Involutive Categories and Monoids, with a GNS-correspondence

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Abstract

This paper develops the basics of the theory of involutive categories and shows that such categories provide the natural setting in which to describe involutive monoids. It is shown how categories of Eilenberg-Moore algebras of involutive monads are involutive, with conjugation for modules and vector spaces as special case. The core of the so-called Gelfand-Naimark-Segal (GNS) construction is identified as a bijective correspondence between states on involutive monoids and inner products. This correspondence exists in arbitrary involutive symmetric monoidal categories.

1 Introduction

In general an involution is a certain endomap i for which $i \circ i$ is the identity. The inverse operation of a group is a special example. But there are also monoids with such an involution, such as for instance the free monoid of lists over some set, with list reversal as involution.

An involution can also be defined on a category. It then consists of an endofunctor $\mathbf{C} \to \mathbf{C}$, which is typically written as $X \mapsto \overline{X}$. It should satisfy $\overline{\overline{X}} \cong X$. Involutive categories occur in the literature, for instance in [6, 1], but have not been studied very extensively. This paper will develop the basic elements of such a theory of involutive categories. Its main technical contribution is a bijective correspondence between states $M \to I$ on an involutive monoid M and inner products $\overline{M} \otimes M \to I$, relating fundamental notions in the mathematical modeling of quantum phenomena.

We should note that involutive categories as we understand them here are different from dagger categories (which have an identity-on-objects functor $(-)^{\dagger} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ with $f^{\dagger\dagger} = f$) and also from $*$-autonomous categories (which have a duality $(-)^{*} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ given by a dualising object D as in $X^{*} = X \multimap D$). In both these cases one has contravariant functors, whereas involution $\overline{(-)} \colon \mathbf{C} \to \mathbf{C}$ is a covariant functor. The relation between involution, dagger and duality for Hilbert spaces is described in [2, §§4.1, 4.2]: each can be defined in terms of the other two.

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Involutive categories and involutive monoids are related: just like the notion of a monoid is formulated in a monoidal category, the notion of involutive monoid requires an appropriate notion of involutive monoidal category. This is in line with the “microcosm principle”, formulated by Baez and Dolan [4], and elaborated in [12, 11, 10]: it involves “outer” structure (like monoidal structure $1 \xrightarrow{I} \mathbf{C} \xleftarrow{\otimes} \mathbf{C} \times \mathbf{C}$ *on* a category $\mathbf{C}$) that enables the definition of “inner” structure (like a monoid $I \xrightarrow{0} M \xleftarrow{+} M \otimes M$ *in* $\mathbf{C}$). We briefly illustrate how this connection between involutive monoids and involutive categories arises.

Consider for instance the additive group $\mathbb{Z}$ of integers with minus $-$ as involution. In the category $\mathbf{Sets}$ of ordinary sets and functions between them we can describe minus as an ordinary endomap $-\colon \mathbb{Z} \to \mathbb{Z}$. The integers form a partially ordered set, so we may wish to consider $\mathbb{Z}$ also as involutive monoid in the category $\mathbf{PoSets}$ of partially ordered sets and monotone functions. The problem is that minus reverses the order: $i \leq j \Rightarrow -i \geq -j$, and is thus not a map $\mathbb{Z} \to \mathbb{Z}$ in $\mathbf{PoSets}$. However, we can describe it as a map $(\mathbb{Z}, \geq) \to (\mathbb{Z}, \leq)$ in $\mathbf{PoSets}$, using the reversed order ($\geq$ instead of $\leq$) on the integers. This order reversal forms an involution $\overline{(-)}\colon \mathbf{PoSets} \to \mathbf{PoSets}$ on the “outer” category, which allows us to describe the involution “internally” as $-\colon \overline{\mathbb{Z}} \to \mathbb{Z}$ in $\mathbf{PoSets}$.

As said, this paper introduces the basic steps of the theory of involutive categories. It introduces the category of “self-conjugate” objects, and shows how involutions arise on categories of Eilenberg-Moore algebras of an “involutive” monad. This general construction includes the important example of conjugation on modules and vector spaces, for the multiset monad associated with an involutive semiring. It allows us to describe abstractly an involutive monoid in such categories of algebras. Pre C^*-algebras (without norm) are such monoids.

