STRONGLY COMPACT CLOSED QUANTUM SEMANTICS

bOB cOECKE @ MFPS XXI

Oxford University Computing Laboratory

STORY LINE

Entanglement as Quantum Information Flow

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; cs/0207057
- The Logic of Entanglement. Coecke (2003) PRG-RR; quant-ph/0402014

Abstract QIF := Strong Compact Closure

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) LiCS'04; quant-ph/0402130
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

Proof Nets and Diagrammatic Calculus

- A Categorical Quantum Logic. Abramsky & Duncan (2004) QPL'04.
- Dagger Compact Closed Categories and CPMs. Selinger (2005) QPL'05.
- Abstract scalars, loops, free traced and SCCCs. Abramsky (2005).

Quantum Mechanics from SCC

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) LiCS'04; `quant-ph/0402130`

- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

Phases & Projective Quantum Axiomatics

- De-linearizing Linearity: Projective Quantum Axiomatics from Strong Compact Closure. Coecke (2005) QPL'05.

Towards Open System Axiomatics

- Dagger Compact Closed Categories and Completely Positive Maps. Selinger (2005) QPL'05.

STRONG COMPACT CLOSURE

Symmetric Monoidal Tensor with

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A \to B} \mapsto f^{\dagger}_{B \to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$;

$$\begin{array}{ccccc}
A & \xleftarrow{\ \simeq\ } & \mathrm{I} \otimes A & \xleftarrow{\ \eta^{\dagger}_{A^*} \otimes 1_A\ } & (A \otimes A^*) \otimes A \\
\uparrow{\scriptstyle 1_A} & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\ \simeq\ ]{} & A \otimes \mathrm{I} & \xrightarrow[\ 1_A \otimes \eta_A\ ]{} & A \otimes (A^* \otimes A)
\end{array}$$

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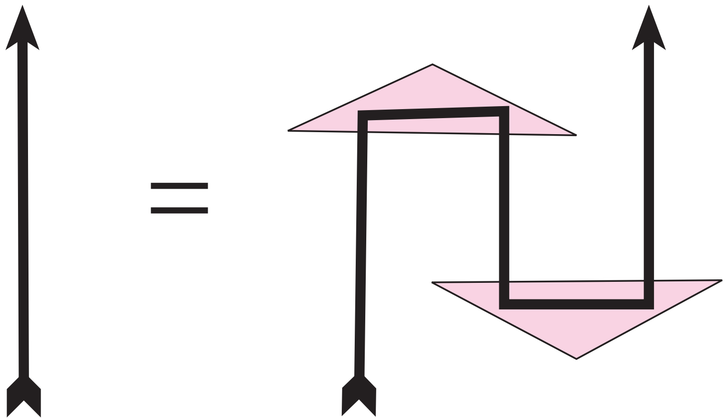

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$$
\begin{array}{ccccccc}
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\Big\uparrow {\scriptstyle 1_A} & & & & & & \Big\uparrow {\scriptstyle 1_{A^*} \otimes \sigma_{A,A}} \\
A & \xrightarrow[\simeq]{} & \mathrm{I} \otimes A & \xrightarrow[\eta_A \otimes 1_A]{} & (A^* \otimes A) \otimes A & \xrightarrow[\simeq]{} & A^* \otimes (A \otimes A)
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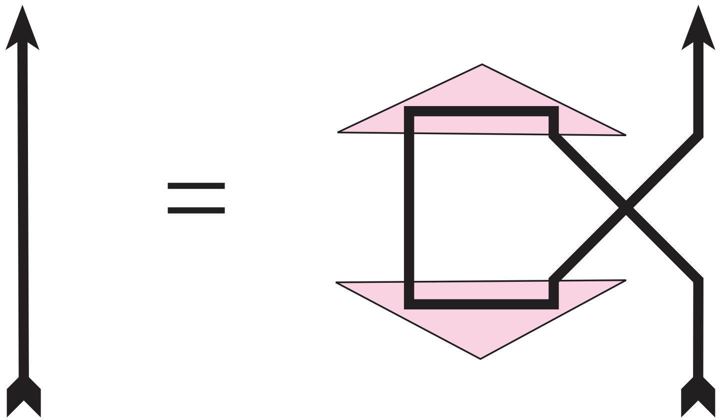

STAR STRUCTURE

A contravariant $\otimes$-involution

$$f_{A\to B} \mapsto f^{*}_{B^*\to A^*}$$

called (**upper**) **star** arises as

$$\begin{array}{ccccc}
A^* & \xleftarrow{\ \simeq\ } & A^*\otimes \mathrm{I} & \xleftarrow{1_{A^*}\otimes \eta^{\dagger}_{B^*}} & A^*\otimes B\otimes B^* \\
{\scriptstyle f^*}\uparrow & & & & \uparrow{\scriptstyle 1_{A^*}\otimes f\otimes 1_{B^*}} \\
B^* & \xrightarrow[\ \simeq\ ]{} & \mathrm{I}\otimes B^* & \xrightarrow[\eta_A\otimes 1_{B^*}]{} & A^*\otimes A\otimes B^*
\end{array}$$

$\Rightarrow$ $*$-autonomy with $\otimes^* = \otimes$ i.e. **compact closure**.

