qudits. The challenge is again about getting meaningful axioms.

5.3 ZW calculus for semiring

The ZW calculus is complete for R**bit**, for some commutative ring R. Rather than extending it for **dit**, we can ask what if we have a commutative semiring instead.

Having semirings instead of rings has a major implication: the crossing do not exist. There is no need for the crossing anyway; the normal form doesn't have a crossing. However, the bialgebra structure of the black vertices is gone, and that is a huge blow to the proof of the completeness since the proof relies heavily on the fact that the vertices can pass through each other relatively freely. However, we can introduce a pseudo-bialgebra:



The white vertices on the right-hand side are all connected to a central black vertex, which the black vertex acts as a control making sure no two wires have the state $|1\rangle$ flowing through them.

The completeness of this calculus should go through roughly in the same manner. However, we need to check through carefully before making any conclusion.

The motivation for studying this is its application to a wider range of problems. Take for example, in [11], the authors presented a diagrammatic calculus that is very similar to the ZW calculus for R**bit** where R is the semiring of two-element Boolean algebra. It will be interesting to translate the axioms in the paper into the ZW language and find the similarities.

We will conclude by stating an example we have slipped in subtly. The Example 1 at the start is a ZW calculus for (countably) infinite dimension over the two-element

Boolean algebra. This has another name, called *resource calculus*, which the motivation behind is to axiomatise the PROP of open Petri nets of [12]; this PROP axiomatises the stateless part of the PROP in [12]. A complete axiomatisation has been given, and indeed a "division" axiom has to be included as suggested in the question when explaining the concept of completeness in the example. This calculus is explored in Robin Piedeleu Ph.D. thesis on Picturing Resources in Concurrency, which is not available yet when this thesis was written. It is interesting to further generalise this to any arbitrary semiring.

Chapter 6

Conclusion

I will just say some simple concluding words. The ZW and ZX calculi are complete. The proof for the completeness for ZW calculus is to perform operations that are similar to matrix operations, that is, we have multiplication and addition of ring elements via plugging and convolution of the fermionic black monoid, and we have matrix multiplication by the bialgebra structure of the black (co)monoid. The ZX calculus is complete by translating to the ZW calculus, which reminds me of the idea of the proof of the Turing completeness of a machine by simulating another Turing complete machine. Now, it's time to find some applications for these two calculi.

There are some work in abstracting error correction algorithms and something along those lines using the ZX calculus [21], and not much have been done with the ZW calculus since it is relatively new. More research can be done in these areas, like having measurements and probabilistic mixing in the fermionic ZW calculus.

I see the ZX and ZW calculi as an intermediary step to further abstraction. I feel that it is still relatively low level programming language when compared to our modern programming language in the classical counterpart. I see a lot of potential in this step, since now we have a natural way of expressing classical operations in the ZX calculus with the triangle vertex. Using assistance software like Quantomatic [42] is definitely a direction to do so. Future quantum programming language may be based on diagrammatic calculi that is not the ZX or ZW calculi, but they are definitely a prototype for future theories.

Lastly, generalising these two calculi to other systems is my next interest. Generalising to qudits systems may advance our understanding of general quantum computing, or even quantum mechanics in general, and mixing different dimensions is a good exercise to understand how to connect different quantum computers running on different architecture. Furthermore, it will be also nice to apply these diagrammatic techniques to solve other real world problems like in the case of Petri nets. Experience and lesson learnt from solving multiple different problems may be give us a better understanding of the whole diagrammatic business.

There are a lot more to be done in the field of diagrammatic calculus, and I see a lot of potential in it. Let's hope that more resources and manpower can be pooled in to help advance this.

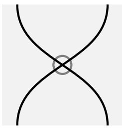
Appendix A

Axioms of ZW calculi

A.1 ZW calculus

The generators

Let R be a commutative ring. The ZW calculus is a self dual, compact closed PROP with the following generators T_{zw} and interpretation i_{zw}:

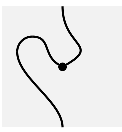
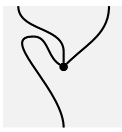
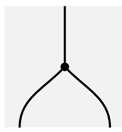
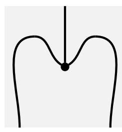
 $\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, j\ k\,\rangle\langle\, k\ j\,|,$

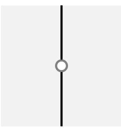
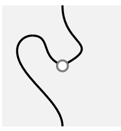
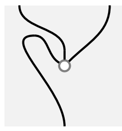
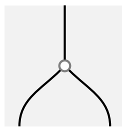
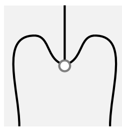
 $\mapsto |\,001\,\rangle + |\,010\,\rangle + |\,100\,\rangle,$  $\mapsto |\,01\,\rangle + |\,10\,\rangle,$

 $\mapsto |\,000\,\rangle + |\,111\,\rangle,$  $\mapsto |\,00\,\rangle + r|\,11\,\rangle,$

for $r \in R$.

It doesn't matter how we define the partial transpositions of the black and white vertices. One possible definition is the following:

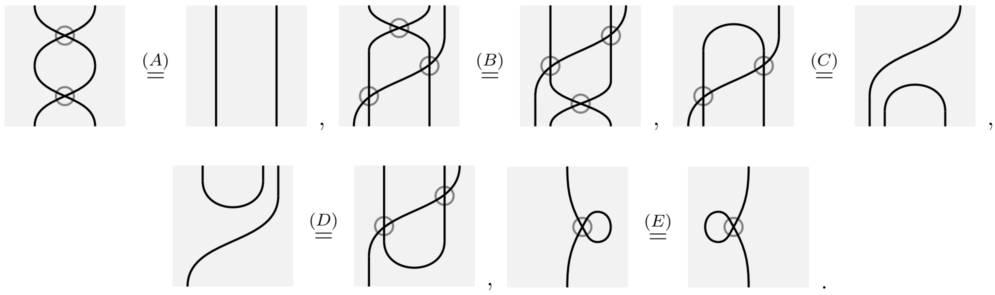
 $:=$  ,  $:=$  ,  $:=$  ,

 $:=$  ,  $:=$  ,  $:=$  .

