Computational Complexity of Geometric Quantum Logic*

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Abstract

We propose a new approach towards the question of whether the equational theory of ortholattices related to Quantum Mechanics is decidable or not: By determining the growth of the algorithmic complexity of the word problem over d-dimensional Hilbert lattices for $d = 1, 2, 3, \ldots$ In case $d = 1$, the (complement of the) word problem amounts to the Boolean satisfiability problem underlying the millennium question "$\mathcal{P} = \mathcal{NP}$?" We show the case $d = 2$ to be $\mathcal{NP}$-complete as well. For fixed $d \geq 3$ and building on Hagge et.al (2005,2007,2009), we reveal quantum satisfiability as polytime-equivalent to the real feasibility of a multivariate quartic polynomial equation: a problem well-known complete for the counterpart of $\mathcal{NP}$ in the Blum-Shub-Smale model of computation lying (probably strictly) between classical $\mathcal{NP}$ and PSPACE. We finally address the problem over *in*definite finite dimensions.

1 Introduction

Quantum logic has been motivated by the physical effects exhibited by elementary particles and their mathematical description on Hilbert space whose closed subspaces correspond to 0/1-observables [BiVN36, Mack63]. The goal is to capture the underlying hard functional analysis into algebraic properties to be studied in the concrete geometric setting [Maye98, Maye07, Wilc09] or to be explored in an axiomatic setting [Megi09] known as *synthetic approach*.

Computer science has been introduced to the quantum world by FEYNMAN and grown into a flourishing and ambitious field: a *quantum computer* is considered essentially the very kind of abstract device that can execute SHOR's famous 'algorithm' for factoring integers in polynomial time. However, other means of harnessing operations on quantum states [CDS01, ACP04, Kieu03, Zieg05] and observables [Pyka00, Ying05, PM07a] for computational purposes have been suggested as well.

1.1 Motivation

The *word problem* for free algebraic structures in an equationally defined class is the question of whether two terms f, g are equivalent modulo the defining laws. In other words,

*extended abstract; full version available at `http://arXiv.org/abs/1004.1696` given a class $\mathcal{S}$ of structures, one asks whether there is a decision procedure for the equational theory of $\mathcal{S}$, i.e. deciding which identities $f = g$ hold in all members of $\mathcal{S}$ under all substitutions. If so, terms can be simplified; and detecting and applying such simplifications automatically is at the core of computer algebra systems. The first structures to be related to quantum logic were the ortholattices $\mathrm{Gr}(\mathcal{H})$ of closed subspaces of Hilbert spaces: in the modular finite dimensional version $\mathrm{Gr}(\mathbb{F}^d)$ [BiVN36] and the non-modular but orthomodular infinite dimensional version [Mack63]. Here, identities may be reduced to the form $f = 0$. In the synthetic context, decidability is known for the word problem for free ortholattices [BrKa73] but remains an open challenge in the orthomodular [Herr87] as well as in the modular-ortho case (cf. [HMR05]). In the geometric context of (modular) projection lattices of finite von Neumann algebra factors, decidability has been shown in [Herr10].

For each fixed-dimensional ortholattice $\mathrm{Gr}(\mathbb{F}^d)$, the first-order theory is decidable [DHMW05, SECTION 3]; but over the class of *all* $\mathrm{Gr}(\mathbb{F}^d)$, $d \in \mathbb{N}$, it is not [Lips74]. The present work starts exploring more closely the algorithmic properties and obstacles (in the sense of computational complexity) of the finite-dimensional case; in particular the interplay of the two natural 'negations' of "$f = 0$ identically": $f > 0$ for some substitution resp. $f = 1$ identically. We hope that this will gain new insights that eventually permit to attack the infinite-dimensional case as well.