Once this setting has been established we take a special look at the famous Gelfand-Naimark-Segal (GNS) construction [3]. It relates C^*-algebras and Hilbert spaces, and shows in particular how a state $A \to \mathbb{C}$ on a C^*-algebra gives rise to an inner product on A. Using conjugation as involution, the latter can be described as a map $\overline{A} \otimes A \to \mathbb{C}$ that incorporates the sesquilinearity requirements in its type (including conjugate linearity in its first argument). The final section of this paper gives the essence of this construction in the form of a non-trivial bijective correspondence between such states and inner products in categorical terms, using the language of involutive categories and monoids.

2 Involutive categories

Definition 2.1 *A category* $\mathbf{C}$ *will be called* involutive *if it comes with a ‘involution’ functor* $\mathbf{C} \to \mathbf{C}$*, written as* $X \mapsto \overline{X}$*, and a natural isomorphism* $\iota\colon X \xrightarrow{\cong} \overline{\overline{X}}$ *satisfying*

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\iota_{\overline{X}}} & \overline{\overline{\overline{X}}} \\ \| & & \| \\ \overline{X} & \xrightarrow{\overline{\iota_X}} & \overline{\overline{\overline{X}}} \end{array} \qquad (1)$$

Each category is trivially involutive via the identity functor. This trivial involution is certainly useful. The category $\mathbf{PoSets}$ is involutive via order reversal. This applies also to categories of, for instance, distributive lattices or Boolean algebras. The category $\mathbf{Cat}$

of (small) categories and functors is also involutive, by taking opposites of categories. Next, consider the category $\mathit{Vect}_{\mathbb{C}}$ of vector spaces over the complex numbers $\mathbb{C}$. It is an involutive category via conjugation. For a vector space $V \in \mathit{Vect}_{\mathbb{C}}$ we define $\overline{V} \in \mathit{Vect}_{\mathbb{C}}$ with the same vectors as V, but with adapted scalar multiplication $s \cdot_{\overline{V}} v = \overline{s} \cdot_V v$, for $s \in \mathbb{C}$ and $v \in V$, where $\overline{s} = a - ib$ is the conjugate of the complex number $s = a + ib \in \mathbb{C}$.

The following is the first of a series of basic observations.

Lemma 2.1 *The involution functor of an involutive category is self-adjoint: $\overline{(-)} \dashv \overline{(-)}$. As a result, involution preserves all limits and colimits that exist in the category.*

Definition 2.2 *A functor $F\colon \mathbf{C} \to \mathbf{D}$ between two involutive categories is called involutive if it comes with a natural transformation (or distributive law) ν with components $F(\overline{X}) \to \overline{F(X)}$ commuting appropriately with the isomorphisms $X \cong \overline{\overline{X}}$, as on the left below.*

$$\begin{array}{ccccc} F(X) & = & = & = & F(X) \\ {\scriptstyle F(\iota_X)}\downarrow\cong & & & & \cong\downarrow{\scriptstyle \iota_{F(X)}} \\ F(\overline{\overline{X}}) & \xrightarrow{\nu_{\overline{X}}} & \overline{F(\overline{X})} & \xrightarrow{\overline{\nu_X}} & \overline{\overline{F(X)}} \end{array} \qquad \begin{array}{ccc} F(\overline{X}) & \xrightarrow{\sigma_{\overline{X}}} & G(\overline{X}) \\ {\scriptstyle \nu^F}\downarrow & & \downarrow{\scriptstyle \nu^G} \\ \overline{F(X)} & \xrightarrow{\overline{\sigma_X}} & \overline{G(X)} \end{array} \tag{2}$$

A natural transformation $\sigma\colon F \Rightarrow G$ between two involutive functors $F, G\colon \mathbf{C} \rightrightarrows \mathbf{D}$ is called involutive if it commutes with the associated ν's, as on the right above. This yields a 2-category $\mathbf{ICat}$ of involutive categories, functors and natural transformations.

This 2-categorical perspective is useful, for instance because it allows us to see immediately what an involutive adjunction or monad is, namely one in which the functors and natural transformations involved are all involutive.

Lemma 2.2 *If F is an involutive functor via $\nu\colon F(\overline{X}) \to \overline{F(X)}$, then this ν is automatically an isomorphism.*

3 Self-conjugates

Definition 3.1 *For an involutive category $\mathbf{C}$, let $SC(\mathbf{C})$ be the category of self-conjugates in $\mathbf{C}$. Its objects are maps $j\colon \overline{X} \to X$ making the triangle below commute.*

$$\begin{array}{ccc} \overline{\overline{X}} & \xrightarrow{\overline{j}} & \overline{X} \\ & {\scriptstyle \iota_X^{-1}}\searrow & \downarrow{\scriptstyle j} \\ & & X \end{array} \qquad \qquad \begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

It is not hard to see that such a map is j is necessarily an isomorphism, with inverse $\overline{j} \circ \iota_X\colon X \to \overline{\overline{X}} \to \overline{X}$.

