Reduced Density Matrix in Spin Models

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Abstract

We consider reduced density matrix of a large block of consecutive spins in the ground states of a Heisenberg spin chain on an infinite lattice. We derive the spectrum of the density matrix using expression of Rényi entropy in terms of modular functions. The eigenvalues λ_n form exact geometric sequence. For example, for strong magnetic field $\lambda_n = Cexp(-\pi\tau_0 n)$, here $\tau_0 > 0$ and $C > 0$ depend on anisotropy and magnetic field. Different eigenvalues are degenerated differently. The largest eigenvalue is unique, but degeneracy g_n increases sub-exponentially as eigenvalues diminish: $g_n \sim exp(\pi\sqrt{n/3})$. For weak magnetic field expressions are similar.

Keywords: Entanglement, Entropy, Entanglement Spectrum, Reduced Density Matrix, XY Model, Partitions Theory

1 Introduction

Entanglement is a peculiar feature of a quantum system, which distinguishes it from a classical one. While the notion of entanglement has been introduced since the dawn of quantum mechanics, only recently physicists have employed a quantitative approach, mostly inspired by the progresses in quantum information and Bethe ansatz.

The most studied quantity is the *Von Neumann entropy*, which is the quantum analog of the Shannon entropy and measures the amount of (quantum) information stored in a system A described by a density matrix ρ_A:

$$S(\rho_A) \equiv -\mathrm{Tr}(\rho_A \ln \rho_A) \ . \tag{1}$$

We shall consider simplest case of a pure system U described by a wave function $|\Psi\rangle$. The system U is composed by the union of two disjoint subsystems A and B, meaning $U \equiv A \cup B$. The reduced density matrix of subsystem A obtained by tracing out the B degrees of freedom $\rho_A \equiv \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$ and the Von Neumann entropy of a subsystem is a measure of entanglement between the two subsystems [1] [1].

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[1] note that $S(\rho_A) = S(\rho_B)$

Characterizing the entanglement with a single number is definitely appealing, but it can hardly capture its complexity. For this reason, other entanglement measures have been introduced, such as, the *Renyi entropy* [2, 3, 5, 6, 7]:

$$S_R(\rho_A, \alpha) \equiv \frac{1}{1-\alpha} \ln \mathrm{Tr}(\rho_A^\alpha), \qquad\qquad 1 > \alpha > 0\ . \tag{2}$$

Note that in the limit $\alpha \to 1$, the Renyi entropy recovers the Von Neumann entropy:

$$\lim_{\alpha \to 1} S_R(\rho_A, \alpha) = S(\rho_A).$$

We calculated the *Renyi entropy* of large block of spins for XY spin chain [12, 9] and studied its analytical continuation into the complex plane of parameter α.

Another quantity that has recently attracted a lot of interest is the spectrum of the reduced density matrix, which is now referred to in the literature as the entanglement spectrum, after [8]. The knowledge of the entanglement spectrum fixes the density matrix up to unitarity transformation, we can say that it fixes the state of the block. It is also clear that the Renyi entropy and the entanglement spectrum are also very closely related. In fact, if we know ζ-function of ρ_A:

$$\zeta_{\rho_A}(\alpha) \equiv \mathrm{tr}\rho_A^\alpha = \sum_{n=0}^{\infty} g_n \lambda_n^\alpha\ . \tag{3}$$

Originally the Renyi entropy is defined for $0 < \alpha \leq 1$, but analytical expression which we derived for it helped to construct analytic continuation to the whole complex plane of α. at all α then we can find eigenvalues λ_n, which have the meaning of probabilities ($0 < \lambda_n < 1$) and their multiplicities g_n. From the definition of the Renyi entropy (2) we see that

$$\zeta_{\rho_A}(\alpha) = \mathrm{e}^{(1-\alpha) S_R(\rho_A, \alpha)}\ . \tag{4}$$

The aim of this paper is to use this relation to calculate the spectrum of ρ_A for the anisotropic XY model, for which analytical expressions for the Renyi entropy are known explicitly [12, 9]. The XY model is one of the simplest integrable models (since quasiparticle excitations are essentially free fermions), while still having an interesting and non-trivial phase diagram. Our approach is based on the paper [4], where Von Neumann entropy of block of spins in the ground state of the XY model was represented as a determinant of Toeplitz matrix. We used Riemann-Hilbert problem to evaluated the Toeplitz determinant for large block of spins [10] Analytical properties of Von Neumann entropy was studied in [13]. The method of [10] was generalized in [14] for more general quantum spins. The evaluation of Renyi Entropy for the XY model, which was done in [12, 9], uses again the Fisher-Hartwig formulae and the Riemann-Hilbert approach. Although this is the first time that these exact results are used to access the full spectrum of ρ_A, general behaviors has already been known, because of the underlying free fermionic structure.

The degeneracy of the eigenvalues is given by the number of ways a given energy can be realized by different excitations. This is essentially a partitioning problem and it has been addressed already several years ago in an effort toward the optimization of Density Matrix Renormalization Group approaches [17]. Our exact approach will agree with results of [17].

