FROM COHERENCE TO QUANTUM

bOB cOECKE
Oxford University
se10.comlab.ox.ac.uk:8080/BobCoecke/Home_en.html
(or Google Bob Coecke)

Impact des Catégories – Paris – Oct. 2005

Prologue, ...

Why categories?

The practicing physicist's answer:

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It's the algebra of practicing physics!

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It's the algebra of practicing physics!

or, even more precise:

A symmetric monoidal category is the algebra of practicing physics!

Bénabou, J. (1963) ***Categories avec multiplication.***

Kinds of systems

$$A\,,\ B\,,\ C\,,\ \ldots$$

- e.g. qubit, n qubits, electron, atom, data, ...

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Operations/experiments on systems

$$A \xrightarrow{f} A\,,\ A \xrightarrow{g} B\,,\ B \xrightarrow{h} C\,,\ \ldots$$

- e.g. preparation, acting force field, measurement, ...

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Composition of operations

$$A \xrightarrow{g\circ f} C \quad := \quad A \xrightarrow{f} B \xrightarrow{g} C$$

'Doing nothing'-operations

$$A \xrightarrow{1_A} A\,,\ B \xrightarrow{1_B} B\,,\ C \xrightarrow{1_C} C\,,\ \ldots$$

Definition. A **category** $\mathbf{C}$ consits of

- Objects $A, B, C, \ldots$
- Morphisms $f, g, h, \ldots \in \mathbf{C}(A, B)$ for each pair A, B
- Associative composition of morphisms i.e.

$$f \in \mathbf{C}(A,B)\ ,\ g \in \mathbf{C}(B,C) \ \Rightarrow\ g \circ f \in \mathbf{C}(A,C)$$

$$\boxed{(h \circ g) \circ f = h \circ (g \circ f)}$$

- An identity morphism $1_A \in \mathbf{C}(A, A)$ for each A i.e.

$$\boxed{f \circ 1_A = 1_B \circ f = f}$$

A trans-disciplinary argument:

LOGIC & PROOF THEORY
Propositions
Proofs

PROGRAMMING
Data Types
Programs

PHYSICAL PRACTICE
Physical System
Physical Operation

Key features of a category:

- **Types**
- **Compositionality**
- **Structure lives on operations**

Outline, ...

- Scientific practice is symmetric monoidal

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- Scientific practice is not always cartesian

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- From coherence to neo-CCC's with Mac Lane & Kelly

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- Application: Quantum mechanics in kindergarten

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- Application: Quantum mechanics in kindergarten

- Reconsideration: Quantum logic which actually works

The symmetric monoidal structure of scientific practice

Compoundness via parallel composition

Two systems/operations can be considered as one whole:

$$A \otimes B \qquad\qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

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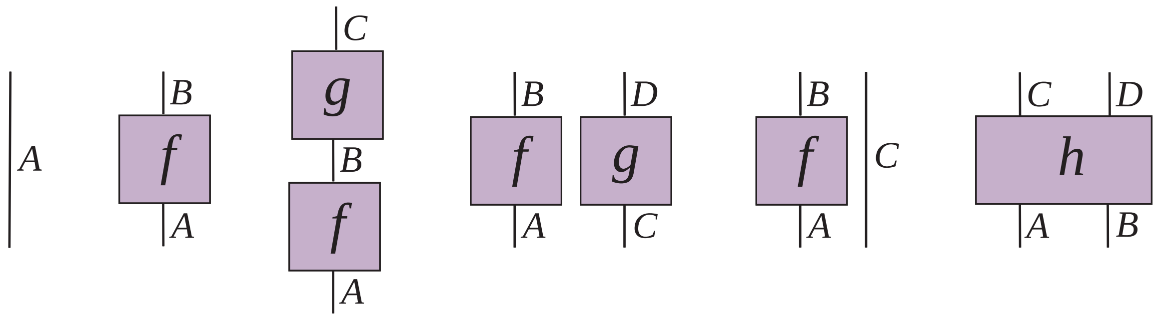
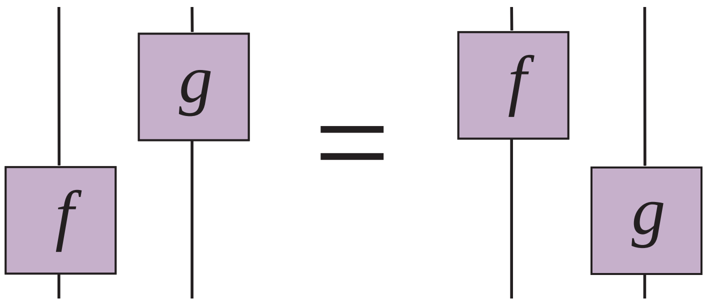
Graphical representation:



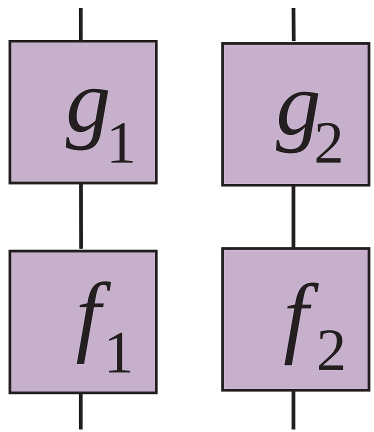
Locality of operations

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes 1_{A_2}} & B_1 \otimes A_2 \\ {\scriptstyle 1_{A_1} \otimes g}\downarrow & & \downarrow{\scriptstyle 1_{B_1} \otimes g} \\ A_1 \otimes B_2 & \xrightarrow[f \otimes 1_{B_2}]{} & B_1 \otimes B_2 \end{array}$$



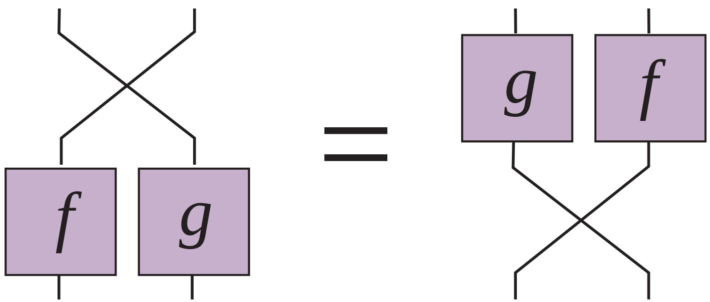
Locality of operations

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$



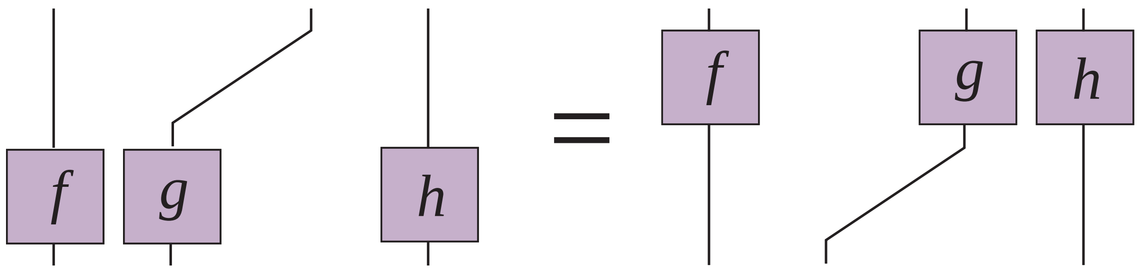
Swapping systems/operations

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes g} & B_1 \otimes B_2 \\ \Big\downarrow \sigma_{A_1,A_2} & & \Big\downarrow \sigma_{B_1,B_2} \\ A_2 \otimes A_1 & \xrightarrow[g \otimes f]{} & B_2 \otimes B_1 \end{array}$$



Colocating systems/operations

$$
\begin{array}{ccc}
(A_1 \otimes A_2) \otimes A_3 & \xrightarrow{(f \otimes g) \otimes h} & (B_1 \otimes B_2) \otimes B_3 \\
\Big\downarrow \alpha_{A_1,A_2,A_3} & & \Big\downarrow \alpha_{B_1,B_2,B_3} \\
A_1 \otimes (A_2 \otimes A_3) & \xrightarrow[f \otimes (g \otimes h)]{} & B_1 \otimes (B_2 \otimes B_3)
\end{array}
$$



Creating/destroying systems

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$\mathrm{I} :=$ 'no system' i.e. $A \otimes \mathrm{I} \simeq A \simeq \mathrm{I} \otimes A$

