Ensemble Steering, Weak Self-Duality, and the Structure of Probabilistic Theories

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1 Introduction

The development of quantum information theory has rekindled interest in the possibility of characterizing quantum theory in operational or information-theoretic terms. It has become clear that many properties of quantum systems, e.g., the existence and basic properties of entangled states, are much better understood as generically non-classical, rather than specifically quantum, phenomena, in the sense that they arise in arbitrary non-classical probabilistic theories [8, 2, 3, 4, 10, 14, 15, 20]. There is therefore a premium on identifying operationally meaningful properties of bipartite quantum states that are *not* generic in this way.

A property of entangled quantum states that struck Schrödinger as especially peculiar and "discomforting" [18] is the fact that an observer controlling one component of such a state can *steer* the other system into any statistical ensemble for its (necessarily, mixed) marginal state, simply by choosing to measure a suitable observable [18, 12]. What Schrödinger found discomforting is now understood to be an important information theoretic feature of quantum mechanics. This became clear when Bennett and Brassard [9], in the same paper that introduced quantum key distribution, considered a natural quantum scheme for another important cryptographic primitive, bit commitment, and showed that ensemble steering can be used to break it. In this paper, we connect the possibility of ensemble steering with two very special geometric properties shared by finite-dimensional classical and quantum state spaces. First, such state spaces are *self-dual*: their cones of (un-normalized) effects are *canonically* isomorphic to their dual cones of (un-normalized) effects, meaning that the isomorphism defines an inner product. Secondly, they are *homogeneous*: their groups of order-isomorphisms act transitively on the interiors of their positive cones. These two properties come close to characterizing finite-dimensional quantum and classical state spaces: by a theorem of Koecher [16] and Vinberg [19], finite-dimensional homogeneous, self-dual cones are precisely the cones of positive elements of formally real

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Jordan algebras. Given this, two further assumptions—local tomography and the existence of qubits—give QM uniquely. Here, we establish that in any probabilistic theory in which universal self-steering is possible (meaning that for each system in the theory, every state is the marginal of a state of two copies of the system, that steers it) state spaces must be homogeneous and *weakly* self-dual, meaning that the cones of un-normalized states and un-normalized effects must be isomorphic, but perhaps not not canonically so. This reduces the gap between the generic "self-steering" theory and quantum mechanics, largely to that between weak and strong self-duality.

The full version of the work summarized in this abstract is available on the `quant-ph` e-print archive at `http://arxiv.org/abs/0912.5532` .

2 Ordered linear spaces formalism for theories

All vector spaces in what follows are finite dimensional. This allows us to identify a vector space V with its double dual V^{**}. An *ordered linear space* (OLS) is a real vector space V equipped with a partial ordering compatible with the linear structure (i.e., $x \leq y \Rightarrow x + z \leq y + z$ and $x \leq y \implies tx \leq ty$ for all $x, y, z \in V,\ t \in \mathbb{R}_+$) Any such ordering is determined by the the pointed convex cone[1] V_+, called the *positive cone*, of vectors x with $0 \leq x$, since $x \leq y$ iff $y - x \in V_+$; conversely, any pointed convex cone induces an ordering on V in this way.

If a pointed convex cone is also generating (i.e., spans V, so that $V = V_+ - V_+$) and closed, it is called *regular*. Henceforth, we mean by "ordered linear space" one whose positive cone is regular.

A linear map $\varphi : V \to W$ is *positive* iff it is order-preserving, equivalently $\varphi(V_+) \subseteq \varphi(W_+)$. An *order isomorphism* is a linear map such that $\varphi(V_+) = W_+$. An order-isomorphism from an OLS to itself is an *order-automorphism*. The set $\mathcal{L}_+(V, W)$ of positive linear mappings from V to W is a pointed, closed convex cone in the space $\mathcal{L}(V, W)$ of all linear maps from V to W; where V and W are finite-dimensional, this cone is also generating. When $W = \mathbb{R}$, we write $\mathcal{L}_+(V, \mathbb{R})$ as V_+^*, called the *dual cone* to V. An *order unit in* an ordered linear space V is a vector $u \in V_+$ such that for every $x \in V_+$ there is some positive scalar t with $x \leq tu$; equivalently in finite dimensions, an interior element of A_+. An order unit *on* V is an order unit *in* V^*.

If A and B are ordered linear spaces, there is a natural ordering on their direct sum, namely, $(A \oplus B)_+ = \{x + y | x \in A_+, y \in B_+\}$. An ordered linear space is *irreducible* iff it has no non-trivial decomposition as an ordered direct sum. Every OLS in finite dimension is a direct sum of irreducible ones.