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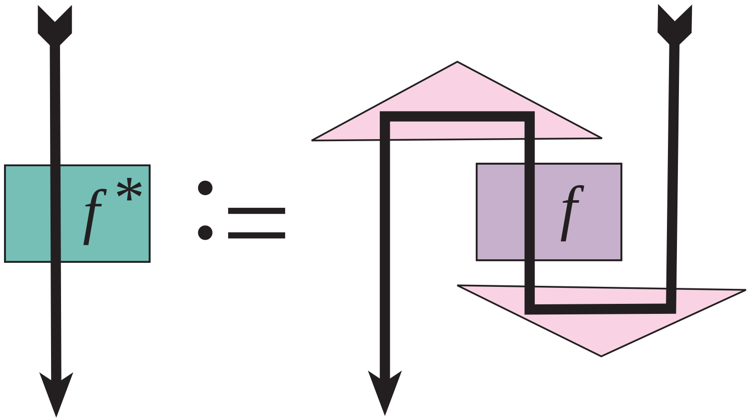

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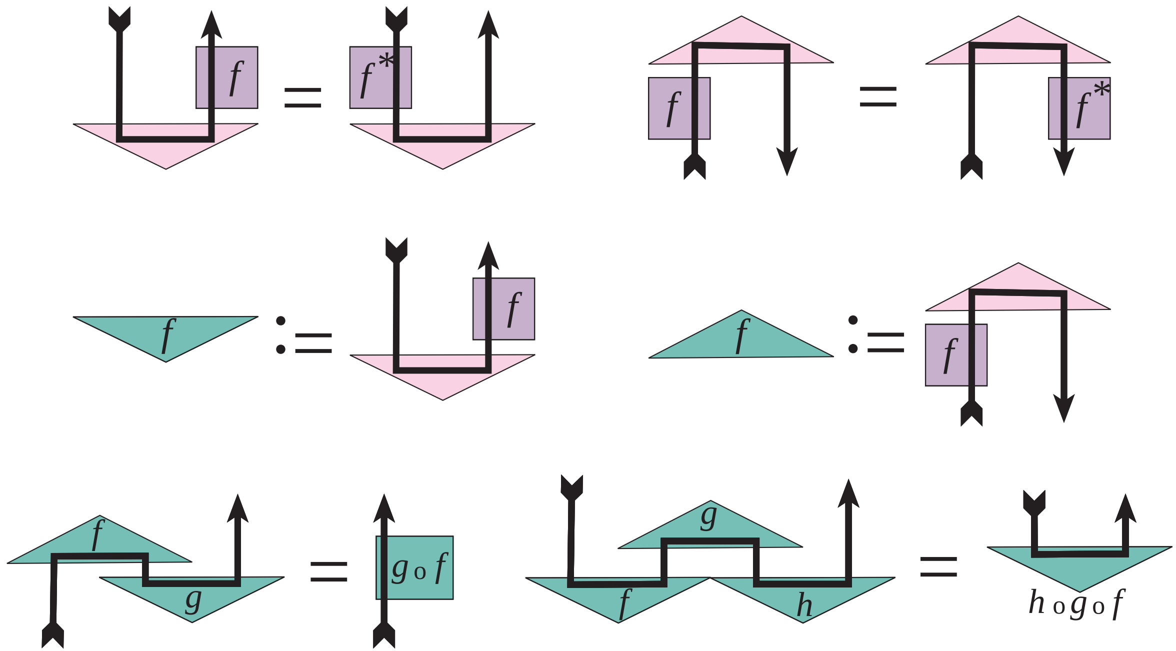
COMPOSITIONALITY



$$\frac{\text{strong compact closure}}{\text{compact closure}} \simeq \frac{\text{inner-product space}}{\text{vector space}}$$

LOWER STAR STRUCTURE

A covariant $\otimes$-involution

$$f^{A\to B} \mapsto f_{*}^{A^*\to B^*}$$

called **lower star** arises as

$$\begin{array}{ccccc}
B^* & \xleftarrow{\ \simeq\ } & B^*\otimes \mathrm{I} & \xleftarrow{1_{B^*}\otimes\eta_{A^*}^{\dagger}} & B^*\otimes A\otimes A^* \\
\uparrow{\scriptstyle f_*} & & & & \uparrow{\scriptstyle 1_{B^*}\otimes f^{\dagger}\otimes 1_{A^*}} \\
A^* & \xrightarrow[\ \simeq\ ]{} & \mathrm{I}\otimes A^* & \xrightarrow[\eta_B\otimes 1_{A^*}]{} & B^*\otimes B\otimes A^*
\end{array}$$

$\Rightarrow$ genuine extra data as compared to compact closure.

