Complementarity in categorical quantum mechanics

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Abstract

We relate notions of complementarity in three layers of quantum mechanics: (i) von Neumann algebras, (ii) Hilbert spaces, and (iii) orthomodular lattices. Taking a more general categorical perspective of which the above are instances, we consider dagger monoidal kernel categories for (ii), so that (i) become (sub)endohomsets and (iii) become subobject lattices. By developing a 'point-free' definition of copyability we link (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean subalgebras.

1 Introduction

Complementarity is a supporting pillar of the Copenhagen interpretation of quantum mechanics. Unfortunately, Bohr's own formulation of the principle remained imprecise and flexible [19], and to date there is no consensus on a clear mathematical definition. Here, we understand it, roughly, to mean that complete knowledge of a quantum system can only be attained through examining all of its possible classical subsystems [8]. Notice that, perhaps unlike Bohr's own, this interpretation concerns *all* classical contexts, leading to a weaker notion of binary complementarity than usual. To avoid clashes with the various existing terminologies and their connotations, and to emphasize the distinction between talking about *two* (totally) incompatible classical contexts (as Bohr typically did), and mentioning all of them, we will speak of *partially complementary* classical contexts only when considering *two* of them. Only taken all together, (pairwise partially complementary) classical contexts give complete information, and we call them *completely complementary*. This paper considers instances of this interpretation of complementarity with regard to three aspects of quantum mechanics.

(i) The observables of a quantum system form a von Neumann algebra. In this setting, complete complementarity is customarily taken to mean that one has to look at all commutative von Neumann subalgebras [14, 20, 3].

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(ii) The states of a quantum system are unit vectors in a Hilbert space, which can be coordinatized by choosing any orthonormal basis. Here, complete complementarity may be interpreted as saying that it takes measurements in all possible orthonormal bases (of many identical copies of a system) to determine its state (perfectly) [22], as in quantum state tomography [15].

(iii) The measurable properties of a quantum system form an orthomodular lattice. Complete complementarity translates to this perspective as stating that the lattice structure is determined by all Boolean sublattices [17, 11].

In fact, we will take a more general perspective, as all three layers, separately, have recently been studied categorically.

(i) The set of commutative von Neumann subalgebras of a von Neumann algebra gives rise to a topos of set-valued functors, whose intuitionistic internal logic sheds light on the the original noncommutative algebra in so far as complete complementarity is concerned [10, 7].

(ii) The category of Hilbert spaces can be abstracted to a dagger monoidal category, in which much of quantum mechanics can still be formulated [1]. In this framework, orthonormal bases are characterized as so-called classical structures [5, 6, 2].

(iii) Orthomodular lattices can be obtained as kernel subobjects in a so-called dagger kernel category [9]. This paper considers Boolean sublattices systematically, in the tradition of *e.g.* [12].

We will take the view that of these three layers, (ii) is the primitive one, which the others derive from. Indeed, our main results are in categories that are simultaneously dagger monoidal categories and dagger kernel categories. We give definitions of partial and complete complementarity for (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean sublattices of the orthomodular lattice of kernels. By developing a notion of copyability, we obtain a bijective correspondence between partially complementary classical structures and partially complementary Boolean sublattices. Also, we characterize categorically what partially complementary commutative von Neumann subalgebras correspond to in terms of classical structures in the category of Hilbert spaces. The plan of the paper is as follows: Sections 2, 3 and 4 study layers (ii), (iii) and (i) respectively. Conclusions are then drawn in Section 5. The author is grateful to Samson Abramsky, Ross Duncan, Klaas Landsman, and Jamie Vicary for useful pointers and discussions.

2 Classical structures

Definition 1 A *classical structure* in a dagger symmetric monoidal category $\mathbf{D}$ is a commutative semigroup $\delta\colon X \to X \otimes X$ that satisfies $\delta^\dagger \circ \delta = \mathrm{id}$ and the following so-called H*-axiom: there is an involution $*\colon \mathbf{D}(I,X)^{\mathrm{op}} \to \mathbf{D}(I,X)$ such that $\delta^\dagger \circ (x^* \otimes \mathrm{id}) = (x^\dagger \otimes \mathrm{id}) \circ \delta$.

This terse definition suffices for this paper, because we will not explicitly use much of a classical structure except its type and the fact that it is dagger monic; for more information we refer to a forthcoming article [2].

The goal of this section is to find out when kernels $k\colon K \to X$ are 'compatible' with a given classical structure. To do so, we develop a notion of copyability that has to be 'point-free' because K is typically not the monoidal unit I.