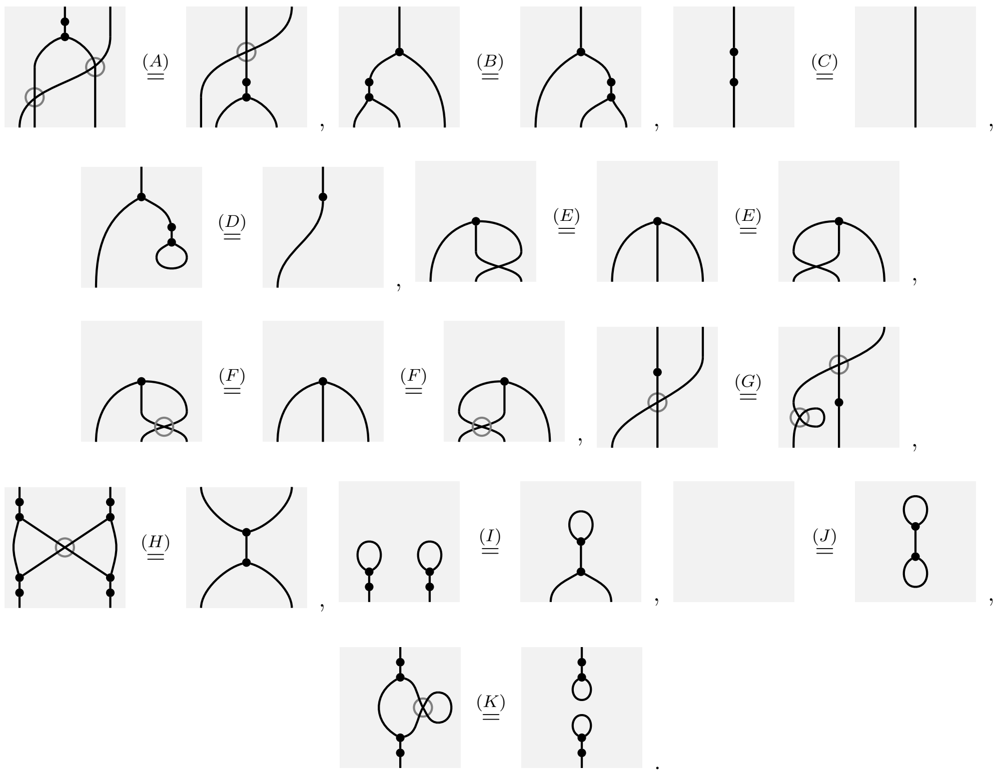
We will not explicitly list the definition of partial transpositions for the other calculi as it should be clear from how the diagrams are depicted.

The axioms

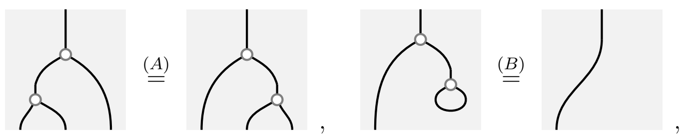
1. The following are the axioms for the crossings:

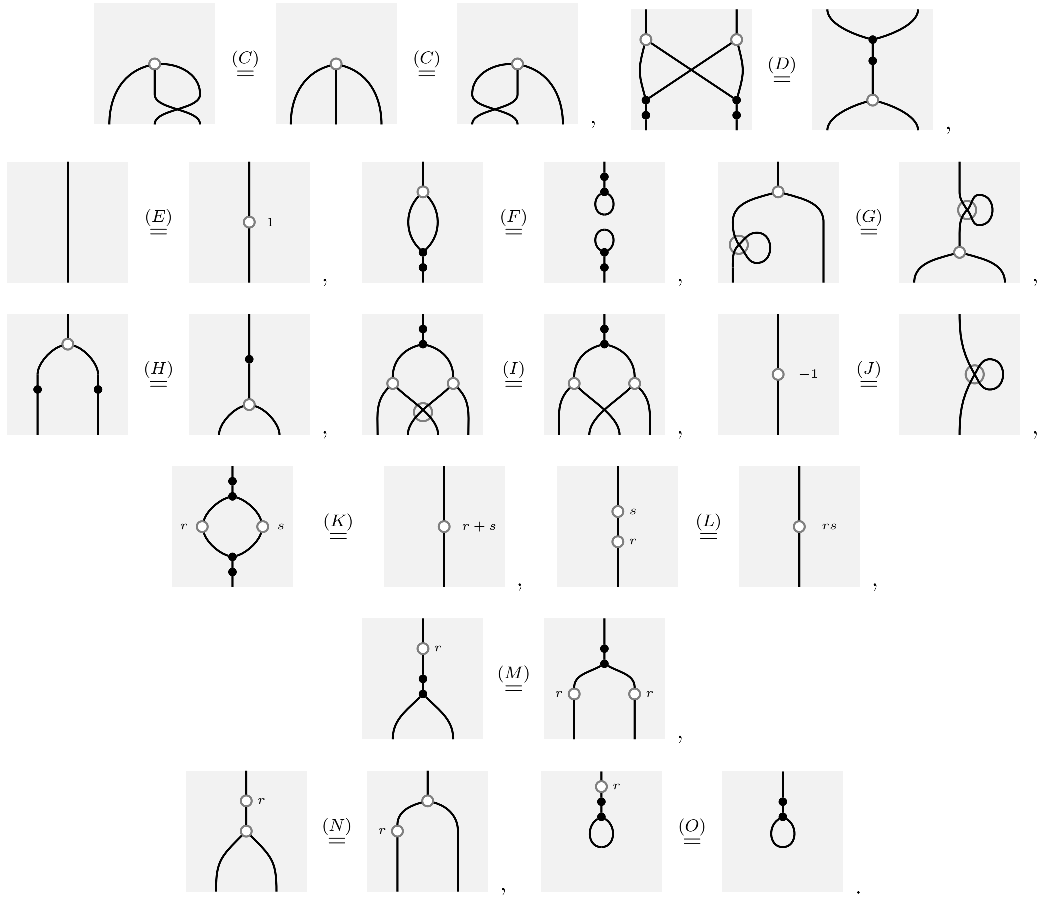


2. The following are the axioms for the black vertices:



3. Given a commutative ring R, the following are the axioms for the white vertices:

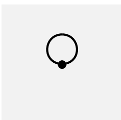
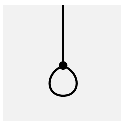


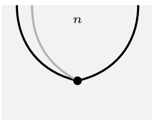


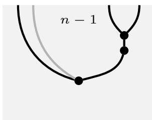
for $r, s \in R$.