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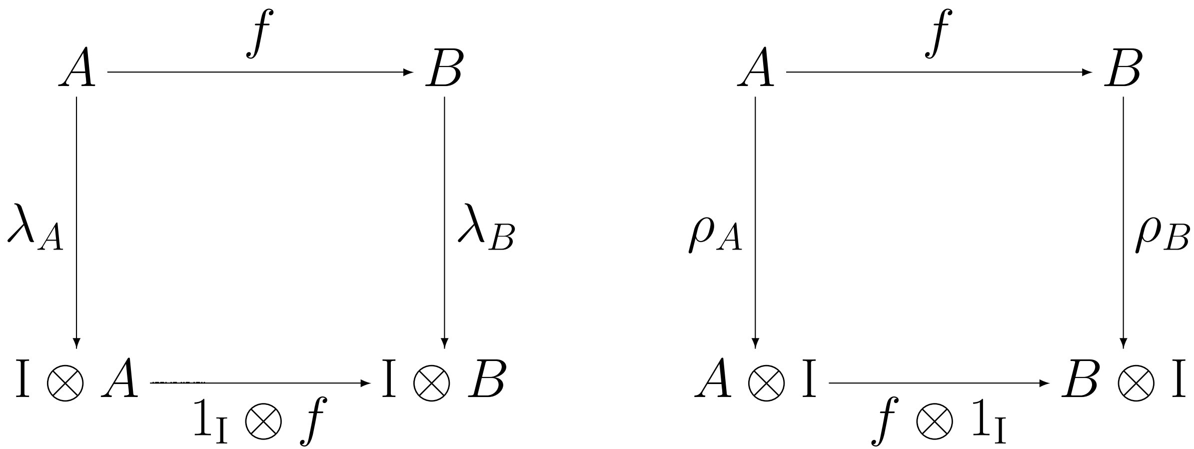

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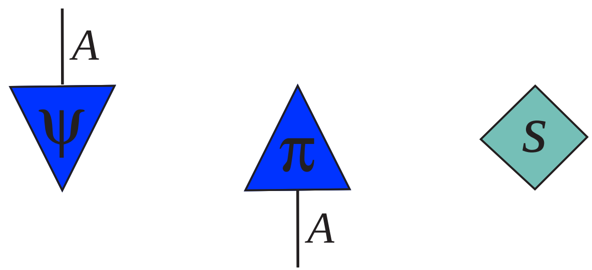

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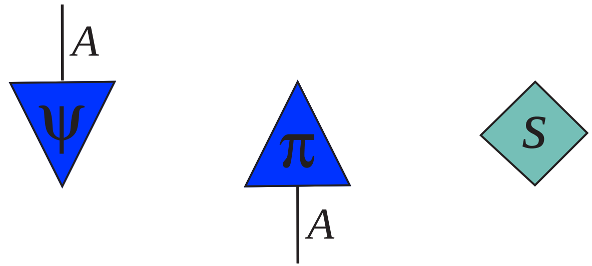

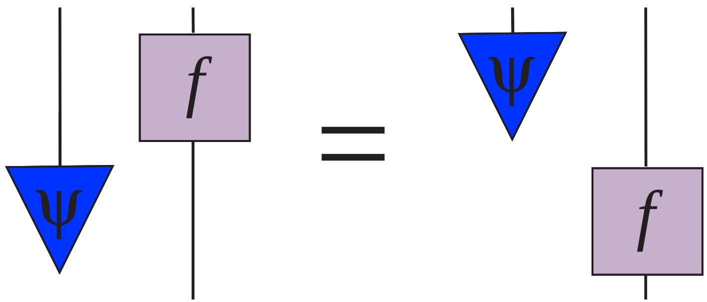

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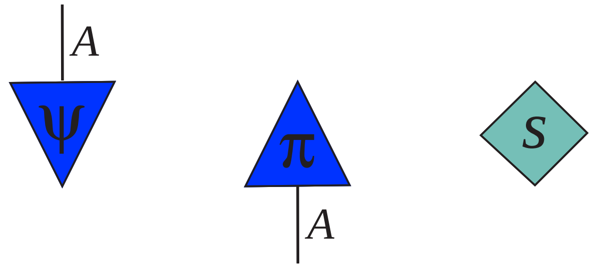
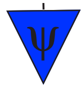


