kindergarten quantum mechanics

bOb cOEckE — Oxford University

se10.comlab.ox.ac.uk:8080/BobCoecke/Home_en.html

(or Google ''Bob Coecke'')

PROLOGUE

“BEAM ME UP SCOTTY!”

“HOW DO I DO THAT?”

"HERE'S THE CODE"

The physicist's description

Alice has an 'unknown' qubit $|\phi\rangle$ and wants to send it to Bob. They have the ability to communicate classical bits, and they share an entangled pair in the EPR-state, that is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which Alice produced by first applying a $\mathsf{Hadamard}$-gate $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the first qubit of a qubit pair in the ground state $|00\rangle$, and by then applying a CNOT-gate, that is

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |01\rangle \quad |10\rangle \mapsto |11\rangle \quad |11\rangle \mapsto |10\rangle,$$

then **she sends the first qubit of the pair to Bob**.

To teleport her qubit, Alice first performs a bipartite measurement on the unknown qubit and her half of the entangled pair in the Bell-base, that is

$$\{|0x\rangle + (-1)^z|1(1-x)\rangle \mid x, z \in \{0, 1\}\},$$

where we denote the four possible outcomes of the measurement by xz. Then **she sends the 2-bit outcome xz to Bob using the classical channel**.

Then, if $x = 1$, Bob performs the unitary operation $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on its half of the shared entangled pair, and he also performs a unitary operation $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on it if $z = 1$. Now **Bob's half of the initially entangled pair is in state $|\phi\rangle$**.

The physicist's proof

In the case that the measurement outcome of the Bell-base measurement is xz, for

$$\mathrm{P}_{xz} := \langle 0x + (-1)^z 1(1-x)|-\rangle|0x + (-1)^z 1(1-x)\rangle$$

we have to apply $\mathrm{P}_{xz} \otimes \mathsf{id}$ to the input state

$$|\phi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,.$$

Setting $|\phi\rangle := \phi_0|0\rangle + \phi_1|1\rangle$ we rewrite the input as

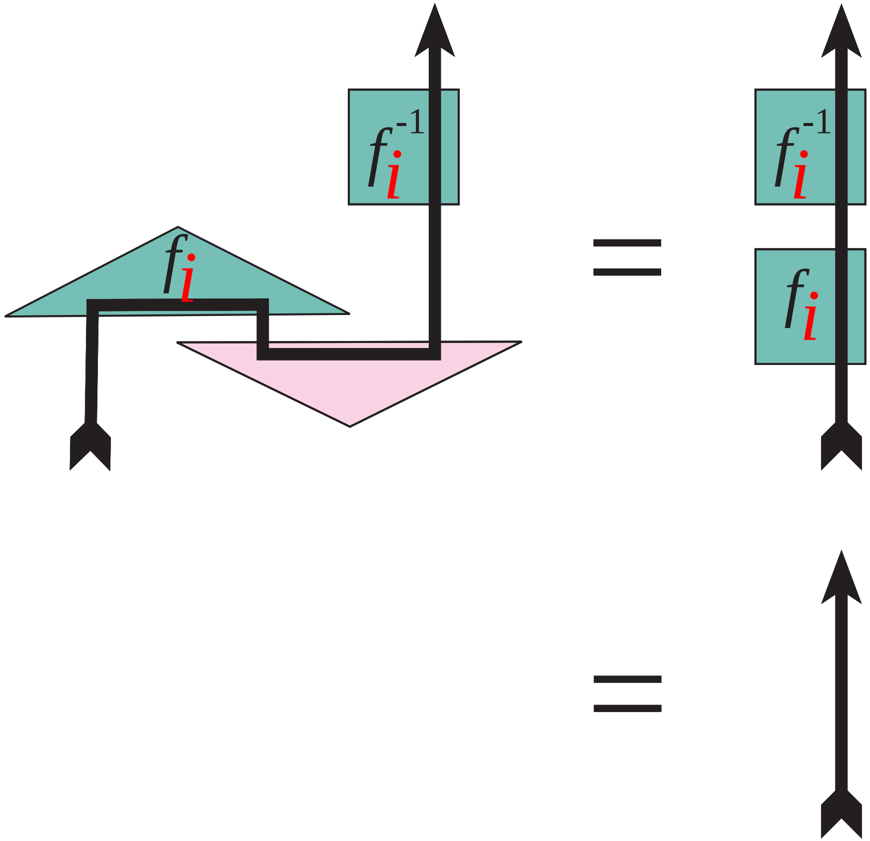
$$\frac{1}{\sqrt{2}}(\phi_0|000\rangle + \phi_0|011\rangle + \phi_1|100\rangle + \phi_1|111\rangle) = \frac{1}{\sqrt{2}}(\phi_0\sum_{x=0,1}|0xx\rangle + \phi_1\sum_{x=0,1}|1(1-x)(1-x)\rangle)$$

and application of $\mathrm{P}_{xz} \otimes \mathsf{id}$ then yields

$$\frac{1}{\sqrt{2}}|0x + (-1)^z 1(1-x)\rangle \otimes (\phi_0|x\rangle + (-1)^z\phi_1|1-x\rangle).$$

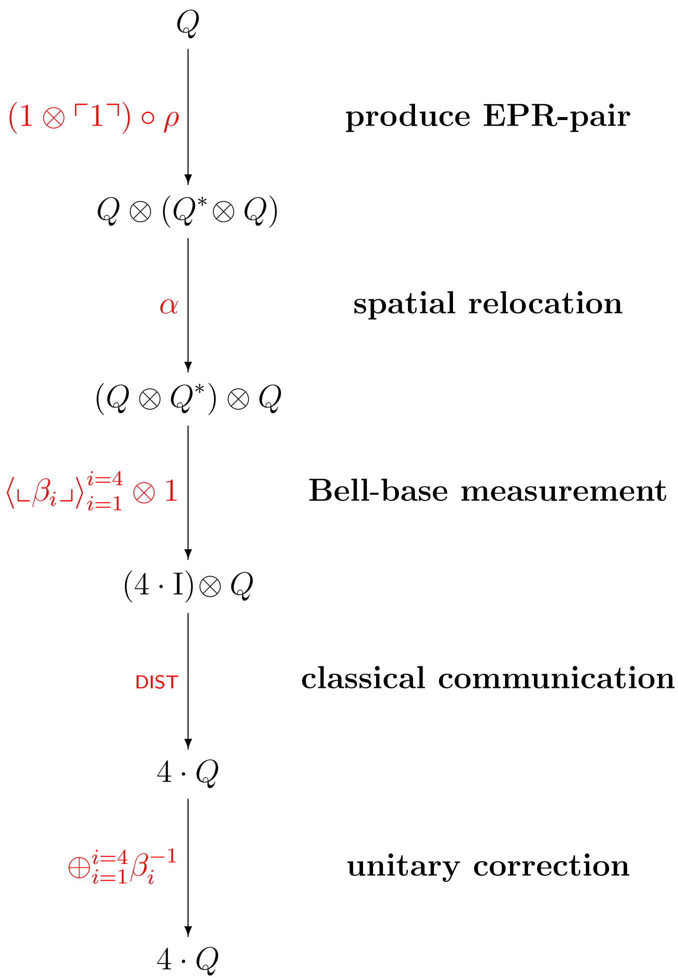
There are four cases concerning the unitary corrections U_{xz} which have to be applied. For $x = z = 0$ the third qubit is $\phi_0|0\rangle + \phi_1|1\rangle = |\phi\rangle$. **If** $x = 0$ **and** $z = 1$ it is $\phi_0|0\rangle - \phi_1|1\rangle$ which after applying $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ becomes $|\phi\rangle$. **If** $x = 1$ it is $\phi_0|1\rangle + (-1)^z\phi_1|0\rangle$ which after applying $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ brings us back to the previous two cases, what completes this proof. $\square$

Our description and proof



... purely syntactically, ...

$$\rho_Q \,;\, 1 \otimes \ulcorner 1 \urcorner \,;\, \alpha \,;\, \langle \llcorner \beta_i \lrcorner \rangle_{i=1}^{i=4} \otimes 1 \,;\, \mathsf{DIST} \,;\, \bigoplus_{i=1}^{i=4} \beta_i^{-1}$$


Q
$(1 \otimes \ulcorner 1 \urcorner) \circ \rho$
produce EPR-pair
$Q \otimes (Q^* \otimes Q)$
α
spatial relocation
$(Q \otimes Q^*) \otimes Q$
$\langle \llcorner \beta_i \lrcorner \rangle_{i=1}^{i=4} \otimes 1$
Bell-base measurement
$(4 \cdot \mathrm{I}) \otimes Q$
DIST
classical communication
$4 \cdot Q$
$\oplus_{i=1}^{i=4} \beta_i^{-1}$
unitary correction
$4 \cdot Q$

THE CHALLENGE

Why did discovering quantum teleportation take 60 year?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\mathbf{GOOD\ QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\mathbf{GOOD\ QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{"}\mathbf{GOOD\ QM}\text{"}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I'll present it to you.

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\textbf{GOOD QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\textbf{HIGH-LEVEL language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I'll present it to you.

Isn't it absurdly abstract coming from you guys?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“GOOD QM”}}{\texttt{von Neumann QM}} \simeq \frac{\text{HIGH-LEVEL language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I'll present it to you.

Isn't it absurdly abstract coming from you guys?

Claim: It could be taught in **kindergarten**!

THE ACTS

Prehistory

Hilbert space Quantum Formalism [von Neumann 1932]

Prehistory

Hilbert space Quantum Formalism [von Neumann 1932]

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." [von Neumann 1935]

Rédei, M. (1997) ***Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead).*** **Studies in History and Philosophy of Modern Physics 27, 493–510.**

1. **Analyse** quantum compoundness.

⇒ A notion of **quantum information flow** emerges.

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; `cs/0207057`
- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

1. **Analyse** quantum compoundness.

$\Rightarrow$ A notion of **quantum information flow** emerges.

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; `cs/0207057`
- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. **Axiomatize** quantum compoundness.

$\Rightarrow$ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

1. **Analyse** quantum compoundness.

$\Rightarrow$ A notion of **quantum information flow** emerges.

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; `cs/0207057`
- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. **Axiomatize** quantum compoundness.

$\Rightarrow$ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

$\Rightarrow$ ... & **quantum logic** & **open systems/CPM's**!

- De-linearizing Linearity: Projective Quantum Axiomatics from SCC. Coecke (2005) QPL'05; `quant-ph/0506134`.
- †-CCC's and Completely Positive Maps. Selinger (2005) QPL'05.

THE FEATURES

EXPLICIT OPERATIONALISM

Processes/operations

$$f : A \longrightarrow B \quad g : B \longrightarrow C \quad h : A \longrightarrow A$$

are the **typed primitive data** where $A, B, C, \ldots$ are the types of the **systems** involved.

Sequential composition is primitive

$$f \circ g : A \longrightarrow C \quad \text{for} \quad f : A \longrightarrow \underline{B} \ \& \ g : \underline{B} \longrightarrow C$$

Parallel composition is a primitive

$$f \otimes g : A \otimes C \longrightarrow B \otimes D \quad \text{for} \quad f : A \longrightarrow B \ \& \ g : C \longrightarrow D$$

CATEGORY THEORY!

CATEGORY THEORY!

Audience: "Seriously, you don't expect us to learn that?"

CATEGORY THEORY!

Audience: "Seriously, you don't expect us to learn that?"

Bob: "**No! Of course not!**"

CATEGORY THEORY!

Audience: “Seriously, you don’t expect us to learn that?”

Bob: “**No! Of course not!**”

“We are gonna go far back in time, ...
to the time you were all still at kindergarten, ...”

CATEGORY THEORY!

Audience: “Seriously, you don’t expect us to learn that?”

Bob: “**No! Of course not!**”

“We are gonna go far back in time, ...
to the time you were all still at kindergarten, ...”

“**We’re gonna draw pictures!**”

CATEGORY THEORY!

Audience: "Seriously, you don't expect us to learn that?"

Bob: "**No! Of course not!**"

"We are gonna go far back in time, ...
to the time you were all still at kindergarten, ..."

"**We're gonna draw pictures!**"

The sheer magic of the kind of category theory we need here is that it formally justifies its own formal absence.

THE NEW FORMALISM

Language and calculus: purely graphical

Language and calculus: purely graphical

Behind the scene: categorical algebra

Language and calculus: purely graphical

Behind the scene: categorical algebra

Concrete model: Hilbert space QM, ...