The condensed axioms

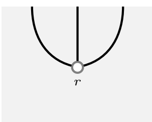
The condensed generators are

 :=  $\mapsto 0,$:=  $\mapsto |\,1\,\rangle,$



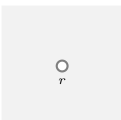
:= 

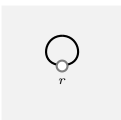
$$\mapsto \sum_{k=1}^{n} |\,\underbrace{0\dots0}_{k-1}1\underbrace{0\dots0}_{n-k}\rangle,$$

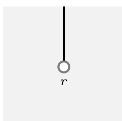


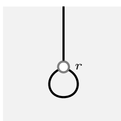
:= 

$\mapsto |\,000\,\rangle + r|\,111\,\rangle,$

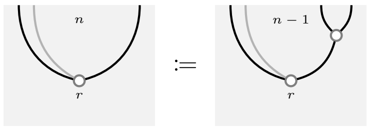


:= 

$\mapsto 2 + r,$ 

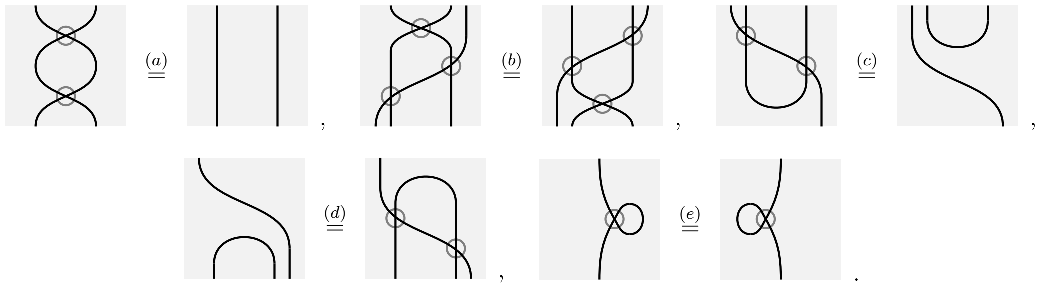
:= 

$\mapsto |\,0\,\rangle + r|\,1\,\rangle,$

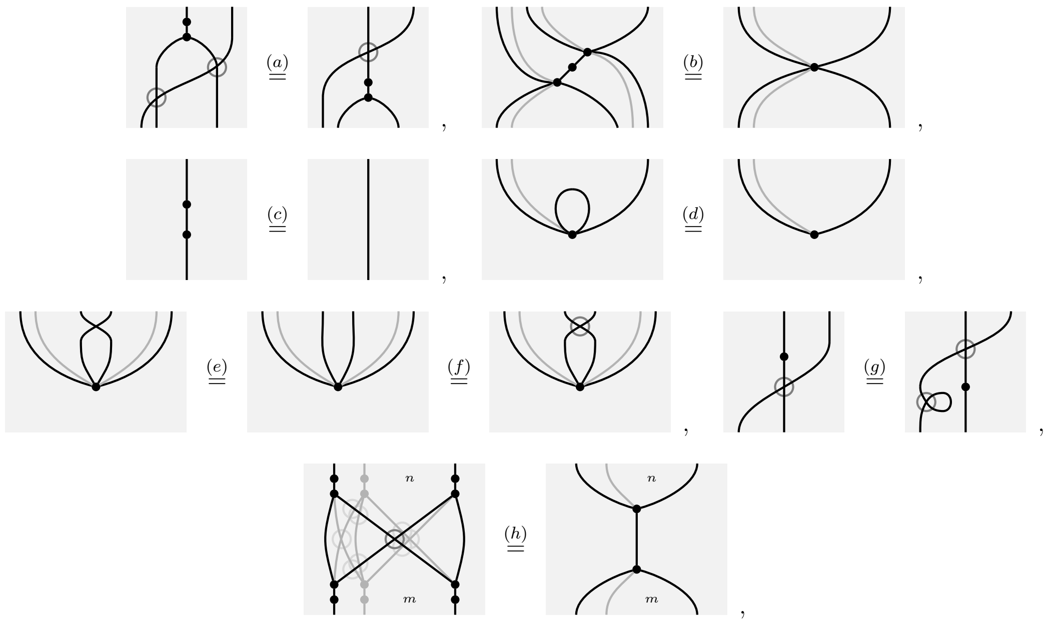


$\mapsto |\underbrace{0\dots0}_{n}\rangle + r|\underbrace{1\dots1}_{n}\rangle.$

1. The following are the axioms for the crossings:

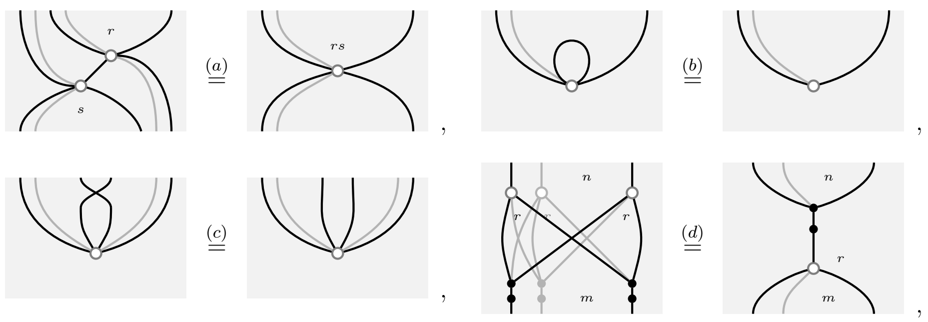


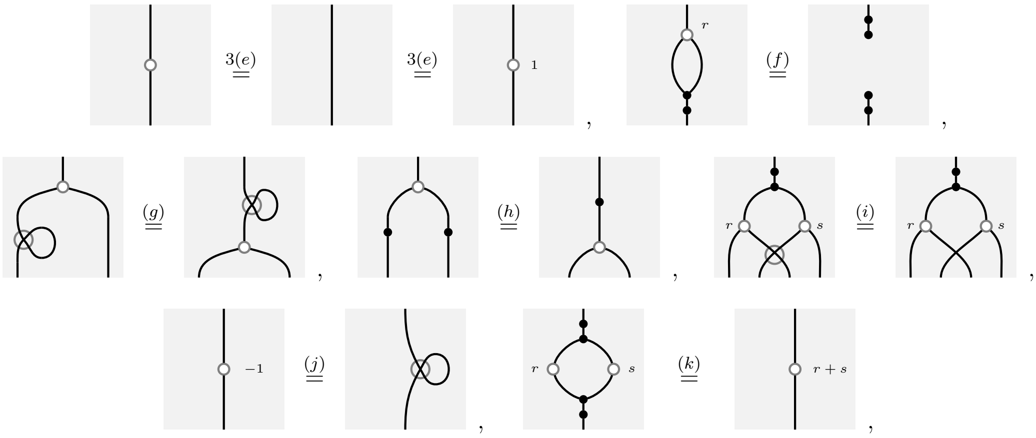
2. The following are the axioms for the black vertices:



for all $m, n \in \mathbb{N}$.