1.2 The Theory of Computation

studies the capabilities and limitations of digital computers. More precisely, complexity theory explores the problems solvable within given bounds on the time/memory asymptotically granted for their algorithmic solution; and computability theory investigates the case without (or 'infinite') such bounds. Formally, a set L of finite (w.l.o.g. binary) strings amounts to the *decision problem* of determining, for any given finite binary string $\bar{x} \in \{0,1\}^* := \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, whether it belongs to L or not; and this problem is considered *decidable* (in asymptotic time $t(n)$ and space $s(n)$, where $s,t : \mathbb{N} \to \mathbb{N}$ denote functions) if some Turing machine can, for every $\bar{x} \in \{0,1\}^n$ and every $n \in \mathbb{N}$ and after $\mathcal{O}\big(t(n)\big)$ steps and using at most $\mathcal{O}\big(s(n)\big)$ bits of memory, correctly report which of $\bar{x} \in L$ or $\bar{x} \notin L$ holds. Note that there are at most countably many algorithms but uncountably many subsets L of $\{0,1\}^*$; hence most problems are in fact undecidable.

$\mathcal{P}$ is by definition the class of *polynomial-time* decision problems i.e. those solvable within time $\mathcal{O}(n^k)$ for some $k \in \mathbb{N}$; similarly EXP for problems solvable in time $2^{\mathcal{O}(n^k)}$, $k \in \mathbb{N}$. PSPACE denotes the class of those solvable within polynomial space. And $\mathcal{NP}$ consists of those decision problems admitting a polynomial-time *verification*, i.e. problems of the form

$$L \quad = \quad \big\{\vec{x} \in \{0,1\}^n \;\big|\; n \in \mathbb{N},\ \exists \vec{y} \in \{0,1\}^{n^k} : (\vec{x},\vec{y}) \in P\big\}$$

with $k \in \mathbb{N}$ and $P \in \mathcal{P}$. Obviously, $\mathcal{P} \subseteq \mathcal{NP}$; and $\mathcal{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP}$ are not too hard to see as well. It remains a big open challenge which equalities holds. In any case, the Cook-Levin Theorem shows the *Boolean satisfiability problem* to be a hardest one in the class $\mathcal{NP}$: SAT =

$$\big\{\langle\varphi\rangle \;\big|\; m \in \mathbb{N}, \varphi \ m\text{-variate Boolean formula}, \exists y_1,\dots,y_m \in \{0,1\} : \varphi(y_1,\dots,y_m) = 1\big\}$$

is easily verified in time polynomial in the length of the binary encoding $\langle\varphi\rangle$ of φ; and every other problem $L \in \mathcal{NP}$ can be *reduced in polynomial-time* to SAT (written "$L \preceq$ SAT") in the following sense: There exists a function $f : \{0,1\}^* \to \{0,1\}^*$ computable in polynomial time such that it holds: $\vec{x} \in L \Leftrightarrow f(\vec{x}) \in$ SAT. In particular if SAT $\in \mathcal{NP}$ turns out decidable in polynomial time, then so does entire $\mathcal{NP}$. For further reading, we refer to [Papa94].

1.3 Satisfiability in Geometric Quantum Logic

As opposed to the synthetic view, we focus on the explicit modular ortholattices of subspaces of $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively (and are thus perhaps even closer to the origins of quantum logic).

Definition 1 *Fix $d \in \mathbb{N}$ and let $\mathbb{F} \subseteq \mathbb{C}$ denote a field (popular cases being e.g. $\mathbb{F} = \mathbb{C}$ itself, $\mathbb{F} = \mathbb{A}$ algebraic numbers, $\mathbb{F} = \mathbb{R}$ real numbers, $\mathbb{F} = \mathbb{Q}$ rationals, and $\mathbb{F} = \mathbb{A} \cap \mathbb{R}$ algebraic reals).*

a) The quantum logic *of $\mathbb{F}^d$ consists of the set $\mathrm{Gr}(\mathbb{F}^d) := \bigcup_{i=0}^{d} \mathrm{Gr}_i(\mathbb{F}^d)$ of all subspaces of $\mathbb{F}^d$ of dimension† $0 \le i \le d$ (i.e. the Grassmannian) equipped with the connectives*

$$\neg : \mathrm{Gr}_i(\mathbb{F}^d) \to \mathrm{Gr}_{d-i}(\mathbb{F}^d), \quad P \mapsto P^\perp := \{\vec{y} \in \mathbb{F}^d : 0 \overset{!}{=} \langle \vec{y}, \vec{x}\rangle := \sum_j y_j^* \cdot x_j \ \forall \vec{x} \in p\}$$