2.1 Kernels and tensor products

Fix a category $\mathbf{D}$, and assume it to be a dagger symmetric monoidal category [1] and a dagger kernel category [9] simultaneously, which additionally satisfies

$$\ker(f) \otimes \ker(g) = \ker(f \otimes \mathrm{id}) \wedge \ker(\mathrm{id} \otimes g)$$

for all morphisms f and g.The categories $\mathbf{Hilb}$ and $\mathbf{Rel}$ both satisfy the above relationship between tensor products and kernels. Some coherence properties follow easily from the assumptions:

$$\begin{aligned} \ker(f) \otimes 0 &= 0, & 0 \otimes \ker(g) &= 0, \\ \ker(f) \otimes \mathrm{id} &= \ker(f \otimes \mathrm{id}), & \mathrm{id} \otimes \ker(g) &= \ker(\mathrm{id} \otimes g), \\ \ker(f) \otimes \mathrm{id} = 0 &\Leftrightarrow \ker(f) = 0, & \mathrm{id} \otimes \ker(g) = 0 &\Leftrightarrow \ker(g) = 0. \end{aligned}$$

Notice that requiring $\ker(f \otimes g) = \ker(f) \otimes \ker(g)$ would have been too strong, for then $\ker(f) \otimes \mathrm{id} = \ker(f) \otimes \ker(0) = \ker(f \otimes 0) = \ker(0) = \mathrm{id}$ for any f. Nevertheless, one does always have $\ker(f \otimes f) = \ker(f) \otimes \ker(f)$; we leave the proof to the reader because of space restrictions.

2.2 Copyability

Throughout this section we fix a classical structure $\delta\colon X \to X \otimes X$.

Definition 2 A morphism $k\colon K \to X$ is called *copyable* (along δ) when

$$\delta \circ P(k) = P(k \otimes k) \circ \delta,$$

where we write $P(k) = k \circ k^\dagger$.

This definition of copyability relates to copyability of vectors as used in [4] as follows. For a unit vector x in $H \in \mathbf{Hilb}$, the following are equivalent:

- the morphism $\mathbb{C} \to H$ defined by $1 \mapsto x$ is copyable;
- there is a phase $z \in \mathbb{C}$ with $|z| = 1$ such that $\delta(x) = z \cdot (x \otimes x)$;
- there is a unit vector $x' \in H$ with $P(x) = P(x')$ and $\delta(x') = x' \otimes x'$.

Example 3 In any dagger kernel category with tensor products satisfying the coherence set out in Section 2.1, zero morphisms and identity morphisms are always copyable. These two kernels are called the *trivial* kernels.

Example 4 In the category $\mathbf{Hilb}$ of Hilbert spaces, a classical structure δ corresponds to the choice of an orthonormal basis (e_i) [2], whereas a kernel corresponds to a (closed) linear subspace [9]. A kernel is copyable if and only if it is the linear span of a subset of the orthonormal basis.

Example 5 In the category $\mathbf{Rel}$ of sets and relations, a classical structure δ on X corresponds to (a disjoint union of) Abelian group structure(s) on X [16], and a kernel corresponds to a subset $K \subseteq X$ [9]. Unfolding definitions, we find that a kernel is copyable if and only if $x \in K \wedge y \in K \Leftrightarrow x \cdot y \in K$. One direction of this equivalence implies that K is a subsemigroup. Fixing $k \in K$, we see that for any $x \in X$ there is $y = x^{-1} \cdot k$ such that $x \cdot y \in K$. Therefore, the other direction implies that $x \in K$. That is, $K = X$. We conclude that the only copyable kernels in $\mathbf{Rel}$ are the trivial ones.

As it turns out, copyability is an algebraic formulation for the existence of classical substructure, as follows.

Proposition 6 *A dagger monic k is copyable if and only if there is a (unique) morphism δ_k making the following diagram commute:*

$$\begin{array}{ccccc} X & \xrightarrow{k^\dagger} & K & \xrightarrow{k} & X \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_k} & & \downarrow{\scriptstyle \delta} \\ X \otimes X & \xrightarrow[k^\dagger \otimes k^\dagger]{} & K \otimes K & \xrightarrow[k \otimes k]{} & X \otimes X. \end{array}$$

If k is a copyable dagger monic, δ_k is a classical structure.