A morphism $f\colon (X, j_X) \to (Y, j_Y)$ in $SC(\mathbf{C})$ is a map $f\colon X \to Y$ in $\mathbf{C}$ making the above rectangle commute. There is thus an obvious forgetful functor $SC(\mathbf{C}) \to \mathbf{C}$.

What we call a self-conjugate object is called a star object in [6]. By the self-adjointness of Lemma 2.1 a self-conjugate $\overline{X} \to X$ may also be described as $X \to \overline{X}$. Sometimes we call an object X a self-conjugate when the map $\overline{X} \stackrel{\cong}{\Rightarrow} X$ involved is obvious from the context. In linear algebra, with $\overline{X}$ given by conjugation (see before Lemma 2.1), a map of the form $\overline{X} \to Y$ is called an 'antilinear' or 'conjugate linear' map.

Lemma 3.1 *For an involutive category* $\mathbf{C}$*, the category* $SC(\mathbf{C})$ *of self-conjugates is again involutive, via:*

$$\overline{\left(\overline{X} \xrightarrow{j} X\right)} \stackrel{def}{=} \left(\overline{\overline{X}} \xrightarrow{\overline{j}} \overline{X}\right). \tag{3}$$

and the forgetful functor $SC(\mathbf{C}) \to \mathbf{C}$ *is an involutive functor, via the identity natural transformation (as '*ν*' in Definition 2.2).*

Example 3.1 *Recall that the category* $\mathbf{PoSets}$ *of posets and monotone functions is involutive via the reversed (opposite) order:* $\overline{(X, \leq)} = (X, \geq)$*. The integers* $\mathbb{Z}$ *are then self-conjugate, via minus* $-\colon \overline{\mathbb{Z}} \stackrel{\cong}{\Rightarrow} \mathbb{Z}$*. Also the positive rational and real numbers* $\mathbb{Q}_{>0}$ *and* $\mathbb{R}_{>0}$ *are self-conjugates in* $\mathbf{PoSets}$*, via* $x \mapsto \frac{1}{x}$*. Similarly, for a Boolean algebra* B*, negation* $\neg$ *yields a self-conjugate* $\neg\colon \overline{B} \stackrel{\cong}{\Rightarrow} B$ *in the category of Boolean algebras. There are similar self-conjugates via orthosupplements* $(-)^{\perp}$ *in orthomodular lattices [13] and effect algebras [9].*

In $\mathbf{Cat}$ *a self-conjugate is given by a self-dual category* $\mathbf{C}^{\mathrm{op}} \cong \mathbf{C}$*.*

Recall the conjugation on vector spaces. Suppose $V \in \mathit{Vect}_{\mathbb{C}}$ *has a basis* $(v_i)_{i \in I}$*. Then we can define a self-conjugate* $\overline{V} \stackrel{\cong}{\Rightarrow} V$ *by* $x = \left(\sum_i x_i v_i\right) \longmapsto \left(\sum_i \overline{x_i} v_i\right)$*.*

Finally, if a category $\mathbf{C}$ *is considered with trivial involution* $\overline{X} = X$*, then* $SC(\mathbf{C})$ *contains the self-inverse endomaps* $j\colon X \to X$*, with* $j \circ j = \mathrm{id}_X$*.*

We first take a closer look at these trivial involutions.

Lemma 3.2 *Let* $\mathbf{C}$ *be an ordinary category, considered as involutive with trivial involution* $\overline{X} = X$*. Assuming binary coproducts* $+$ *and products* $\times$ *exist in* $\mathbf{C}$*, there are left and right adjoints to the forgetful functor:*

$$\begin{array}{c} SC(\mathbf{C}) \\ {\scriptstyle X \mapsto 2\times X = X+X} \left(\dashv \Big\downarrow \dashv\right) {\scriptstyle X \mapsto X^2 = X \times X} \\ \mathbf{C} \end{array}$$

using the swap maps $[\kappa_2, \kappa_1]\colon X + X \stackrel{\cong}{\Rightarrow} X + X$ *and* $\langle \pi_2, \pi_1 \rangle\colon X \times X \stackrel{\cong}{\Rightarrow} X \times X$ *as self-conjugates.*

Lemma 3.3 *Let* $\mathbf{C}$ *be an involutive category;* $SC(\mathbf{C})$ *inherits all limits and colimits that exist in* $\mathbf{C}$*, and the forgetful functor* $SC(\mathbf{C}) \to \mathbf{C}$ *preserves them.*

For the record we note the following (see [18, 7] for background).