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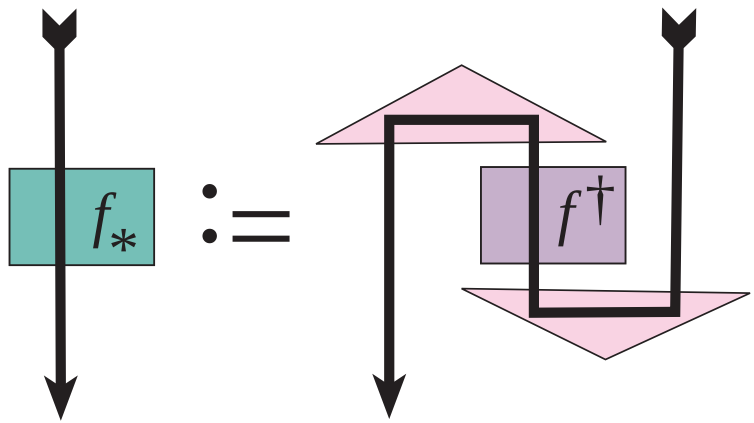

$\Rightarrow$ genuine extra data as compared to compact closure.

TRACE STRUCTURE

A **Joyal-Street-Verity (partial) trace**

$$f_{C\otimes A\to C\otimes B} \;\mapsto\; \mathrm{Tr}_C(f)_{A\to B}$$

arises as

$$\begin{array}{ccccccc}
B & \xleftarrow{\;\simeq\;} & \mathrm{I}\otimes B & \xleftarrow{\;\eta_C^\dagger\otimes 1_B\;} & (C^*\otimes C)\otimes B & \xleftarrow{\;\simeq\;} & C^*\otimes(C\otimes B) \\
\uparrow {\scriptstyle \mathrm{Tr}_C(f)} & & & & & & \uparrow {\scriptstyle 1_{C^*}\otimes f} \\
A & \xrightarrow[\;\simeq\;]{} & \mathrm{I}\otimes A & \xrightarrow[\;\eta_C\otimes 1_A\;]{} & (C^*\otimes C)\otimes A & \xrightarrow[\;\simeq\;]{} & C^*\otimes(C\otimes A)
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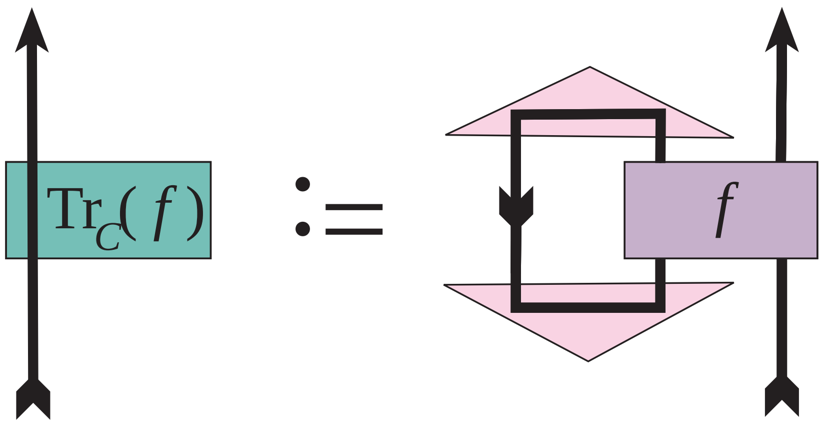

TRACE STRUCTURE

A corresponding **full trace**

$$h_{A \to A} \;\mapsto\; \mathrm{Tr}(h)_{\mathrm{I} \to \mathrm{I}}$$

arises as

$$\begin{array}{ccc}
\mathrm{I} & \xleftarrow{\;\eta_A^\dagger\;} & A^* \otimes A \\
{\scriptstyle \mathrm{Tr}(h)} \uparrow & & \uparrow {\scriptstyle 1_{A^*} \otimes h} \\
\mathrm{I} & \xrightarrow[\;\eta_A\;]{} & A^* \otimes A
\end{array}$$

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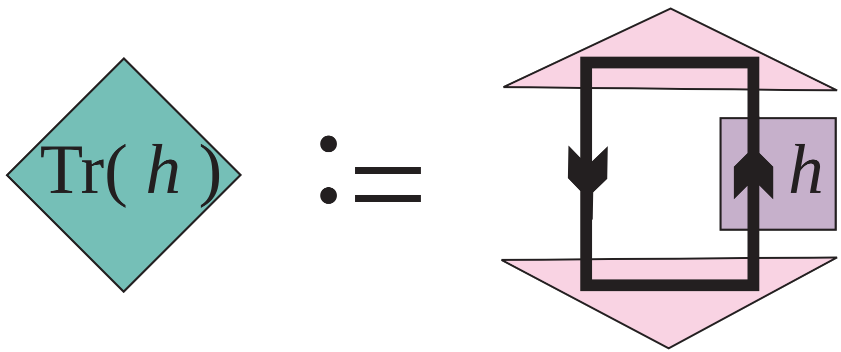


$\Rightarrow$ all endomorphisms carry a value of type $\mathrm{I} \to \mathrm{I}$.