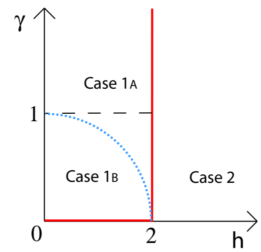

Figure 1: Phase diagram of the anisotropic XY model in a constant magnetic field (only $\gamma \geq 0$ and $h \geq 0$ shown). The three cases 2, 1A, 1B, considered in this paper, are clearly marked. The critical phases ($\gamma = 0$, $h \leq 2$ and $h = 2$) are drawn in bold lines (red, online). The boundary between cases 1A and 1B, where the ground state is given by two degenerate product states, is shown as a dotted line (blue, online). The Ising case ($\gamma = 1$) is also indicated, as a dashed line.

2 Quantum entropies for the XY model

The anisotropic XY spin chain is defined by the Hamiltonian

$$H = - \sum_{j=-\infty}^{\infty} \left[(1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y + h\sigma_j^z \right] \ , \tag{5}$$

where γ is the anisotropy parameter, σ_j^x, σ_j^y and σ_j^z are the Pauli matrices and h is the magnetic field. The Hamiltonian is clearly symmetric under the transformations $\gamma \to -\gamma$ or $h \to -h$, therefore we can consider just the quadrant $\gamma > 0$ and $h \geq 0$ without loss of generality. The system is gapped in the bulk of the phase diagram and has two phase transitions where the spectrum becomes critical: at $\gamma = 0, |h| < 2$ one has the XX model (universality of free fermions on a lattice) and $|h| = 2$ is the critical magnetic field of the Ising phase transition, see Figure 1.

The model was solved in [18, 21, 22, 23]. It is known that its correlation functions can be calculated using methods of Toeplitz determinants and Riemann-Hilbert problems. These techniques were applied in [10, 12, 13, 9] to the study of the quantum entropies.

The density matrix of the unique ground state $|GS\rangle$ of the model is given by $\rho_{AB} = |GS\rangle\langle GS|$. The reduced density matrix of a subsystem A is $\rho_A = Tr_B(\rho_{AB})$. We take the subsystem A to be a block of n consecutive spins (system B is the state of the rest of the chain) and consider the double scaling limit $1 << n << N = \infty$, where N is the total number of sites in the chain, which we take to be infinite. Matrix elements of ρ_A are correlation functions, see formulae (17) and (18) of [12].

The Rényi entropy (2) converges to the von Neumann entropy (1) for $\alpha \to 1$, therefore we can concentrate just on the former quantity. Its analytical expressions were derived in [9] and they can be written as

$$S_R(\rho_A, \alpha) = \frac{1}{6}\ \frac{\alpha}{1-\alpha}\ \ln(k\ k') + \frac{1}{3}\ \frac{1}{1-\alpha}\ \ln\left(\frac{\theta_3^2(0|\alpha \mathrm{i}\tau_0)}{\theta_2(0|\alpha \mathrm{i}\tau_0)\ \theta_4(0|\alpha i\tau_0)} \right) + \frac{1}{3}\ln 2\ , \tag{6}$$

for $h > 2$ and

$$S_R(\rho_A,\alpha) = \frac{1}{6}\ \frac{\alpha}{1-\alpha}\ \ln\left(\frac{k'}{k^2}\right)+\frac{1}{3}\ \frac{1}{1-\alpha}\ \ln\left(\frac{\theta_2^2(0|\alpha i\tau_0)}{\theta_3(0|\alpha i\tau_0)\ \theta_4(0|\alpha i\tau_0)}\right)+\frac{1}{3}\ln 2\ , \quad (7)$$

for $h < 2$; where

$$\tau_0 \equiv \frac{I(k')}{I(k)}\ , \qquad k' = \sqrt{1-k^2}\ , \quad (8)$$

$I(k)$ is the complete elliptic integral of the first kind,

$$I(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad (9)$$

and

$$\theta_j(z|\tau) := \theta_j(z,q)\ , \quad q = \mathrm{e}^{\pi \mathrm{i}\tau} \quad j = 1,2,3,4, \quad (10)$$

are the elliptic theta functions defined by the following Fourier series ($|q| < 1$)

$$\theta_1(z,q) = \mathrm{i}\sum_{m=-\infty}^{\infty}(-1)^m q^{\left(\frac{2m-1}{2}\right)^2}\mathrm{e}^{2\mathrm{i}z\left(m-\frac{1}{2}\right)}\ , \quad (11)$$

$$\theta_2(z,q) = \sum_{m=-\infty}^{\infty} q^{\left(\frac{2m-1}{2}\right)^2}\mathrm{e}^{2\mathrm{i}z\left(m-\frac{1}{2}\right)}\ , \quad (12)$$

$$\theta_3(z,q) = \sum_{m=-\infty}^{\infty} q^{n^2}\mathrm{e}^{2\mathrm{i}zm}\ , \quad (13)$$

$$\theta_4(z,q) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2}\mathrm{e}^{2\mathrm{i}zm}\ . \quad (14)$$