Lemma 3.4 *The mapping* $\mathbf{C} \mapsto SC(\mathbf{C})$ *is a 2-functor* $\mathbf{ICat} \to \mathbf{ICat}$*, and even a 2-comonad.*

4 Involutive monoidal categories

Definition 4.1 *An* involutive monoidal category *or an* involutive symmetric monoidal category, *abbreviated as IMC or ISMC, is a category* $\mathbf{C}$ *which is both involutive and (symmetric) monoidal in which involution* $\overline{(-)}\colon \mathbf{C} \to \mathbf{C}$ *is a (symmetric) monoidal functor and* $\iota\colon \mathrm{id} \Rightarrow \overline{\overline{(-)}}$ *is a monoidal natural transformation.*

The fact that involution is a (symmetric) monoidal functor means that there are (natural) maps $\zeta\colon I \to \overline{I}$ and $\xi\colon \overline{X} \otimes \overline{Y} \to \overline{X \otimes Y}$ commuting with the monoidal isomorphisms $\alpha\colon X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$, $\lambda\colon I \otimes X \xrightarrow{\cong} X$, $\rho\colon X \otimes I \to X$, and also with the swap map $\gamma\colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ in the symmetric case. That the isomorphism ι is monoidal means that we have commuting diagrams:

$$\begin{array}{ccccc} I & = & = & = & I \\ \| & & & & \downarrow{\scriptstyle \iota} \\ I & \xrightarrow{\zeta} & \overline{I} & \xrightarrow{\overline{\zeta}} & \overline{\overline{I}} \end{array} \qquad \begin{array}{ccccc} X \otimes Y & = & = & = & X \otimes Y \\ {\scriptstyle \iota\otimes\iota}\downarrow & & & & \downarrow{\scriptstyle \iota} \\ \overline{\overline{X}} \otimes \overline{\overline{Y}} & \xrightarrow{\xi} & \overline{\overline{X} \otimes \overline{Y}} & \xrightarrow{\overline{\xi}} & \overline{\overline{X \otimes Y}} \end{array} \tag{4}$$

Like in Lemma 2.2 we get isomorphism for free.

Lemma 4.1 *In an IMC the involution functor* $\overline{(-)}$ *is automatically strong monoidal: the maps* $\zeta\colon I \to \overline{I}$ *and* $\xi\colon \overline{X} \otimes \overline{Y} \to \overline{X \otimes Y}$ *are necessarily isomorphisms.*

In the category *Vect*$_{\mathbb{C}}$ of vector spaces over the complex number the tensor unit I is $\mathbb{C} \in$ *Vect*$_{\mathbb{C}}$. The above map $\zeta\colon \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{C}}$ is simply conjugation of complex numbers.

Remark 4.2 *The notion of 'bar category' introduced in [6] is similar to the above notion of IMC (or ISMC), but is subtly different: by definition, bar categories have isomorphisms* $\overline{X \otimes Y} \xrightarrow{\cong} \overline{Y} \otimes \overline{X}$. *The object reversal involved makes sense in a non-symmetric setting. But in the present context all our examples are symmetric, and many results rely on symmetry, so we often assume it and thus have no difference with [6].*

In order to be complete we also have to define the following.

Definition 4.3 *A functor* $F\colon \mathbf{C} \to \mathbf{D}$ *between IMC's is called* involutive monoidal *if it is both involutive, via* $\nu\colon F(\overline{X}) \to \overline{F(X)}$, *and monoidal, via* $\zeta^F\colon I \to F(I)$ *and* $\xi^F\colon F(X) \otimes F(Y) \to F(X \otimes Y)$, *and these natural transformations* ν, ζ^F, ξ^F *interact appropriately with* ζ, ξ *from*(4)*, as in:*