PROOF The first claim is a matter of unfolding definitions. For the second claim, we say that f is a dagger retract of g if there are dagger monics a and b with $b \circ f = g \circ a$ and $b^\dagger \circ g = f \circ a^\dagger$. Notice that if f and f' are both dagger retracts of g (along the same a and b), then $f = b^\dagger \circ b \circ f = b^\dagger \circ g \circ a = f' \circ a^\dagger \circ a = f'$.

Now, if k is a copyable dagger monic, then it follows from Proposition 6 that δ_k is a dagger retract of δ, and $\delta_k^\dagger$ is a dagger retract of $\delta^\dagger$. Therefore, δ_k is associative, commutative, and is dagger monic. For example, to verify commutativity, notice that $\gamma_k \colon K \otimes K \to K \otimes K$ is a dagger retract of $\gamma \colon X \otimes X \to X \otimes X$. Since dagger retracts compose, this means that $\gamma_k \circ \delta_k$ and δ_k are both dagger retracts (along the same morphisms) of $\gamma \circ \delta = \delta$. Hence $\gamma_k \circ \delta_k = \delta_k$. The other algebraic properties are verified similarly (including the Frobenius equation). We are left to check the H*-axiom. Let $x \colon I \to K$. Since δ satisfies the H*-axiom, there is $(k \circ x)^* \colon I \to X$ such that $\delta^\dagger \circ ((k \circ x)^* \otimes \mathrm{id}) = ((k \circ x)^\dagger \otimes \mathrm{id}) \circ \delta$. Now put $x^* = k^\dagger \circ (k \circ x)^* \colon I \to K$. Then:

$$\begin{aligned} \delta_k^\dagger \circ (x^* \otimes \mathrm{id}) &= \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ ((k \circ x)^* \otimes \mathrm{id}) \circ k \\ &= k^\dagger \circ \delta^\dagger \circ ((k \circ x)^* \otimes \mathrm{id}) \circ k \\ &= k^\dagger \circ ((k \circ x)^\dagger \otimes \mathrm{id}) \circ \delta \circ k \\ &= (x^\dagger \otimes \mathrm{id}) \circ \delta_k. \end{aligned}$$

Hence δ_k satisfies the H*-axiom, too. □

As a corollary, we find that a dagger monic k is copyable if and only if its domain carries a classical structure δ_k and k is simultaneously a (non-unital) monoid homomorphism and a (non-unital) comonoid homomorphism. It stands to reason to define categories of classical structures to have such morphisms; this also matches [13, 2.4.4]. We end this section with our first definition of partial complementarity.

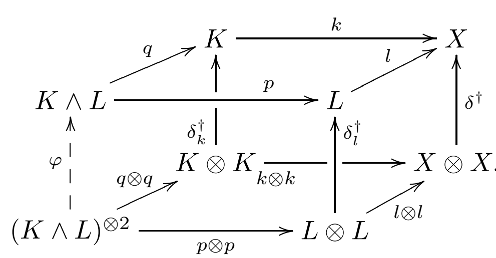
Definition 7 Two classical structures are *partially complementary* if no nontrivial kernel is simultaneously copyable along both.

3 Boolean subalgebras of orthomodular lattices

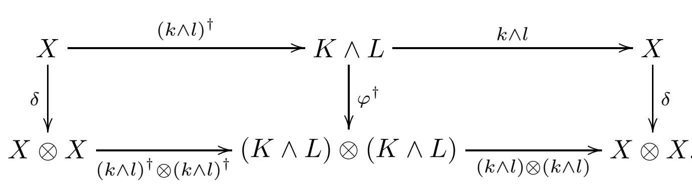
This section concerns level (iii) of the Introduction. We will prove that kernels that are copyable along δ form a Boolean subalgebra of the orthomodular lattice of all kernel subobjects of X.

Lemma 8 *The copyable kernels form a sub-meetsemilattice of* $\mathrm{KSub}(X)$.