SCALARS

Scalars $s : \mathrm{I} \to \mathrm{I}$ commute in any Sym. Mon. Cat. and

$$s \circ t = \lambda_{\mathrm{I}}^{-1} \circ (s \otimes t) \circ \lambda_{\mathrm{I}}$$

Scalar multiplication

$$s \bullet f := \lambda_B^{-1} \circ (s \otimes f) \circ \lambda_A : A \to B\,.$$

satisfies

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

The **inner-product** of $\psi, \phi : \mathrm{I} \longrightarrow A$ is a scalar

$$\langle \psi \mid \phi \rangle := \psi^\dagger \circ \phi : \mathrm{I} \longrightarrow \mathrm{I}$$

cf. Dirac's notation

$$\texttt{bra} := \langle \psi | \qquad \texttt{ket} := | \phi \rangle \qquad \texttt{bra-ket} := \langle \psi \mid \phi \rangle$$

Adjointness implies

$$\langle f^\dagger \circ \psi \mid \phi \rangle = \langle \psi \mid f \circ \phi \rangle$$

Unitarity means $U = \left(U^\dagger\right)^{-1} : A \longrightarrow B$ and implies

$$\langle U \circ \psi \mid U \circ \phi \rangle = \langle \psi \mid \phi \rangle$$

1. **FdHilb** $\Rightarrow$ **von Neumann's formalism**

2. **Rel** $\Rightarrow$ **qualitative quantum reasoning**

3. **Finitary objects in a biproduct category**

- $\mathrm{Ob} := \{\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n} \mid n \in \mathbb{N}\}$ for some $\mathrm{I} \in \mathrm{Ob}(\mathbf{BP})$
- $\mathrm{Hom}(n, m) = n \times m$ matrices in $\mathbf{BP}(\mathrm{I}, \mathrm{I})$
- $(\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n}) \otimes (\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{m}) := \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n \times m}$
- $(-)^{\dagger} := (-)^{T}$ and $\eta_n := \Delta : \mathrm{I} \to \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n \times n}$
- non-trivial $(-)_*$ from involution on $\mathbf{BP}(\mathrm{I}, \mathrm{I})$

$\Rightarrow$ **SCC is a component of linearity**

PROJECTORS

Positivity:

$$g \circ g^\dagger : A \to A \qquad\qquad (\text{with } \; g : C \to A)$$

Projector iff

$$g^\dagger \circ g = 1_A \qquad\qquad (\Rightarrow \; \text{idempotence})$$

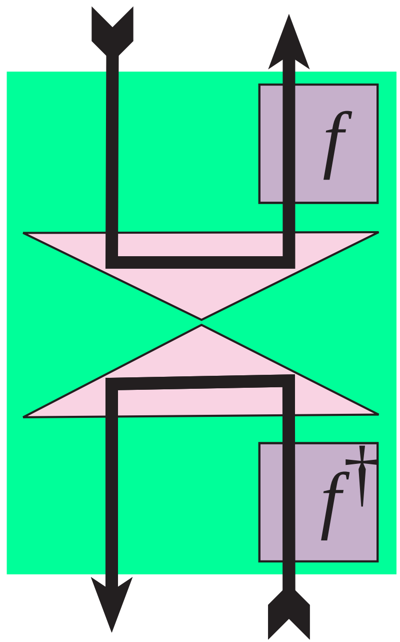
Complete projector iff

$$C = \mathrm{I} \qquad\qquad (\text{e.g. } \; g := \ulcorner f \urcorner : \mathrm{I} \to A_1^* \otimes A_2)$$

Complete bipartite projector:

$$\mathrm{P}_f := (1_{A_1^*} \otimes f) \circ \eta \circ \eta^\dagger \circ (1_{A_1^*} \otimes f^\dagger)$$

THE FACE OF A PROJECTOR



$$\mathrm{P}_f := (1_{A_1^*} \otimes f) \circ \eta \circ \eta^\dagger \circ (1_{A_1^*} \otimes f^\dagger)$$

QUANTUM MECHANICS

System of type A $:=$ Object A

Composite of A and B $:=$ Tensor $A \otimes B$

Process of type $A \to B$ $:=$ Morphism $f : A \to B$

State of A $:=$ Element $\psi : \mathrm{I} \to A$

Evolution of A $:=$ Unitary $U : A \to A$

Measurement on A $:=$ Projectors $\{\mathrm{P}_i : A \to A\}_i$

- Data $:=$ $\nu \in \{i\}_i$
- Dynamics $:=$ $\psi \mapsto \mathrm{P}_\nu \circ \psi$
- Probability $:=$ $\psi^\dagger \circ \mathrm{P}_\nu \circ \psi : \mathrm{I} \to \mathrm{I}$

GOOD FAMILIES

The following issues remain to be clarified:

1. **Which families of what are allowed?**
2. **How to turn such a family in a morphisms?**

Once these are clarified we can describe and prove correctness of standard protocols (cf. A & C @ LiCS'04).