$$\wedge, \vee : \mathrm{Gr}(\mathbb{F}^d) \times \mathrm{Gr}(\mathbb{F}^d) \to \mathrm{Gr}(\mathbb{F}^d), \ \ \wedge : (P,Q) \mapsto P \cap Q, \ \ \vee : (P,Q) \mapsto P + Q.$$

b) A quantum logic formula *f over variables $X_1, \ldots, X_n$ is a well-formed term over X_i, $\neg$, $\wedge$, and $\vee$. (We sometimes write $f(\vec{X})$ to emphasize the role of the variables $(X_1, \ldots, X_n) =: \vec{X}$.)*

c) For a fixed ortholattice $\mathcal{L}$, a formula $f(X_1, \ldots, X_n)$ gives rise to a mapping $f : \mathcal{L}^n \to \mathcal{L}$ via $(Y_1, \ldots, Y_n) \mapsto f(Y_1, \ldots, Y_n)$. In case $f(\vec{Y}) = 1$, we say that $f(\vec{Y})$ evaluates to `true` *(over $\mathcal{L}$); $f(\vec{Y}) \neq 0$ means that $f(\vec{Y})$ is* weakly true *(i.e. not* `false`*).*

d) A formula $f(X_1, \ldots, X_n)$ is (strongly) satisfiable *over $\mathcal{L}$ if there exist $Y_1, \ldots, Y_n \in \mathcal{L}$ such that $f(\vec{Y}) = 1$. It is* weakly *satisfiable if $f(\vec{Y}) \neq 0$ for some $Y_1, \ldots, Y_n \in \mathcal{L}$.*

1.4 Blum-Shub-Smale (BSS) Model of Real Number Computation

Note that the Turing machine is inherently limited to computability and complexity considerations over discrete structures like $\{0,1\}^*$ or, encoded in binary, integers $\mathbb{N}$ and fractions thereof: $\mathbb{Q}$. For investigations involving algebraic numbers, BLUM, SHUB, and SMALE have proposed a generalized abstract machine (which had, independently and under the name real-RAM, been underlying most algorithms in Computational Geometry) capable

†We adopt the affine notion of dimension because it coincides with the height of the induced lattice in contrast to the projective dimension

of reading, storing, comparing, and arithmetically operating on real numbers exactly and in on step each [BSS89]. This gives rise to the class $\mathcal{P}^0_{\mathbb{R}}$ of real problems—i.e. subsets of $\mathbb{R}^* = \bigcup_{n\in\mathbb{N}} \mathbb{R}^n$—decidable by a BSS machine in time polynomial in n. (Superscript "0" indicates that, as opposed to the prevalent convention, our BSS programs may use no pre-stored constants other than 0 and 1.) Now $\mathcal{NP}^0_{\mathbb{R}}$ is defined as above as the class of real problems polynomial-time verifiable in the sense that they are of the form $\{\vec{x} \in \mathbb{R}^n \mid n \in \mathbb{N},\ \exists \vec{y} \in \mathbb{R}^{n^k} : (\vec{x}, \vec{y}) \in \mathbb{P}\}$ with $k \in \mathbb{N}$ and $\mathbb{P} \in \mathcal{P}^0_{\mathbb{R}}$. And the question "$\mathcal{P}^0_{\mathbb{R}} = \mathcal{NP}^0_{\mathbb{R}}$?" has turned out as as notorious as its original [FoKo00]. Note that, as opposed to the discrete case, the BSS-decidability of all problems in $\mathcal{NP}^0_{\mathbb{R}}$ is not obvious: Naively, there are uncountably many putative witnesses $\vec{y}$ to check for $(\vec{x}, \vec{y}) \in \mathbb{P}$. In fact an effectivization of TARSKI's quantifier elimination comes into play here; and the stronger statement "$\mathcal{NP}^0_{\mathbb{R}} \subseteq \mathsf{EXP}^0_{\mathbb{R}}$" requires even more sophisticated arguments. For later reference, we record here the following