The elliptic parameter $k = k(\gamma, h)$ is defined in the different regions of the phase diagram as

$$k \equiv \begin{cases} \sqrt{(h/2)^2+\gamma^2-1}\ /\ \gamma\ , & \text{Case 1a: } 4(1-\gamma^2) < h^2 < 4; \\ \sqrt{(1-h^2/4-\gamma^2)/(1-h^2/4)}\ , & \text{Case 1b: } h^2 < 4(1-\gamma^2); \\ \gamma\ /\ \sqrt{(h/2)^2+\gamma^2-1}\ , & \text{Case 2 : } h > 2. \end{cases} \quad (15)$$

Alternatively, we can write the Rényi in terms of the λ - *modular function* (see [9]) as

$$S_R = \begin{cases} \frac{1}{6}\ \frac{\alpha}{1-\alpha}\ \ln(k\ k') - \frac{1}{12}\ \frac{1}{1-\alpha}\ \ln\Big[\lambda(\alpha\mathrm{i}\tau_0)\,(1-\lambda(\alpha\mathrm{i}\tau_0))\Big] + \frac{1}{3}\ln 2 & h > 2 \\ \frac{1}{6}\ \frac{\alpha}{1-\alpha}\ \ln\left(\frac{k'}{k^2}\right) - \frac{1}{12}\ \frac{1}{1-\alpha}\ \ln\left[\frac{1-\lambda(\alpha\mathrm{i}\tau_0)}{\lambda^2(\alpha\mathrm{i}\tau_0)}\right] + \frac{1}{3}\ln 2 & h < 2 \end{cases} . \quad (16)$$

The modular function is defined as ($\Im\tau > 0$)

$$\lambda(\tau) = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} \equiv k^2(\tau)\ , \quad \text{and} \quad 1-\lambda(\tau) = \frac{\theta_4^4(0|\tau)}{\theta_3^4(0|\tau)} \equiv {k'}^2(\tau)\ , \quad (17)$$

We note that the basic modular parameter $k \equiv k(\gamma, h)$ defined in (15) coincide with the value of the function $k(\tau)$ at $\tau = \mathrm{i}\tau_0$.

A third representation of the Renyi entropy in terms of $q-$series will be useful to determine the multiplicities of the reduced density matrix eigenvalues [9]:

$$S_R(\rho_A, \alpha) = \begin{cases} \dfrac{1}{12}\dfrac{\alpha}{1-\alpha}\ \ln\left(\dfrac{k^2 k'^2}{16q}\right) + \dfrac{2}{1-\alpha}\ln\displaystyle\prod_{m=0}^{\infty}\left[1+q^{(2m+1)\alpha}\right] & h>2 \\ \dfrac{1}{6}\dfrac{\alpha}{1-\alpha}\ \ln\left(\dfrac{16qk'}{k^2}\right) + \dfrac{2}{1-\alpha}\ln\displaystyle\prod_{m=1}^{\infty}\left[1+q^{2m\,\alpha}\right] + \ln 2 & h<2 \end{cases}, \tag{18}$$

where

$$q \equiv \mathrm{e}^{-\pi\tau_0} = \mathrm{e}^{-\pi I(k')/I(k)}\ . \tag{19}$$

3 Spectrum of ρ_A

Using the expressions for the Renyi entropy we just listed, we now want to determine the eigenvalues λ_n $(0 < \lambda_n < 1)$ the operator ρ_A and their multiplicities g_n, through its momentum function (3) using (4). Using (18) we have

$$\zeta_{\rho_A}(\alpha) = \begin{cases} \mathrm{e}^{\alpha\left(\frac{\pi\tau_0}{12}+\frac{1}{6}\ln\frac{k\,k'}{4}\right)}\displaystyle\prod_{m=0}^{\infty}\left(1+q_\alpha^{2m+1}\right)^2 & h>2 \\ 2\mathrm{e}^{\alpha\left(-\frac{\pi\tau_0}{6}+\frac{1}{6}\ln\frac{k'}{4k^2}\right)}\displaystyle\prod_{m=1}^{\infty}\left(1+q_\alpha^{2m}\right)^2 & h<2 \end{cases}, \tag{20}$$

where

$$q_\alpha \equiv \mathrm{e}^{-\alpha\pi\tau_0} = q^\alpha\ . \tag{21}$$

To use these expression, we will need some results on q-series and elementary notions of the theory of partitions.

Let us concentrate first on the case $h > 2$. Classical arguments of the theory of partitions (see e.g. [24]) tell us that

$$\prod_{n=0}^{\infty}\left(1+q^{2n+1}\right) = \sum_{n=0}^{\infty} p_{\mathcal{O}}^{(1)}(n) q^n, \tag{22}$$

where $p_{\mathcal{O}}^{(1)}(0) = 1$ and $p_{\mathcal{O}}^{(1)}(n)$, for $n > 1$, denotes the number of partitions of n into distinct positive *odd* integers, i.e.

$$\begin{aligned} p_{\mathcal{O}}^{(1)}(n) \quad \equiv \quad & \#\big\{(m_1,\ldots,m_k) : m_j = 2r_j+1, \quad m_1 > m_2 > \ldots > m_k, \\ & n = m_1 + m_2 + \ldots + m_k\big\}\ . \end{aligned} \tag{23}$$