(i) comprises classical info-flow, **(ii)** quantum info-flow is explicit, **(iii)** all kinds are reflected in the types.

GOOD FAMILIES

The following issues remain to be clarified:

1. **Which families of what are allowed?**
2. **How to turn such a family in a morphisms?**

This involves adding/requiring **additive structure**, which itself involves both formal and physical choices.

Other SCC features play a crucial role here.

ABSTRACT GLOBAL PHASES

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \ , \ s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \implies f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

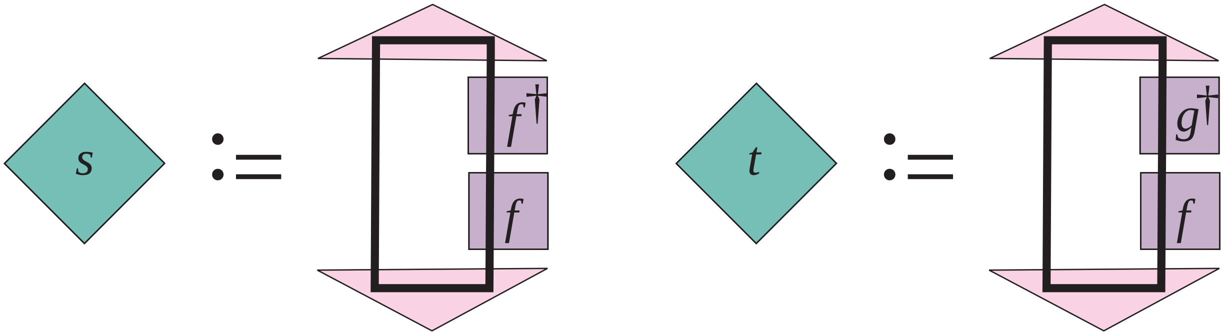
$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t \ : \ s \bullet f = t \bullet g \ , \ s \circ s^\dagger = t \circ t^\dagger$$

e.g.

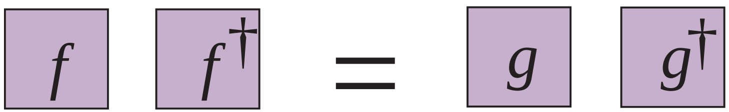
$$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \qquad \text{and} \qquad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

$\sharp 1 \quad s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$ and $t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$