PROOF The bottom element 0 is always copyable by Example 3. So we have to prove that if k and l are copyable kernels, then so is $k \wedge l$. Recall that $k \wedge l$ is defined as the pullback. Together with the assumption that k and l are copyable, this means that the top, back, right and bottom face of the following cube commute:



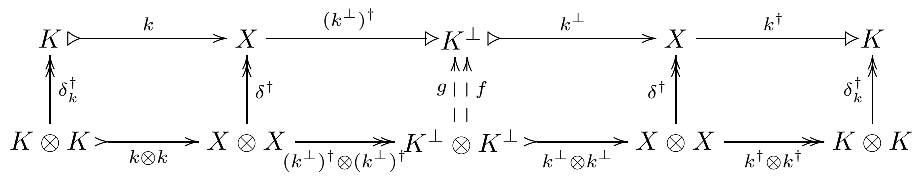
Hence $l \circ \delta_l^\dagger \circ (p \otimes p) = k \circ \delta_k^\dagger \circ (q \otimes q)$. Therefore, by the universal property of pullbacks, there exists a dashed morphism φ making the left and front sides of the above cube commute. Using the fact that p and q are dagger monic, we deduce $\varphi = (k \wedge l)^\dagger \circ \delta^\dagger \circ ((k \wedge l) \otimes (k \wedge l))$. This means that the left square in the following diagram commutes:



The right square is seen to commute analogously—take daggers of all the vertical morphisms in the cube. Therefore the whole rectangle commutes. In other words, $\delta \circ P(k \wedge l) = (P(k \wedge l) \otimes P(k \wedge l)) \circ \delta$, that is, $k \wedge l$ is copyable. $\square$

Lemma 9 *The copyable kernels form an orthocomplemented sublattice of the orthomodular lattice* $\mathrm{KSub}(X)$.

PROOF We have to prove that if k is a copyable kernel, then so is $k^\perp = \ker(k^\dagger)$.



Since k is copyable, we have $k^\dagger \circ \delta^\dagger \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (0 \otimes 0) = 0$, so that the dashed arrow f in the above diagram exists, making the square to its right commute. Since $k^\perp$ is dagger monic, f must equal $\mathrm{coker}(k) \circ \delta^\dagger \circ (\ker(k^\dagger) \otimes \ker(k^\dagger))$.

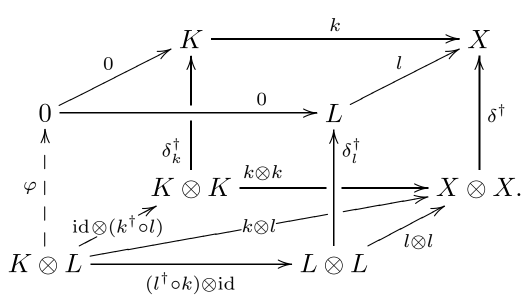
Similarly, it follows from copyability of k that $(k^\perp)^\dagger \circ \delta_k^\dagger \circ (k \otimes k) = 0$, so that the dashed arrow g exists. Since g must be $\mathrm{coker}(k) \circ \delta^\dagger \circ (\mathrm{coker}(k)^\dagger \otimes \mathrm{coker}(k)^\dagger)$, we see that f and g coincide. Hence the rectangle composed of the middle two squares commutes. Taking its dagger yields a commutative diagram showing that $k^\perp$ is copyable. □

Notice that if the classical structure had a unit ε, the previous result would have been impossible if we had additionally demanded $\varepsilon \circ P(k) = \varepsilon$ for k to be copyable, since then $\varepsilon = \varepsilon \circ P(k^\perp) = \varepsilon \circ P(k) \circ P(k^\perp) = \varepsilon \circ P(k \wedge k^\perp) = \varepsilon \circ 0 = 0$. Compare [2].

Lemma 10 *[9, Theorem 1] An orthocomplemented sublattice* L *of* $\mathrm{KSub}(X)$ *is Boolean if and only if* $k \wedge l = 0$ *implies* $l^\dagger \circ k = 0$ *for all* $k, l \in L$. □

Theorem 11 *The copyable kernels form a Boolean subalgebra of the orthomodular lattice* $\mathrm{KSub}(X)$.

PROOF By the previous lemmas, it suffices to prove that if $k \wedge l = 0$ for copyable kernels k and l, then $l^\dagger \circ k = 0$. So let k and l be copyable kernels and suppose $k \wedge l = 0$. Say $k = \ker(f)$ and $l = \ker(g)$. Then $(f \otimes \mathrm{id}) \circ (k \otimes l) = (f \circ k) \otimes l = 0 \otimes l = 0$, so that $k \otimes l \leq \ker(f \otimes \mathrm{id}) = k \otimes \mathrm{id} \leq (k \otimes \mathrm{id}) \wedge (\mathrm{id} \otimes k) = k \otimes k$. Similarly, $k \otimes l \leq l \otimes l$. Therefore the bottom, top, back and right faces of the following cube commute:



The universal property of the pullback formed by the top face yields the dashed morphism φ making the left and front faces commute. Hence $\delta_l^\dagger \circ ((l^\dagger \circ k) \otimes \mathrm{id}) = 0$. But then, as k

and l are copyable:

$$\begin{aligned} l^\dagger \circ k &= l^\dagger \circ \delta^\dagger \circ (k \otimes k) \circ \delta_k \\ &= \delta_l^\dagger \circ (l^\dagger \otimes l^\dagger) \circ (k \otimes k) \circ \delta_k \\ &= \delta_l^\dagger \circ ((l^\dagger \circ k) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (l^\dagger \circ k)) \circ \delta_k = 0. \end{aligned}$$ □

The following definition expresses the standard view in (order-theoretic) quantum logic that Boolean subalgebras of orthomodular lattices are regarded as embodying complete complementarity. It is precisely what is needed to make Theorem 13 true.

Definition 12 Two Boolean subalgebras of an orthomodular lattice are called *partially complementary* when they have trivial intersection.

Theorem 13 *Two classical structures are partially complementary if and only if their collections of copyable kernels are partially complementary.* □

Hence we have linked, fully abstractly, partial complementarity in the order-theoretic sense to partial complementarity in the sense of classical structures.

4 Von Neumann algebras

Finally, this section advances to level (i) of the Introduction. We instantiate the dagger monoidal kernel category $\mathbf{D}$ to be $\mathbf{Hilb}$. For any object $H \in \mathbf{Hilb}$, the endohomset $A = \mathbf{Hilb}(H, H)$ is then a type I von Neumann algebra. At this level, the notion of complete complementarity is formalized by considering all commutative von Neumann subalgebras C of A. We denote the collection of all such subalgebras of A by $\mathcal{C}(A)$. Let us recall some facts about this situation.

(a) The set $\mathrm{Proj}(A) = \{p \in A \mid p^\dagger = p = p^2\}$ of projections is a complete, atomic, atomistic orthomodular lattice [18, p85].

(b) There is an order isomorphism $\mathrm{Proj}(A) \cong \mathrm{KSub}(H)$ [9, Proposition 12].

(c) Any von Neumann algebra is generated by its projections [18, 6.3], so in particular $C = \mathrm{Proj}(C)''$.

(d) Since C is a subalgebra of A, also $\mathrm{Proj}(C)$ is a sublattice of $\mathrm{Proj}(A)$.

(e) Because C is commutative, $\mathrm{Proj}(C)$ is a Boolean algebra [18, 4.16].

The following lemma draws a conclusion of interest from these facts.

Lemma 14 *Commutative von Neumann subalgebras C of $A = \mathbf{Hilb}(H, H)$ are in bijective correspondence with Boolean subalgebras of* $\mathrm{KSub}(H)$. □

We now set out to establish the relation between commutative subalgebras of A and classical structures on H.

Lemma 15 *An orthocomplemented sublattice L of $\mathrm{KSub}(H)$ is Boolean if and only if the following equivalent conditions hold:*

- *there exists a classical structure on the greatest element of L along which every element of L is copyable;*
- *there exists a classical structure on H along which every element of L is copyable.*

PROOF Necessity is established by Theorem 11. For sufficiency, let L be a Boolean sublattice of $\mathrm{KSub}(H)$. Since $\mathrm{KSub}(H)$ is complete by (a) above, $\bigvee L$ exists. By atomicity (a), $\bigvee L$ is completely determined by the set of atoms a_i below it. By definition of atoms, $a_i \wedge a_j = 0$ when $i \neq j$. Because L is Boolean, it follows from Lemma 10 that a_i and a_j are orthogonal. Also, because $\mathbf{Hilb}$ is simply well-pointed, the kernels a_i correspond to one-dimensional subspaces [9, Lemma 11]. That is, the a_i give an orthonormal basis for (the domain of) the greatest element of L (which can be extended to an orthonormal basis of H). This, in turn, induces a classical structure δ on L (or H) [2]. Finally, Example 4 shows that the kernels a_i, and hence all $l \in L$, are copyable along δ. □

Theorem 16 *For the von Neumann algebra $A = \mathbf{Hilb}(H, H)$:*

$$\mathcal{C}(A) \cong \{L \subseteq \mathrm{KSub}(H) \mid L \textit{ orthocomplemented sublattice}, \\ \exists_{\delta\colon 1_L \to 1_L \otimes 1_L} \forall_{l \in L} [l \textit{ copyable along } \delta]\}.$$

PROOF This is just a combination of Lemma 14 and Lemma 15. □

The previous theorem implies that for any classical structure δ on H, there is an induced commutative von Neumann subalgebra $C \in \mathcal{C}(A)$ corresponding to the lattice L of all copyable kernels. The following definition and corollary finish the connections of partial complementarity across the three levels discussed in the Introduction.

