Some steps towards noncommutative Gel'fand duality

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Abstract

Gel'fand duality is one of the central mathematical insights of the last century [7]. Each abelian C^*-algebra $\mathcal{A}$ gives rise to a compact or locally compact Hausdorff space, the Gel'fand spectrum $\Sigma^{\mathcal{A}}$. Conversely, each compact or locally compact Hausdorff space X determines an abelian C^*-algebra $C(X)$ of continuous functions. In quantum theory, *noncommutative* C^*-algebras play a central rôle, but we are still lacking a good notion of spectrum for these algebras. Such a spectrum would be a suitable noncommutative space and would provide quantum theory with a geometrical underpinning that is absent so far. In previous work [5, 6], it was shown that the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ associated with an arbitrary unital C^*- or von Neumann algebra $\mathcal{A}$ has many properties of a spectrum. Here we show that the assignment $\mathcal{A} \mapsto \underline{\Sigma}^{\mathcal{A}}$ is functorial in a suitable sense and can be seen as the first half of a noncommutative version of Gel'fand duality. We show that for abelian algebras, our construction reduces to ordinary Gel'fand duality. Moreover, it is shown how the group of unitary operators in a von Neumann algebra is faithfully represented by automorphisms of the (set of subobjects of the) spectral presheaf.

1 Introduction

There is a well-developed strand of research aiming to describe the spectra of nonabelian C^*-algebras in terms of noncommutative topology. This approach has first been developed in functional analysis by Akemann [1, 2], Giles and Kummer [8] in the late 1960s and early 1970s, and later on in the form of quantales by Mulvey [9], Borceux, Rosický [3, 4] and others. On the other hand, the topos approach to the formulation of physical theories [6] has led to the consideration of certain spaces without points associated with nonabelian C^*- and von Neumann algebras. The object in question is the spectral presheaf. Let $\mathcal{A}$ be a C^*-algebra, and let $\underline{\Sigma}^{\mathcal{A}}$ be its spectral presheaf (see definition below). Here, we show that this assignment is functorial in a suitable sense and prove some consequences. We consider arbitrary $*$-homomorphisms between unital C^*-algebras.

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2 The spectral presheaf

Let $\mathcal{A}$ be an arbitrary unital C^*-algebra, and let $\mathcal{V}(\mathcal{A})$ be the set of commutative, unital C^*-subalgebras of $\mathcal{A}$ such that the unit element in each $C \in \mathcal{V}(\mathcal{A})$ is the unit element in $\mathcal{A}$. $\mathcal{V}(\mathcal{A})$ is partially ordered under inclusion. The *spectral presheaf* $\underline{\Sigma}^{\mathcal{A}}$ of $\mathcal{A}$ is the contravariant, $\mathbf{Set}$-valued functor on $\mathcal{V}(\mathcal{A})$ defined

(a) on objects: for all $C \in \mathcal{V}(\mathcal{A})$, let $\underline{\Sigma}^{\mathcal{A}}_C$ be the Gel'fand spectrum of C, i.e., the space of characters of C, equipped with the Gel'fand topology;

(b) on arrows: for all inclusions $i_{C'C}$, let $\underline{\Sigma}^{\mathcal{A}}(i_{C'C}) : \underline{\Sigma}^{\mathcal{A}}_C \to \underline{\Sigma}^{\mathcal{A}}_{C'}$ be the function that sends each character λ to its restriction $\lambda|_{C'}$ to the smaller algebra. This function is well-known to be continuous and surjective.

An analogous definition exists for arbitrary von Neumann algebras $\mathcal{N}$. In this case, we consider the poset $\mathcal{V}(\mathcal{N})$ of abelian, counital *von Neumann* subalgebras and define the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$ over this category.

3 The main result

In fact, we want to assign to a given C^*-algebra $\mathcal{A}$ not just its spectral presheaf, but the *topos* $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ of presheaves over the context category $\mathcal{V}(\mathcal{A})$, together with the distinguished object $\underline{\Sigma}^{\mathcal{A}} \in Ob(\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}})$. (Analogously for von Neumann algebras $\mathcal{N}$.) The reason is the following:

Theorem 1 *Each $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras induces a geometric morphism $\Phi : \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}}$ between the associated topoi such that the inverse image functor $\Phi^* : \mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ maps the spectral presheaf $\underline{\Sigma}^{\mathcal{B}}$ of $\mathcal{B}$ to an object $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$ in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$. Using Gel'fand duality at each stage, we obtain a subobject $(\mathcal{G} \circ \Phi^*)(\underline{\Sigma}^{\mathcal{B}})$ of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of $\mathcal{A}$.*