Definition 2.1. *An* abstract state space *is a pair (A, u_A) where A is an ordered linear space and u_A is a distinguished order-unit on A. We refer to a positive element of A with $u_A(\alpha) = 1$ as a* normalized state. *The set of all normalized states is a compact convex subset of A_+, which we denote by Ω_A.* [2]

[1] A *convex cone* in a real vector space V is a convex set $K \subseteq V$ closed under multiplication by non-negative scalars. If $K \cap -K = \{0\}$, the cone is said to be *pointed*.

[2] The set Ω_A is a *base* for the positive cone A_+: a convex set S such that every non-zero $\alpha \in A_+$ is a positive scalar multiple of a unique vector in S. In the case $S = \Omega_A$), the vector is $\alpha/u_A(\alpha)$. Indeed, what we are calling

We also need some way to describe the results of measurements performed on a system.

Definition 2.2. *An* effect *on an abstract state space* (A, u_A) *is a positive functional* $a \in A^*$ *such that* $a \leq u_A$*—equivalently,* $a \in A^*$ *is an effect iff* $0 \leq a(\alpha) \leq 1$ *for every normalized state* $\alpha \in \Omega_A$.

Every measurement outcome will correspond to an effect on A. We make the further assumption here that every effect represents a measurement outcome. Accordingly, a discrete *observable* on A is a family $\{a_x\}_{x\in X}$ of effects, indexed by a finite set X (a "value space"), with $\sum_{x\in X} a_x(\alpha) = 1$ for all $\alpha \in \Omega$, i.e., with $\sum_x a_x = u_A$.

The formalism sketched above accommodates composite systems. Let A and B be abstract state spaces, with order-units $u_A \in A^*$, $u_B \in B^*$ and normalized state spaces Ω_A and Ω_B. We write $A \otimes_{\max} B$ for the space of bilinear forms on $A^* \times B^*$, ordered by the cone of forms nonnegative on products $a \otimes b$ of positive elements (i.e. $a \in A^*_+$, $b \in B^*_+$), with order unit $u_A \otimes u_B$. We write $A \otimes_{\min} B$ for the same space, ordered by the (generally, much smaller) cone generated by the product states $\alpha \otimes \beta$, where $\alpha \in A_+$ and $\beta \in B_+$.

States in $A \otimes_{\max} B$ satisfy a natural no-signaling condition, that the marginal states of A and B are well-defined. Moreover [15, 20] the maximal tensor product captures all non-signaling states—at least, insofar as we regard bipartite states as determined by joint probability assignments to pairs of local measurement outcomes. This last assumption is called *local tomography* or *local observability*. We shall adopt it here as a working assumption.

More generally, we can consider the space $(A^* \otimes B^*)^*$ of bilinear forms on $A^* \times B^*$, equipped with any cone lying between the minimal and maximal ones, as "a tensor product" of A and B. In what follows, we shall write AB, generically, for such a composite.

For present purposes we understand by the term *physical theory* a class of abstract state spaces, closed under the formation of such a product, so as to allow the representation of composite systems. A more complete treatment of this idea might take a theory to be a *category* of abstract state spaces, with morphisms corresponding to the processes allowed by the theory; see [5], [6] and [7], the latter of which is abstracted in the present preproceedings.

3 Weak self-duality

A bipartite state on a composite system AB, represented by a positive bilinear form $\omega : A^* \times B^* \to \mathbb{R}$, can also be represented by a positive map $\hat{\omega} : A^* \to B = B^{**}$ defined by $\hat{\omega}(a)(b) = \omega(a, b)$. We have $\hat{\omega}(u_A) = \omega^B$, the B marginal of ω. Also the adjoint map $\hat{\omega}^* : B^* \to A^{**} = A$ represents the same state, evaluated in the opposite order, i.e, $\hat{\omega}^*(b)(a) = \hat{\omega}(a)(b) = \omega(a, b)$. Hence, $\hat{\omega}^*(u_B) = \omega^A$.

Definition 3.1. An abstract state space A is *weakly self-dual* iff there exists an order-isomorphism $\eta : A^* \simeq A$.

Definition 3.2. A bipartite state ω in $A \otimes_{\max} B$ is an *isomorphism state* iff $\hat{\omega} : A^* \to B$ is an order isomorphism.

an abstract state space is essentially the same thing as a ordered vector space with a distinguished cone-base, i.e., what we might call a (fi nite-dimensional) cone-base space.

The existence of a composite containing isomorphism states is far from guaranteeing the weak self-duality of A or B; indeed, for *any* state space B, there is a state space A whose positive cone is isomorphic to the dual of B's, hence for which $A \otimes_{\max} B$ contains isomorphism states. But the existence of an isomorphism state *in* $A \otimes A$ obviously *does* imply that A is weakly self-dual. We call such a state an *automorphism state*.

