Modal quantum theory

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Abstract

We present a discrete model theory similar in structure to ordinary quantum mechanics, but based on a finite field instead of complex amplitudes. The interpretation of this theory involves only the "modal" concepts of possibility and necessity rather than quantitative probability measures. Despite its simplicity, our model theory includes entangled states and has versions of both Bell's theorem and the no cloning theorem.

Modal quantum theory

In quantum theory, the states of physical systems are represented by vectors in a complex Hilbert space. The complex scalars serve as probability amplitudes, quantities whose squared magnitudes are the probabilities of measurement outcomes. Other types of quantum theory have sometimes been considered, based on real or quaternionic amplitudes [1, 2]. Though the quantum mechanics of nature does not appear to be real or quaternionic, these alternate mathematical formalisms shed light on the structure of the actual quantum theory (which we will here abbreviate AQT).

Here we will explore the properties of another type of "toy model" of quantum theory using scalars drawn from a finite field $\mathcal{F}$. The simplest example is based on the two-element field $\mathbb{Z}_2$, but many other choices are possible. Our toy model lacks much of the mathematical paraphernalia of complex Hilbert spaces. For instance, there is no natural inner product and thus no concept of "orthogonality" between vectors. Nevertheless, we will find that the theory is well-defined; that it has a sensible interpretational framework; and that entanglement and many other aspects of AQT have analogues in the theory.

The interpretation of AQT involves quantitative probabilities, but our interpretation of finite-field theories is more primitive, involving only the distinction between *possible* and *impossible* events. Suppose $\mathcal{E}$ is the set of outcomes of some experiment. In AQT, a given quantum state would yield a probability distribution over the elements of $\mathcal{E}$. But our new theory will only designate a non-empty subset $\mathcal{R} \subseteq \mathcal{E}$, the set of possible results, without distinguishing more or less likely elements of the set. Any outcome not in $\mathcal{R}$ is taken to be impossible, and if $\mathcal{R}$ only contains a single element r, then we may say that r is "certain" or "necessary".

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This distinction between "possible", "impossible" and "necessary" events is exactly the distinction used in modal logic [3]. Thus, we will refer to our finite-field quantum theories as *modal quantum theory*, or MQT.

For a finite field $\mathcal{F}$, the MQT state of a system is a non-null vector $[\psi]$ in a finite-dimensional vector space $\mathcal{V}$, which is isomorphic to $\mathcal{F}^d$ for some dimension d. A measurement on the system corresponds to a basis set $A = \{[a]\}$ for $\mathcal{V}$, where each basis element $[a]$ is associated with an outcome a of the measurement procedure. (Note that, in the absence of an inner product, there is no requirement in MQT that the basis elements be orthogonal.) Every state vector $[\psi]$ can be written

$$[\psi] = \sum_a \psi_a \, [a] \, , \tag{1}$$

where the coefficients ψ_a are scalars in $\mathcal{F}$. The measurement outcome a is possible if and only if $\psi_a \neq 0$. For the basis A and state $[\psi]$, the set of possible measurement results is thus

$$\mathcal{R}\left(A|\,[\psi]\right) = \{a : \psi_a \neq 0\}. \tag{2}$$

The simplest type of MQT has $\mathcal{F} = \mathbb{Z}_2$, and the simplest MQT system has state space dimension $d = 2$. The resulting example may be called a *mobit*. A mobit has three states: basis states $[0]$ and $[1]$, and a single superposition state $[\sigma] = [0] + [1]$. In fact, any one of these states is a superposition of the other two, and so any pair of the states is a basis for the vector space. We define three modal observables, which we will call X, Y and Z, associated with the three possible basis sets. For each measurement, we can conveniently label the two outcomes by $+$ and $-$. That is,

$$\begin{array}{lll} [+_z] = [0] & [+_x] = [1] & [+_y] = [\sigma] \\ [-_z] = [1] & [-_x] = [\sigma] & [-_y] = [0] \end{array} . \tag{3}$$

Finally, we can outline a framework for describing the time evolution of a system in MQT. Time must be regarded as a sequence of discrete intervals. Just as in AQT, the "coherent" time evolution of a system over one of these intervals is represented by a linear transformation T of the state vector. Thus, if $[a] \rightarrow [a'] = T\,[a]$ and $[b] \rightarrow [b'] = T\,[b]$ then

$$[a] + [b] \rightarrow T\left([a] + [b]\right) = T\,[a] + T\,[b] = [a'] + [b'] \, . \tag{4}$$

Since the zero vector is not a physical state, we require that $T\,[a] \neq 0$ for any state $[a]$. This means that the kernel of T is trivial, so that T is invertible.