Hence (20) for $h > 2$ becomes

$$\zeta_{\rho_A}(\alpha) = \mathrm{e}^{\alpha\left(\frac{\pi\tau_0}{12}+\frac{1}{6}\ln\frac{k\,k'}{4}\right)}\sum_{n=0}^{\infty} a_n q_\alpha^n\ , \tag{24}$$

where,

$$a_0 = 1 \, , \qquad a_n = \sum_{l=0}^{n} p_{\mathcal{O}}^{(1)}(l) p_{\mathcal{O}}^{(1)}(n-l) \tag{25}$$

Since

$$q_\alpha^n = \left(\mathrm{e}^{-\pi\tau_0 n}\right)^\alpha , \tag{26}$$

we conclude that

$$\zeta_{\rho_A}(\alpha) = \sum_{n=0}^{\infty} a_n \lambda_n^\alpha, \qquad \lambda_n = \mathrm{e}^{-\pi\tau_0 n + \frac{\pi\tau_0}{12} + \frac{1}{6}\ln\frac{k\,k'}{4}}. \tag{27}$$

Comparing the last equation with (3) we arrive at the following theorem:

Theorem 3.1. *Let the magnetic field $h > 2$. Then, the eigenvalues of the reduced density matrix ρ_A are given by*

$$\lambda_n = \mathrm{e}^{\frac{1}{6}\ln\frac{k\,k'}{4} - \pi\frac{I(k')}{I(k)}\left[n - \frac{1}{12}\right]}, \qquad n = 0, 1, 2, \ldots \tag{28}$$

and the corresponding multiplicities $g_n = a_n$ are defined by (25).

The case $h < 2$ is treated in a very similar way. Instead of (22) we use another combinatorial identity,

$$\prod_{n=1}^{\infty}\left(1 + q^{2n}\right) = \sum_{n=0}^{\infty} p_{\mathcal{N}}^{(1)}(n) q^{2n}, \tag{29}$$

where $p_{\mathcal{N}}^{(1)}(0) = 1$ and $p_{\mathcal{N}}^{(1)}(n)$, for $n > 1$, denotes the number of partitions of n into distinct positive integers, i.e.

$$\begin{aligned} p_{\mathcal{N}}^{(1)}(n) \quad \equiv \quad & \#\big\{(m_1, \ldots, m_k) : m_1 > m_2 > \ldots > m_k > 0, \\ & n = m_1 + m_2 + \ldots + m_k\big\}. \end{aligned} \tag{30}$$

It is worth noticing that (see e.g. [24])

$$p_{\mathcal{N}}^{(1)}(n) = p_{\mathcal{O}}(n) \, , \tag{31}$$

where $p_{\mathcal{O}}(n)$ denotes the partitions of n into positive *odd* integers:

$$p_{\mathcal{O}}(n) \equiv \#\big\{(m_1, \ldots, m_k) : m_j = 2r_j + 1, \quad n = m_1 + m_2 + \ldots + m_k\big\}. \tag{32}$$

Here we assume that m_k form an increasing sequence. The analog of equation (24), with the help of (29), now reads as

$$\begin{aligned} \zeta_{\rho_A}(\alpha) \quad &= \quad 2\mathrm{e}^{\alpha\left(-\frac{\pi\tau_0}{6} + \frac{1}{6}\ln\frac{k'}{4k^2}\right)} \sum_{n=0}^{\infty} b_n q_\alpha^{2n} \\ &= \quad 2\sum_{n=0}^{\infty} b_n \lambda_n^\alpha, \qquad \lambda_n = \mathrm{e}^{-2\pi\tau_0 n - \frac{\pi\tau_0}{6} + \frac{1}{6}\ln\frac{k'}{4k^2}}, \end{aligned} \tag{33}$$

where

$$b_0 = 1\ , \qquad b_n = \sum_{l=0}^{n} p_{\mathcal{N}}^{(1)}(l) p_{\mathcal{N}}^{(1)}(n-l)\ . \tag{34}$$

Finally, comparing (33) with equation (3) we arrive at the analog of Theorem 3.1 for the case $h < 2$:

Theorem 3.2. *Let the magnetic field $h < 2$. Then, the eigenvalues of the reduced density matrix ρ_A are given by the equation,*

$$\lambda_n = \mathrm{e}^{\frac{1}{6}\ln\frac{k'}{4k^2} - 2\pi\frac{I(k')}{I(k)}\left[n+\frac{1}{12}\right]}, \qquad n = 0, 1, 2, \dots \tag{35}$$

and the corresponding multiplicities $g_n = 2b_n$ where the integers b_n are determined by (34).

4 Asymptotics of g_n

Consider first the case $h > 2$. Following the usual methodology, we introduce the generating function

$$f(z) := \sum_{n=0}^{\infty} g_n z^n\ . \tag{36}$$

This function is holomorphic in the unit disc, $\tau_0 > 0$. Indeed, we have from (24) that,

$$f(z) = \mathrm{e}^{-\alpha\left(\frac{1}{6}\ln\frac{k\,k'}{4} + \frac{\pi\tau_0}{12}\right)}\zeta_{\rho_A}(\alpha), \quad \alpha = -\frac{1}{\pi\tau_0}\ln z. \tag{37}$$

Statement of holomorphicity then follows from the first equation in (20).