3. Given a commutative ring R, the following are the axioms for the white vertices:





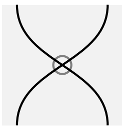
for $r, s \in R$, $m, n \in \mathbb{N}$ such that either $m = n = 0$ or $m > 0$.

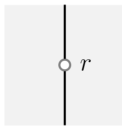
To better suit different rings, for instance the complex numbers $\mathbb{C}$ to model **Qubit** and its Clifford+T fragment $\mathbb{Z}\left[\frac{1}{2}\right]$ **bit**, a small modification to the ZW calculus can be done to reflect the ring structures. Please refer to main text for more elaborations.

A.2 Fermionic ZW calculus (FZW)

The generators

Let R be a commutative ring with a multiplicative inverse to 2, written as $\frac{1}{2}$. The fermionic ZW calculus is a self dual, compact closed PROP with the following generators T_{fzw} and interpretation i_{fzw}:

 $\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, j\ k \,\rangle\langle\, k\ j \,|$,



, $\mapsto |\,0\,\rangle\langle\,0\,| + r|\,1\,\rangle\langle\,1\,|$,

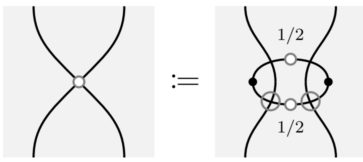
 $\mapsto |\,001\,\rangle + |\,010\,\rangle + |\,100\,\rangle$,

 $\mapsto |\,01\,\rangle + |\,10\,\rangle$,

for $r \in R$. The partial transpositions are defined as before.

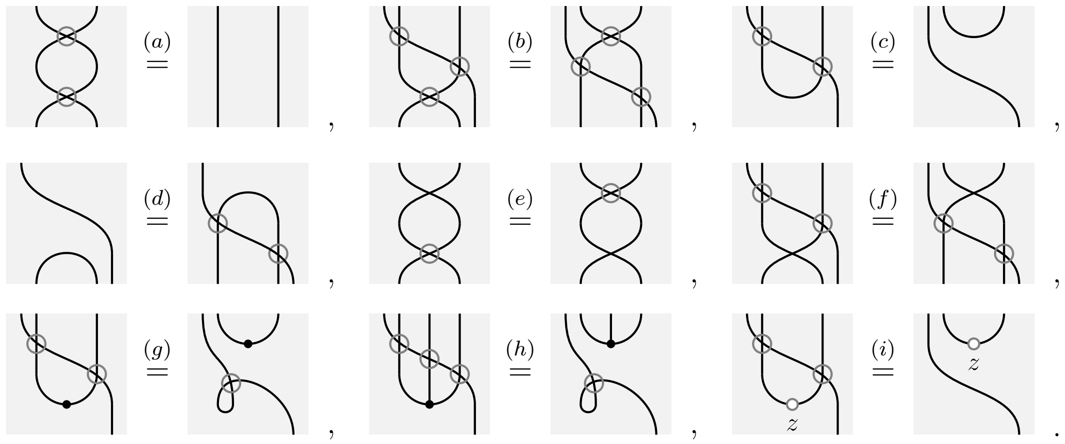
The axioms

Defining a quaternary white vertex:

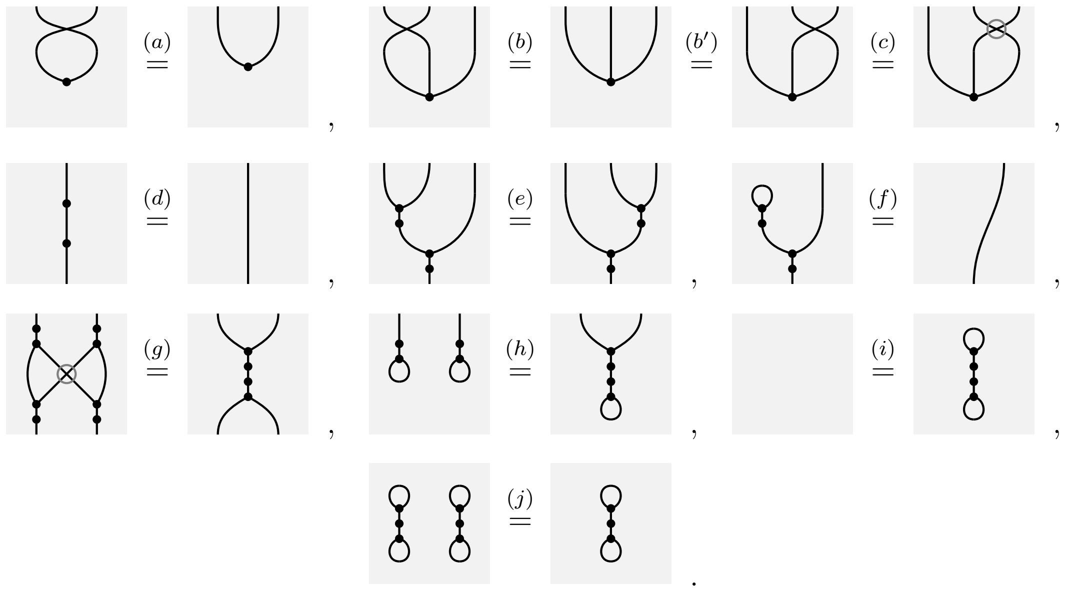


.