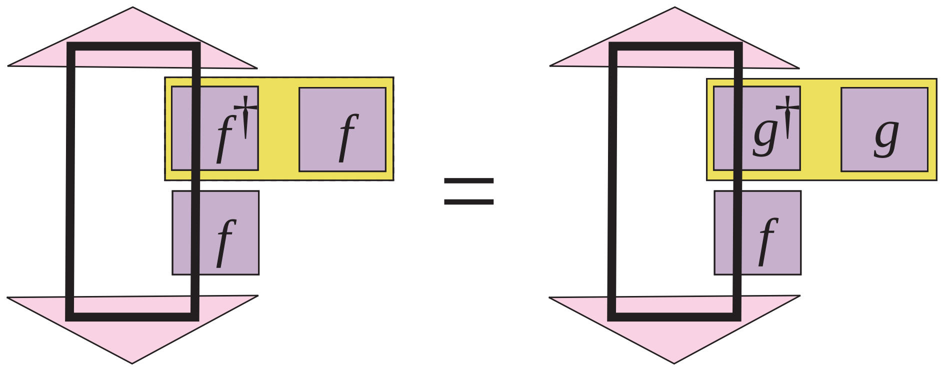


$\sharp 2 \quad f \otimes f^\dagger = g \otimes g^\dagger$



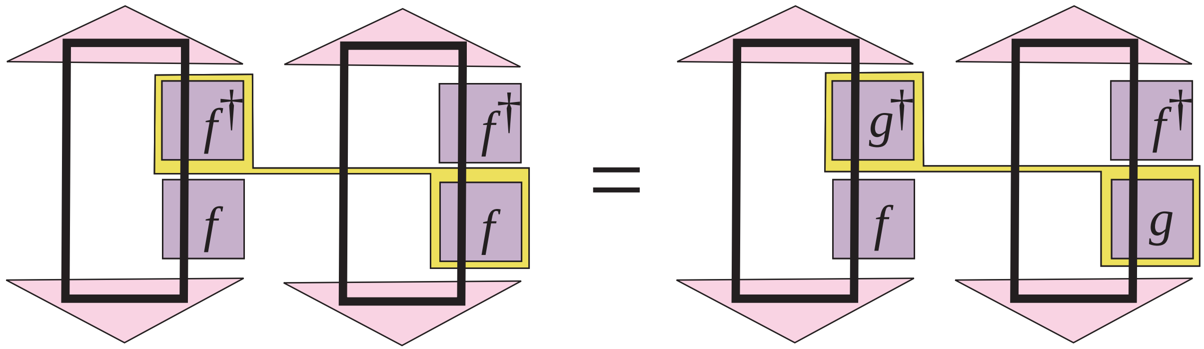
Proof.

$\sharp 3 \quad s \bullet f = t \bullet g$ with $s/t := (\ulcorner f/g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$



Proof.

$\sharp 4 \quad s \circ s^\dagger = t \circ t^\dagger$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



Hilbert-Schmidt norm of linear map

$$|f| = \sqrt{\sum_i \langle f(e_i) \mid f(e_i) \rangle}$$

exists in SCC as **squared Hilbert-Schmidt norm**

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

$$\begin{aligned} ||f||(1) &= \left(\eta^\dagger \circ (1 \otimes f)^\dagger \circ (1 \otimes f) \circ \eta\right)(1) \\ &= \left(\eta^\dagger \circ (1 \otimes (f^\dagger \circ f))\right)\left(\sum e_i \otimes e_i\right) \\ &= \eta^\dagger \left(\sum e_i \otimes f^\dagger(f(e_i))\right) \\ &= \sum \langle e_i \mid f^\dagger(f(e_i))\rangle = \sum \langle f(e_i) \mid f(e_i)\rangle \,. \end{aligned}$$

The corresponding inner-product

$$\langle f \mid g\rangle = \sum_i \langle f(e_i) \mid g(e_i)\rangle$$

exists in SCC as **Hilbert-Schmidt inner-product**

$$\langle f \mid g\rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$$

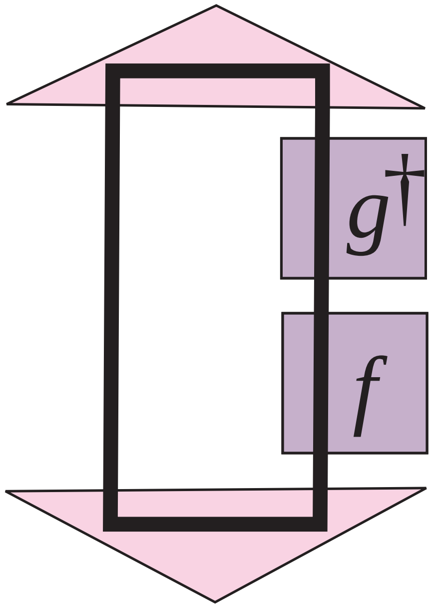
and generalizes 'the one on states' since

$$(\ulcorner \psi \urcorner)^\dagger \circ \ulcorner \phi \urcorner = \psi^\dagger \circ \phi$$

$\Rightarrow$ canonical norm on morphisms for SCC

$\Rightarrow$ inner-product on morphisms for SCC

PORTRAIT OF H. & S.



Exercise. Prove $\langle f \mid g\rangle^\dagger = \langle f^\dagger \mid g^\dagger\rangle$ (hint: portrait).

A CALCULATION

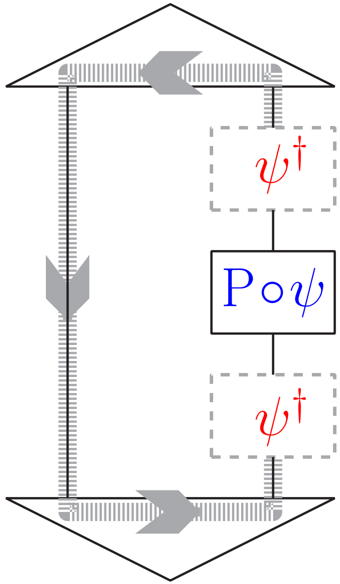
Density operator representation cf. `ket-bra` $|\psi\rangle\langle\psi|$

$$\rho := \psi \circ \psi^\dagger : A \to A$$

$$\begin{aligned}
\psi^\dagger \circ (\mathrm{P} \circ \psi) &= \eta_{\mathrm{I}}^\dagger \circ (1_{\mathrm{I}^*} \otimes \psi^\dagger) \circ (1_{\mathrm{I}^*} \otimes (\mathrm{P} \circ \psi)) \circ \eta_{\mathrm{I}} \\
&= \eta_A^\dagger \circ ((\psi^\dagger)^* \otimes 1_A) \circ (1_{\mathrm{I}^*} \otimes (\mathrm{P} \circ \psi)) \circ \eta_{\mathrm{I}} \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes (\mathrm{P} \circ \psi)) \circ ((\psi^\dagger)^* \otimes 1_{\mathrm{I}}) \circ \eta_{\mathrm{I}} \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes (\mathrm{P} \circ \psi)) \circ (1_{A^*} \otimes \psi^\dagger) \circ \eta_A \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes ((\mathrm{P} \circ \psi) \circ \psi^\dagger)) \circ \eta_A \\
&= \mathrm{Tr}(\mathrm{P} \circ \rho)
\end{aligned}$$