Theorem 3.3. *Let A be an irreducible ordered linear space. Then automorphisms of A lie on extremal rays of the cone $\mathcal{L}_+(A, A)$ of positive maps from A to A.*

Simple examples show that automorphisms need not be extremal in reducible cones.

Corollary 3.4. *If A is an irreducible abstract state space, then isomorphism states (if any exist) are pure in $A \otimes_{max} B$.*

4 Purification

An important fact about quantum states is that they can be *purified*: any state is the marginal of a pure bipartite state. One would like to know to what extent this is true more generally.

Given an abstract state space (A, u), we can turn A^* into an abstract state space by using any interior *state* $\alpha_o \in A_+$ as the order unit. We shall write $A^\blacklozenge$, generically, for such a state space (A^*, α_o), leaving the (non-canonical) choice of α_o tacit. Then the identity map $A \to A$ can be interpreted as an isomorphism (hence, by Theorem 3.3, pure if A is irreducible) state in $A^\blacklozenge \otimes_{\max} A$ having α_o as its A-marginal. In this sense, every state interior to A has a purification. In general, however, the "ancilla" $A^\blacklozenge$ in terms of which $\alpha_o \in A$ is purified, depends on α_o.

Theorem 4.1. *The following are equivalent:*

(a) A is homogeneous;

(b) Every normalized state in the interior of A_+ is the A-marginal of an isomorphism state in $B \otimes_{max} A$, where B is any (fixed) state space order-isomorphic to A^*.

This gives us a physical interpretation of homogeneity: first, that the various "dual" abstract state spaces $A^\blacklozenge = (A^*, \alpha_o)$ are all isomorphic, not only as ordered linear spaces but *as abstract state spaces* , and second, as telling us that in the irreducible case, all interior states of A can be purified to isomorphism states using a *fixed* ancilla, namely, any choice of $A^\blacklozenge$.

Corollary 4.2. *For any irreducible state space A, the following are equivalent:*

(a) A is weakly self-dual and homogeneous;

(b) Every normalized state in the interior of A_+ is the marginal of an isomorphism state in $A \otimes_{max} A$.

5 Steering

By an *ensemble* for a state $\beta \in B$, we mean a finite set of $\beta_i \in B_+$ such that $\sum_i \beta_i = \beta$. Note that we defined ensembles not as lists of probabilities and associated normalized states, but as lists of unnormalized states; the two definitions are equivalent, as the norms $u_B(\beta_i)$ of the β_i encode the probabilities. If instead $\sum_i \beta_i \leq \omega^B$ is required, we have a *subensemble* for ω^B.

Definition 5.1. A bipartite state $\omega \in A \otimes_{\text{max}} B$ is *steering for its B marginal* iff, for every ensemble (convex decomposition) $\omega^B = \sum_i \beta_i$, where β_i are un-normalized states of B, there exists an observable $E = \{x_i\}$ on A with $\beta_i = \hat{\omega}(x_i)$. We say that ω is *bisteering* iff it's steering for both marginals.

Note that a state steering for its B marginal is not necessarily pure. It follows almost immediately from the definition, that if ω is steering for its B-marginal, $\hat{\omega}(A_+)$ is a face of B_+. Indeed, we have

Lemma 5.2. *If ω is steering, then $\hat{\omega}(A_+) = \text{Face}(\omega^B)$; the converse, however, does not hold.*

The condition that ω be steering for its B-marginal places a very strong and subtle constraint on $\hat{\omega}$. If X and Y are partially ordered sets, an order-preserving surjection $p : X \to Y$ is a *quotient map* iff, for all $y_1, y_2 \in X$, $y_1 \leq y_2$ iff $y_i = p(x_i)$ for some $x_1 \leq x_2$ in X. We shall say that p is a *strong quotient map* iff it has the property that every chain $y_1 \leq y_2 \leq \cdots \leq y_n$ in Y is the image of some chain $x_1 \leq x_2 \leq \cdots \leq x_n$ in X, i.e., $y_1 = p(x_1), y_2 = p(x_2), ..., y_n = p(x_n)$. A strong quotient map is a quotient map, but the converse is, in general, false.

Theorem 5.3. *Let ω be a bipartite state in AB. Then ω is steering for its B marginal iff $\hat{\omega} : [0, u_A] \to [0, \omega^B]$ is a strong quotient map of ordered sets.*

We suspect, but so far have been unable to prove, that a quotient map of order-intervals $[0, u] \to [0, v]$ is necessarily a strong quotient. An obvious *sufficient* condition for $\hat{\omega} : [0, u_A] \to [0, \omega^B]$ to be a quotient map of ordered sets is for there to exist an affine section $\sigma : [0, \omega^B] \to [0, u^A]$. However, examples show that this is not necessary for steering.