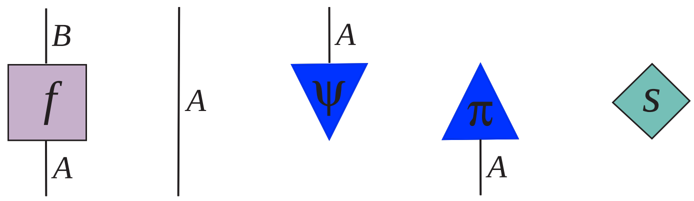
Language and calculus: purely graphical

Behind the scene: categorical algebra

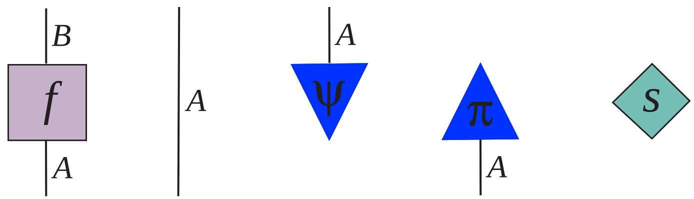
Concrete model: Hilbert space QM, ...
and also many others, ...

Not assumed: some number field, any kind of matrix calculus, vectors and sums thereof, elements of objects/types (cf. state space) and corresponding mappings, ...

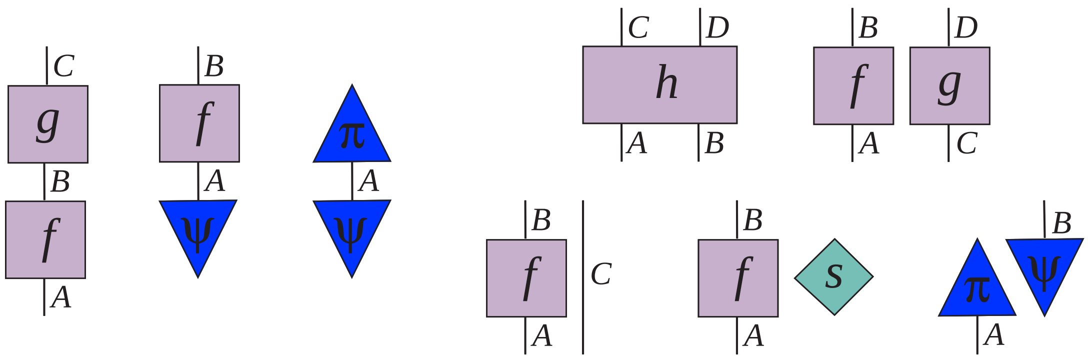
Primitive data:



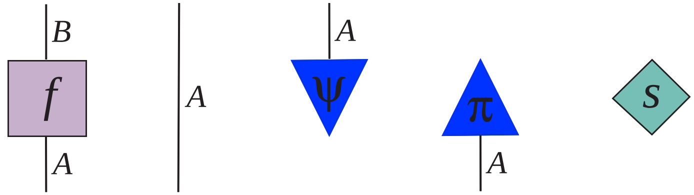
Primitive data:



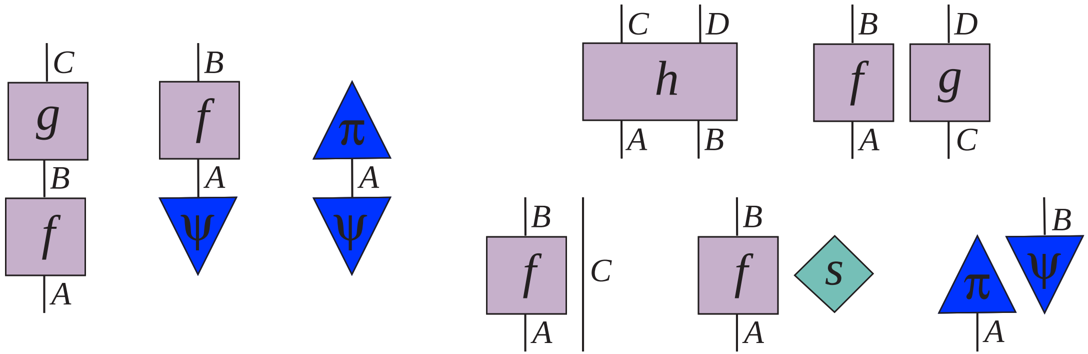
Sequential and parallel composition:



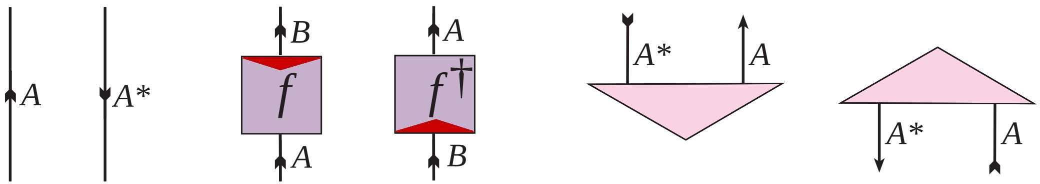
Primitive data:



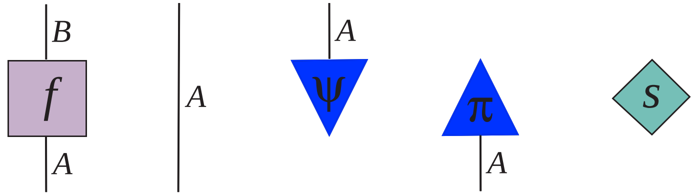
Sequential and parallel composition:



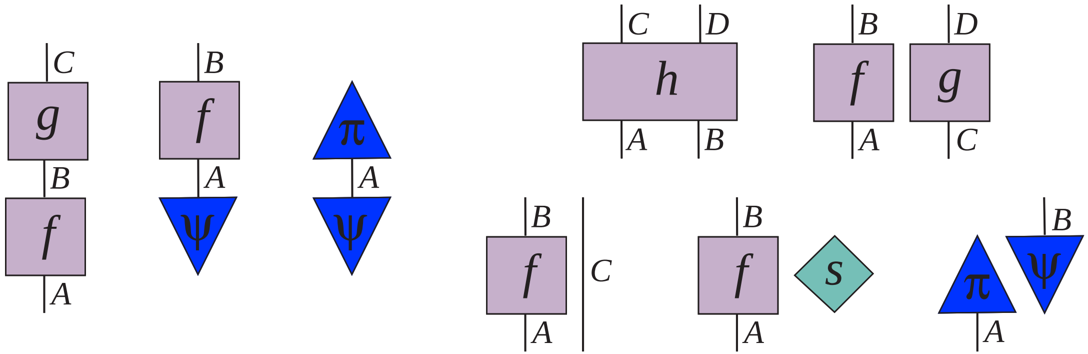
Duals, adjoints and EPR-states:



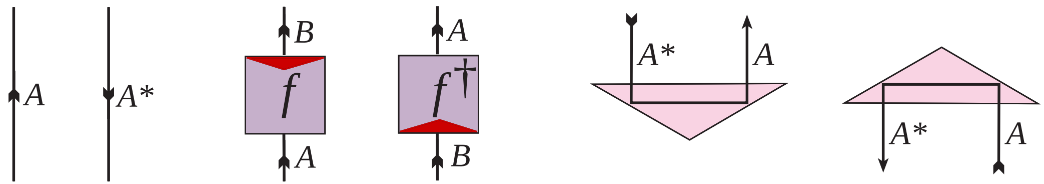
Primitive data:



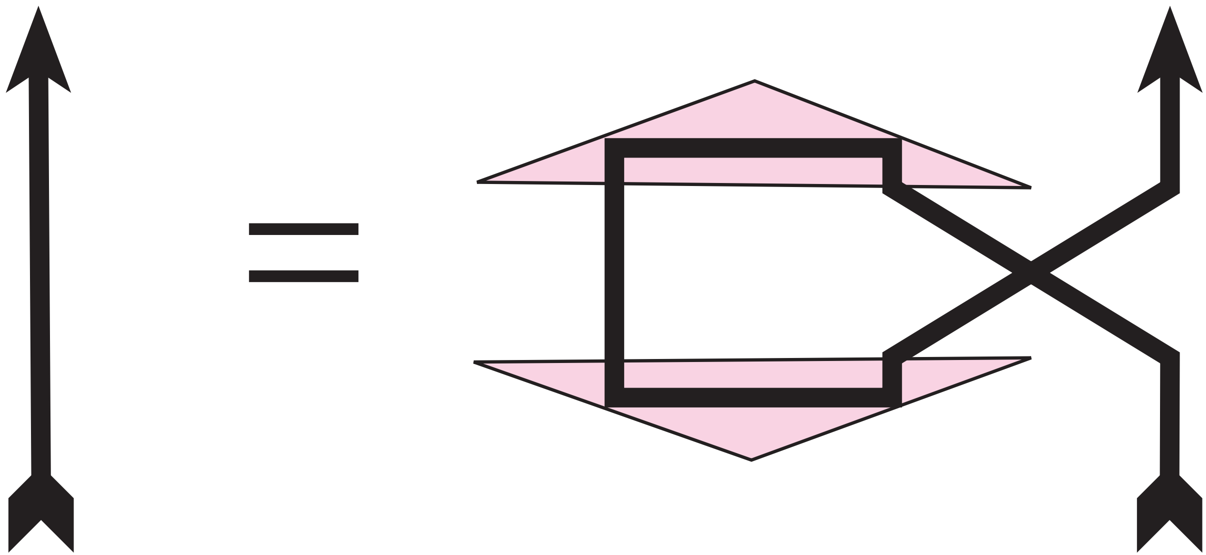
Sequential and parallel composition:



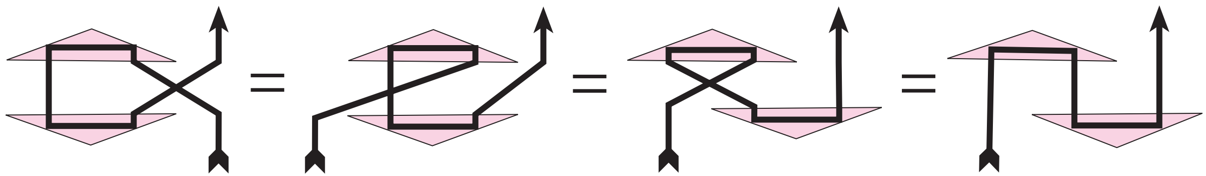
Duals, adjoints and EPR-states:



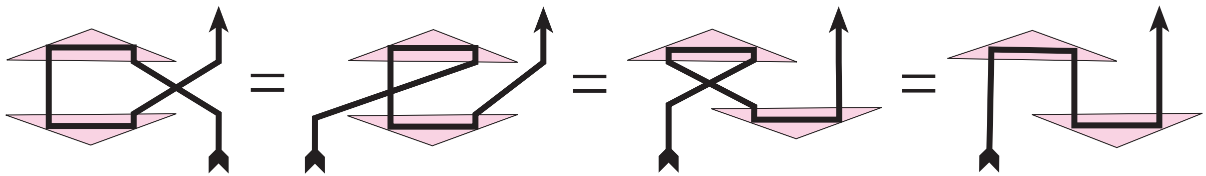
THE SOLE AXIOM



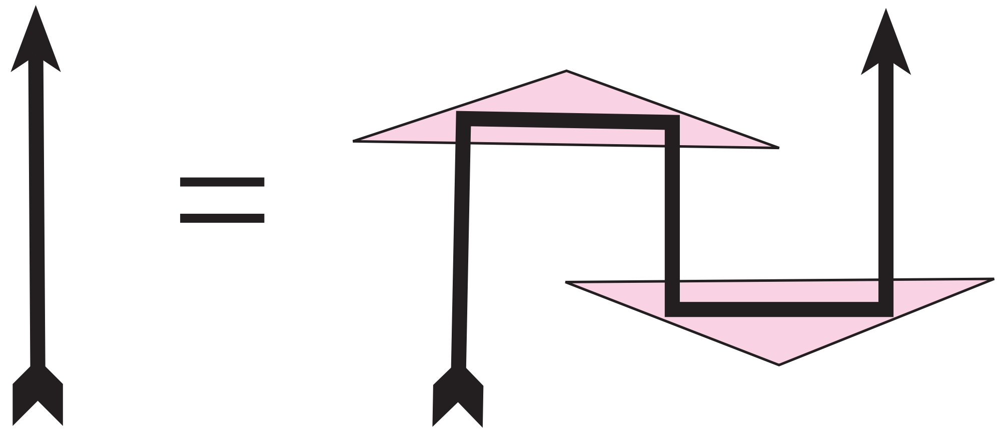
Since



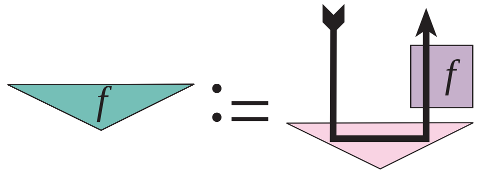
Since

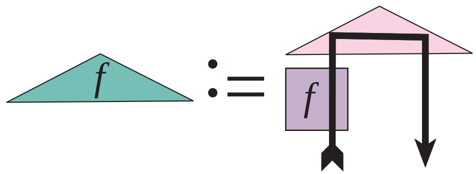


the axiom is equivalent to

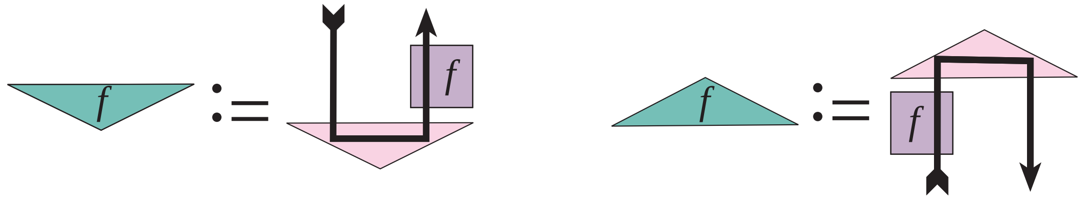


When setting

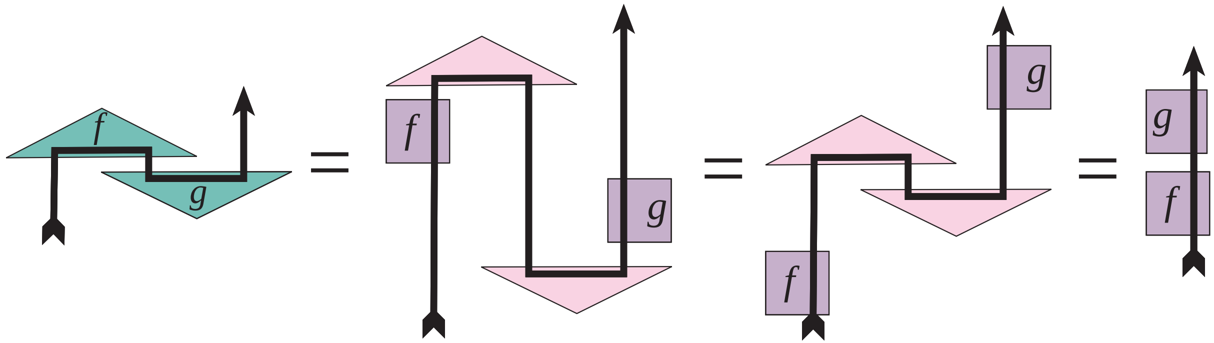




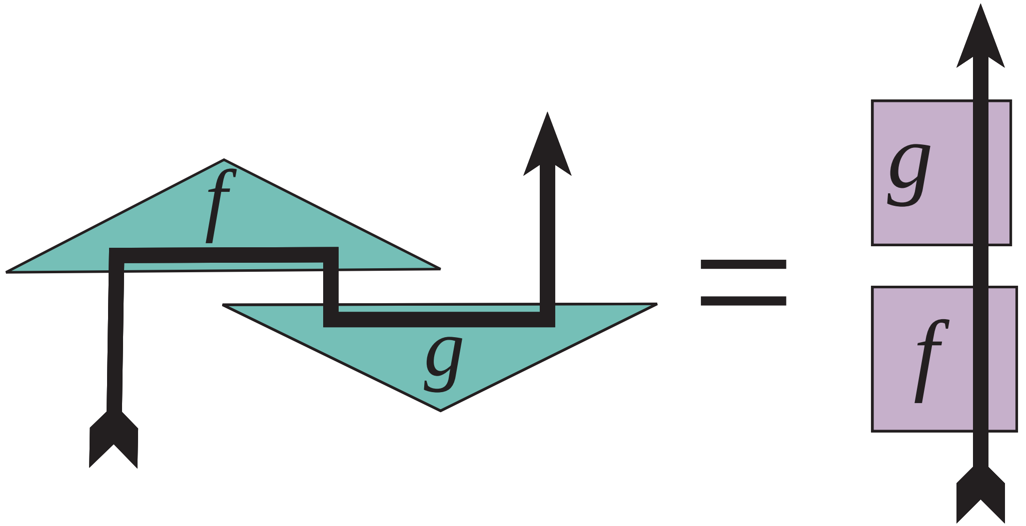
When setting



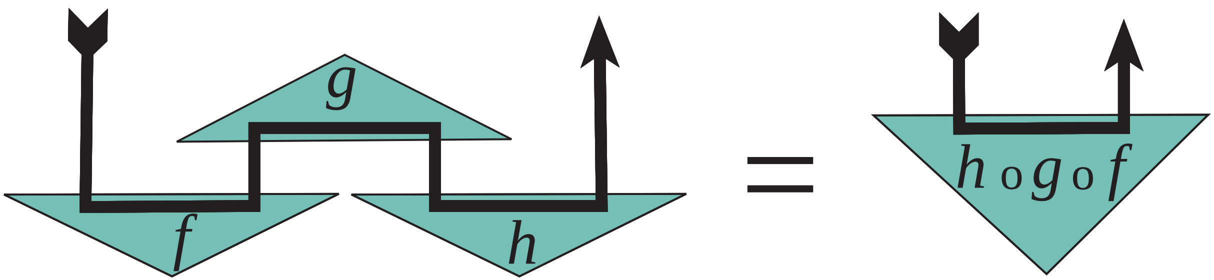
we obtain



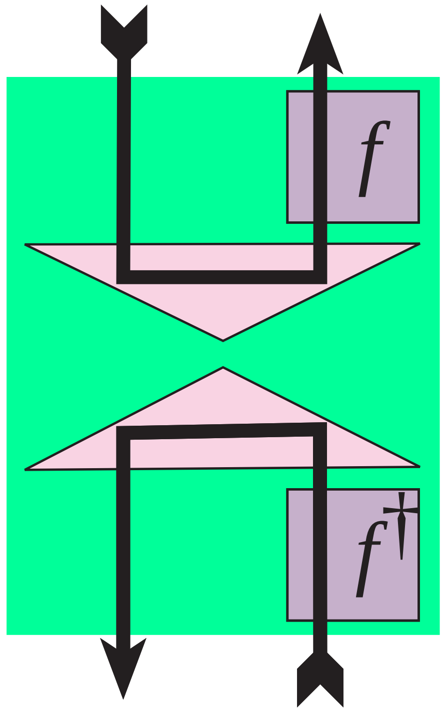
COMPOSITIONALITY



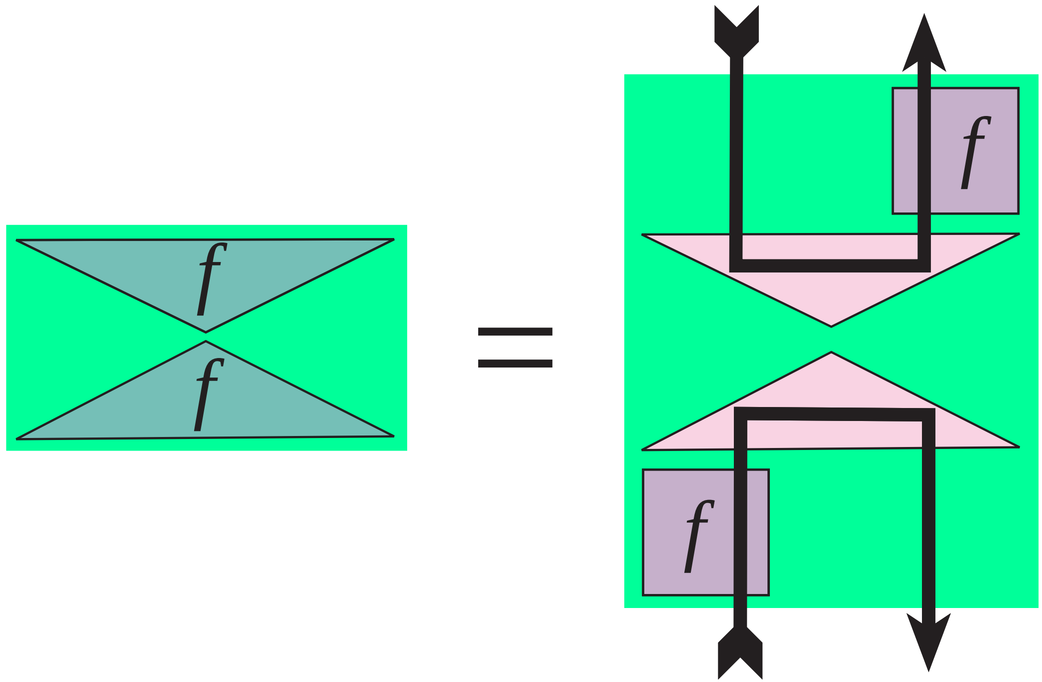
COMPOSITIONALITY bis



BIPARTITE PROJECTOR

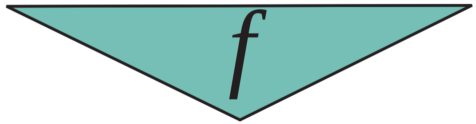


$$\mathrm{P}_f : A^* \otimes B \to A^* \otimes B$$

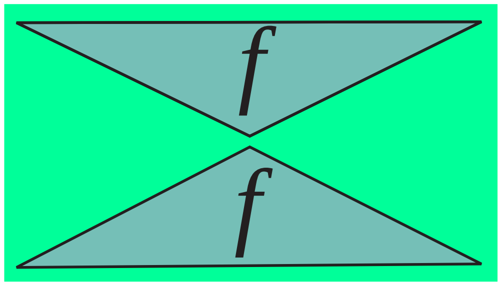


$$\mathrm{P}_f : A \otimes B^* \to A^* \otimes B$$

The concepts of bipartite state



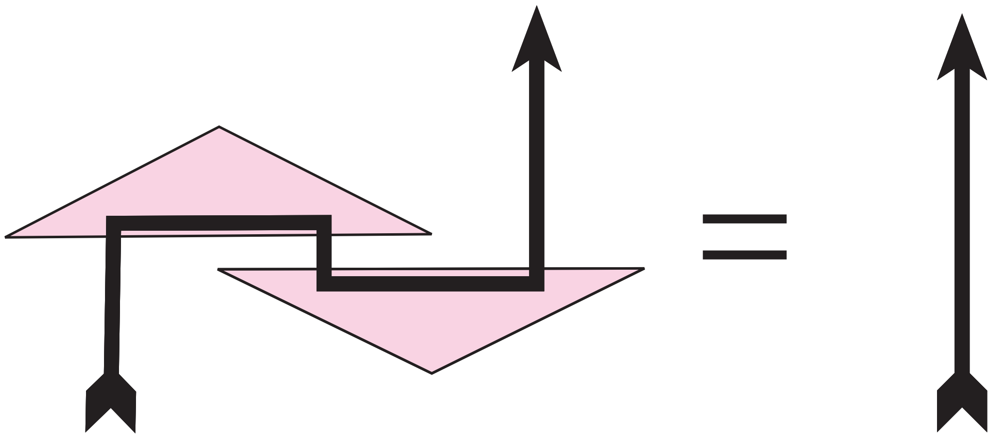
and of bipartite projector



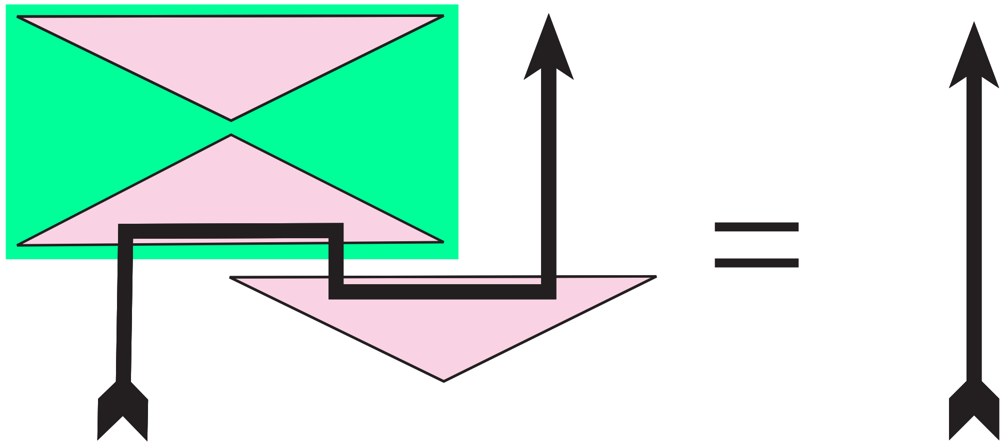
yield the following corrolaries ...