$$\begin{array}{ccccc} I & \xrightarrow{\zeta^F} & F(I) & \xrightarrow{F(\zeta)} & F(\overline{I}) \\ \| & & & & \downarrow{\scriptstyle \nu} \\ I & \xrightarrow{\zeta} & \overline{I} & \xrightarrow{\overline{\zeta^F}} & \overline{F(I)} \end{array} \qquad \begin{array}{ccccc} F(\overline{X}) \otimes F(\overline{Y}) & \xrightarrow{\xi^F} & F(\overline{X} \otimes \overline{Y}) & \xrightarrow{F(\xi)} & F(\overline{X \otimes Y}) \\ {\scriptstyle \nu\otimes\nu}\downarrow & & & & \downarrow{\scriptstyle \nu} \\ \overline{F(X)} \otimes \overline{F(Y)} & \xrightarrow{\xi} & \overline{F(X) \otimes F(Y)} & \xrightarrow{\overline{\xi^F}} & \overline{F(X \otimes Y)} \end{array}$$

It should then be obvious what an involutive symmetric *monoidal functor is.*

An involutive monoidal natural transformation $\sigma\colon F \Rightarrow G$ *between two involutive monoidal functors is both involutive and monoidal.*

Hence also in this case we have 2-categories $\mathbf{IMCat}$ and $\mathbf{IMSCat}$ of involutive (symmetric) monoidal categories. The following is the main result of this section.

Proposition 4.2 *A category $SC(\mathbf{C})$ inherits (symmetric) monoidal structure from $\mathbf{C}$. As a result, the forgetful functor $SC(\mathbf{C}) \to \mathbf{C}$ is an involutive (symmetric) monoidal functor. In case $\mathbf{C}$ is monoidal closed, then so is $SC(\mathbf{C})$ and $SC(\mathbf{C}) \to \mathbf{C}$ preserves the exponent $\multimap$.*

5 Involutive Monoids

Now that we have the notion of involutive category as ambient category, we can define the notion of involutive monoid in this setting, in the style of [12, 11, 10].

We start with some preliminary observations. Let $M = (M, \cdot, 1)$ be an arbitrary monoid (in $\mathbf{Sets}$), not necessarily commutative. An involution on M is a special endofunction $M \to M$ which we shall write as superscript negation x^-, for $x \in M$. It satisfies $x^{--} = x$ and $1^- = 1$. The interaction of involution and multiplication may happen in two ways: either in a "reversing" manner, as in $(x \cdot y)^- = y^- \cdot x^-$, or in a "non-reversing" manner: $(x \cdot y)^- = x^- \cdot y^-$. Obviously, in a commutative monoid there is no difference between a reversing or non-reversing involution.

As we have argued in the first section via the example of integers in $\mathbf{PoSets}$, a proper formulation of the notion of involutive monoid requires an involutive category, so that the monoid involution can be described as a map $\overline{M} \to M$.

Definition 5.1 *Let $\mathbf{C}$ be an involutive symmetric monoidal category. An involutive monoid in $\mathbf{C}$ consists of a monoid $I \xrightarrow{u} M \xleftarrow{m} M \otimes M$ in $\mathbf{C}$ together with an involution map $\overline{M} \xrightarrow{j} M$ satisfying $j \circ \overline{u} \circ \zeta = u$ and $j \circ \overline{j} = \iota^{-1}$, and, one of the following diagrams:*

"reversing"

$$\begin{array}{ccccc} \overline{M}\otimes\overline{M} & \xrightarrow{\xi} & \overline{M\otimes M} & \xrightarrow{\overline{m}} & \overline{M} \\ {\scriptstyle j\otimes j}\downarrow & & & & \downarrow{\scriptstyle j} \\ M\otimes M & \xrightarrow[\cong]{\gamma} & M\otimes M & \xrightarrow{m} & M \end{array}$$

"non-reversing"

$$\begin{array}{ccccc} \overline{M}\otimes\overline{M} & \xrightarrow{\xi} & \overline{M\otimes M} & \xrightarrow{\overline{m}} & \overline{M} \\ {\scriptstyle j\otimes j}\downarrow & & & & \downarrow{\scriptstyle j} \\ M\otimes M & & \xrightarrow{\qquad m \qquad} & & M \end{array}$$

One may call M a simple *involutive monoid if $\mathbf{C}$'s involution $\overline{(-)}$ is the identity.*

A morphism of involutive monoids $M \to M'$ is a morphism of monoids $f\colon M \to M'$ satisfying $f \circ j = j' \circ \overline{f}$. This yields two subcategories $\mathbf{rIMon}(\mathbf{C}) \hookrightarrow \mathbf{Mon}(\mathbf{C})$ and $\mathbf{IMon}(\mathbf{C}) \hookrightarrow \mathbf{Mon}(\mathbf{C})$ of reversing and non-reversing involutive monoids. There is also a commutative version, forming a (full) subcategory. $\mathbf{ICMon}(\mathbf{C}) \hookrightarrow \mathbf{IMon}(\mathbf{C})$.