This shows that each morphism ϕ of C^*-algebras induces a morphism in the opposite direction between the associated spectral presheaves. Hence, the assignment $\mathcal{A} \mapsto \underline{\Sigma}^{\mathcal{A}}$ is functorial. The map from $\underline{\Sigma}^{\mathcal{B}}$ to $\underline{\Sigma}^{\mathcal{A}}$ constructed in the proof below is the generalisation of the continuous map $\Sigma^{\mathcal{B}} \to \Sigma^{\mathcal{A}}$, $\lambda \to \lambda \circ \varphi$ between the Gel'fand spectra induced by a $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between abelian C^*-algebras. (Note that we have contravariance.)

Proof. Let $C \in \mathcal{V}(\mathcal{A})$ be an abelian subalgebra of $\mathcal{A}$. It is straightforward to show that $\phi(C)$ is an abelian C^*-subalgebra of $\mathcal{B}$. Clearly, if $C' \subset C$, then $\phi(C') \subseteq \phi(C)$, hence ϕ induces a morphism

$$\begin{aligned} \tilde{\phi} : \mathcal{V}(\mathcal{A}) &\longrightarrow \mathcal{V}(\mathcal{B}) \\ C &\longmapsto \phi(C) \end{aligned}$$

of posets, i.e., an order-preserving map. The posets $\mathcal{V}(\mathcal{A})$ and $\mathcal{V}(\mathcal{B})$ are the base categories of the topoi $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}}$, so $\tilde{\phi}$ induces a geometric morphism Φ : $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}}$. The inverse image functor Φ^* is given by

$$\begin{aligned} \Phi^* : \mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}} &\longrightarrow \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}} \\ \underline{P} &\longmapsto \underline{P} \circ \tilde{\phi}. \end{aligned}$$

In particular, $\Phi^*(\underline{\Sigma}^{\mathcal{B}}) = \underline{\Sigma}^{\mathcal{B}} \circ \tilde{\phi}$, and we obtain

$$\forall C \in \mathcal{V}(\mathcal{A}) : \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C = (\underline{\Sigma}^{\mathcal{B}} \circ \tilde{\phi})_C = \underline{\Sigma}^{\mathcal{B}}_{\tilde{\phi}(C)}.$$

Now we can apply ordinary Gel'fand duality: for each $C \in \mathcal{V}(\mathcal{A})$, we have an arrow $\phi|_C : C \to \phi(C)$ between abelian C^*-algebras, which determines a continuous function $\mathcal{G}_{\phi|_C} : \Sigma_{\phi(C)} \to \Sigma_C$ given by $\lambda \mapsto \lambda \circ \phi|_C$. Using the fact that $\underline{\Sigma}^{\mathcal{B}}_{\tilde{\phi}(C)} = \Sigma_{\phi(C)}$ and $\underline{\Sigma}^{\mathcal{A}}_C = \Sigma_C$, we define

$$\forall C \in \mathcal{V}(\mathcal{A}) : \mathcal{G}_{\phi|_C}(\Phi^*(\underline{\Sigma}^{\mathcal{B}})_C) = \mathcal{G}_{\phi|_C}(\underline{\Sigma}^{\mathcal{B}}_{\tilde{\phi}(C)}) = \{\lambda \circ \phi|_C \mid \lambda \in \underline{\Sigma}^{\mathcal{B}}_{\phi}(C)\} \subseteq \underline{\Sigma}^{\mathcal{A}}_C.$$

It is easy to check that the components $\mathcal{G}_{\phi|_C}$ ($C \in \mathcal{V}(\mathcal{A})$) assemble into a natural transformation $\mathcal{G} : \Phi^*(\underline{\Sigma}^{\mathcal{B}}) \to \underline{\Sigma}^{\mathcal{A}}$ and that the image of $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$ under $\mathcal{G}$ is a subobject of $\underline{\Sigma}^{\mathcal{A}}$. We write this subobject as $(\mathcal{G} \circ \Phi^*)(\underline{\Sigma}^{\mathcal{B}})$. ■

For von Neumann algebras $\mathcal{M}, \mathcal{N}$, the appropriate morphisms are weakly continuous $*$-homomorphisms $\phi : \mathcal{M} \to \mathcal{N}$. Such a ϕ induces an order-preserving map $\tilde{\phi} : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{N})$ between the posets of abelian von Neumann subalgebras, which in turn determines a geometric morphism $\Phi : \mathbf{Set}^{\mathcal{V}(\mathcal{M})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}$. The composite $\mathcal{G} \circ \Phi^*$ maps $\underline{\Sigma}^{\mathcal{N}}$ to $\underline{\Sigma}^{\mathcal{M}}$.