$$
\begin{aligned}
\psi^\dagger \circ \mathrm{P} \circ \psi &= \eta_{\mathrm{I}}^\dagger \circ (1_{\mathrm{I}^*} \otimes \psi^\dagger \circ (\mathrm{P} \circ \psi)) \circ \eta_{\mathrm{I}} \\
&= \ldots \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes ((\mathrm{P} \circ \psi) \circ \psi^\dagger)) \circ \eta_A = \mathrm{Tr}(\mathrm{P} \circ \rho)
\end{aligned}
$$

$\psi^\dagger \circ \mathrm{P} \circ \psi \quad \rightsquigarrow$

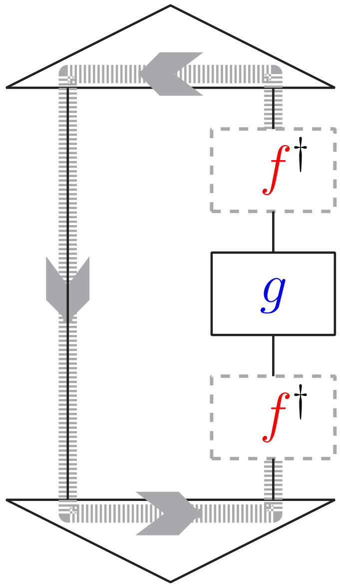


$\rightsquigarrow \quad \mathrm{Tr}(\mathrm{P} \circ \rho)$

PORTRAIT: SOLUTION

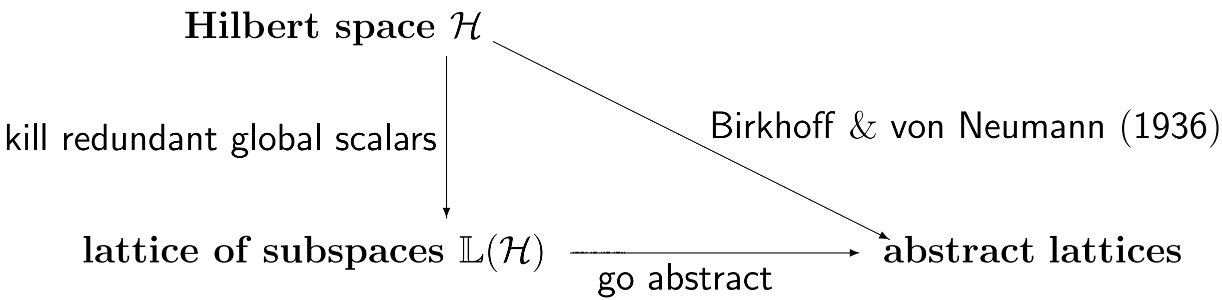
$$\langle f \mid g\rangle^\dagger = \left(\eta^\dagger \circ f^\dagger \circ g \circ \eta\right)^\dagger = \eta^\dagger \circ g^\dagger \circ f \circ \eta = \langle g \mid f\rangle$$

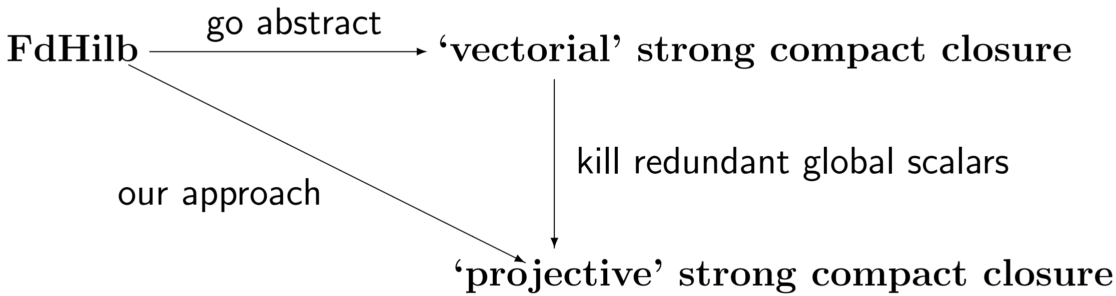
$\langle g \mid f\rangle \quad \rightsquigarrow$



$\rightsquigarrow \quad \langle f^\dagger \mid g^\dagger\rangle$

PROJECTIVE vs VECTORIAL





KILLING GLOBAL PHASES

Construction. SCCC $\mathbf{C}$ yields SCCC $WProj(\mathbf{C})$:

- The objects of $WProj(\mathbf{C})$ are those of $\mathbf{C}$
- $WProj(\mathbf{C})(A, B) := \{f \otimes f^\dagger \mid f \in \mathbf{C}(A, B)\}$
- $(f \otimes f^\dagger) \,\bar{\circ}\, (g \otimes g^\dagger) := (f \circ g) \otimes (f \circ g)^\dagger$

Proposition.

$$f \otimes f^\dagger = g \otimes g^\dagger \iff f \otimes f_* = g \otimes g_* \iff \mathrm{P}_f = \mathrm{P}_g$$

Example.

$$(WProj \circ WProj)(\mathbf{FdHilb}) \simeq WProj(\mathbf{FdHilb})$$

ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^{\dagger} = g \otimes g^{\dagger} \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

iff

$$\psi \circ \psi^{\dagger} = \phi \circ \phi^{\dagger} \implies \psi = \phi$$

iff

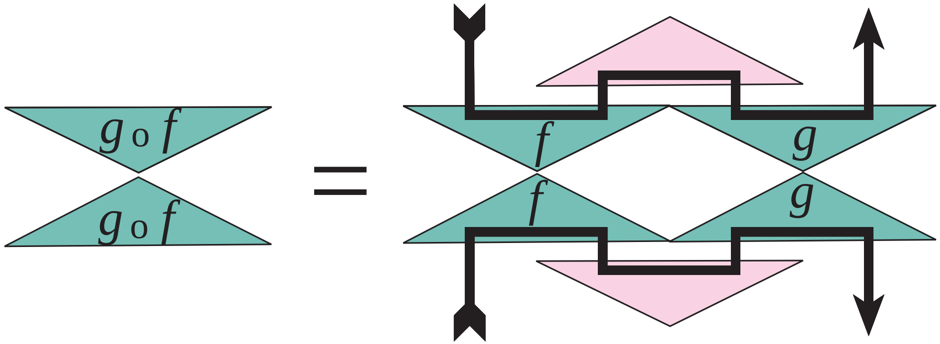
Equal Preparations Produce Equal States

OPEN SYSTEMS AND CPMs

Composition in

$$\{\mathrm{P}_f : A^* \otimes B \to A^* \otimes B\} \simeq \{f \otimes f^\dagger : A \otimes B \to B \otimes A\}$$

is



By closedness

$$\{\psi \circ \psi^\dagger \mid \psi : \mathrm{I} \to A^* \otimes B\} = \{\mathrm{P}_f \mid f : A \to B\}$$

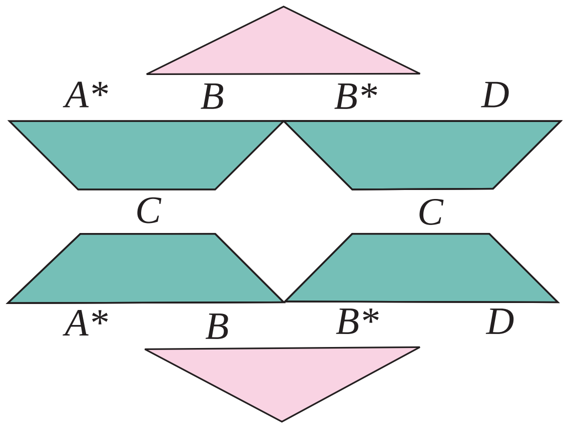
Replacing

$$\{\psi \circ \psi^\dagger : A^* \otimes B \to A^* \otimes B \mid \psi : \mathrm{I} \to A^* \otimes B\}$$

by

$$\{g \circ g^\dagger : A^* \otimes B \to A^* \otimes B \mid g : C \to A^* \otimes B\}$$

in $WProj(\mathbf{C})$ yields **Selinger's CPM-construction** which turns any SCCC into an SCCC of open systems.



GOOD FAMILIES

1. Closed systems. Assume sum type $\bigoplus_i A_i$ and (pseudo-)projections $p_j : \bigoplus_i A_i \longrightarrow A_j$ then

$$\mathrm{P}_j := \pi_j^\dagger \circ \pi_j \qquad M := \left(\bigoplus_i \pi_i^\dagger\right) \circ U : A \longrightarrow \bigoplus_i A$$

where $\pi_j := p_j \circ U$ for any $U : A \longrightarrow \bigoplus_i A_i$ unitary.

2. Open systems. Assume enrichment in monoids, add (bi)products freely, then for $\{\mathrm{P}_i\}_i$ all positive with

$$\sum_i \mathrm{P}_i = 1_A \qquad M := \langle \mathrm{P}_i \rangle_i : A \to \bigoplus_i A$$

CONCLUSION

Doing Quantum Semantics

$$\Downarrow$$

Doing Strong Compact Closure

Loads of things remain to be done!