The involution map $j\colon \overline{M} \to M$ of an involutive monoid is of course a self-conjugate—see Definition 3.1—and thus an isomorphism. In fact, we have the following result.

Lemma 5.1 *Involutive monoids (of the non-reversing kind) are ordinary monoids in the category of self-conjugates: the categories $\mathbf{IMon}(\mathbf{C})$ and $\mathbf{Mon}(SC(\mathbf{C}))$ are the same. Similarly in the commutative case, $\mathbf{ICMon}(\mathbf{C}) = \mathbf{CMon}(SC(\mathbf{C}))$.*

This lemma suggests a pattern for defining an involutive variant of certain categorical structure, namely by defining this structure in the category of self-conjugates.

Example 5.1 *As we have observed before, the category* $\mathbf{PoSets}$ *of posets and monotone functions is involutive, via order-reversal* $\overline{(X, \leq)} = (X, \geq)$. *The poset* $\mathbb{Z}$ *of integers forms an involutive monoid in* $\mathbf{PoSets}$, *with minus* $-\colon \overline{\mathbb{Z}} \to \mathbb{Z}$ *as involution. Also, the positive rationals* $\mathbb{Q}_{>0}$ *or reals* $\mathbb{R}_{>0}$ *with multiplication* $\cdot$, *unit* 1, *and inverse* $(-)^{-1}$ *form involutive monoids in* $\mathbf{PoSets}$.

In the category $\mathbf{Cat}$ *of categories, with finite products as monoidal structure, a monoid is a strictly monoidal category. If such a category* $\mathbf{C}$ *has a dagger* $\dagger\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ *that commutes with these tensors (in the sense that* $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, *see e.g. [2]) then* $\mathbf{C}$ *is an involutive monoid in* $\mathbf{Cat}$.

Inside such a dagger symmetric (not necessarily strict) monoidal category $\mathbf{C}$ *with dagger* $(-)^\dagger\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ *the homset of scalars* $I \to I$ *is a commutative involutive monoid, with involution* $s^- = s^\dagger$.

The tensor unit $I \in \mathbf{C}$ *in an arbitrary involutive category* $\mathbf{C}$ *is a commutative involutive monoid object, with involution* $\zeta^{-1}\colon \overline{I} \to I$.

Free involutive monoids on a set V *are given by the set* $(2 \times V)^\star$ *of lists of "signed" elements from* V. *The involution of the non-reversing version changes signs, whereas the reversing version also reverses the lists.*

6 Involutions and algebras

This section briefly discusses involutions on monads and will focus on algebras of such monads. Familiarity with the basics of the theory of monads is assumed, see *e.g.* [5, 17, 16]. An involutive monad is a monad in the 2-category $\mathbf{ICat}$ of involutive categories. It thus involves an ordinary monad (T, η, μ) together with a distributive law $\nu\colon T(\overline{X}) \to \overline{T(X)}$.

We start with the main example, namely the "multiset" monad. Let S be an involutive commutative semiring, *i.e.* a commutative semiring with an endomap $(-)^-\colon S \to S$ that is a semiring homomorphism with $s^{--} = s$. An obvious example is the set $\mathbb{C}$ of complex numbers with conjugation $\overline{a + ib} = a - ib$. Similarly, the Gaussian rational numbers (with $a, b \in \mathbb{Q}$ in $a + ib$) form an involutive semiring, albeit not a complete one. The multiset monad $\mathcal{M}_S\colon \mathbf{Sets} \to \mathbf{Sets}$ associated with S is defined on a set X as:

$$\mathcal{M}_S(X) \quad = \quad \{\varphi\colon X \to S \mid \mathit{supp}(\varphi) \text{ is finite}\},$$

see *e.g.* [8]. The category of algebras of this monad is the category $\mathbf{Mod}_S$ of modules over S. This monad is monoidal / commutative, because S is commutative. It is involutive, with involution $\nu\colon \mathcal{M}_S(X) \to \mathcal{M}_S(X)$ given by $\nu(\sum_i s_i x_i) = \sum_i s_i^- x_i$. Here we use $\mathbf{Sets}$ as trivial involutive category, with the identity as involution.

