Strict algebraic models of weak ω-categories

Sanjeevi Krishnan*

Naval Research Laboratory

Extended Abstract

1 Overview

As clean and potentially convenient higher categorical foundations for studying spacetime manifolds and extending TQFT's, we present algebras over a certain monoidal monad $cubcat$ on cubical sets as models of weak ω-categories and homomorphisms of such algebras up to cubical homotopies as models of weak ω-functors. Examples of $cubcat$-algebras include "singular cubical sets" of spacetime manifolds, compact oriented cobordisms, cubical nerves of small categories, and Kan complexes. Just as strict ω-fold groupoids model weak homotopy types of spaces, $cubcat$-algebras model weak homotopy types of spaces equipped with causal structure. Although we give a flavor for some of the technical details in §2 and indicate some current directions in §3, we assume no special expertise in category theory or topology during our talk.

2 Cubcat-Algebras

Just as small categories amount to reflexive digraphs equipped with a suitable composition operation, we can define higher weak categories to be cubical sets equipped with coherent compositions of higher cubes. We define cubical sets in §2.1; define composable configurations of cubes and requisite coherence conditions for composition in the form of a monad $cubcat$ and identify $cubcat$-algebras in nature in §2.2; present a convenient coherence theorem in §2.3; and sketch a spacetime-geometric interpretation of $cubcat$-algebras in §2.4.

2.1 Cubical sets

Digraphs, presheaves over the category $\square_1$ presented by arrows

$$\delta_-, \delta_+ : [0] \to [1], \quad \sigma : [1] \to [0]$$

subject to the relations $\sigma\delta_- = \sigma\delta_+ = \mathrm{id}_{[0]}$, generalize to higher dimensional *cubical sets*, presheaves over the sub-monoidal category of the Cartesian monoidal category $\mathscr{C}$ of small

*sanjeevi.krishnan@gmail.com

categories and functors between them generated by $\square_1$. We write $\hat{\square}$ for the category of cubical sets and *cubical functions*, natural transformations between cubical sets. A *cubical nerve* functor $N : \mathscr{C} \to \hat{\square}$ sends each functor $F : C \to D$ to

$$\mathscr{C}(-, F) : \mathscr{C}(-, C)_{\restriction \square^{\mathrm{op}}} \to \mathscr{C}(-, D)_{\restriction \square^{\mathrm{op}}}.$$

2.2 Cubical pasting schemes

All possible coherent composable configurations of cubes in a cubical set C form another cubical set $cubcat(C)$ defined as follows. We write $\Diamond\!\Diamond$ for the smallest subcategory of $\mathscr{C}$ containing $\square$ such that for all natural numbers n, $\Diamond\!\Diamond$-map $\alpha : [1]^{\otimes n} \to \mathfrak{p}$ preserving minima and $\Diamond\!\Diamond$-map $\beta : [1]^{\otimes n} \to \mathfrak{q}$ preserving maxima, there exist dotted $\Diamond\!\Diamond$-diagrams making the square

$$\begin{array}{ccc} \mathfrak{p} & \dashrightarrow & \mathfrak{r} \\ {\scriptstyle \beta}\uparrow & & \vdots\uparrow \\ [1]^{\otimes n} & \xrightarrow[\alpha]{} & \mathfrak{q}, \end{array} \tag{1}$$

a pushout in $\mathscr{C}$. We write $cubcat$ for the functor

$$cubcat(-_1)(-_2) = \int_{\Diamond\!\Diamond}^{\mathfrak{p}} \hat{\square}(\Diamond\!\Diamond[-_2, \mathfrak{p}]_{\restriction \square^{\mathrm{op}}}, -_1) \cdot N\,\mathfrak{p} : \hat{\square} \to \hat{\square}$$

and μ for the natural transformation $\mathrm{id}_{\hat{\square}} \to cubcat$ induced from the inclusions $\square \hookrightarrow \Diamond\!\Diamond$ and $\square \hookrightarrow \mathscr{Q}$. We henceforth regard $cubcat$ as a monad defined as follows.

Proposition 2.1. *There exists a unique natural transformation*

$$cubcat^2 \to cubcat : \hat{\square} \to \hat{\square} \tag{2}$$

turning $cubcat$ into a monad having unit μ.

Algebras over $cubcat$ are ubiquitous. We recall that *singular cubical sets of directed spaces* [1] are cubical sets whose n-hypercubes are locally monotone maps from ordered topological hypercubes into spaces equipped with temporal structure, *Kan cubical sets* are fibrant objects in a Quillen model category modelling classical weak homotopy types, and *cubical nerves* are just cubical sets of the form $N\mathscr{G}$ for all small categories $\mathscr{G}$.

Theorem 2.2. *The following cubical sets underlie $cubcat$-algebras.*

1. *Singular cubical sets of directed spaces.*

2. *Cubical nerves of small categories*

3. *Kan cubical sets*

These examples suggest that $cubcat$-algebras (1) model weak compositions, (2) generalize categories, and (3) satisfy the Homotopy Hypothesis in some sense.

2.3 A coherence theorem

The monad $cubcat$ encodes pasting schemes and weak associativity conditions into a single structure. It suffices to check that a potential composition operation on a cubical set be unital in order to conclude that it defines a $cubcat$-algebraic multiplication.

Theorem 2.3. *For each cubical set C, every retraction*

$$cubcat(C) \to C$$

of μ_C turns C into a $cubcat$-algebra.

As an application, we illustrate how compact oriented cobordisms form a $cubcat$-algebra whose structure map corresponds to gluing.

2.4 Weak directed types

We sketch a construction of a geometric realization

$$\rceil - \lfloor : \hat{\square} \to \mathscr{S}$$

from $\hat{\square}$ to a category $\mathscr{S}$ of directed spaces and localizations $\bar{h}\hat{\square}$ and $\bar{h}\hat{\mathscr{S}}$ of $\hat{\square}$ and $\mathscr{S}$ with respect to “weak equivalences.” In particular, a weak equivalence of cubical sets is a cubical function ψ such that the induced cubical function $\hom_{\otimes}(\psi, Z)$ of mapping cubical sets passes to a bijection on connected components for each $cubcat$-algebra Z. The following theorem expands upon previous cubical approximation theorems [3] in directed topology.

Theorem 2.4. *The adjunction $\rceil - \lfloor \dashv \mathrm{sing}$ passes to an equivalence*

$$\bar{h}\hat{\square} \leftrightarrows \bar{h}\mathscr{S}.$$

We speculate on how such a result might facilitate the construction of hypothetical $cubcat$-theoretic, and hence ω-dimensional, analogues of TQFT’s as homotopy classes of directed maps of directed spaces.

3 Current work

Current work in progress includes: comparing $cubcat$-algebras with other models of higher categories (such as algebras over the initial globular monad-with-contraction [2]); relating the geometry of spacetimes with categorical properties of associated singular cubical sets (as part of joint work with Keye Martin); and investigating “n-strictifications” of $cubcat$-algebras as analogues of nth homotopy groups of based spaces. Time permitting, we discuss current progress and conjectures in these areas.

References

[1] M. Grandis, *Directed algebraic topology: models of irreversible worlds*, Cambridge University Press, 2009, pp. 449.

[2] T. Leinster, *Higher Operads, Higher Categories*, preprint posted at `arXiv:math/0305049v1`.

[3] S. Krishnan, *Cubical approximation for directed topology*, preprint posted at `http://mathsci.sungv.info`.