The function $f(z)$ has a singularity at $z = 1$. In order to see this, we deduce from (4) the representation for $f(z)$ in terms of the entropy $S_R(\rho_A, \alpha)$

$$f(z) = \mathrm{e}^{(1-\alpha)S_R(\rho_A,\alpha) - \alpha\left(\frac{1}{6}\ln\frac{k\,k'}{4} + \frac{\pi\tau_0}{12}\right)}, \qquad \alpha = -\frac{1}{\pi\tau_0}\ln z\ . \tag{38}$$

In [9], using the explicit formulae (6) and (16), and the modular properties of the λ - function,

$$\lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau), \qquad \lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau)-1}, \tag{39}$$

it was obtained that

$$S_R(\rho_A, \alpha) \quad = \quad \frac{1}{\alpha(1-\alpha)}\frac{\pi}{12}\frac{I(k)}{I(k')} + \frac{\alpha}{1-\alpha}\frac{1}{6}\ln\frac{k\,k'}{4} + \mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{\alpha\tau_0}}\right)\ , \qquad \alpha \to 0, \quad -\frac{\pi}{2} < \arg\alpha < \frac{\pi}{2}\ . \tag{40}$$

Hence,

$$f(z) = \mathrm{e}^{-\frac{\pi^2}{12\ln z} + \frac{1}{12}\ln z + \mathrm{O}\left(\mathrm{e}^{\pi^2/\ln z}\right)}, \qquad z \to 1, \quad |z| < 1\ . \tag{41}$$

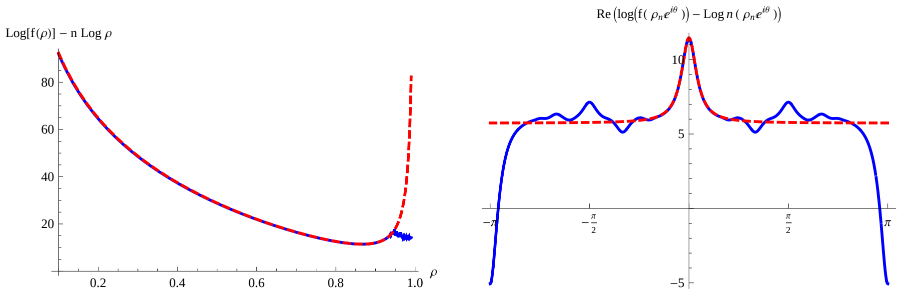

Figure 2: Case $h > 2$: plots of the logarithm of the generating function in the Cauchy integral (42) as a function of the radius ρ (left panel) and of the angular phase θ at the saddle point radius ρ_n given by (45) (right panel; only real part shown). The plots show the comparison between the exact expression (38) (continuous line, blue on-line) and its asymptotic approximation (41) (dashed line, red on-line) at $n = 40$.

The coeffi cients g_n are given by the Cauchy formula,

$$g_n = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1-\epsilon} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z\ , \tag{42}$$

From this formula, the large n asymptotics of g_n can be rigorously obtained by Hardy-Ramanujan-Rademacher circle method (see e.g. [26]) using modular properties (39) of the λ-function. According to the circle method, the leading contribution to integral (42) comes from the neighborhood[2] of the point $1-\epsilon$. This fact can be also demonstrated by plotting the function $g(z) = \ln f(z) - n \ln z$, see Figure 2. Therefore, we can replace the explicit formula (42) by the estimate,

$$g_n \simeq \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z \simeq \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\mathrm{e}^{-G(z)}}{z}\,\mathrm{d}z\ , \quad n \to \infty, \tag{43}$$

where $\mathcal{L} = \{|z| = \rho_n,\ |\arg z| < \delta \le -\ln \rho_n\}$, and

$$G(z) \equiv \frac{\pi^2}{12 \ln z} + \left(n - \frac{1}{12}\right) \ln z\ . \tag{44}$$

We determine ρ_n as the stationary point of $G(z)$, i.e.:

$$\left.\frac{dG(z)}{dz}\right|_{z=\rho_n} = 0 \qquad \Rightarrow \qquad \rho_n = \mathrm{e}^{-\frac{\pi}{\sqrt{12n-1}}} \equiv \mathrm{e}^{-\epsilon_n}\ . \tag{45}$$

[2]The implementation of the circle method in its full power would yield the Hardy-Ramanujan-Rademacher type expansions for the multiplicities g_n (cf. [26] where the classical case of $p_{\mathcal{N}}(n)$ is considered). In this paper, we are only concerned with the leading behavior of g_n, and to this end we only need the localization of the integral near the point $z = 1-\epsilon$. The rigorous proof of this property of integral (42) is not at all trivial. Indeed, it needs again the modular properties of the function $f(z)$.

It should be also mentioned that there are more general techniques of the asymptotic analysis of the partitions, such as Meinardus theorem (see e.g. [24]; see also [25]). These techniques do not exploit the modular properties of the corresponding generating functions, however, unlike the circle method, they only provide the leading terms of the asymptotics. The Meinardus theorem, as it is stated in [24], is not directly applicable to generating function (36).