DESIGNING PROTOCOLS

$\frac{1}{4}$th-TELEPORTATION

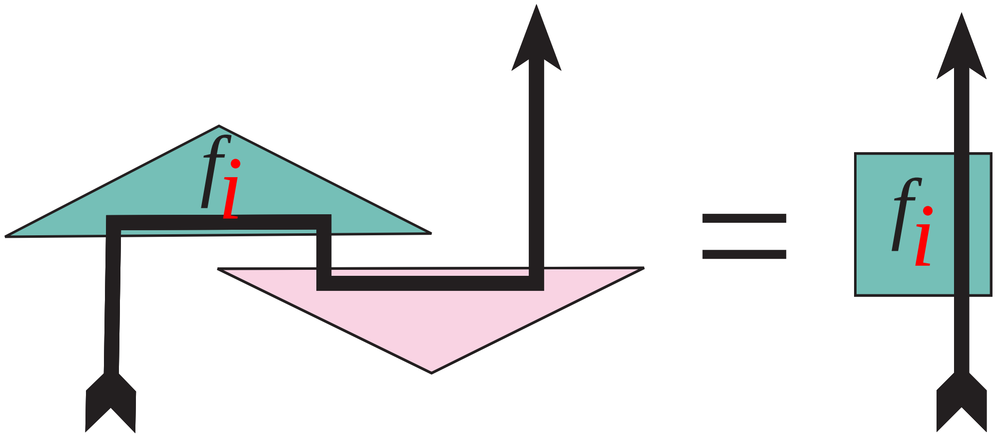


$\frac{1}{4}$th- TELEPORTATION



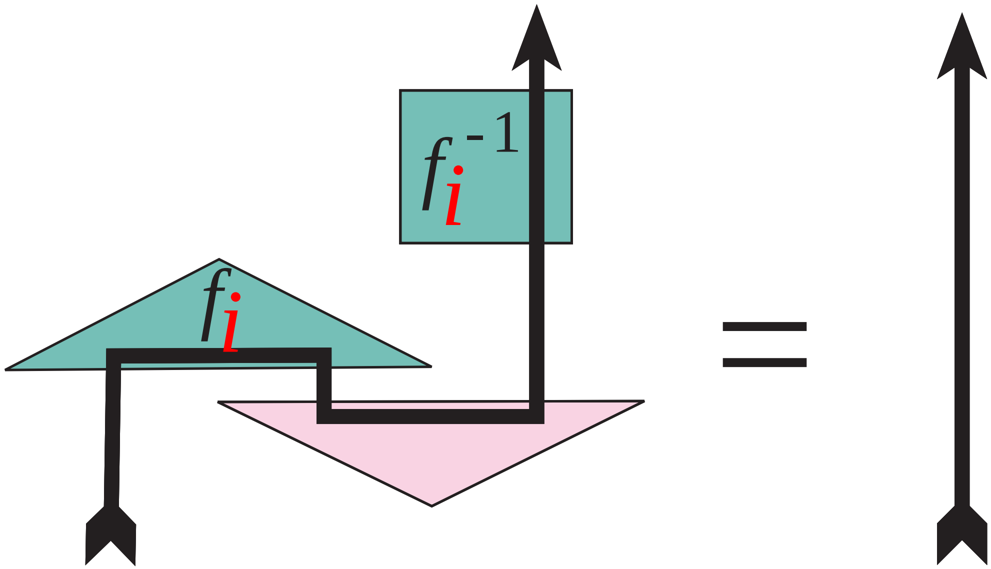
since $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$

FULL TELEPORTATION



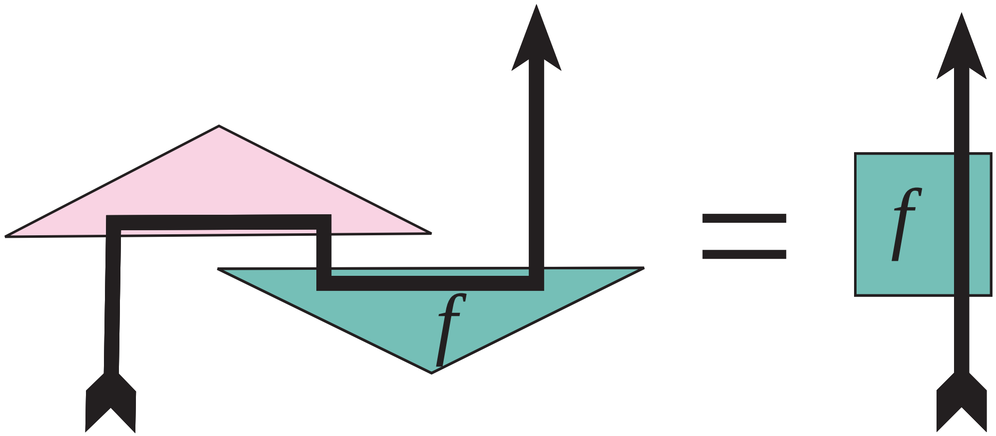
for $1 \leq i \leq 4$

FULL TELEPORTATION



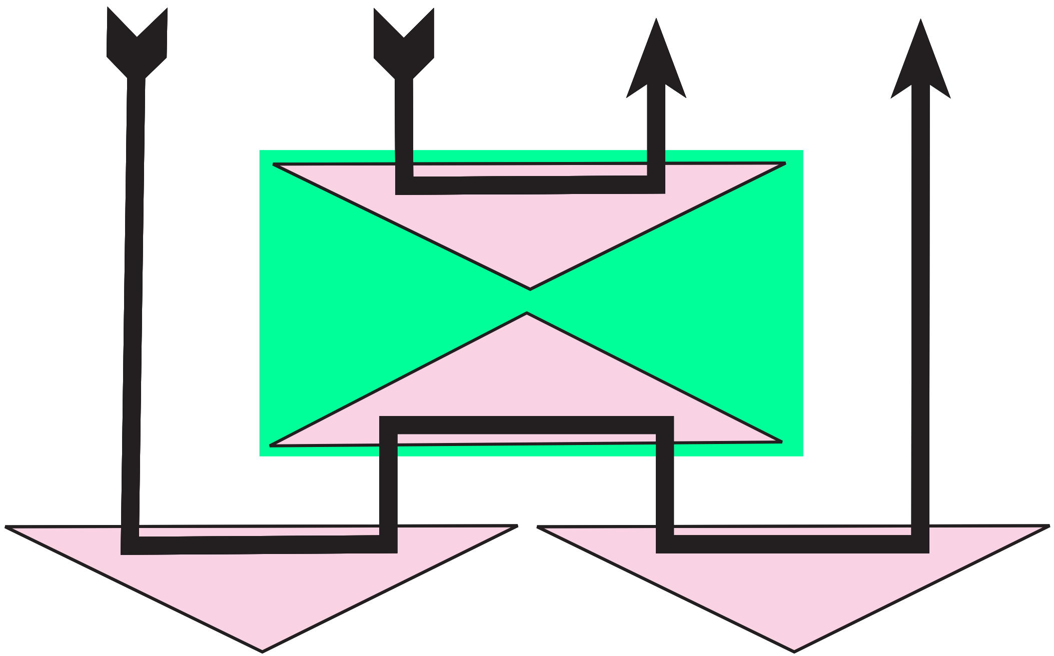
for $1 \leq i \leq 4$

LOGIC GATE TELEPORTATION

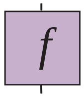


since $f \circ \mathrm{id} = f$

ENTANGLEMENT SWAPPING



The von Neumann-Dirac model



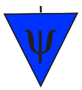
$: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map

$: \mathbb{C} \longrightarrow \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

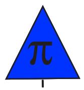
$: \mathbb{C} \longrightarrow \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

$\mathcal{H}^*$:= conjugate Hilbert space of $\mathcal{H}$

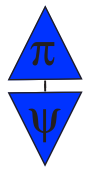
$f^\dagger$:= linear adjoint of f



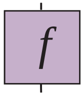
$= |\psi\rangle$



$= \langle\phi|$ for $\pi := \phi^\dagger$



$= \langle\phi \mid \psi\rangle$



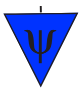
$: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map

$: \mathbb{C} \longrightarrow \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

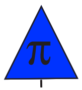
$: \mathbb{C} \longrightarrow \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

$\mathcal{H}^*$:= conjugate Hilbert space of $\mathcal{H}$

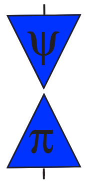
$f^\dagger$:= linear adjoint of f



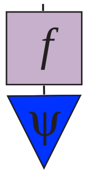
$= |\psi\rangle$



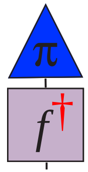
$= \langle\phi|$ for $\pi := \phi^\dagger$



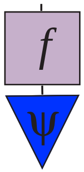
$= |\psi\rangle\langle\phi|$

$| \; f \circ \psi \rangle =$ 

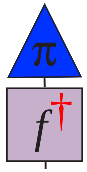
$= f \circ \psi$

$\langle f \circ \phi \; | =$ 

$= \phi^{\dagger} \circ f^{\dagger}$

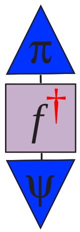
$| \; f \circ \psi \rangle =$ 

$= f \circ \psi$

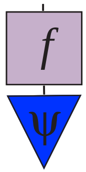
$\langle f \circ \phi \; | =$ 

$= \phi^{\dagger} \circ f^{\dagger}$

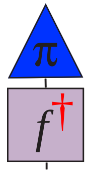
Adjointness implies

$\langle f \circ \phi \; | \; \psi \rangle =$ 

$= \langle \phi \; | \; f^{\dagger} \circ \psi \rangle$

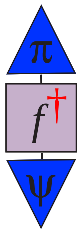
$| f \circ \psi \rangle =$ 

$= f \circ \psi$

$\langle f \circ \phi | =$ 

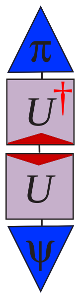
$= \phi^{\dagger} \circ f^{\dagger}$

Adjointness implies

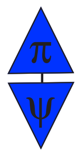
$\langle f \circ \phi \mid \psi \rangle =$ 

$= \langle \phi \mid f^{\dagger} \circ \psi \rangle$

Unitarity means $U^{-1} = U^{\dagger}$ so

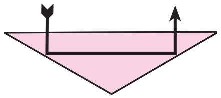
$\langle U \circ \phi \mid U \circ \psi \rangle =$ 

$=$ 

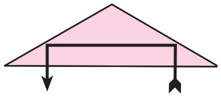
$=$ 

$= \langle \phi \mid \psi \rangle$

EPR-states and their adjoints:



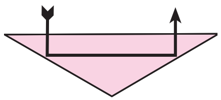
$$: \mathbb{C} \longrightarrow \mathcal{H}^* \otimes \mathcal{H} :: \ 1 \ \mapsto \ \Big| \sum_i e_i \otimes e_i \Big\rangle$$

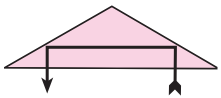


$$: \mathcal{H}^* \otimes \mathcal{H} \longrightarrow \mathbb{C} :: \Phi \mapsto \Big\langle \sum_i e_i \otimes e_i \ \Big| \ \Phi \Big\rangle$$

$$:: \phi_1 \otimes \phi_2 \mapsto \big\langle \phi_1 \ \big| \ \phi_2 \big\rangle$$

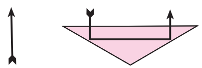
EPR-states and their adjoints:

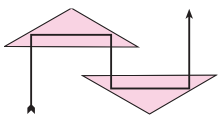
 $: \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \left| \sum_i e_i \otimes e_i \right\rangle$

 $: \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C} :: \Phi \mapsto \left\langle \sum_i e_i \otimes e_i \,\middle|\, \Phi \right\rangle$