$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle (\psi \otimes 1_A) \circ \lambda_A} \downarrow & & \downarrow {\scriptstyle (\psi \otimes 1_B) \circ \lambda_B} \\
C \otimes A & \xrightarrow[\quad 1_C \otimes f \quad]{} & C \otimes B
\end{array}
$$

Creating/destroying systems

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
\downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda_B} \\
\mathrm{I} \otimes A & \xrightarrow[\quad 1_{\mathrm{I}} \otimes f \quad]{} & \mathrm{I} \otimes B \\
\downarrow{\scriptstyle \psi \otimes 1_A} & \color{magenta}{Bifunct.} & \downarrow{\scriptstyle \psi \otimes 1_B} \\
C \otimes A & \xrightarrow[\quad 1_C \otimes f \quad]{} & C \otimes B
\end{array}
$$

- **State** :=



$: \mathrm{I} \longrightarrow A$

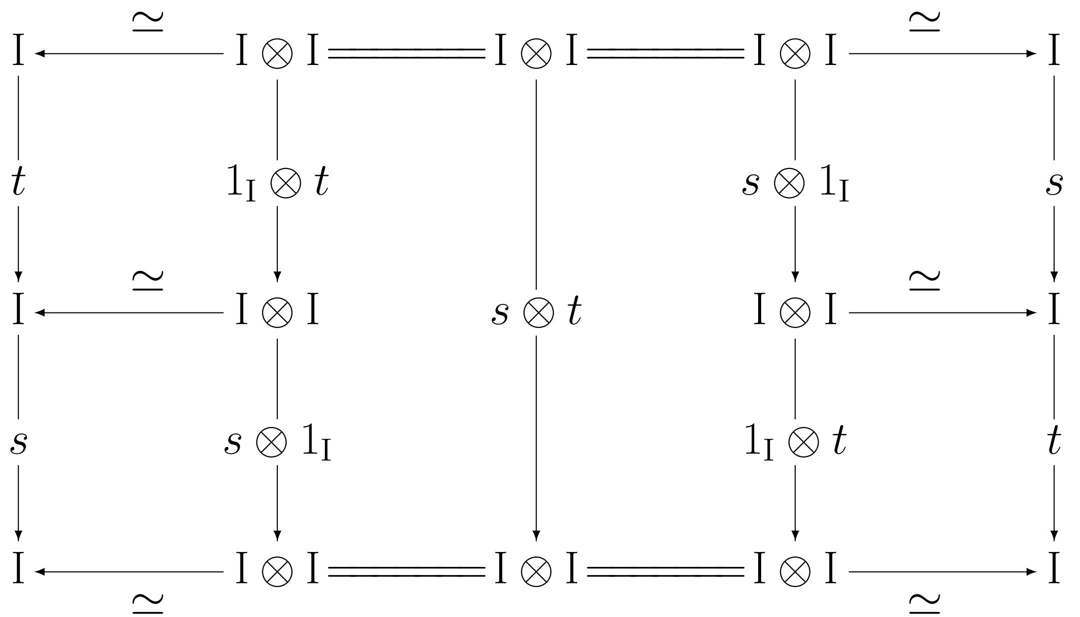
- **Scalar/value** :=

$: \mathrm{I} \longrightarrow \mathrm{I}$

- **State space** := $\mathbf{C}(\mathrm{I}, A)$

- **Scalar monoid**:= $\mathbf{C}(\mathrm{I}, \mathrm{I})$

Thm. Commutativity of the scalar monoid



Kelly & Laplaza, M. L. (1980) ***Coherence for CCCs.*** JPAA **19**.

Scalar multiplication comes for free

$$s \bullet f := A \xrightarrow{\simeq} A \otimes \mathrm{I} \xrightarrow{f \otimes s} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

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Thm.

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

Scalar multiplication comes for free

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Thm.