For an involutive monad T on an involutive category $\mathbf{C}$ we can consider two liftings, namely of the monad T to self-dualities $\mathit{SC}(\mathbf{C})$ following Lemma 3.4, or of $\mathbf{C}$'s involution $\overline{(-)}$ to algebras $\mathit{Alg}(T)$, as in the following two diagrams.

$$\begin{array}{ccc} \mathit{SC}(\mathbf{C}) & \xrightarrow{\mathit{SC}(T)} & \mathit{SC}(\mathbf{C}) \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{T} & \mathbf{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathit{Alg}(T) & \xrightarrow{\overline{(-)}} & \mathit{Alg}(T) \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{\overline{(-)}} & \mathbf{C} \end{array} \tag{5}$$

The lifting on the left yields a new monad $\mathit{SC}(T)$ because lifting in Lemma 3.4 is 2-functorial. The lifting on the right arises because an involutive monad involves a distributive law commuting with unit and multiplication. Explicitly, it is given by:

$$\overline{\big(T(X) \xrightarrow{a} X\big)} \stackrel{\text{def}}{=} \Big(T(\overline{X}) \xrightarrow{\nu_X} \overline{T(X)} \xrightarrow{\overline{a}} \overline{X}\Big). \tag{6}$$

Proposition 6.1 *Suppose T is an involutive monad on an involutive category $\mathbf{C}$. The category $\mathit{Alg}(T)$ is then also involutive via* (6)*, and:*

1. *$\mathit{Alg}(\mathit{SC}(T)) = \mathit{SC}(\mathit{Alg}(T))$, for which we sometimes write $\mathit{IAlg}(T)$;*
2. *the canonical adjunction $\mathit{Alg}(T) \leftrightarrows \mathbf{C}$ is an involutive one.*

In a next step we would like to show that these categories of algebras of an involutive monoidal monad are also involutive monoidal categories. The monoidal structure is given by the standard construction of Anders Kock [15, 14]. Tensors of algebras exist in case certain colimits exist. This is always the case with monads on sets, due to a result of Linton's, see [5, § 9.3, Prop. 4].

Theorem 6.2 *Suppose T is an involutive monoidal monad on an involutive monoidal category $\mathbf{C}$; assume the category $\mathit{Alg}(T)$ of algebras has enough coequalisers to make it monoidal. The category $\mathit{Alg}(T)$ is then also involutive monoidal, and the canonical adjunction $\mathit{Alg}(T) \leftrightarrows \mathbf{C}$ is an involutive monoidal one. This result extends to* symmetric *monoidal structure, and also to closure (with exponents $\multimap$).*

The construction (6) gives for an involutive commutative semiring S an involution on the category $\mathbf{Mod}_S$ of S-modules, which maps a module X to its conjugate space $\overline{X}$, with the same vectors but with scalar multiplication in $\overline{X}$ given by: $s \cdot_{\overline{X}} x = s^{-} \cdot_X x$.

Conjugate modules often occur in the context of Hilbert spaces. The category $\mathbf{Hilb}$ is indeed an involutive category, via this conjugation. Hence one can consider for instance involutive monoids in $\mathbf{Hilb}$. They are sometimes called (unital) H^*-algebras.

7 The core of the GNS-construction

In this final section we wish to apply the theory developed so far to obtain what can be considered as the core of the (unital version of the) Gelfand-Naimark-Segal (GNS) construction [3], giving a bijective correspondence between states on C^*-algebras and certain sesquilinear maps. Roughly, for an involutive monoid A in the category $\mathbf{Mod}_S$ of modules, a state $f\colon A \to S$ gives rise to an inner product $\langle - \,|\, - \rangle\colon \overline{A} \otimes A \to S$ by $\langle a \,|\, b\rangle = f(a^{-} \cdot b)$, where $\cdot$ is the multiplication of the monoid A. Notice that using the involution $\overline{(-)}$ in the domain $\overline{A} \otimes A$ of the inner product gives a neat way of handling conjugation in the condition $\langle s \cdot a \,|\, b\rangle = s^{-} \cdot \langle a \,|\, b\rangle$, where this last $\cdot$ is the (scalar) multiplication of the semiring S (which is the tensor unit in $\mathbf{Mod}_S$).