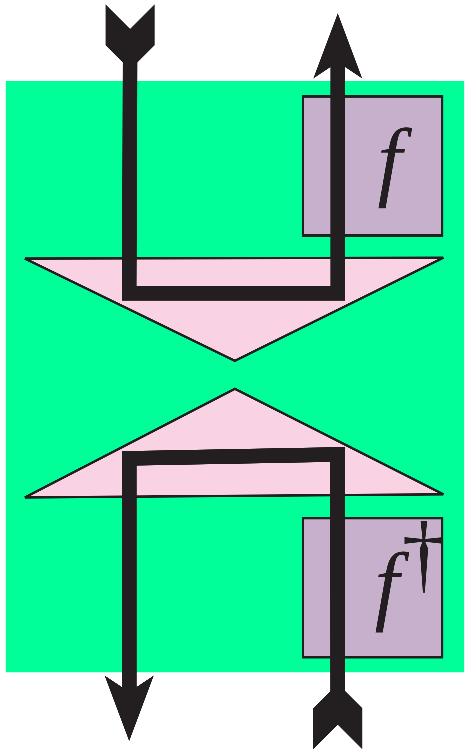
$:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 \mid \phi_2 \rangle$

We verify the axiom:

 $= (-) \otimes \left(\sum_i e_i \otimes e_i\right) = \sum_i (- \otimes e_i) \otimes e_i$

 $= \sum_i \langle - \mid e_i \rangle \cdot e_i \ = \ \mathsf{id}$

Exercise. Verify that in Hilbert space bipartite projectors on one-dimensional subspaces indeed factor as



Theorem proving

A key role is played by

$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \quad \simeq \quad \mathcal{H}_1 \to \mathcal{H}_2$$

i.e. bipartite states $\Psi \in \mathcal{H}_1^* \otimes \mathcal{H}_2$ are representable by linear functions $f : \mathcal{H}_1 \to \mathcal{H}_2$ and vice versa. Indeed

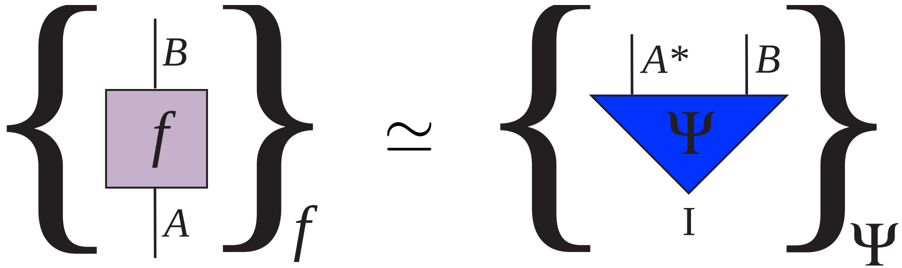
$$\Psi = \sum_{ij} m_{ij} \, |ij\rangle \quad \stackrel{\simeq}{\longleftrightarrow} \quad \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kn} \end{pmatrix}$$

$$\stackrel{\simeq}{\longleftrightarrow} \quad f = \sum_{ij} m_{ij} \, |j\rangle\langle i|$$

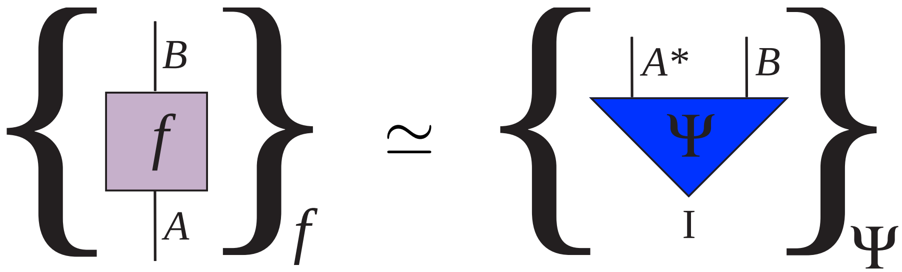
e.g.

$$|00\rangle + |11\rangle \quad \stackrel{\simeq}{\longleftrightarrow} \quad \mathsf{id} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

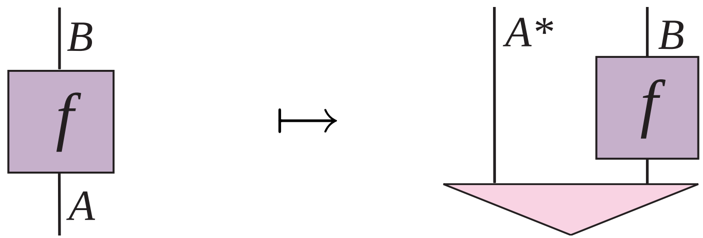
PROCESSES $\simeq$ 2-STATES



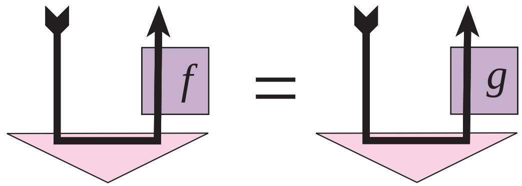
PROCESSES $\simeq$ 2-STATES



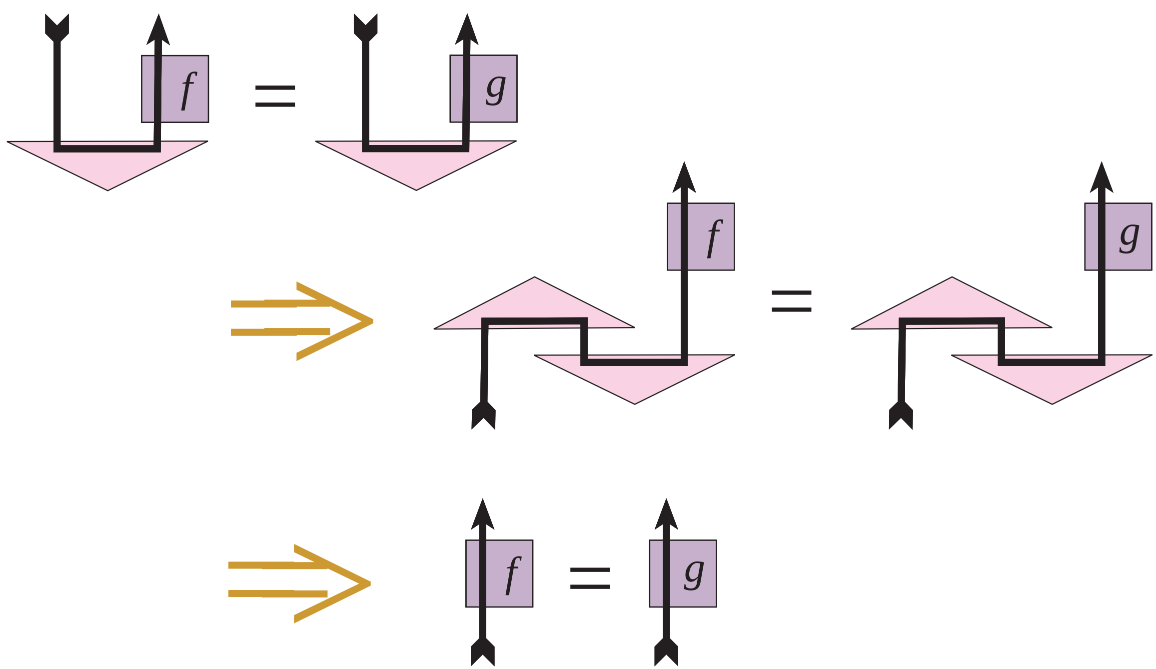
for the bijection $f \mapsto \ulcorner f \urcorner$ i.e.



Proof of injectivity.

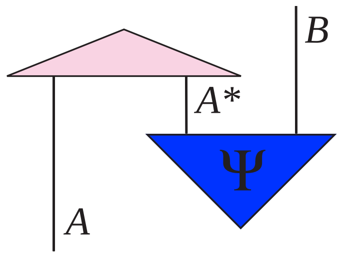


Proof of injectivity.

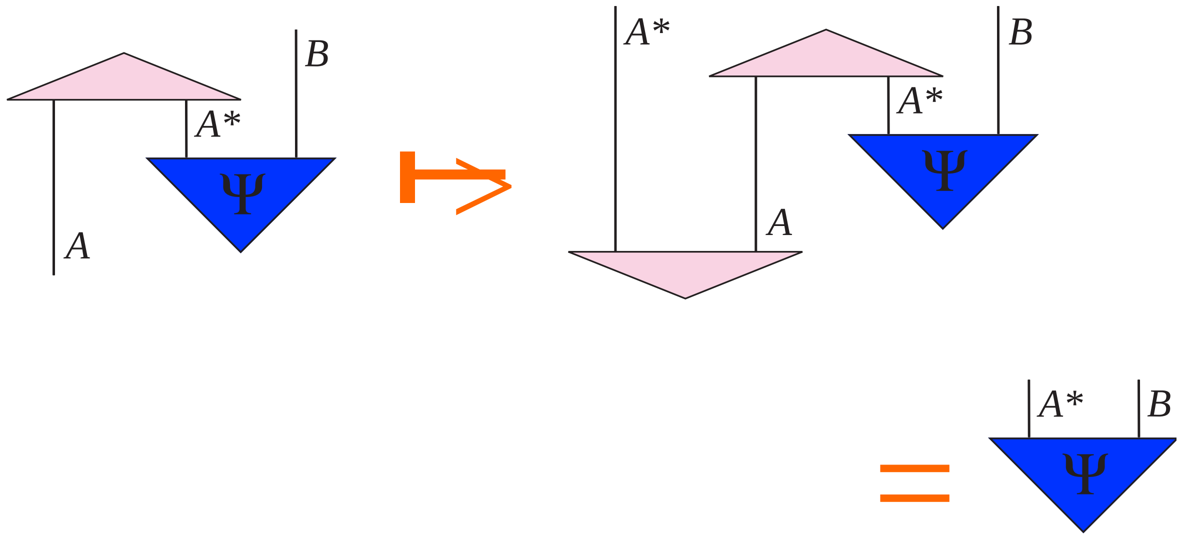


Proof of surjectivity.

Proof of surjectivity.



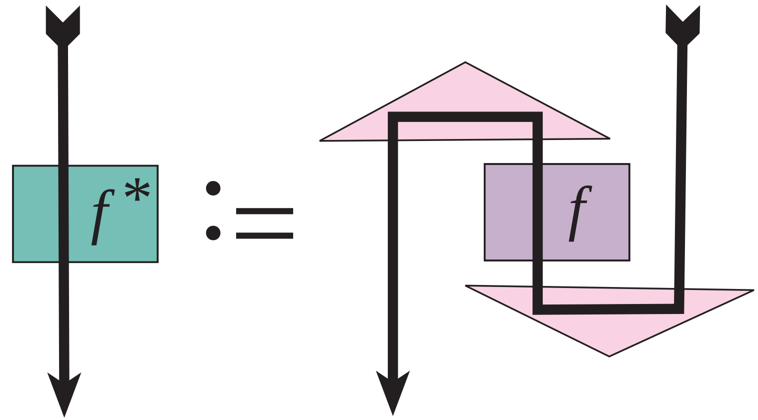
Proof of surjectivity.