Switching then to polar coordinate $z = \mathrm{e}^{-\epsilon_n + \mathrm{i}\theta}$, we can rewrite $G(z)$ in the form,

$$G(z) = \frac{\pi^2}{12}(-\epsilon_n + \mathrm{i}\theta)^{-1} + \left(n - \frac{1}{12}\right)(-\epsilon_n + \mathrm{i}\theta)$$

$$= -\frac{\pi}{\sqrt{3}}\sqrt{n - \frac{1}{12}} + \frac{\pi^2}{12\epsilon_n^3}\theta^2\left(1 + \mathrm{O}\left(\frac{\theta}{\epsilon_n}\right)\right), \quad z \in \mathcal{L}. \tag{46}$$

It can be shown, using again the circle method, that there exists a positive κ_0 such that the following choice of the parameter δ in the definition of the arc $\mathcal{L}$ is consistence with estimate (43):

$$\delta = \epsilon_n^{1+2\kappa}, \quad 0 < \kappa < \kappa_0. \tag{47}$$

Using this specification of δ we re-write estimate (46) as

$$G(z) = -\frac{\pi}{\sqrt{3}}\sqrt{n - \frac{1}{12}} + \frac{\pi^2}{12\epsilon_n^3}\theta^2\Big(1 + \mathrm{O}\left(\epsilon_n^{2\kappa}\right)\Big),$$

$$= -\frac{\pi}{\sqrt{3}}\sqrt{n - \frac{1}{12}} + \frac{\pi^2}{12\epsilon_n^3}\theta^2\Big(1 + \mathrm{O}\left(n^{-\kappa}\right)\Big), \quad 0 < \kappa < \kappa_0, \quad n \to \infty, \quad z \in \mathcal{L}. \tag{48}$$

In its turn, this estimate yields the following representation for $G(z)$ on $\mathcal{L}$,

$$G(z) = -\frac{\pi}{\sqrt{3}}\sqrt{n - \frac{1}{12}} + \frac{\pi^2}{12\epsilon_n^3}t^2(\theta), \tag{49}$$

where $t(\theta)$ is a function holomorphic in the neighborhood of the interval $[-\delta, \delta]$ and satisfying the estimates,

$$t(\theta) = \theta\Big(1 + \mathrm{O}\left(n^{-\kappa}\right)\Big), \quad \frac{dt}{d\theta} = 1 + \mathrm{O}\left(n^{-\kappa}\right), \tag{50}$$

$$0 < \kappa < \kappa_0, \quad n \to \infty, \quad \theta \in [-\delta, \delta].$$

From (49) we have that

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \frac{\mathrm{e}^{-G(z)}}{z}\,\mathrm{d}z = \mathrm{e}^{\frac{\pi}{\sqrt{3}}\sqrt{n-\frac{1}{12}}}\frac{1}{2\pi}\int_{-\delta}^{\delta} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}t^2(\theta)}\mathrm{d}\theta. \tag{51}$$

At the same time, equations (50) allow us to use $t = t(\theta)$ as a new integration variable and transform (51) into the asymptotic formula,

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}} \frac{\mathrm{e}^{-G(z)}}{z}\,\mathrm{d}z = \mathrm{e}^{\frac{\pi}{\sqrt{3}}\sqrt{n-\frac{1}{12}}}\frac{1}{2\pi}\int_{t(-\delta)}^{t(\delta)} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}t^2}\frac{d\theta}{dt}\mathrm{d}t$$

$$= \mathrm{e}^{\frac{\pi}{\sqrt{3}}\sqrt{n-\frac{1}{12}}}\frac{1}{2\pi}\left[\int_{-\delta}^{\delta} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}t^2}\mathrm{d}t\Big(1 + \mathrm{O}\left(n^{-\kappa}\right)\Big) + \mathrm{O}\left(n^{-1}\right)\right], \quad n \to \infty, \tag{52}$$

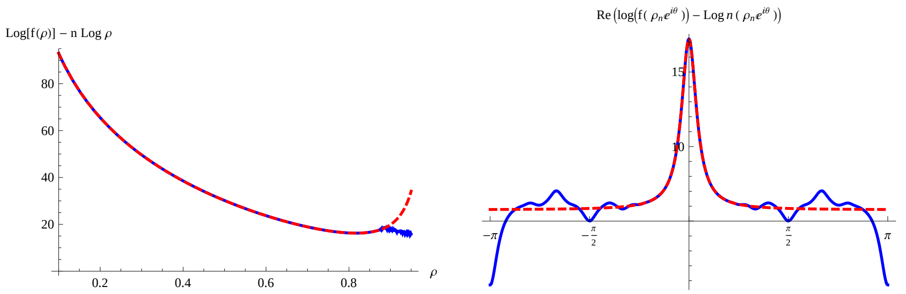

Figure 3: Case $h < 2$: plots of the logarithm of the generating function in the Cauchy integral (42) as a function of the radius ρ (left panel) and of the angular phase θ at the saddle point radius ρ_n given by (61) (right panel; only real part shown). The plots show the comparison between the exact expression (57) (continuous line, blue on-line) and its asymptotic approximation (59) (dashed line, red on-line) at $n = 40$.