This induced inner product $\langle a \,|\, b\rangle = f(a^{-} \cdot b)$ satisfies two special properties that we capture abstractly below, namely: $\langle u \,|\, - \rangle = \langle - \,|\, u\rangle$ and $\langle a \cdot b \,|\, c\rangle = \langle a \,|\, b^{-} \cdot c\rangle$. These two properties appear as conditions (a) and (b) in the following result. Most commonly the inner product is described as a map $p\colon \overline{M}\otimes M \to I$ with the tensor unit I as codomain, but the correspondence in the next result holds for an arbitrary self-conjugate X instead of I. Thus is will be formulated more generally.

Theorem 7.1 *Let $M = (M, m, u, j)$ be a reversing involutive monoid in an involutive symmetric monoidal category (ISMC) $\mathbf{C}$ and let $j_X\colon \overline{X} \to X$ be a self-conjugate. Consider the following two properties of a map $p\colon \overline{M}\otimes M \to X$.*

(a) Sameness when restricted to units:

$$\begin{array}{ccccccc}
\overline{M} & \xrightarrow[\cong]{\rho^{-1}} & \overline{M}\otimes I & \xrightarrow{\mathrm{id}\otimes u} & \overline{M}\otimes M & \xrightarrow{p} & X \\
{\scriptstyle j}\downarrow & & & & & & \uparrow{\scriptstyle p} \\
M & \xrightarrow[\cong]{\lambda^{-1}} & I\otimes M & \xrightarrow{\zeta\otimes\mathrm{id}} & \overline{I}\otimes M & \xrightarrow{\overline{u}\otimes\mathrm{id}} & \overline{M}\otimes M
\end{array}$$

(b) Shifting of multiplications:

$$\begin{array}{ccccccc}
(\overline{M}\otimes\overline{M})\otimes M & \xrightarrow{\xi\otimes\mathrm{id}} & \overline{(M\otimes M)}\otimes M & \xrightarrow{\overline{m}\otimes\mathrm{id}} & \overline{M}\otimes M & \xrightarrow{p} & X \\
{\scriptstyle \gamma\otimes\mathrm{id}}\downarrow{\scriptstyle\cong} & & & & & & \uparrow{\scriptstyle p} \\
(\overline{M}\otimes\overline{M})\otimes M & \xrightarrow[\cong]{\alpha^{-1}} & \overline{M}\otimes(\overline{M}\otimes M) & \xrightarrow{\mathrm{id}\otimes(j\otimes\mathrm{id})} & \overline{M}\otimes(M\otimes M) & \xrightarrow{\mathrm{id}\otimes m} & \overline{M}\otimes M
\end{array}$$

Then there is a bijective correspondence between maps in SC$(\mathbf{C})$,

$$\frac{M \xrightarrow{f} X}{\overline{M}\otimes M \xrightarrow[p]{} X \ \text{ satisfying (a) and (b)}} \qquad (7)$$

where $\overline{M}\otimes M$ is provided with the "twist" conjugate t defined as:

$$t \overset{def}{=} \Big(\overline{\overline{M}\otimes M} \xrightarrow{\overline{\mathrm{id}\otimes\iota_M}} \overline{\overline{M}\otimes\overline{\overline{M}}} \xrightarrow{\overline{\xi}} \overline{\overline{M\otimes\overline{M}}} \xrightarrow{\iota^{-1}} M\otimes\overline{M} \xrightarrow{\gamma} \overline{M}\otimes M\Big).$$

Proof Verification of this correspondence involves many details, but here we present only the correspondence (7). Given $f\colon M \to X$ in SC$(\mathbf{C})$, we define

$$\widehat{f} \overset{\text{def}}{=} \Big(\overline{M}\otimes M \xrightarrow{j\otimes\mathrm{id}} M\otimes M \xrightarrow{m} M \xrightarrow{f} X\Big).$$

Conversely, given $p\colon \overline{M}\otimes M \to X$ in SC$(\mathbf{C})$ we take:

$$\widehat{p} = \Big(M \xrightarrow{\lambda^{-1}} I\otimes M \xrightarrow{\zeta\otimes\mathrm{id}} \overline{I}\otimes M \xrightarrow{\overline{e}\otimes\mathrm{id}} \overline{M}\otimes M \xrightarrow{p} X\Big). \qquad \square$$

As said, this result only captures the heart of the GNS construction [3]; it ignores the analytic aspects. The whole construction additionally involves suitable quotients, in order to identify points a, b with $\langle a \,|\, b\rangle = 0$, and completions, in order to get a complete metric space, and thus a Hilbert space.

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