It follows from Theorem 5.3 that the ordering of $\text{Face}(\omega^B) = \hat{\omega}(A_+)$ is exactly the quotient linear ordering induced by the linear surjection $\hat{\omega}$, i.e., $\beta_1 \leq \beta_2$ in $\text{Face}(\omega^B)$ iff $\beta_i = \hat{\omega}(a_i)$ for some $a_1, a_2 \in A$ with $a_1 \leq a_2$. We also have:

Corollary 5.4. *Let ω be steering for ω^B, where ω^B is interior to B_+, so that $\text{Face}(\omega^B) = B_+$. If $\hat{\omega}$ is injective (non-singular), then $\hat{\omega}$ is an order isomorphism. If B_+ (and therefore A_+) is irreducible, therefore, by Theorem 3.3, it is pure in $A \otimes_{max} B$.*

In other words, if A and B have the same dimension, then the states that are steering for an interior marginal are precisely the isomorphism states (and hence steering for both marginals). We are now in a position to make good on the claim made in the introduction.

Definition 5.5. *A probabilistic theory* supports universal steering *if, for every system B in the theory and every state $\beta \in B$, there exists a system A_β and a bipartite state ω in*

$A_\beta \otimes B$ *that steers its* B*-marginal* $\omega^B = \beta$. *A theory* supports uniform universal steering *if, for every system* B *in the theory, there exists a system* A_B *such that for every state* $\beta \in A$, *there exists a state* ω *in* $A_B \otimes B$ *that steers its* B*-marginal* β. *A probabilistic theory* supports universal self-steering *if, for every system* A *in the theory, every state* $\alpha \in A$ *can be represented as a marginal of some bipartite state on two copies of* A*—that is, some state* $\omega \in AA$*—steering for that marginal. (That is, it supports uniform universal steering with* $A_B \simeq A$*.)*

Corollary 5.4, combined with Theorem 4.1, establish

Proposition 5.6. *In any theory that supports universal uniform steering, every irreducible, finite-dimensional state space in the theory is homogeneous.*

In light of Corollary 4.2, we also have

Proposition 5.7. *In any theory that supports universal self-steering, every irreducible, finite-dimensional state space in the theory is homogeneous and weakly self-dual.*

If a theory supports universal self-steering, and also has the property that every direct summand of a state space is again a state space belonging to the theory (a reasonable requirement, at least in finite dimensional settings), then every finite-dimensional state space in the theory is a direct sum of homogeneous, weakly self-dual factors, hence, homogeneous and weakly self-dual.[3]

An interesting question is to what extent the gap between universal steering and uniform universal steering is a genuine one. One might investigate this question by looking for examples of state spaces for which each state can be steered, but that are not homogeneous.

6 Conclusion

We have shown that the state spaces of any probabilistic theory that allows for uniform universal ensemble steering, in the sense that for every system A in the theory, there's another system B such that every state on system A can be steered by some state in the composite BA, are homogeneous. We say that system B *steers system* A, in this case. If we require systems to be *self*-steering (i.e., that each system A steer *itself*), they must be homogeneous and weakly self-dual. If one could motivate the stronger assumption that these state spaces are *strongly* self-dual, then by the Koecher–Vinberg theorem [16, 19], these state spaces would be those of formally real Jordan algebras. Then by the Jordan–von Neumann–Wigner classification theorem [13] their normalized state spaces must be convex direct sums of sets affinely isomorphic to the unit-trace elements in the cones of positive semidefinite matrices in a real, complex, or quaternionic full matrix algebra, or to Euclidean balls, or to the unit-trace 3×3 positive semidefinite matrices over the octonions.

From here, our standing assumption of local tomography (that bipartite states are determined by the probabilities they assign to product effects) restricts the possibilities further. A theorem of Hanche-Olsen [11] asserts that any JB-algebra A whose vector-space tensor

[3] It is easily seen that direct sums of homogeneous or weakly self-dual cones are, respectively, homogeneous or weakly self-dual.

product with the self-adjoint part of $M_2(\mathbb{C})$—can be made into a JB tensor product, is isomorphic to the self-adjoint part of a (complex) C^*-algebra. In other words, it is essentially quantum-mechanical. As we will establish elsewhere, Hanche-Olsen's requirements for a JB tensor product impose on the cones associated with the three JB-algebras in question, exactly the operational requirements we've imposed on a composite of state spaces. Therefore Hanche-Olsen's result implies that if a homogeneous, self-dual state space has a *locally tomographic* homogeneous, self-dual composite with a three-dimensional ball—i.e., the state space of a qubit, then it is the state space of a C^*-algebra — so, a direct sum of the state spaces of standard quantum theory.

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