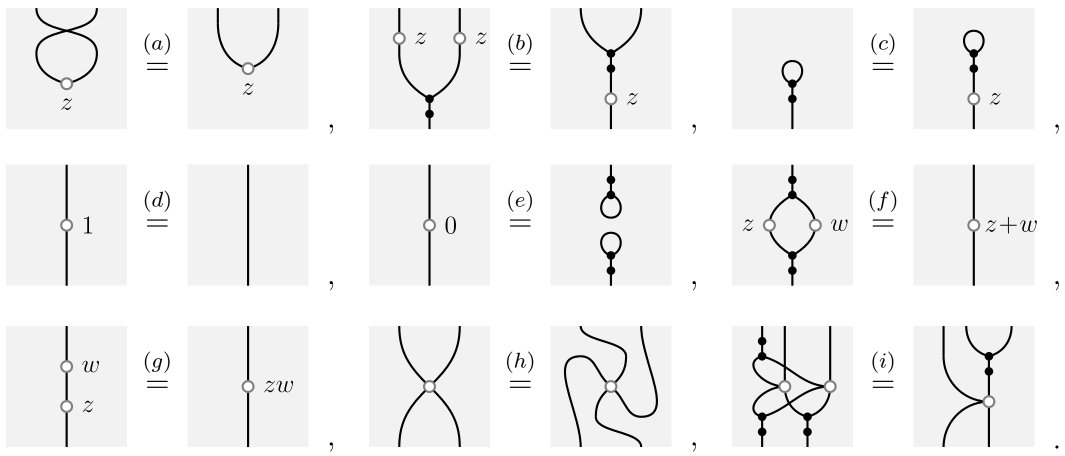
1. The following are the axioms for the fermionic swap:



2. The following are the axioms for the black vertices:



3. The following are the axioms for the white vertices:



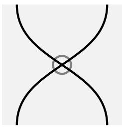
for $w, z \in R$.

A condensed version of the fermionic ZW calculus is presented as propositions in Proposition 50 and 51.

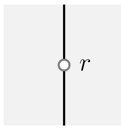
A.3 Even ZW calculus (EZW)

The generators

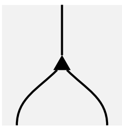
Let R be a commutative ring with a multiplicative inverse to 2, written as $\frac{1}{2}$. The even ZW calculus is a self dual, compact closed PROP with the following generators T_{ezw} and interpretation i_{ezw}:



$\mapsto \sum_{j,k=0}^{1} (-1)^{jk} |\, j\ k \,\rangle\langle\, k\ j \,|,$



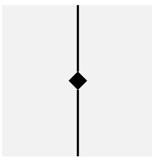
$, \quad \mapsto |\, 0 \,\rangle\langle\, 0 \,| + r|\, 1 \,\rangle\langle\, 1 \,|,$



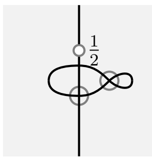
$\mapsto |\, 0 \,\rangle\langle\, 00 \,| + |\, 1 \,\rangle\langle\, 01 \,| + |\, 1 \,\rangle\langle\, 10 \,|.$

for $r \in R$.

Defining a binary odd projector:

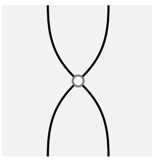


$:=$

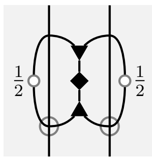


$\mapsto |\, 1 \,\rangle\langle\, 1 \,|,$

and the quaternary white vertex:



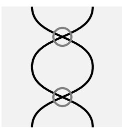
$:=$



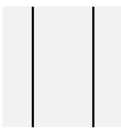
.

The axioms

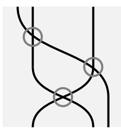
1. The following are the axioms for the fermionic swap:



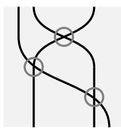
$\overset{(a)}{=}$



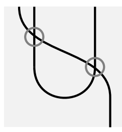
,



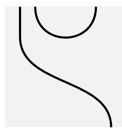
$\overset{(b)}{=}$



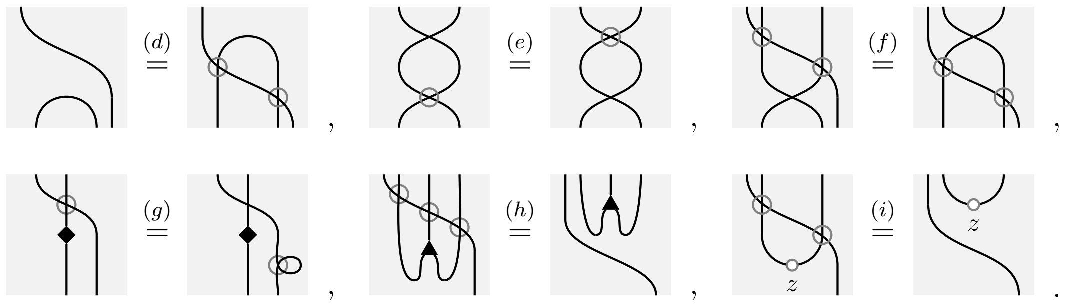
,



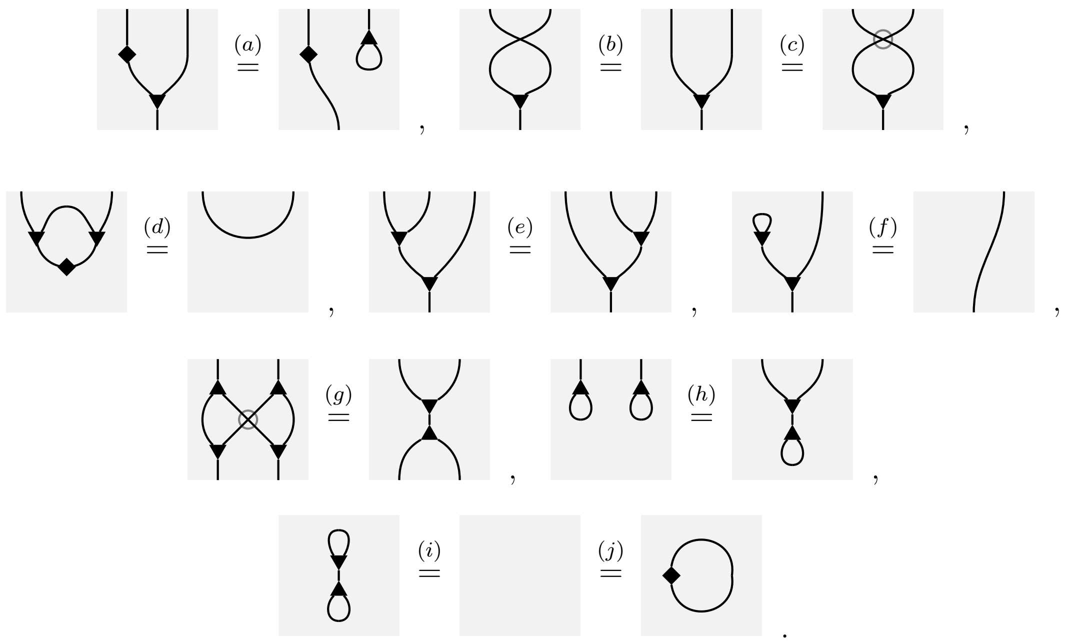
$\overset{(c)}{=}$



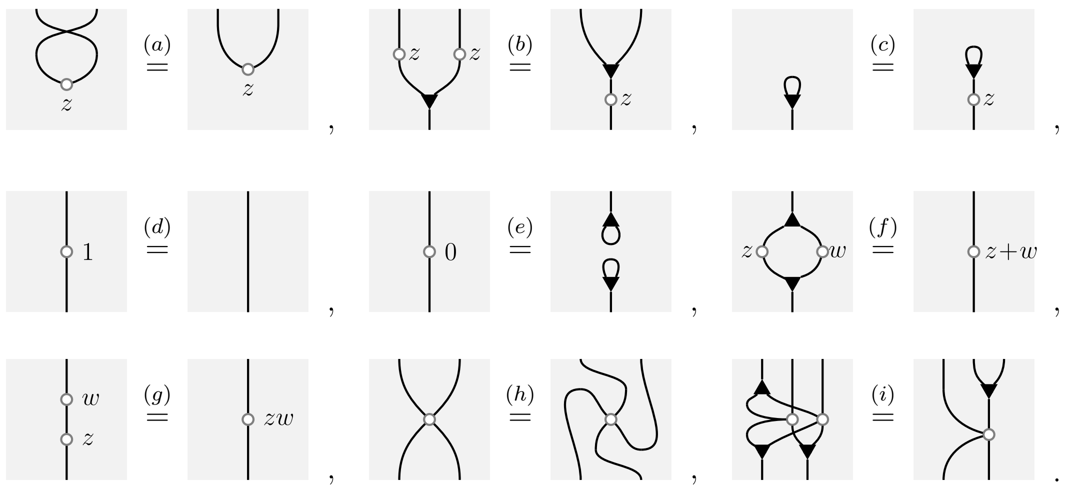
,



2. The following are the axioms for the black vertices:



3. The following are the axioms for the white vertices:



for $w, z \in R$.

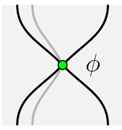
Appendix B

Axioms of ZX calculi

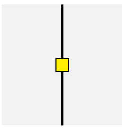
B.1 ZX calculus

The generators

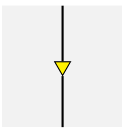
The ZX calculus has the following set of generators T_{zx} with the interpretation i_{zx}:



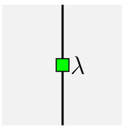
$$\mapsto |\,0\dots0\,\rangle\langle\,0\dots0\,| + e^{i\phi}|\,1\dots1\,\rangle\langle\,1\dots1\,|,$$



$$\mapsto \frac{1}{\sqrt{2}}(|\,0\,\rangle\langle\,0\,| + |\,0\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,0\,| - |\,1\,\rangle\langle\,1\,|),$$

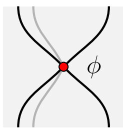


$$\mapsto |\,0\,\rangle\langle\,0\,| + |\,0\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,1\,|,$$

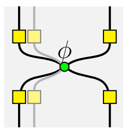


$$\mapsto |\,0\,\rangle\langle\,0\,| + \lambda|\,1\,\rangle\langle\,1\,|,$$

where $\phi \in [0, 2\pi)$, and real $\lambda > 0$. The red vertex is defined in the same way:



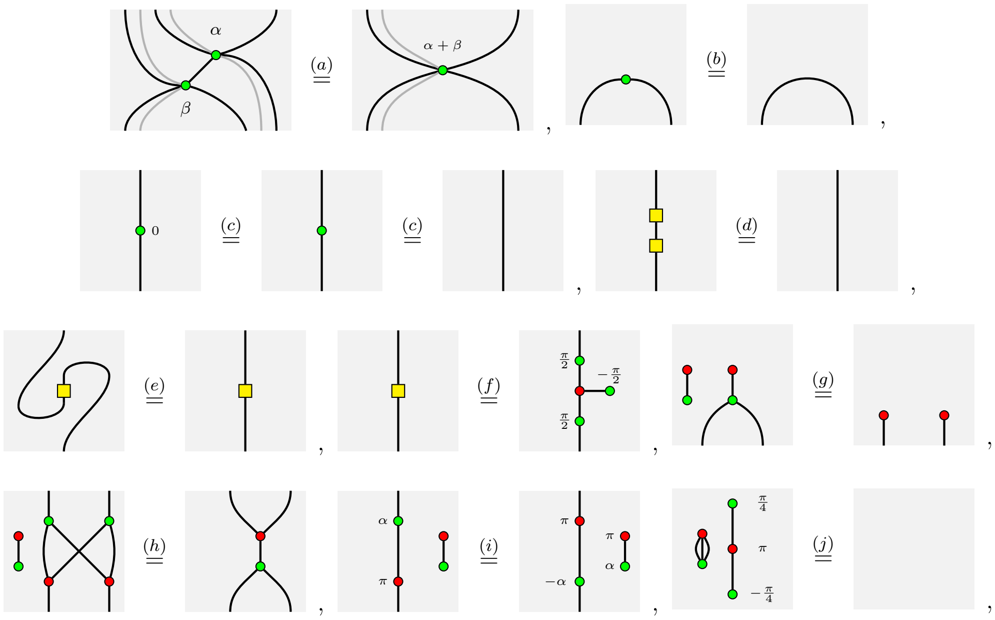
$:=$



.