Fact 2 *a) Given (the entries of) real $n \times n$-matrices A, B, a BSS machine executing Gaußian elimination can in time $\mathcal{O}(n^3)$ calculate matrices $C_\wedge, C_\vee, C_\neg \in \mathbb{R}^{n\times n}$ with* $\mathrm{range}(C_\wedge) = \mathrm{range}(A) \wedge \mathrm{range}(B)$, $\mathrm{range}(C_\vee) = \mathrm{range}(A) \vee \mathrm{range}(B)$, *and* $\mathrm{range}(C_\neg) = \neg\,\mathrm{range}(A)$.

b) The following problem is a BSS-counterpart to SAT in the sense that it is $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$-complete: Given a multivariate polynomial of total degree 4 with coefficients from $\{0, \pm1, \pm2\}$, does it admit a real root?

Here $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}}) := \{\mathbb{L} \cap \{0,1\}^* : \mathbb{L} \in \mathcal{NP}^0_{\mathbb{R}}\}$ denotes the restriction of complexity class $\mathcal{NP}^0_{\mathbb{R}}$ to languages over (i.e. to inputs being) bit strings.

The computational problem in Item b) is denoted 4FEAS in [BCSS98]. For a proof cf. e.g. [MeMi97].

It is easy to see that $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$ contains the classical $\mathcal{NP}$; moreover, a collection of highly celebrated results has established $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}}) \subseteq$ PSPACE [Grig88, Cann88, HRS90, Rene92]. As of today, this remains the best known upper bound—although it is believed far from optimal regarding that both the Turing and BSS polynomial hierarchy lie within PSPACE as well [CuGr97].

2 Results

Since $\mathrm{Gr}(\mathbb{F}^1) = \{0, 1\}$, the classical (i.e. Boolean) satisfiability problem amounts to (weak or, equivalently, strong) satisfiability over 1D quantum logic. We extend the Cook-Levin Theorem ("SAT is $\mathcal{NP}$-complete") to the 2D case:

Theorem 3 *Let $\mathbb{F}^2$ denote any 2-dimensional inner product space.*

a) [Weak] satisfiability over $\mathrm{Gr}(\mathbb{F}^2)$ of a given formula f is in $\mathcal{NP}$, i.e. can be verified in polynomial time.

b) Boolean satisfiability is polynomial-time reducible to [weak] 2D satisfiability, i.e. is $\mathcal{NP}$-hard.

Naively, there are infinitely many putative witnesses $Y_1, \ldots, Y_n \in \mathrm{Gr}(\mathbb{F}^2)$ to make f evaluate to `true` [non-`false`] in Claim a). However in the 2D case, the sub-ortholattice spanned by any such an assignment is embeddable into $\mathcal{MO}_n$, the height 2 ortholattice with $2n$ atoms; hence it suffices to verify satisfiability of f over this linear-size lattice. For Claim b), we encode Booleans into $\mathcal{MO}_n$ by requiring Y_i, Y_j to pairwise commute: formula $f(Y_1, \ldots, Y_n)$ is satisfiable over $\{0, 1\}$ iff $f(Y_1, \ldots, Y_n) \wedge \bigwedge_{j<j} C(Y_i, Y_j)$ is satisfiable over $\mathrm{Gr}(\mathbb{F}^d)$, independent of d.

2.1 Three and Higher (but fixed) Dimension

Here, quantum satisfiability turns out as $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$-complete:

Theorem 4 *Fix $d \geq 3$ and let $\mathbb{F}$ denote one of the fields $\mathbb{C}, \mathbb{R}, \mathbb{A}$, or $\mathbb{A} \cap \mathbb{R}$.*

a) [*Weak*] *satisfiability over $\mathrm{Gr}(\mathbb{F}^d)$ of a given formula f can be verified by a BSS machine in polynomial time, i.e. is in $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$.*

b) 4FEAS is $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$-complete, i.e. polynomial-time reducible to [*weak*] *satisfiability over $\mathrm{Gr}(\mathbb{F}^d)$.*

Item a) is an improvement over The mere decidability observed in [DHMW05, SECTION 3] based on Tarski's quantifier elimination. The authors there also applied Tarski-Seidenberg to conclude that [weak] satisfiability over $\mathrm{Gr}(\mathbb{C}^d)$ is equivalent to [weak] satisfiability over $\mathrm{Gr}(\mathbb{A}^d)$; similarly for $\mathrm{Gr}(\mathbb{R}^d)$ and $\mathrm{Gr}\left((\mathbb{A} \cap \mathbb{R})^d\right)$.