No additional restriction (such as unitarity in AQT) on the time evolution operator T is motivated by the general framework of MQT. We will generally suppose that any invertible linear transformation of state vectors corresponds to a possible time evolution of the system.

Entangled states

In AQT, the Hilbert space describing a composite system is the tensor product of the Hilbert spaces describing the individual subsystems. The same rule applies to the vector spaces in MQT. In general, a composite system may have both product states and non-product (entangled) states. Since the state spaces in MQT are discrete, we can calculate the numbers of product and entangled states for a given pair of systems. We find that every composite system has both product and entangled states, and that as the subsystem state space dimensions become large, the entangled states greatly outnumber the product states.

Consider a pair of mobits, for which $\mathcal{F} = \mathbb{Z}_2$. There are 15 allowed state vectors for the pair, all representing distinct states of the system. Nine of these are product states and six are entangled.

One particular entangled state of two mobits has especially elegant properties: $[S] = [0,1] + [1,0]$. Using Equation 3, we can write this state with respect to the X, Y and Z bases for the two mobits:

$$[S] = [+_z, -_z] + [-_z, +_z] = [+_x, -_x] + [-_x, +_x] = [+_y, -_y] + [-_y, +_y] . \quad (5)$$

If the same measurement is made on each mobit, the results of those measurements must necessarily disagree. (In this way, the MQT state $[S]$ is analogous to the "singlet" state $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ of a pair of spins in AQT.)

If different measurements are made on the two mobits, the correspondences are somewhat different in form. Consider a joint measurement of (Z, X). We can write

$$[S] = [+_z, +_x] + [-_z, +_x] + [-_z, -_x] . \quad (6)$$

Thus, every joint result is possible for this measurement except for $(+_z, -_x)$. Other combinations of measurements lead to similar results, which can be obtained by a cyclic permutation of X, Y and Z.

In AQT, Bell showed that the correlations between entangled quantum systems were incompatible with any local hidden variable theory [4]. The same is true for the correspondences between entangled mobits in the $[S]$ state defined above. We suppose that experimenters Alice and Bob share such a pair, and that X, Y and Z measurements are possible on each one.

In a hidden variable theory, we imagine that the MQT state $[S]$ corresponds to a set Λ of possible values of a hidden variable. We further imagine that the hidden variable controls the outcomes of the possible measurements on the mobits in a completely local way. That is, for any particular value $\lambda \in \Lambda$, the set of possible results of Alice's measurement depends only on λ and her own choice of measurement, not any measurement choices or results for Bob's mobit. Let $\mathcal{R}_\lambda(E)$ be the set of possible results of a measurement of E for the hidden variable value λ. Our locality assumption means that, given A and B measurements for Alice and Bob and a particular λ value,

$$\mathcal{R}_\lambda(A, B) = \mathcal{R}_\lambda(A) \times \mathcal{R}_\lambda(B) , \quad (7)$$

the simple Cartesian product of separate sets $\mathcal{R}_\lambda(A)$ and $\mathcal{R}_\lambda(B)$.The MQT set of possible results should therefore be

$$\mathcal{R}(A, B|\,[S]) = \bigcup_{\lambda \in \Lambda} \mathcal{R}_\lambda(A) \times \mathcal{R}_\lambda(B) . \quad (8)$$