Assume now that [3]

$$0 < \kappa < \min\left\{\frac{1}{4}, \kappa_0\right\}. \tag{53}$$

Then, the integral in the right hand side of (52) can be estimated as follows,

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}\theta^2}\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}\theta^2}\mathrm{d}\theta - \frac{1}{\pi}\int_{\delta}^{\infty} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}\theta^2}\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}\theta^2}\mathrm{d}\theta - \epsilon_n^{3/2}\frac{1}{\pi}\int_{\delta\epsilon_n^{-3/2}}^{\infty} \mathrm{e}^{-\frac{\pi^2}{12}\theta^2}\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\pi^2}{12\epsilon_n^3}\theta^2}\mathrm{d}\theta + \mathrm{O}\left(n^{-\infty}\right)$$

$$= \frac{1}{2\pi}\sqrt{\frac{12\epsilon_n^3}{\pi}} + \mathrm{O}\left(n^{-\infty}\right) = 2^{-3/2}3^{-1/4}\left(n - \frac{1}{12}\right)^{-3/4} + \mathrm{O}\left(n^{-\infty}\right) \tag{54}$$

Estimates (43), (52), and (54) yield the following asymptotic formula for the multiplicities g_n of the eigenvalues of the reduced density matrix for $h > 2$.

$$g_n \simeq 2^{-3/2}3^{-1/4}n^{-3/4}\mathrm{e}^{\pi\sqrt{\frac{n}{3}}}, \qquad n \to \infty\,. \tag{55}$$

Turning now to the $h < 2$ case, using (33) and remembering that $g_n = 2b_n$, we have

$$\zeta_{\rho_A}(\alpha) = \mathrm{e}^{\alpha\left(\frac{1}{6}\ln\frac{k'}{4k^2} - \frac{\pi\tau_0}{6}\right)} f\left(\mathrm{e}^{-2\pi\tau_0\alpha}\right), \tag{56}$$

with $f(z)$ defined as in (36). As before, using (4) we can express $f(z)$ in terms of the Renyi Entropy $S_R(\rho_A, \alpha)$:

$$f(z) = \mathrm{e}^{(1-\alpha)S_R(\rho_A,\alpha) - \alpha\left(\frac{1}{6}\ln\frac{k'}{4k^2} - \frac{\pi\tau_0}{6}\right)}, \qquad \alpha = -\frac{1}{2\pi\tau_0}\ln z\,. \tag{57}$$

[3]It is worth noticing, that under condition (53) the term $\mathrm{O}\left(n^{-1}\right)$ in (52) becomes in fact $\mathrm{O}\left(n^{-\infty}\right)$ (exponentially small)

Again, we are interested in the neighbors of the point $z \sim 1$ (see Figure 3, where we can use the asymptotics derived in [9]:

$$S_R(\rho_A,\alpha) \;=\; \frac{1}{\alpha(1-\alpha)}\frac{\pi}{12}\frac{I(k)}{I(k')}+\frac{\alpha}{1-\alpha}\frac{1}{6}\ln\frac{k'}{4k^2}+\mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{\alpha\tau_0}}\right) , \qquad \alpha\to 0,\quad -\frac{\pi}{2}<\arg\alpha<\frac{\pi}{2}\;. \tag{58}$$

Using (58) in (57) we have (cf. (41))

$$f(z)=\mathrm{e}^{-\frac{\pi^2}{6\ln z}-\frac{1}{12}\ln z+\mathrm{O}\left(\mathrm{e}^{2\pi^2/\ln z}\right)}, \qquad z\to 1,\quad |z|<1\;. \tag{59}$$

The calculation proceeds exactly as before, where in (43) we now have

$$G(z)\equiv\frac{\pi^2}{6\ln z}+\left(n+\frac{1}{12}\right)\ln z\;, \tag{60}$$

and the saddle point $z=\rho_n$ is:

$$\left.\frac{dG(z)}{dz}\right|_{z=\rho_n}=0 \qquad\Rightarrow\qquad \rho_n=\mathrm{e}^{-\frac{\pi\sqrt{2}}{\sqrt{12n+1}}}\equiv\mathrm{e}^{-\epsilon_n}\;. \tag{61}$$

Instead of (52) we now have,

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{L}}\frac{\mathrm{e}^{-G(z)}}{z}\,\mathrm{d}z=\mathrm{e}^{\pi\sqrt{\frac{2}{3}}\left(n+\frac{1}{12}\right)}\left[\frac{1}{2\pi}\int_{-\delta}^{\delta}\mathrm{e}^{-\frac{\pi^2}{6\epsilon_n^3}\theta^2}\mathrm{d}\theta\Big(1+\mathrm{O}\left(n^{-\kappa}\right)\Big)+\mathrm{O}\left(n^{-1}\right)\right], \tag{62}$$

with δ defined by exactly the same equation (47). Choosing κ as in (53), we can again approximate the integral in the right hand side of (62) by a complete Gaussian integral (cf. (54):

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}\mathrm{e}^{-\frac{\pi^2}{6\epsilon_n^3}\theta^2}\mathrm{d}\theta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\pi^2}{6\epsilon_n^3}\theta^2}\mathrm{d}\theta+\mathrm{O}\left(n^{-\infty}\right)$$

$$=\frac{1}{2\pi}\sqrt{\frac{6\epsilon_n^3}{\pi}}+\mathrm{O}\left(n^{-\infty}\right)=2^{-5/4}3^{-1/4}\left(n+\frac{1}{12}\right)^{-3/4}+\mathrm{O}\left(n^{-\infty}\right) \tag{63}$$

Estimates (43), (62), and (63) yield the following asymptotic formula [4]for the multiplicities g_n of the eigenvalues of the reduced density matrix for $h<2$.

$$g_n\simeq 2^{-5/4}3^{-1/4}n^{-3/4}\mathrm{e}^{\pi\sqrt{\frac{2}{3}\,n}}\;, \qquad n\to\infty\;. \tag{64}$$

[4]It is worth noticing that for the case $h<2$ the generating function $f(z)$ can be easily transformed to the one satisfying the conditions of the Menardus theorem and hence asymptotics (64) can be also obtained by using the Menardus theorem.