The mapping $\mathcal{G} \circ \Phi^*$ is a composite of the inverse image part of a geometric morphism, i.e., and arrow between from the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{B})^{\mathrm{op}}}$ to the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$, and an arrow $\mathcal{G}$ in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$. As matters stand, $\mathcal{G}$ is not yet a functor from $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ to itself, because it is only defined by its action on $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$. Since this is all we need here (and $\mathcal{G}$ is not applied to any other object in $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$), we are at liberty to extend the definition of $\mathcal{G}$ to other objects and arrows in $\mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ such that it becomes a functor. Another option is to consider the spectral presheaves $\underline{\Sigma}^{\mathcal{A}}$ and $\underline{\Sigma}^{\mathcal{B}}$ topos-externally as locales in $\mathbf{Set}$. By very similar arguments, a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ induces a morphism from $\underline{\Sigma}^{\mathcal{B}}$ to $\underline{\Sigma}^{\mathcal{A}}$.

The map $\mathcal{G}_\phi : \Sigma^{\mathcal{B}} \to \Sigma^{\mathcal{A}}$ between the Gel'fand spectra of abelian C^*-algebras $\mathcal{A}, \mathcal{B}$ induced by a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is continuous. It remains to be worked out how the notion of continuity generalises to the nonabelian situation: in which sense, if any, can the composite $\mathcal{G} \circ \Phi^* : \underline{\Sigma}^{\mathcal{B}} \to \underline{\Sigma}^{\mathcal{A}}$ be regarded as continuous? (It is known that the spectral presheaves each carry a distinguished family of subobjects that can be seen as opens.)

4 Reduction to ordinary Gel'fand duality

If $\mathcal{A}$ is an abelian C^*-algebra, we expect to get back ordinary Gel'fand duality. This does not quite happen, though: the poset $\mathcal{V}(\mathcal{A})$ contains all abelian C^*-subalgebras of $\mathcal{A}$, so $\mathcal{A}$

itself is the top element of $\mathcal{V}(\mathcal{A})$ (if $\mathcal{A}$ is abelian), but is not the only element. Accordingly, the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ will contain the Gel'fand spectrum of $\mathcal{A}$, but also the spectra of all its subalgebras. This can easily be remedied if we consider the poset $\mathcal{V}^Z(\mathcal{A})$ of those abelian subalgebras that contain the center Z of $\mathcal{A}$: for $\mathcal{A}$ abelian, $Z = \mathcal{A}$, and the poset $\mathcal{V}^Z(\mathcal{A})$ contains only $\mathcal{A}$. Hence, $\mathbf{Set}^{\mathcal{V}^Z(\mathcal{A})^{\mathrm{op}}} = \mathbf{Set}$, and we are in the usual situation that the spectrum of the abelian algebra $\mathcal{A}$ is an object in $\mathbf{Set}$, i.e., it is a set (with additional structure).

The choice of $\mathcal{V}^Z(\mathcal{A})$ as the poset of abelian subalgebras and base category of the topos still makes sense if $\mathcal{A}$ is nonabelian. Then $\mathcal{V}^Z(\mathcal{A})$ of course contains more than one element, and the spectral presheaf lives in a topos different from $\mathbf{Set}$.

5 The action of the unitary group

Let $\hat{U} \in \mathcal{A}$ be a unitary operator. Then

$$\begin{aligned} l_{\hat{U}} : \mathcal{A} &\longrightarrow \mathcal{A} \\ \hat{A} &\longmapsto \hat{U}\hat{A}\hat{U}^{-1} \end{aligned}$$

is a $*$-homomorphism from $\mathcal{A}$ to itself. Of course, unitary operators are of central importance in quantum theory, both for the description of time evolution and to express covariance properties. $l_{\hat{U}}$ induces an automorphism

$$\begin{aligned} \tilde{l}_{\hat{U}} : \mathcal{V}(\mathcal{A}) &\longrightarrow \mathcal{V}(\mathcal{A}) \\ C &\longmapsto \hat{U}C\hat{U}^{-1} \end{aligned}$$

of the poset of abelian subalgebras and hence a geometric automorphism $L_{\hat{U}} : \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{A})^{\mathrm{op}}}$ of the topos associated with $\mathcal{A}$. The inverse image functor $L^*_{\hat{U}}$ acts on the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ in the following way:

$$\forall C \in \mathcal{V}(\mathcal{A}) : L^*_{\hat{U}}(\underline{\Sigma}^{\mathcal{A}})_C = (\underline{\Sigma}^{\mathcal{A}} \circ \tilde{l}_{\hat{U}})_C = \underline{\Sigma}^{\mathcal{A}}_{\tilde{l}_{\hat{U}}(C)}.$$