The individual sets $\mathcal{R}_\lambda(A)$ are simultaneously defined for all of the measurements that can be made by Alice and Bob. Therefore, we may consider the set

$$\begin{aligned} \mathcal{J} \;\; = \;\; \bigcup_{\lambda \in \Lambda} \; & \mathcal{R}_\lambda(X^{(A)}) \times \mathcal{R}_\lambda(Y^{(A)}) \times \mathcal{R}_\lambda(Z^{(A)}) \\ & \times \mathcal{R}_\lambda(X^{(B)}) \times \mathcal{R}_\lambda(Y^{(B)}) \times \mathcal{R}_\lambda(Z^{(B)}) . \end{aligned} \quad (9)$$

There might be up to $2^6 = 64$ elements in $\mathcal{J}$. However, since $\mathcal{J}$ can only contain elements that agree with the properties of $[S]$, we can eliminate many elements. In fact, when all such restrictions are applied, we find the surprising result that *all* of the elements of $\mathcal{J}$ are eliminated. *No* assignment of definite results to all six possible measurements can possibly agree with the correspondences obtained from the entangled MQT state $[S]$. We therefore conclude that these correspondences are incompatible with any local hidden variable theory.

Mixed states and cloning

In both actual quantum theory and modal quantum theory, a *mixed state* arises when we cannot ascribe a definite quantum state vector to a system. This may happen because several state vectors are possible, or because the system is only a part of a larger system in an entangled state.

Suppose that a system in MQT might be in any one of several possible states $[\psi_1]$, $[\psi_2]$, etc. We collect these together into a set $\mathcal{M}$, which characterizes the mixture of states. A measurement result is possible for the mixture $\mathcal{M}$ if it is possible for at least one of the state vectors in $\mathcal{M}$. For a basis A, we write that

$$\mathcal{R}\left(A|\mathcal{M}\right) = \bigcup_{[\psi]\in\mathcal{M}} \mathcal{R}\left(A|\left[\psi\right]\right). \tag{10}$$

Two mixtures $\mathcal{M}$ and $\mathcal{M}'$ are equivalent provided they lead to exactly the same possible measurement results—i.e., that $\mathcal{R}\left(A|\mathcal{M}\right) = \mathcal{R}\left(A|\mathcal{M}'\right)$ for any basis A. When this is true, we say that $\mathcal{M}$ and $\mathcal{M}'$ correspond to the same mixed state of the system. We denote this mixed state by $\langle\mathcal{M}\rangle = \langle\mathcal{M}'\rangle$.

It can be shown that $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if and only if their vector space spans are equal. We therefore identify the mixed state $\langle\mathcal{M}\rangle$ for the mixture $\mathcal{M}$ with the subspace spanned by $\mathcal{M}$.

How can we arrive at a mixed state for a subsystem of an entangled system in MQT? Suppose systems #1 and #2 have a joint state vector $[\psi^{(12)}]$. Given a basis $A = \{[a]\}$ for system #1, we can write this as

$$[\psi^{(12)}] = \sum_a \left[a, \psi_a\right]. \tag{11}$$

We can take the non-zero states $[\psi_a]$ that appear in this to define a mixture $\mathcal{M}$ for system #2, which defines in turn a mixed state $\langle\mathcal{M}\rangle$. It is straightforward to show that this mixed state for system #2 is independent of the choice of basis A for system #1.

Finally, we note that a no-cloning theorem holds in MQT, and that its proof is virtually identical to that of Wootters and Zurek for AQT [5]. We imagine that a cloning machine that successfully copies input states $[a]$ and $[b]$, a process that can be represented by the evolution of the input, output and machine systems:

$$[a, 0, M_0] \rightarrow [a, a, M_a] \quad \text{and} \quad [b, 0, M_0] \rightarrow [b, b, M_b]. \tag{12}$$

If we now consider the superposition input state $[c] = [a] + [b]$, linearity of the overall evolution means that the final state of input and output is instead either a superposition or mixture of $[a, a]$ and $[b, b]$ (depending on the relation of the final machine states $[M_a]$ and $[M_b]$). In neither case do we obtain the cloned state $[c, c] = [a, a] + [a, b] + [b, a] + [b, b]$. Therefore, any cloning machine in MQT must fail for some input states.

References

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