5 Conclusions

We have calculated the spectrum of the reduced density matrix of the ground state of the anisotropic XY model using the known results on its Renyi entropy [9]. This quantity is now known in the literature as the *entanglement spectrum* [8].

We have confirmed the expectation that, being the model essentially non-interacting, the eigenvalues are equidistant and their multiplicities have simple interpretation in terms of combinatorics and different partitions of integer, see Theorem 3.1 and 3.2. The exact formulae for the eigenvalues have been given in terms of the parameters of the model.

The asymptotic behavior of the multiplicities has been calculated, using the modular properties of Renyi entropy. The leading terms of the asymptotics are given in equations (55) and (64) for strong and weak magnetic field, respectively. Our results agree with the estimates in [17]. However, this is not the log-normal behavior quoted in [16] for the XY model and also taken from [17]. In fact, the log-normal result was achieved by combining (and smearing) the degeneracy with the eigenvalue behavior to give an estimate of the behavior of an effective eigenvalue in a non-integrable system. This estimate is important to implement an efficient DMRG calculation for generic systems.

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References

[1] M. A. Nielsen, and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).

[2] A. Rényi, *Probability Theory*, North-Holland (1970)

[3] S. Abe, and A.K. Rajagopal, Phys. Rev. **A 60**, 3461 (1999).

[4] G.Vidal, J.I. Latorre, E.Rico and A. Kitaev, Phys.Rev.Lett.90:227902,2003

[5] C. H. Bennett, and D. P. DiVincenzo, Nature **404** 247 (2000).

[6] H. E. Brandt, *Quantum Information and Computation IV, Proc. SPIE*, Vol. 6244, Bellingham, Washington (2006) pp. 62440G-1-8.

[7] S. Lloyd, Science **261** 1569 (1993); Science **263** 695 (1994).

[8] s H. Li, and F.D.M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).

[9] F. Franchini, A. R. Its, and V. E. Korepin, J. Phys. A: Math. Theor. **41**, 025302 (2008).

[10] A. R. Its, B.-Q. Jin, and V. E. Korepin, J. Phys. A 38, 2975 (2005), and arXiv:quant-ph/0409027 , 2004

[11] I. Peschel, J. Stat. Mech., P12005 (2004) and arXiv:cond-mat/0410416

[12] B.-Q. Jin, and V. E. Korepin, J. Stat. Phys. **116**, 79 (2004).

[13] F. Franchini, A. R. Its, B.-Q. Jin, and V. E. Korepin, J. Phys. A: Math. Theor. **40**, 8467 (2007).

[14] A. R. Its, F. Mezzadri, and M. Y. Mo, Communications in Mathematical Physics, vol: 284, Pages: 117 - 185 (2008)

[15] A. R. Its, and V. E. Korepin, Journal of Statistical Physics: Volume 137, Issue 5 (2009), Page 1014, DOI 10.1007/s10955-009-9835-9, arXiv:0906.4511.

[16] I. Peschel, V. Eisler, arXiv:0906.1663.

[17] K. Okunishi, Y. Hieida, and Y. Akutsu, Phys. Rev. **E 59**, R6227 (1999).

[18] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961).

[19] Y. Xu, H. Katsura, T. Hirano, V. Korepin Jour. Stat. Phys. vol 133, no. 2, 347-377 (2008), see also arXiv:0801.4397

[20] V. Korepin, Y. Xu http://arxiv.org/pdf/0908.2345

[21] E. Barouch, and B.M. McCoy, Phys. Rev. **A 3**, 786 (1971).

[22] E. Barouch, B.M. McCoy, and M. Dresden, Phys. Rev. **A 2**, 1075 (1970).

[23] D.B. Abraham, E. Barouch, G. Gallavotti and A. Martin-Löf, Phys. Rev. Lett. **25**, 1449 (1970); Studies in Appl. Math. **50**, 121 (1971); *ibid* **51**, 211 (1972).

[24] G. E. Andrews, *The Theory of Partition*, Addison-Wesley Publishing Company (1976) as Vol. 2 in *Encyclopedia of Mathematica and its Applications*.

[25] B. Berndt, *Ramanujan's Notebooks. Part IV* Springer-Verlag, New York (1994)

[26] H. Rademacher, The Annals of Mathematics, Second Series, **44**, no. 3, 416 - 422 (1943)

[27] P. Calabrese, and A. Lefevre, Phys. Rev **A 78**, 032329 (2008).

[28] F. Pollmann, J. E. Moore arXiv:0910.0051