Claim a) now follows from Fact 2a), i.e., by verifying $f\left(\mathrm{range}(A_1), \ldots, \mathrm{range}(A_n)\right) = 1$ [$\neq 0$] for appropriate (real and imaginary parts of) $d \times d$-matrices $A_1, \ldots, A_n$.

For Item b) we combine Fact 2b) with the well-known *von Staudt* embedding of the regular ring $\mathbb{F}$ into the continuous geometry $\mathrm{Gr}(\mathbb{F}^3)$ [Neum60]. In case $\mathbb{F} = \mathbb{C}$, complex conjugation—and in particular the condition for the sought root to be real—can be expressed as quantum logic formula as well.

The equivalence (in the sense of mutual polynomial-time reducibility) between weak and strong satisfiability in *fixed* dimension is based on techniques from [DHMW05, Hagg07, Hagg09], collected in the following

Fact 5 *For an n-variate formula f, let*

$$\mathrm{maxdim}_{\mathbb{F}}(f, d) \quad := \quad \max\left\{ \dim f(\vec{X}) : X_1, \ldots, X_n \in \mathrm{Gr}(\mathbb{F}^d) \right\} .$$

a) If formulas f and g have no variables in common, then

$$\mathrm{maxdim}_{\mathbb{F}}(f \vee g, d) = \max\left\{ d, \mathrm{maxdim}_{\mathbb{F}}(f, d) + \mathrm{maxdim}_{\mathbb{F}}(g, d) \right\} .$$

b) For formulas $f(\vec{X})$ and $g(\vec{Y})$ let the restriction $f(\vec{X})|_{g(\vec{Y})}$ be defined by replacing in f each X_i with $X_i \wedge g(\vec{Y})$ and each $\neg X_i$ with $\neg\left(X_i \wedge g(\vec{Y})\right) \wedge g(\vec{Y})$, where w.l.o.g. $f(\vec{X}, \neg\vec{X})$ is presumed free of negations (de Morgan). Then it holds $\mathrm{maxdim}_{\mathbb{F}}\left(f|_g, d\right) = \mathrm{maxdim}_{\mathbb{F}}\left(f, \mathrm{maxdim}_{\mathbb{F}}(g, d)\right)$.

c) To any $k \in \mathbb{N}$, there exists a formula $\psi_k(\vec{X})$ with $\mathrm{maxdim}_{\mathbb{F}}(\psi_k, d) = \lfloor d/k \rfloor$.

An alternative approach can build on the fact that, given a von Neumann d-frame in $\mathrm{Gr}(\mathbb{F}^d)$, one can define a discriminator [HMR05].

From Theorem 4 we conclude

Corollary 6 *Fix $d, k \geq 3$. Every formula f can in polynomial time be converted into another formula g such that f is [weakly] satisfiable over $\mathrm{Gr}(\mathbb{F}^d)$ iff g is [weakly] satisfiable over $\mathrm{Gr}(\mathbb{F}^k)$.*

2.2 Indefinitely-finite Dimensions

As common in computer science, we use the wildcard $*$ to denote an indefinite but finite dimension:

Theorem 7 *Let $\mathbb{F}$ be as in Theorem 4. Call formula f [weakly] satisfiable over $\mathrm{Gr}(\mathbb{F}^*)$ iff it is [weakly] satisfiable over $\mathrm{Gr}(\mathbb{F}^d)$ for some $d \in \mathbb{N}$.*

a) Weak satisfiability over $\mathrm{Gr}(\mathbb{F}^*)$ of a given formula is in $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$ and in particular decidable in PSPACE.

b) Strong satisfiability over $\mathrm{Gr}(\mathbb{F}^*)$ of a given formula is $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$-hard.

c) Strong satisfiability over $\mathrm{Gr}(\mathbb{F}^*)$ of a given formula is semi-decidable.