A “contravariant” involution

$$f : A \to B \quad \mapsto \quad f^{*} : B^* \to A^*$$

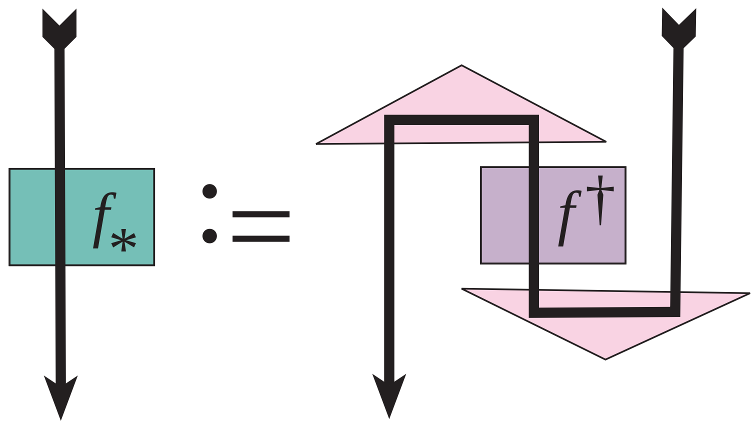
called **upper star** arises as



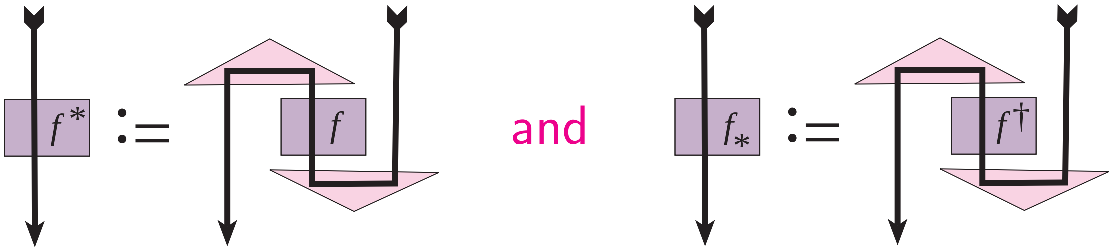
A "covariant" involution

$$f : A \longrightarrow B \quad \mapsto \quad f_{*} : A^{*} \longrightarrow B^{*}$$

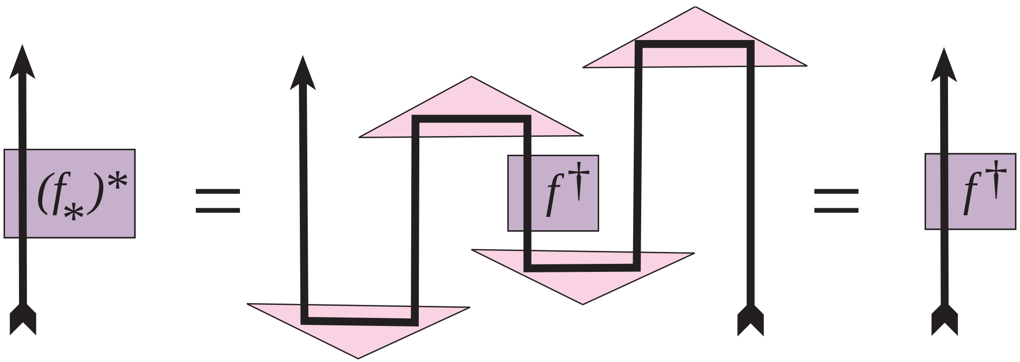
called **lower star** arises as



From



follows



and analogous we can prove that $(f^*)_* = f^\dagger$

FACTORS OF THE ADJOINT

There is a decomposition of the adjoint:

$$f^{\dagger} = (f^{*})_{*} = (f_{*})^{*}$$

FACTORS OF THE ADJOINT

There is a decomposition of the adjoint:

$$f^{\dagger} = (f^{*})_{*} = (f_{*})^{*}$$

In particular, for the Hilbert space model we have

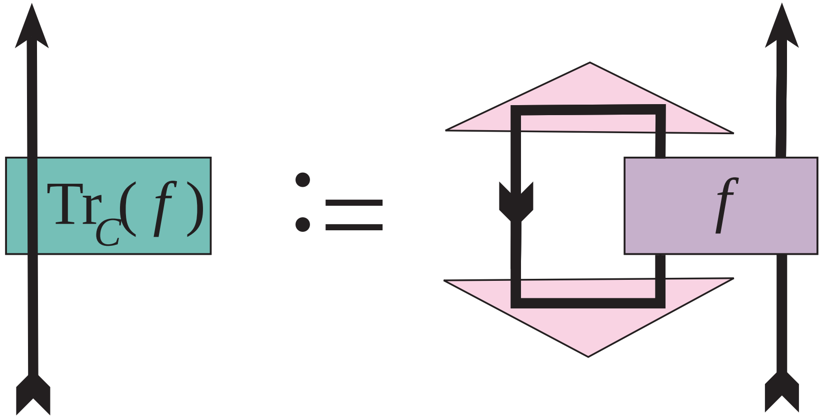
$$(-)^{*} := \textbf{transposition}$$

$$(-)_{*} := \textbf{complex conjugation}$$

A **partial trace**

$$f : C \otimes A \to C \otimes B \quad \mapsto \quad \mathrm{Tr}_C(f) : A \to B$$

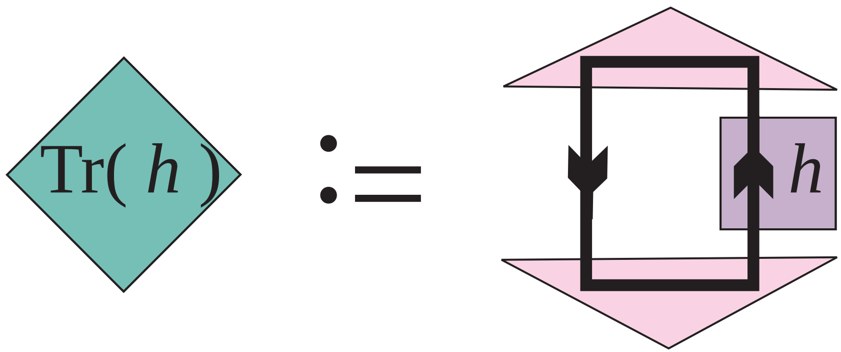
arises as



A corresponding **full trace**

$$h : A \longrightarrow A \quad \longmapsto \quad \mathrm{Tr}(h) : \mathrm{I} \longrightarrow \mathrm{I}$$

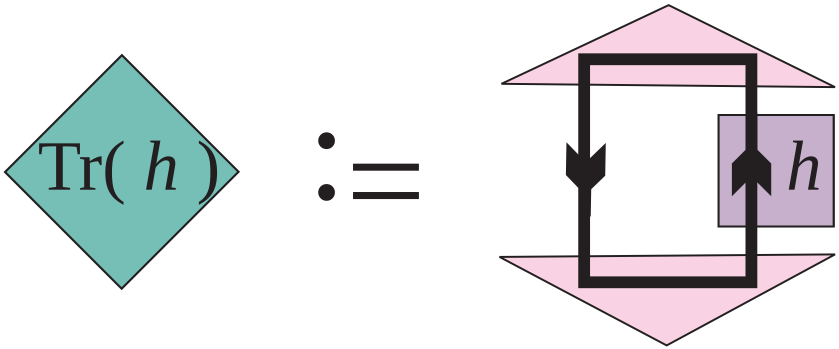
arises as



A corresponding **full trace**

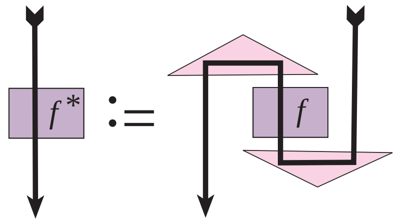
$$h : A \to A \quad \mapsto \quad \mathrm{Tr}(h) : \mathrm{I} \to \mathrm{I}$$

arises as

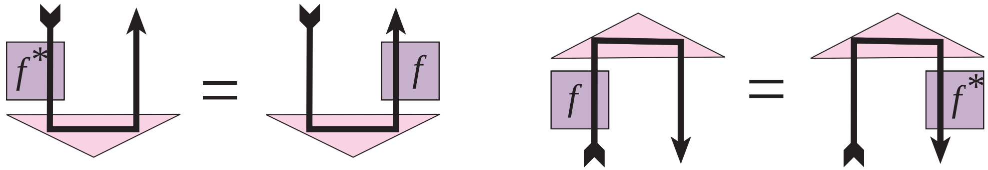


$\Rightarrow h$ "carries a diamond" cf. **probabilistic weight**

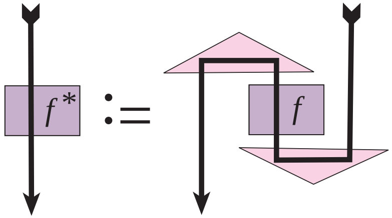
From



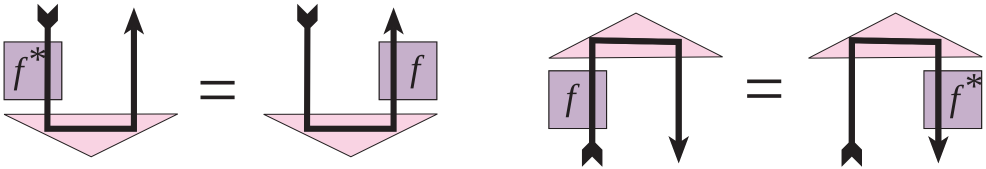
follows



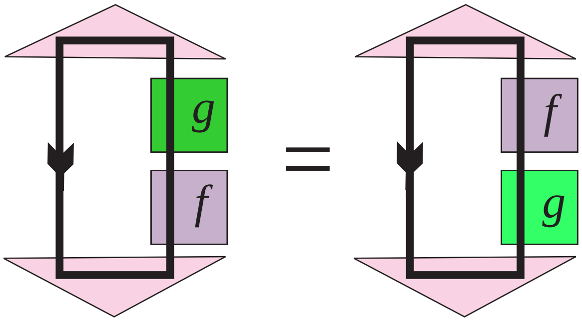
From



follows



and hence

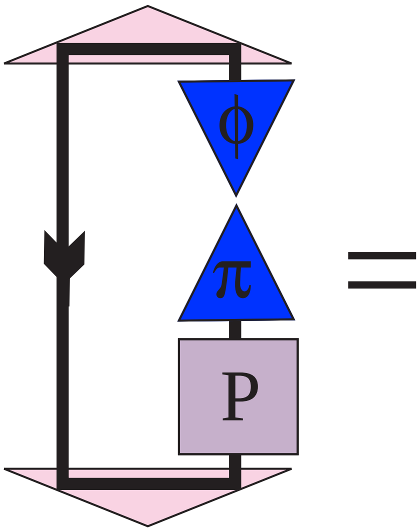


EQUIVALENT BORN RULES

$$\mathsf{Tr}(\rho_\phi \circ \mathrm{P}) \overset{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$

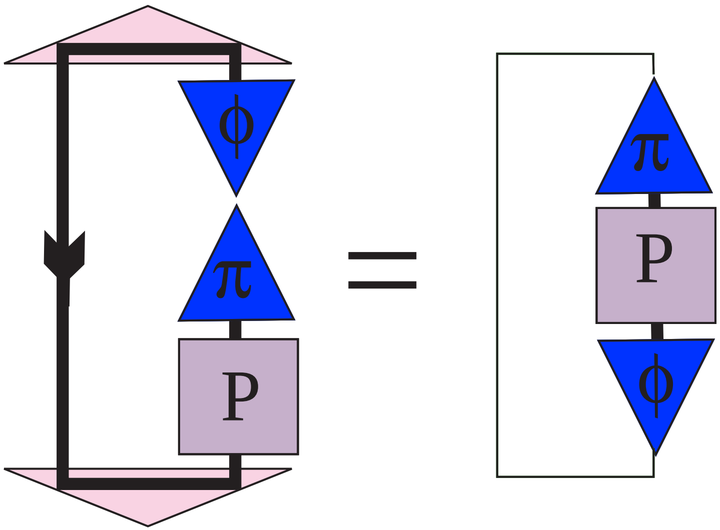
EQUIVALENT BORN RULES

$$\mathrm{Tr}(\rho_\phi \circ \mathrm{P}) \overset{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



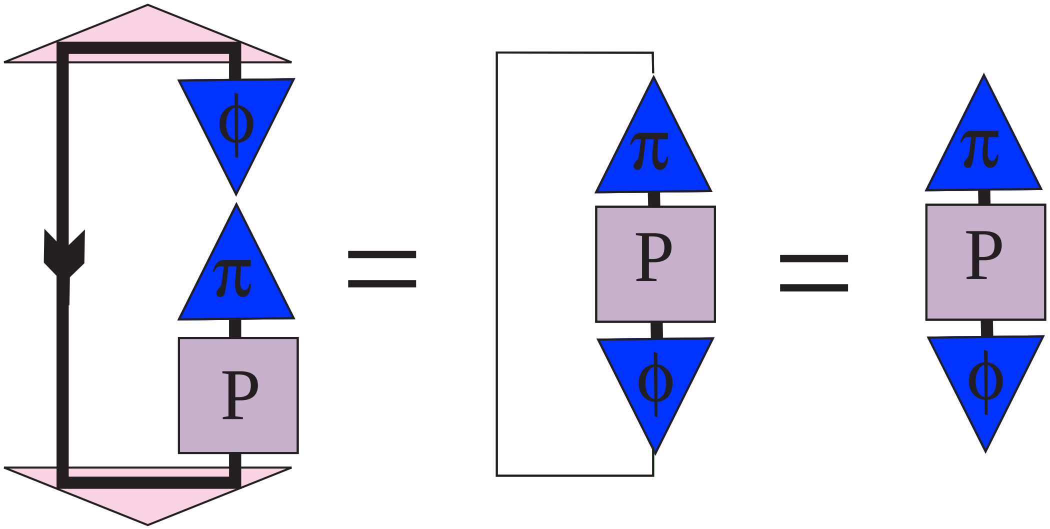
EQUIVALENT BORN RULES

$$\mathsf{Tr}(\rho_\phi \circ \mathrm{P}) \stackrel{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