We apply the same trick as before and use Gel'fand duality: the morphism $l_{\hat{U}}|_C : C \to l_{\hat{U}}(C)$ of abelian C^*-algebras induces a function $\mathcal{G}_{l_{\hat{U}}} : \Sigma_{\phi(C)} \to \Sigma_C$, which allows us to define

$$\mathcal{G}_{l_{\hat{U}}}(L^*_{\hat{U}}(\underline{\Sigma}^{\mathcal{A}})_C) = \mathcal{G}_{l_{\hat{U}}}(\underline{\Sigma}^{\mathcal{A}}_{\tilde{l}_{\hat{U}}})_C = \{\lambda \circ l_{\hat{U}}|_C \mid \lambda \in \underline{\Sigma}^{\mathcal{A}}_{\tilde{l}_{\hat{U}}(C)}\}$$

Clearly, $\mathcal{G}_{l_{\hat{U}}}(L^*_{\hat{U}}(\underline{\Sigma}^{\mathcal{A}})_C)$ can be identified with $\underline{\Sigma}^{\mathcal{A}}_C$. This may seem trivial: we have just mapped each component $\underline{\Sigma}^{\mathcal{A}}_C$ of the spectral presheaf to itself. Yet, this is not actually a problem, since subobjects of $\underline{\Sigma}^{\mathcal{A}}$ are not left invariant, they are 'rotated' by the action of $\mathcal{G}_{l_{\hat{U}}} \circ L^*_{\hat{U}}$ in the appropriate way, as we will show now.

Let $\underline{S}$ be a subobject of $\underline{\Sigma}^{\mathcal{A}}$. In particular, for each component $\underline{S}_C$, we have $\underline{S}_C \subseteq \underline{\Sigma}^{\mathcal{A}}_C$. Then

$$\mathcal{G}_{l_{\hat{U}}}(L^*_{\hat{U}}(\underline{S})_C) = \{\lambda \circ l_{\hat{U}}|_C \mid \lambda \in \underline{S}_{\tilde{l}_{\hat{U}}(C)}\}.$$

Intuitively, this means that the application of $\mathcal{G}_{l_{\hat{U}}} \circ L^*_{\hat{U}}$ to $\underline{S}$ gives a subobject that has the same 'shape' as $\underline{S}$, but is rotated by $\hat{U}$. For each $C \in \mathcal{V}(\mathcal{A})$, the component $\underline{S}_{l_{\hat{U}}(C)}$ at $l_{\hat{U}}(C)$ is moved to become the new component at C.

The inverse transformation is the geometric morphism $L_{\hat{U}^{-1}}$ induced by the unitary $\hat{U}^{-1}$, composed with the arrow $\mathcal{G}_{l_{\hat{U}^{-1}}}$. It is clear that two different unitaries $\hat{U}_1, \hat{U}_2$ induce two different geometric morphisms $L_{\hat{U}_1}, L_{\hat{U}_2}$. We have shown:

Proposition 2 *There is a faithful representation of the unitary group $\mathcal{U}(\mathcal{A})$ of the algebra $\mathcal{A}$ by automorphisms of $\mathrm{Sub}(\underline{\Sigma}^{\mathcal{A}})$, the set of subobjects of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$.*

For a von Neumann algebra $\mathcal{N}$, we can consider the automorphisms $l_{\hat{U}} : \mathcal{N} \to \mathcal{N}$ $(\hat{U} \in \mathcal{U}(\mathcal{N}))$ as well, since they are weakly continuous. Hence, we also get a faithful representation of the unitary group $\mathcal{U}(\mathcal{N})$ by automorphisms of $\mathrm{Sub}(\underline{\Sigma}^{\mathcal{N}})$.

6 Future work

There are many interesting open questions. The first one is if and how the map $\mathcal{G} \circ \Phi^* : \underline{\Sigma}^{\mathcal{B}} \to \underline{\Sigma}^{\mathcal{A}}$ defined above can be seen as continuous. Of course, it is of great interest to see whether there is a functor from spectral presheaves (and the topoi in which they lie) to C^*-algebras, giving the other half of a noncommutative Gel'fand duality. Clearly, this is a highly non-trivial problem and will require new ideas.

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