We have the following

Lemma 8 *Let formula $f(X_1, \ldots, X_n)$ be weakly satisfiable over $\mathrm{Gr}(\mathbb{F}^*)$. Then it is also weakly satisfiable over $\mathrm{Gr}(\mathbb{F}^{n\ell})$ where $\ell = |f|$ denotes the syntactic length of f.*

In particular, the proof of Theorem 4a) carries over to matrices of size $d \times d$ with $d := n \cdot |f|$ polynomial in the input size, thus yielding Theorem 7a). For Theorem 7b) we extend the proof of Theorem 4b) to embed the matrix ring $\mathbb{F}^{d \times d}$ with adjunction as involution into $\mathrm{Gr}(\mathbb{F}^{3d})$; and then encode the additional requirement of all matrices to be symmetric and pairwise commuting: By the spectral theorem, any satisfying assignment in some dimension $3d$ to the thus obtained quantum logic formula corresponds to a d-fold direct product of roots to the original polynomial. Theorem 7c) follows from iteratively trying $d = 1, 2, 3 \ldots$ and applying Theorem 4b) each.

Question 9 *a) Is weak satisfiability over $\mathrm{Gr}(\mathbb{F}^*)$ $\mathcal{NP}$-hard or even $\mathrm{BP}(\mathcal{NP}^0_{\mathbb{R}})$-hard?*

b) Is strong satisfiability over $\mathrm{Gr}(\mathbb{F}^*)$ decidable?

2.3 First-Order Quantum Logic

In dimensions > 1, the connective "$\vee$" behaves like Boolean disjunction for weak truth; but "$\wedge$" is different from Boolean conjunction: $X \wedge Y \neq 0$ may well fail for both $X, Y \neq 0$. Dually, "$\wedge$" behaves in a Boolean way for strong truth but "$\vee$" does not. Furthermore, Boolean negation is different from complement: $X \neq 0 \underset{\not\Rightarrow}{\Leftarrow} \neg X = 0$.

Using Fact 5, Boolean semantics can be expressed in existentially quantified quantum formulas over any fixed dimension; but over indefinitely finite dimensions, the expressiveness of quantum logic becomes skew to that of Boolean logic:

Example 10 *a) For every $d \in \mathbb{N}$ and every $X \in \mathrm{Gr}(\mathbb{F}^d)$, it holds: $X \in \{0,1\}$ iff $C(X,Y) = 1$ for all $Y \in \mathrm{Gr}(\mathbb{F}^d)$.*

b) There is no formula $f(X,\vec{Y})$ such that for all $d \in \mathbb{N}$ it holds: $X \in \{0,1\} \quad \Leftrightarrow \exists\vec{Y} \in \mathrm{Gr}(\mathbb{F}^d) : f(X;\vec{Y}) = 1$; similarly for $f(X;\vec{Y}) \neq 0$.

In other words: universal quantifi cation strictly adds to the expressiveness of quantum logic over $\mathrm{Gr}(\mathbb{F}^*)$; in particular, not every formula is equivalent to a positive primitive one.

This example suggests to study first-order quantum logic, that is, quantum logic formulas with both existential and universal quantifi ers ranging over $\mathrm{Gr}(\mathbb{F}^*)$:

Definition 11 *a) Let $f(\vec{X}_1,\ldots,\vec{X}_k)$ denote a formula in $n_1+\cdots+n_k$ variables. Then the expression*

$$\exists\vec{X}_1\,\forall\vec{X}_2\,\exists\vec{X}_3\ \ldots\ \mathrm{Q}_k\,\vec{X}_k : \quad f(\vec{X}_1,\ldots,\vec{X}_k)$$

is called a Σ_k-formula, where Q_k denotes "$\forall$" for k even and "$\exists$" in case k is odd. Similarly, a Π_k-formula starts with a universal quantifier followed by $k-1$ alternating quantifiers.

b) We say that this Σ_k-formula is `true` *(strongly valid) over $\mathrm{Gr}(\mathbb{F}^d)$ if there exist $X_{1,1},\ldots,X_{1,n_1} \in \mathrm{Gr}(\mathbb{F}^d)$ such that for all $X_{2,1},\ldots,X_{2,n_2} \in \mathrm{Gr}(\mathbb{F}^d)$ there exist $\cdots\cdots$ such that $f(\vec{X}_1,\ldots,\vec{X}_k) = 1$. Similarly for Π_k formulas and for weak validity.*

c) A Σ_k-formula is strongly/weakly valid over $\mathrm{Gr}(\mathbb{F}^*)$ if it is over $\mathrm{Gr}(\mathbb{F}^d)$ for some $d \in \mathbb{N}$. A Π_k-formula is strongly/weakly valid over $\mathrm{Gr}(\mathbb{F}^*)$ if it is over $\mathrm{Gr}(\mathbb{F}^d)$ for every $d \in \mathbb{N}$.