EQUIVALENT BORN RULES

$$\mathrm{Tr}(\rho_\phi \circ \mathrm{P}) \stackrel{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



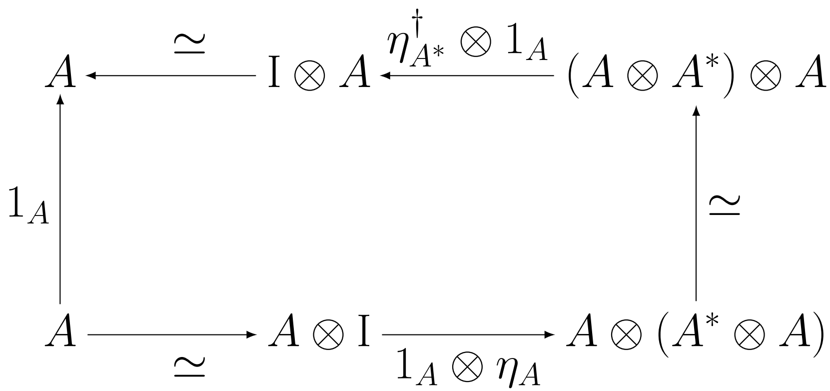
$$\mathbb{C}^* \otimes \mathbb{C} \simeq \mathbb{C}$$

ALGEBRA BEHIND THE SCENE

STRONG COMPACT CLOSURE

Symmetric monoidal bifunctor $-\otimes- : \mathbf{C}\times\mathbf{C} \longrightarrow \mathbf{C}$ and

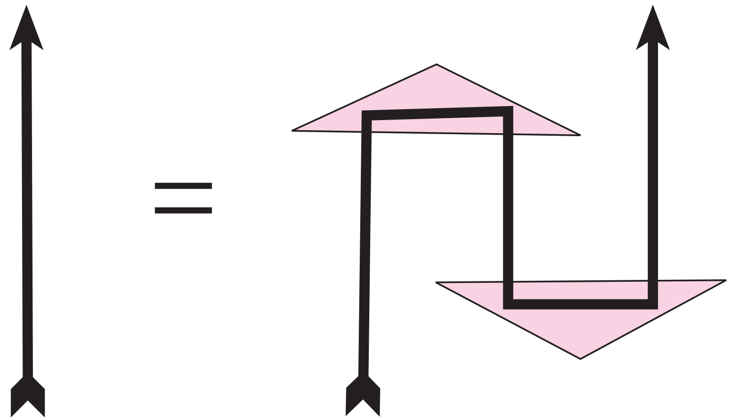
- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A\to B} \mapsto f^\dagger_{B\to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$;



STRONG COMPACT CLOSURE

Symmetric monoidal bifunctor $-\otimes- : \mathbf{C}\times\mathbf{C} \longrightarrow \mathbf{C}$ and

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A\to B} \mapsto f^{\dagger}_{B\to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A,A^*} \circ \eta_A$;

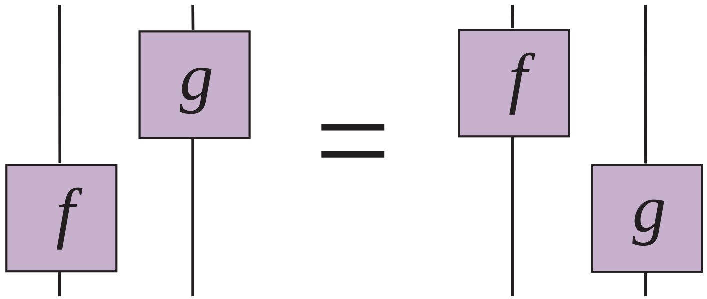


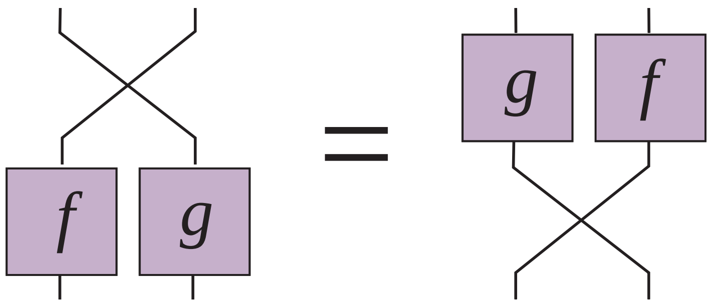
FUNCTORIALITY AND SYMMETRY

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f_1 \otimes \mathsf{id}} & B_1 \otimes A_2 \\ {\scriptstyle \mathsf{id} \otimes f_2}\downarrow & & \downarrow{\scriptstyle \mathsf{id} \otimes f_2} \\ A_1 \otimes B_2 & \xrightarrow[f_1 \otimes \mathsf{id}]{} & B_1 \otimes B_2 \end{array}$$

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2} & B_1 \otimes B_2 \\ {\scriptstyle \sigma_{A_1,A_2}}\downarrow & & \downarrow{\scriptstyle \sigma_{B_1,B_2}} \\ A_2 \otimes A_1 & \xrightarrow[f_2 \otimes f_1]{} & B_2 \otimes B_1 \end{array}$$

FUNCTORIALITY AND SYMMETRY





SCALARS

For $s, t : \mathrm{I} \longrightarrow \mathrm{I}$ where $A \otimes \mathrm{I} \simeq A$ we have

$$s \circ t = \mathrm{I} \xrightarrow{\simeq} \mathrm{I} \otimes \mathrm{I} \xrightarrow{s \otimes t} \mathrm{I} \otimes \mathrm{I} \xrightarrow{\simeq} \mathrm{I}$$

and we can define **scalar multiplication** as

$$s \bullet f := A \xrightarrow{\simeq} A \otimes \mathrm{I} \xrightarrow{f \otimes s} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

s.t.

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

i.e. diamonds can move around freely in ‘time’ and ‘space’

NO-CLONING NO-DELETING

We do not want to be $- \otimes -$ a **categorical product** since that would imply existence of

$$A \xrightarrow{\Delta} A \otimes A \qquad\qquad A \otimes B \xrightarrow{p} A$$

NO-CLONING NO-DELETING

We do not want to be $-\otimes-$ a **categorical product** since that would imply existence of

$$A \xrightarrow{\Delta} A \otimes A \qquad\qquad A \otimes B \xrightarrow{p} A$$

Strong compact closure and cartesian closure turn out to not like each other very much.

GLOBAL PHASES

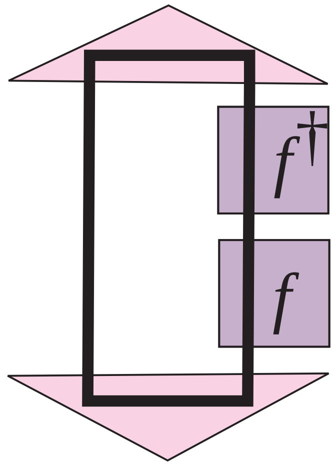
The **squared Hilbert-Schmidt norm**

$$||f|| = \sum_i \langle f(e_i) \mid f(e_i) \rangle$$

exists in the picture formalism as

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

i.e.



The **squared Hilbert-Schmidt norm**

$$||f|| = \sum_i \langle f(e_i) \mid f(e_i)\rangle$$

exists in the picture formalism as

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

$$\begin{aligned} ||f||(1) &= \left(\eta^\dagger \circ (1 \otimes f)^\dagger \circ (1 \otimes f) \circ \eta\right)(1) \\ &= \left(\eta^\dagger \circ (1 \otimes (f^\dagger \circ f))\right)\left(\sum e_i \otimes e_i\right) \\ &= \eta^\dagger\left(\sum e_i \otimes f^\dagger(f(e_i))\right) \\ &= \sum \langle e_i \mid f^\dagger(f(e_i))\rangle \\ &= \sum \langle f(e_i) \mid f(e_i)\rangle\,. \end{aligned}$$

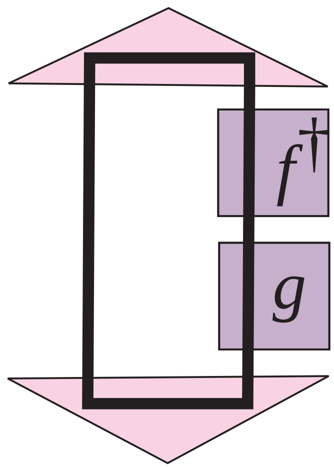
The **Hilbert-Schmidt inner-product**

$$\langle f, g\rangle = \sum_i \langle f(e_i) \mid g(e_i)\rangle$$

exists in the picture formalism as

$$\langle f, g\rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$$

i.e.



and genuinely generalizes the bra-ket inner-product

ABSTRACT GLOBAL PHASES

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ in $\mathbf{FdHilb}$.

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in **FdHilb**.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g\ ,\ s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger$$

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in **FdHilb**.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \implies f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t : s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger$$

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \, , s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \implies f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

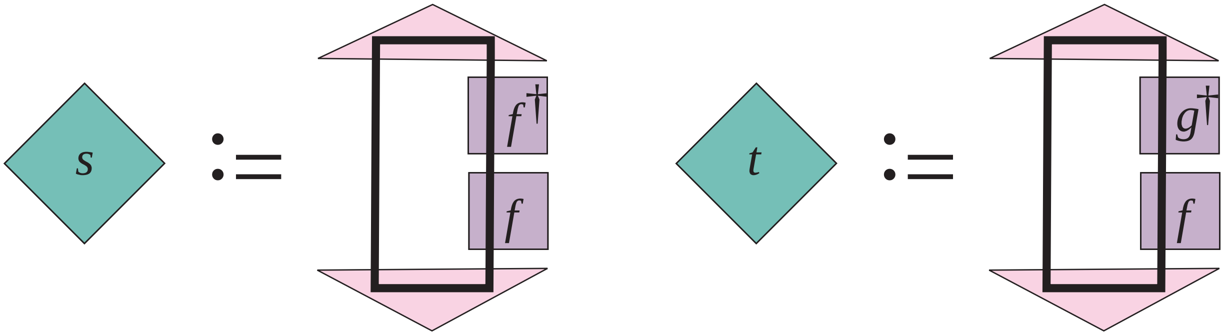
$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t : \; s \bullet f = t \bullet g \, , s \circ s^\dagger = t \circ t^\dagger$$

e.g.

$$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \qquad \text{and} \qquad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

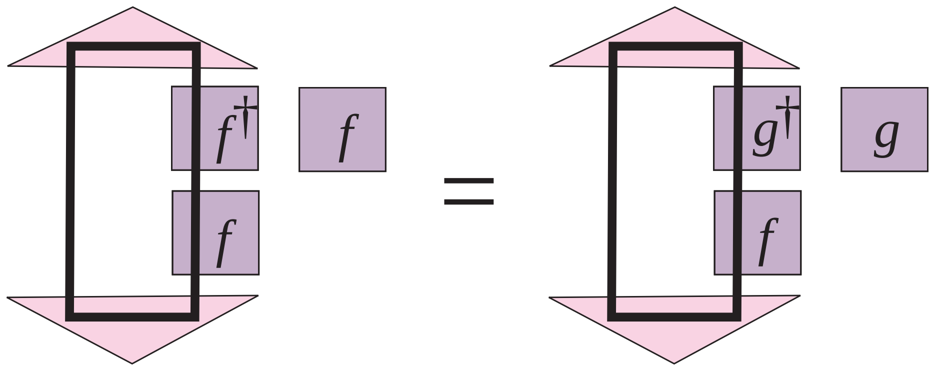
♯1 $s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$ and $t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$



♯2 $f \otimes f^\dagger = g \otimes g^\dagger$

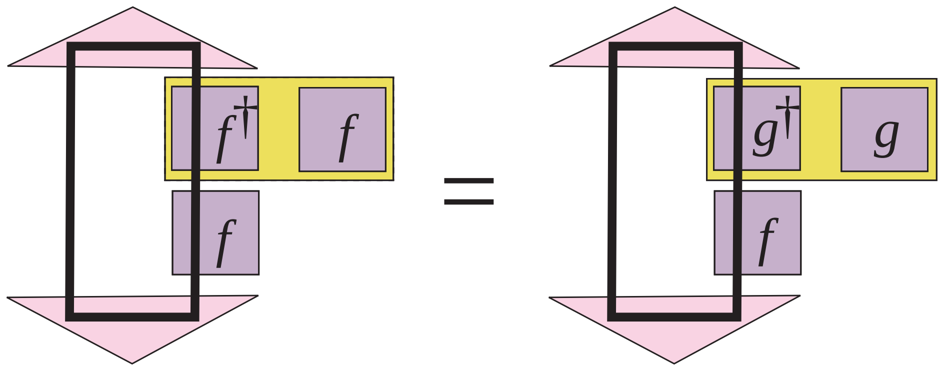
Proof.