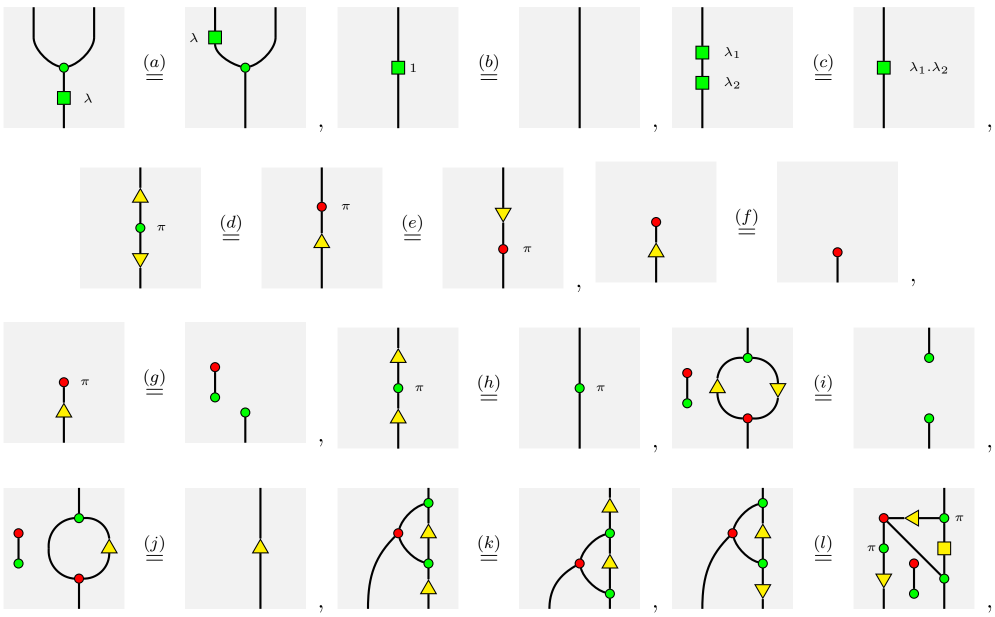
The axioms

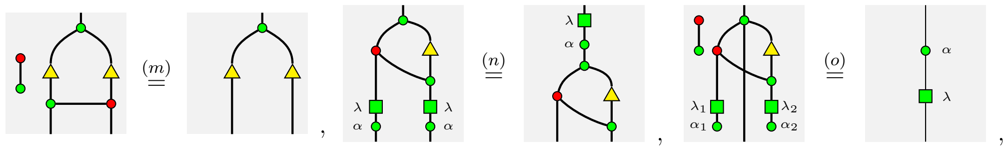
1. The following are the axioms for the traditional generators of the ZX calculus:



   for $\alpha, \beta \in [0, 2\pi)$. It is derivable that the axioms are also true for the interchange of the red and green colours, and for simplicity we will give them the same axiom labels.

2. The following are the axioms for the extended generators of the ZX calculus:





for $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, $\alpha, \alpha_1, \alpha_2 \in [0, 2\pi)$, and in (o), $\lambda e^{i\alpha} = \lambda_1 e^{i\alpha_1} + \lambda_2 e^{i\alpha_2}$.

B.2 Clifford+T ZX calculus, and more

The generators

The generators are pretty much the same as the ZX calculus, only with some restriction on green box and green vertex: the label on the green box is restricted to $0 < \lambda \in \mathbb{Z}[\frac{1}{2}]$, and the angle on the green vertices is restricted to integer multiples of $\frac{\pi}{4}$.

The axioms

The $\mathrm{ZX}_{\frac{\pi}{4}}$ calculus features the same axioms as the ZX calculus, with restricted phases $\alpha = k\frac{\pi}{4}$, $k = 0, 1, \dots 7$, lengths $0 < \lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$; moreover, we will modify the conditions of axiom $2(o)$ to $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\alpha \equiv \alpha_1 \equiv \alpha_2 (\mathrm{mod}\ \pi)$.

Remark 98. In fact, with suitable small modifications, the ZX calculus is complete with respect to R**bit** for any commutative ring containing a subring isomorphic to $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$.

Bibliography

[1] TikZit. `http://tikzit.github.io/`. Accessed: 2018-10-11.

[2] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*. IEEE, 2004.

[3] Miriam Backens. The ZX-calculus is complete for stabilizer quantum mechanics. *New Journal of Physics*, 16(9):093021, 2014.

[4] Miriam Backens. The ZX-calculus is complete for the single-qubit Clifford+T group. *Electronic Proceedings in Theoretical Computer Science*, 172:293–303, 2014.

[5] Miriam Backens and Aleks Kissinger. ZH: A complete graphical calculus for quantum computations involving classical non-linearity. *arXiv preprint arXiv:1805.02175*, 2018.

[6] Miriam Backens, Simon Perdrix, and Quanlong Wang. A simplified stabilizer ZX-calculus. *EPTCS*, 236:1–20, January 2017.

[7] Miriam Backens, Simon Perdrix, and Quanlong Wang. Towards a minimal stabilizer ZX-calculus. *arXiv preprint arXiv:1709.08903*, 2017.

[8] Mari-Carmen Bañuls, J Ignacio Cirac, and Michael M Wolf. Entanglement in fermionic systems. *Physical Review A*, 76(2), 2007.

[9] Filippo Bonchi, Paweł Sobociński, and Fabio Zanasi. Interacting bialgebras are Frobenius. In *International Conference on Foundations of Software Science and Computation Structures*, pages 351–365. Springer, 2014.

[10] Sergey B Bravyi and Alexei Yu Kitaev. Fermionic quantum computation. *Annals of Physics*, 298(1):210–226, 2002.

[11] Roberto Bruni, Ivan Lanese, and Ugo Montanari. A basic algebra of stateless connectors. *Theoretical Computer Science*, 366(1):98–120, 2006.

[12] Roberto Bruni, Hernan Melgratti, Ugo Montanari, and Pawel Sobocinski. Connector algebras for C/E and P/T nets' interactions. *Logical Methods in Computer Science*, 9(3), 2013.

[13] Bob Coecke and Ross Duncan. Interacting quantum observables. In *Automata, Languages and Programming*, pages 298–310. Springer, 2008.