Note that the negation of a weakly valid Σ_k formula is a strongly valid Π_k formula; and the negation of a weakly valid Π_k formula is a strongly valid Σ_k formula. Based on [Lips74], we obtain

Theorem 12 *Strong validity over $\mathrm{Gr}(\mathbb{C}^*)$ of a given Σ_4-formula is generally undecidable to a Turing machine.*

References

[ACP04] V.A. ADAMYAN, C.S. CALUDE, B.S. PAVLOV: "Transcending the limits of Turing computability", pp.119–138 in *Quantum Information Complexity. Proceedings of Meijo Winter School 2003* (T. Hida, K. Saito, S. Si, edts), World Scientifi c, Singapore (2004).

[BCSS98] L. BLUM, F. CUCKER, M. SHUB, S. SMALE: "*Complexity and Real Computation*", Springer (1998).

[Bera85] L. BERAN: "*Orthomodular Lattices*", Springer (1985).

[BiVN36] G. BIRKHOFF, J. VON NEUMANN: "The Logic of Quantum Mechanics", pp.823–843 in *Annals of Mathematics* vol.**37:4** (1936).

[BrKa73] G. BRUNS, G. KALMBACH: "Some remarks on free orthomodular lattices", pp.397–408 in *Proc. Houston Lattice Theory Conference* (1973).

[BSS89] L. BLUM, M. SHUB, S. SMALE: "On a Theory of Computation and Complexity over the Real Numbers: $\mathcal{NP}$-Completeness, Recursive Functions, and Universal Machines", pp.1–46 in *Bulletin of the American Mathematical Society* (AMS Bulletin) vol.**21** (1989).

[Cann88] J. CANNY: "Some Algebraic and Geometric Computations in PSPACE", pp.460–467 in *Proc. 20th annual ACM Symposium on Theory of Computing* (SToC 1988).

[CDS01] C.S. CALUDE, M.J. DINNEEN, K. SVOZIL: "Reflections on Quantum Computing", pp.35–37 in *Complexity* vol.**6(1)**, Wiley (2001).

[CMW00] B. COECKE, D. MOORE, A. WILCE (Editors): "Current research in operational quantum logic", vol.**111** in Series *Fundamental Theories of Physics*, Springer (2000).

[CuGr97] F. CUCKER, D. GRIGORIEV: "On the Power of Real Turing Machines over Binary Inputs", pp.243–254 in *SIAM Journal on Computing* vol.**26:1** (1997).

[DHMW05] J.M. DUNN, T.J. HAGGE, L.S. MOSS, Z. WANG: "Quantum Logic as Motivated by Quantum Computing", pp.353–359 in *Journal of Symbolic Logic* vol.**70:2** (2005).

[EGL07] K. ENGESSER, D.M. GABBAY, D. LEHMANN (Editors): "*Handbook of Quantum Logic and Quantum Structures*", Elsevier (2007).

[FoKo00] H. FOURNIER, P. KOIRAN: "Lower Bounds Are Not Easier over the Reals", pp.832–843 in *Proc. 27th Int. Colloquium on Automata, Languages and Programming* (ICALP 2000), Springer LNCS vol.**1853**.

[Grig88] D.Y. GRIGORIEV: "Complexity of Deciding Tarski Algebra", pp.65–108 in *Journal of Symbolic Computation* vol.**5** (1988).

[Hagg07] T.J. HAGGE: "$QL(\mathbb{C}^n)$ Determines n", pp.1194–1196 in *Journal of Symbolic Logic* vol.**72:4** (2007).