$\sharp 3\quad s \bullet f = t \bullet g$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



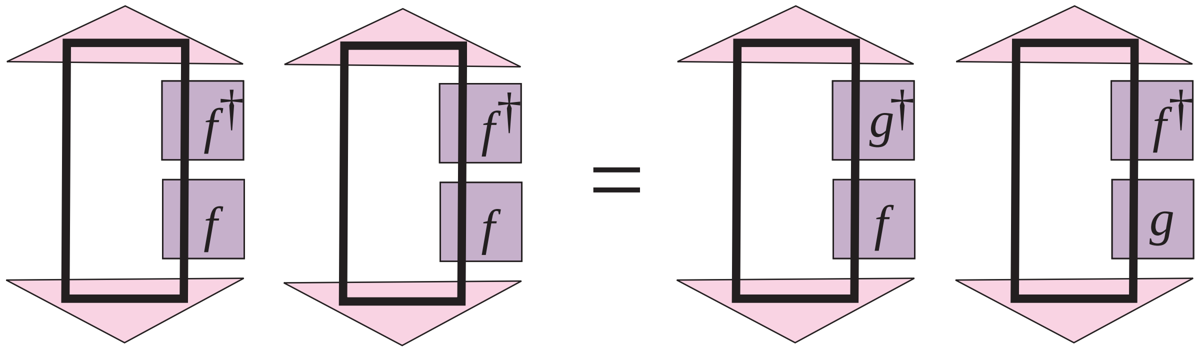
Proof.

$\sharp 3 \quad s \bullet f = t \bullet g$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



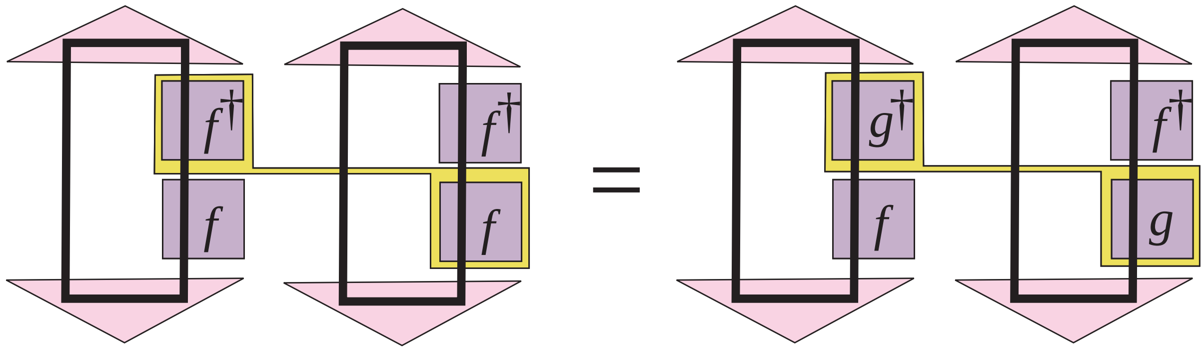
Proof.

$\sharp 4 \quad s \circ s^{\dagger} = t \circ t^{\dagger}$ with $s/t := (\ulcorner f/g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$

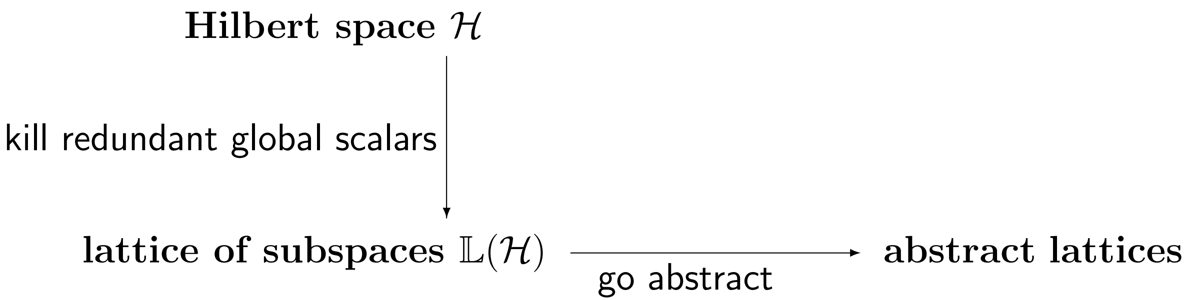


Proof.

$\sharp 4 \quad s \circ s^\dagger = t \circ t^\dagger$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



PROJECTIVE vs VECTORIAL



PROJECTIVE vs VECTORIAL

Hilbert space $\mathcal{H}$

↓ kill redundant global scalars

lattice of subspaces $\mathbb{L}(\mathcal{H})$ → (go abstract) **abstract lattices**

FdHilb → (go abstract) **'vectorial' strong compact closure**

↓ kill redundant global scalars

'projective' strong compact closure

ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^{\dagger} = g \otimes g^{\dagger} \implies f = g$$

ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

iff

$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

ABSENCE OF GLOBAL PHASES

Proposition. $\mathit{WProj}(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

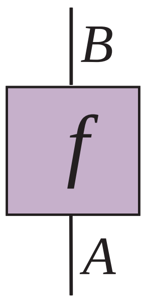
iff

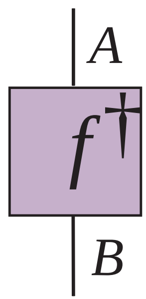
$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

iff

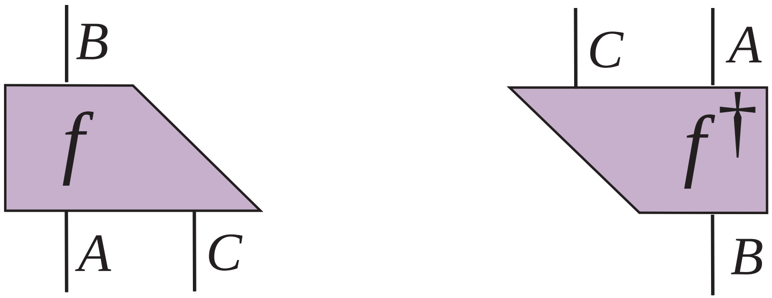
Equal Preparations Produce Equal States

OPEN SYSTEMS AND CPMs

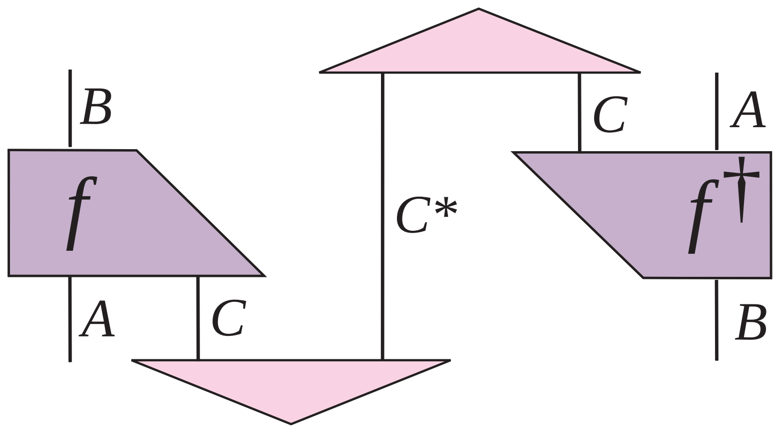




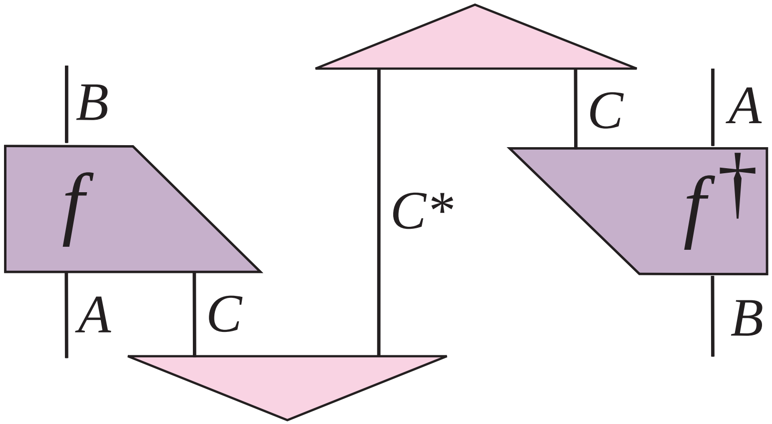
$\Rightarrow$ projective process



$\Rightarrow$ projective process with ancila



$\Rightarrow$ projective process with hidden ancila



In the case of Hilbert spaces and linear maps we exactly obtain **completely positive maps** (Selinger 2005)

FULL CATEGORICAL QM

System of type A $:=$ Object A

Composite of A and B $:=$ Tensor $A \otimes B$

Process of type $A \to B$ $:=$ Morphism $f : A \to B$

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \to B$:= Morphism $f : A \to B$

State of A := Element $\psi : \mathrm{I} \to A$

Evolution of A := Unitary $U : A \to A$

Measurement on A := "Projectors" $\{\mathrm{P}_i : A \to A\}_i$

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \to B$:= Morphism $f : A \to B$

State of A := Element $\psi : \mathrm{I} \to A$

Evolution of A := Unitary $U : A \to A$

Measurement on A := "Projectors" $\{\mathrm{P}_i : A \to A\}_i$

- Data := $\nu \in \{i\}_i$
- Dynamics := $\psi \mapsto \mathrm{P}_\nu \circ \psi$
- Probability := $\psi^\dagger \circ \mathrm{P}_\nu \circ \psi = \mathrm{Tr}(\mathrm{P}_\nu \circ \rho_\psi) : \mathrm{I} \to \mathrm{I}$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \longrightarrow A\}_i$

- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \longrightarrow \ldots$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \longrightarrow A\}_i$
- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \longrightarrow \ldots$

E.g. for $U : A \longrightarrow \bigoplus_i A_i$ unitary let $\pi_j := p_j \circ U$ and

$$\{\mathrm{P}_i := \pi_i^\dagger \circ \pi_i\}_j \qquad M := \langle \mathrm{P}_i \rangle_i : A \longrightarrow \bigoplus_i A\,.$$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \to A\}_i$

- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \to \ldots$

E.g. for $U : A \to \bigoplus_i A_i$ unitary let $\pi_j := p_j \circ U$ and

$$\{\mathrm{P}_i := \pi_i^\dagger \circ \pi_i\}_j \qquad M := \langle \mathrm{P}_i \rangle_i : A \to \bigoplus_i A\,.$$

where $\bigoplus_i$, p_j and $\langle\ \ \rangle_i$ can be provided by **biproducts** as in Abramsky-Coecke (IEEE-LiCS'04).

But, **you can actually pick your favorite** from:

Vector-space style formalism

Projective (+ weights) style formalism

Generalized measurement style formalism

But, **you can actually pick your favorite** from:

Vector-space style formalism

— use biproducts

Projective (+ weights) style formalism

— a weaker additive structure had to be designed

Generalized measurement style formalism

— conceptually interesting ongoing work where interpretational schemes on quantum probability can be formally implemented e.g. Fuchs, Hardy, ...

APPLICATIONS

— "why computer scientists care about this stuff" —

Quantum protocol/program/algorithm design

Quantum protocol/program/algorithm verification

Appropriate semantics for new quantum computational paradigms e.g. one-way (Briegel), teleportation based (Gottesman-Chuang), measurement based in general, topological quantum computing (Kitaev et al.) etc.

Representation Theorem

P. Deligne (1990) **Catégories tannakiennes** —
In: *The Grothendieck Festschrift*.

1. **Analyse** quantum compoundness.

⇒ A notion of **quantum information flow** emerges.

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; `cs/0207057`
- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. **Axiomatize** quantum compoundness.

⇒ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

⇒ ... & **quantum logic** & **open systems/CPM's**!

- De-linearizing Linearity: Projective Quantum Axiomatics from SCC. Coecke (2005) QPL'05; `quant-ph/0506134`.
- †-CCC's and Completely Positive Maps. Selinger (2005) QPL'05.