Definition 17 Two commutative von Neumann subalgebras of $\mathbf{Hilb}(H, H)$ are *partially complementary* when their intersection is the trivial subalgebra $\{z \cdot \mathrm{id} \mid z \in \mathbb{C}\}$.

Corollary 18 *Two classical structures on an object H in $\mathbf{Hilb}$ are partially complementary if and only if they induce partially complementary commutative von Neumann subalgebras of $\mathbf{Hilb}(H, H)$.* □

Unlike $\mathrm{Proj}(A)$, the sublattice $\mathrm{Proj}(C)$ is not atomic for general $C \in \mathcal{C}(A)$; for a counterexample, take $H = L^2([0, 1])$ and $C = L^\infty([0, 1])$. If this does happen to be the case, for example if we restrict the ambient category $\mathbf{D}$ to that of finite-dimensional Hilbert spaces, we can strengthen the characterization of $\mathcal{C}(A)$ in Theorem 16.

Proposition 19 *For a finite-dimensional Hilbert space H and the von Neumann algebra $A = \mathbf{fdHilb}(H, H)$:*

$$\mathcal{C}(A) \cong \{(\delta_i)_{i \in I} \mid \delta_i, \delta_j \textit{ classical structures, partially complementary when } i \neq j, \\ \exists_{\delta\colon H \to H \otimes H} \forall_i \exists_{k_i\colon \delta_i \to \delta} [k_i \textit{ morphism of classical structures}]\}.$$

Hence $\mathcal{C}(A)$ is isomorphic to the collection of cocones in the category of classical substructures on H that are pairwise partially complementary. □

Notice that the characterization of Proposition 19 above has no need for the cumbersome combinatorial symmetry considerations of [10, 1.4.5].

5 Concluding remarks

Observing the similarities across the three levels of quantum mechanics considered, we now propose the following precise formulation of complete complementarity.

> A collection of classical structures is *completely complementary* when its members are pairwise partially complementary and jointly epic.

Compare also [21]. Notice that this formulation is almost information-theoretic.

The view on $\mathcal{C}(A)$ provided by Section 4 holds several promises for the study of functors on $\mathcal{C}(A)$ that we intend to explore further in future work:

- One can consider variations in the study of $\mathbf{Set}$-valued functors on $\mathcal{C}(A)$ by choosing different morphisms on $\mathcal{C}(A)$: *e.g.* inclusions [10], or reverse inclusions [7]. In the above perspective, the natural direction that suggests itself is that of morphisms between classical structures, *i.e.* inclusions. Moreover, a more interesting choice of morphisms based on classical structures (see *e.g.* [5]) could make $\mathcal{C}(A)$ into a category that is not just a partially ordered set.
- The topos of functors on $\mathcal{C}(A)$ can be abstracted away from $\mathbf{Hilb}$ to any dagger monoidal kernel category that satisfies a suitable 'spectral assumption' linking commutative submonoids of endohomsets to classical structures. For example, one could lift Theorem 16 or even Proposition 19 to a definition, and study $\mathbf{Set}$-valued functors on these characterizations of $\mathcal{C}(A)$ in any dagger monoidal kernel category.

  In fact, in this generalized setting, there is no need for the base category to be $\mathbf{Set}$. After all, the basic objects of study of *e.g.* [7] are really partial orders of subobjects in a functor category. This just happens to be a Heyting algebra because the functors take values in the topos $\mathbf{Set}$, but in principle less structured partial orders of subobjects are just as interesting, and perhaps are also justifiable physically.
- One of the weak points of the study of functors on $\mathcal{C}(A)$ to date is that there is no obvious way to study compound systems. That is, there is no obvious (satisfactory) relation between $\mathcal{C}(A \otimes B)$ and $\mathcal{C}(A)$ and $\mathcal{C}(B)$. Considering A as (a submonoid of) an endohomset opens the broader context of a fibred setting in which studying entanglement is possible.

All in all, the above considerations strongly suggest studying fibrations of all classical structures over all objects of a dagger (kernel) monoidal category.

Finally, we remark that we have not used the H*-axiom (or the Frobenius equation) at all in this paper. Apparently, the combination of (copyable) kernels with the dagger monic type $X \to X \otimes X$ of classical structures suffices for these purposes.

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