[14] Bob Coecke, Ross Duncan, Aleks Kissinger, and Quanlong Wang. Strong complementarity and non-locality in categorical quantum mechanics. In *2012 27th Annual IEEE Symposium on Logic in Computer Science*. IEEE, 2012.

[15] Bob Coecke, Bill Edwards, and Robert W Spekkens. Phase groups and the origin of non-locality for qubits. *Electronic Notes in Theoretical Computer Science*, 270(2):15–36, 2011.

[16] Bob Coecke and Aleks Kissinger. The compositional structure of multipartite quantum entanglement. In *Automata, Languages and Programming*, pages 297–308. Springer, 2010.

[17] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes.* Cambridge University Press (CUP), 2017.

[18] Bob Coecke and Dusko Pavlovic. Quantum measurements without sums. In *Chapman & Hall/CRC Applied Mathematics & Nonlinear Science*, pages 559–596. Chapman and Hall/CRC, 2007.

[19] Bob Coecke, Dusko Pavlovic, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 23(03):555–567, 2012.

[20] Giacomo Mauro D'Ariano, Franco Manessi, Paolo Perinotti, and Alessandro Tosini. The Feynman problem and fermionic entanglement: Fermionic theory versus qubit theory. *International Journal of Modern Physics A*, 29(17):1430025, 2014.

[21] Niel de Beaudrap and Dominic Horsman. The ZX calculus is a language for surface code lattice surgery. *arXiv preprint arXiv:1704.08670*, 2017.

[22] Christian Schröder de Witt and Vladimir Zamdzhiev. The ZX calculus is incomplete for quantum mechanics. In *Proceedings of the 11th workshop on Quantum Physics and Logic*, pages 285–292, 2014.

[23] Ross Duncan and Simon Perdrix. Graph states and the necessity of euler decomposition. In *Conference on Computability in Europe*, pages 167–177. Springer, 2009.

[24] Ross Duncan and Simon Perdrix. Pivoting makes the ZX-calculus complete for real stabilizers. *Electronic Proceedings in Theoretical Computer Science*, 171:50–62, 2014.

[25] Wolfgang Dür, Guifre Vidal, and J Ignacio Cirac. Three qubits can be entangled in two inequivalent ways. *Physical Review A*, 62(6), 2000.

[26] Avshalom C Elitzur and Lev Vaidman. Quantum mechanical interaction-free measurements. *Foundations of Physics*, 23(7):987–997, 1993.

[27] Emmanuel G Floratos and TN Tomaras. A quantum mechanical analogue for the q-oscillator. *Physics Letters B*, 251(1):163–166, 1990.

[28] Nicolai Friis, Antony R Lee, and David Edward Bruschi. Fermionic-mode entanglement in quantum information. *Physical Review A*, 87(2), 2013.

[29] Brett Giles and Peter Selinger. Exact synthesis of multiqubit Clifford+T circuits. *Physical Review A*, 87(3):032332, 2013.

[30] Amar Hadzihasanovic. A diagrammatic axiomatisation for qubit entanglement. In *Proceedings of the 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '15, pages 573–584. IEEE, 2015.

[31] Amar Hadzihasanovic. *The algebra of entanglement and the geometry of composition.* DPhil thesis, University of Oxford, 2017.

[32] Amar Hadzihasanovic, Giovanni de Felice, and Kang Feng Ng. A diagrammatic axiomatisation of fermionic quantum circuits. In Hélène Kirchner, editor, *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*, volume 108 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 17:1–17:20, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[33] Amar Hadzihasanovic, Kang Feng Ng, and Quanlong Wang. Two complete axiomatisations of pure-state qubit quantum computing. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 502–511. ACM, 2018.

[34] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Y-calculus: A language for real matrices derived from the ZX-calculus. In *Proceedings of the 17th workshop on Quantum Physics and Logic*, pages 23–57, 2017.

[35] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A Generic Normal Form for ZX-Diagrams and Application to the Rational Angle Completeness. *arXiv preprint arXiv:1805.05296*, 2018.

[36] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 559–568. ACM, 2018.

[37] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Diagrammatic Reasoning Beyond Clifford+T Quantum Mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 569–578. ACM, 2018.

[38] Yang Ji, Yunchul Chung, D Sprinzak, M Heiblum, D Mahalu, and Hadas Shtrikman. An electronic mach–zehnder interferometer. *Nature*, 422(6930):415–418, 2003.

[39] Louis H. Kauffman. *Knots and Physics*. World Scientific Publishing Co. Pte. Ltd., 2001.

[40] Louis H Kauffman. Introduction to virtual knot theory. *Journal of Knot Theory and Its Ramifications*, 21(13):1240007, 2012.

[41] Aleks Kissinger, Alex Merry, and Matvey Soloviev. Pattern graph rewrite systems. *Electronic Proceedings in Theoretical Computer Science*, 143:54–66, 2014.

[42] Aleks Kissinger and Vladimir Zamdzhiev. Quantomatic: A proof assistant for diagrammatic reasoning. In *Automated Deduction - CADE-25*, pages 326–336. Springer International Publishing, 2015.

[43] Emanuel Knill, Raymond Laflamme, and Gerald J Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409(6816):46–52, 2001.

[44] Lucas Lamata, Juan León, D Salgado, and E Solano. Inductive entanglement classification of four qubits under stochastic local operations and classical communication. *Physical Review A*, 75(2):022318, 2007.

[45] Shahn Majid. *A Quantum Groups Primer*. Cambridge University Press, 2002.

[46] Simon Perdrix and Quanlong Wang. Supplementarity is necessary for quantum diagram reasoning. In *41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016)*, volume 58, pages 76–1, 2016.

[47] Benjamin Schumacher and Michael D Westmoreland. Modal quantum theory. *Foundations of Physics*, 42(7):918–925, 2012.

[48] Marlan O Scully and M Suhail Zubairy. *Quantum optics*. Cambridge University Press, 1997.

[49] Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

[50] Peter Selinger. Finite dimensional Hilbert spaces are complete for dagger compact closed categories. *Electronic Notes in Theoretical Computer Science*, 270(1):113–119, 2011.

[51] Renaud Vilmart. A ZX-calculus with triangles for Toffoli-Hadamard, Clifford+T, and beyond. In *Proceedings of the 18th workshop on Quantum Physics and Logic*, 2018.

[52] Peter Woit. *Quantum Theory, Groups and Representations*. Springer International Publishing, 2017.

Index