[Hagg09] T.J. HAGGE: "Quantum logic on finite dimensional Hilbert spaces", presentation at the *Workshop on Quantum Logic Inspired by Quantum Computation* (M. Dunn, L. Moss, Z. Wang organisers), Indiana University, May 2009; `http://www.indiana.edu/~iulg/qliqc/hagge_qliqc_talk.pdf`

[Herr87] C. HERRMANN: "The free orthomodular word problem is solved" (review of the paper by G. KALMBACH), in *Zentralblatt für Mathematik* **Zbl 0585.06004** and *Mathematical Reviews* **MR 87K:06023** (1987).

[Herr09] C. HERRMANN: "Complemented Modular Lattices with Involution and Orthogonal Geometry", pp.339–364 in *Algebra Universalis* vol.**61** (2009).

[Herr10] C. HERRMANN: "On the equational theory of projection lattices of finite von-Neumann factors", *J. Symbolic Logic* (in print).

[HMR05] C. HERRMANN, F. MICOL, M.S. RODDY: "On n-distributive Modular Ortholattices", pp.143–147 in *Algebra Universalis* vol.**53** (2005).

[HRS90] J. HEINTZ, M.-F. ROY, P. SOLERNÓ: "Sur la complexité du principe de Tarski–Seidenberg", pp.101–126 in *Bull. Soc. Math. France* vol.**118** (1990).

[Kalm83] G. KALMBACH: "*Orthomodular Lattices*", Academic Press (1983).

[Kieu03] T. KIEU: "Quantum Algorithm for Hilbert's Tenth Problem", pp.1461–1478 in *International Journal of Theoretical Physics* **42**, Springer (2003).

[Lips74] L. LIPSHITZ: "The Undecidability of the Word Problems for Projective Geometries and Modular Lattices", 171–180 in *Transactions of the American Mathematical Society* vol.**193** (1974).

[Mack63] G.W. MACKEY: "Mathematical Foundations of Quantum Mechanics", *Mathematical Physics Monograph Series* (1963).

[Maye98] R. MAYET: "Some Characterizations of the Underlying Division Ring of a Hilbert Lattice by Automorphisms", pp.109–114 in *International Journal of Theoretical Physics* vol.**37:1** (1998).

[Maye07] R. MAYET: "Equations Holding in Hilbert Spaces", pp.1257–1287 in *International Journal of Theoretical Physics* vol.**45:7** (2007).

[Megi09] N.D. MEGILL: "Quantum Logic Explorer Home Page" at *Metamath*; `http://us.metamath.org/qlegif/mmql.html` (2009).

[MeMi97] K. MEER, C. MICHAUX: "A Survey on Real Structural Complexity Theory", pp.113–148 in *Bulletin of the Belgian Mathematical Society* vol.**4** (1997).

[Neum55] J.VON NEUMANN: "*Mathematical Foundations of Quantum Mechanics*", Princeton University Press (1955).

[Neum60] J.VON NEUMANN: "*Continuous Geometry*", Princeton Math.Series no.**25** (1960).

[Papa94] C. PAPADIMITRIOU: "*Computational Complexity*", Addison-Wesley (1994).

[PM07a] M. PAVIČIĆ, N.D. MEGILL: "Quantum Logic and Quantum Computation", pp.755–792 in [EGL07].

[Pyka00] J. PYKACZ: "Quantum Logic as a Basis for Computations", pp.839–850 in *International Journal of Theoretical Physics* vol.**39:3** (2000).

[Rene92] J. RENEGARD: "On the Computational Complexity and Geometry of the First-order Theory of the Reals", pp.255–352 in *Journal of Symbolic Computation* vol.**13:3** (1992).

[Svoz98] K. SVOZIL: "*Quantum Logic*", Springer (1998).

[Wilc09] A. WILCE: "Four and a Half Axioms for Finite Dimensional Quantum Mechanics", preprint `arXiv.org/abs/0912.5530` (2009).

[Ying05] M. YING: "A Theory of Computation based on Quantum Logic I", pp.134–207 in *Theoretical Computer Science* vol.**344** (2005).

[Zieg05] M. ZIEGLER: "Computational Power of Infinite Quantum Parallelism", pp.2059–2071 in *International Journal of Theoretical Physics* vol.**44:11** (2005).