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

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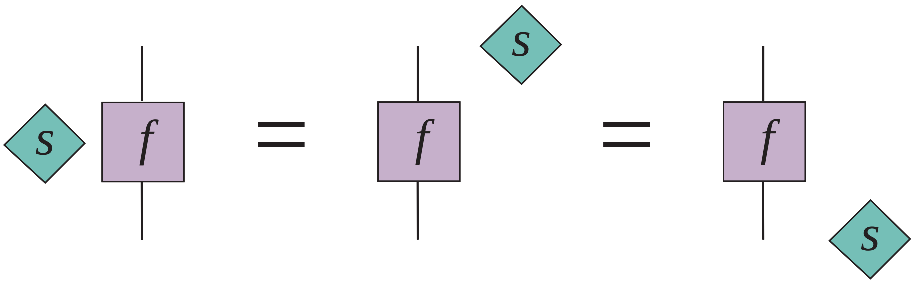

Set vs. Rel

$(\mathbf{Set}, \times)$: functions $f_x : \{*\} \longrightarrow X :: * \mapsto x$

$(\mathbf{Hilb}, \otimes)$: linear functions $f_\psi : \mathbb{C} \longrightarrow \mathcal{H} :: 1 \mapsto \psi$

$(\mathbf{Rel}, \times)$: relations $R \subseteq \{*\} \times X :: * \mapsto Y$

Set vs. Rel

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For $Y_i := X$ iff $i \in Y$ and $Y_i := \emptyset$ iff $i \notin Y$:

$$|\psi\rangle = \sum_{i \in X} \psi_i \cdot |\, i\,\rangle \qquad \leftrightarrow \qquad Y = \bigcup_{i \in X} Y_i \cap \{i\}$$

Set vs. Rel

$(\mathbf{Set}, \times)$:	functions	$f_x : \{*\} \to X :: * \mapsto x$
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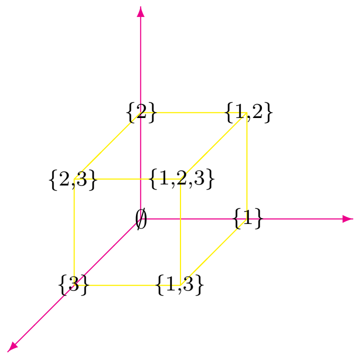

Scientific practice is not always cartesian

internalizing morphisms

cartesianity:

$$\mathbf{C}(A \times B, C) \simeq \mathbf{C}(A, B \Rightarrow C)$$

$\Rightarrow$ Multiplicative (constructive) Logic

cartesian ⇒ diagonal

— the process of copying —

$$\{\Delta_A : A \longrightarrow A \otimes A\}_A$$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \Delta_B} \\ A \otimes A & \xrightarrow[\; f \otimes f \;]{} & B \otimes B \end{array}$$

cartesian ⇒ diagonal

— the process of copying —

Computing: resources are limited ⇒ Barr's $*$-autonomy

Language: not $\neq$ not not ⇒ Lambek-style semantics

Physics: quantum no-cloning theorem ⇒ ?

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$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, (B \otimes C^*)^*)$$

$\Rightarrow$ Multiplicative Linear Logic with Negation

Prop. A symmetric monoidal category which is both a cartesian closed and $*$-autonomous is a preorder.

cartesian $\Rightarrow$ diagonal

— the process of copying —

Not commutative in $(\mathbf{Rel}, \times)$:

$$
\begin{array}{ccc}
\{*\} & \xrightarrow{\{(*,0),(*,1)\}} & \{0,1\} \\
\downarrow{\scriptstyle \{(*,(*,*))\}} & & \downarrow{\scriptstyle \{(0,(0,0)),(1,(1,1))\}} \\
\{(*,*)\} & \xrightarrow[\{(*,0),(*,1)\} \times \{(*,0),(*,1)\}]{} & \{0,1\} \times \{0,1\}
\end{array}
$$

cartesian $\Rightarrow$ diagonal

— the process of copying —

Not commutative in $(\mathbf{Hilb}, \otimes)$:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\;1 \mapsto |0\rangle + |1\rangle\;} & \mathbb{C} \oplus \mathbb{C} \\
\Big\downarrow {\scriptstyle 1 \mapsto 1 \otimes 1} & & \Big\downarrow {\scriptstyle |0\rangle \mapsto |0\rangle \otimes |0\rangle,\; |1\rangle \mapsto |1\rangle \otimes |1\rangle} \\
\mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C} & \xrightarrow[\;1 \otimes 1 \mapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\;]{} & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C})
\end{array}$$

cartesian ⇒ diagonal

— the process of copying —

Not commutative in $(\mathbf{Hilb}, \otimes)$:

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\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\;1 \mapsto |0\rangle + |1\rangle\;} & \mathbb{C} \oplus \mathbb{C} \\
{\scriptstyle 1 \mapsto 1 \otimes 1}\downarrow & & \downarrow {\scriptstyle \begin{array}{l} |0\rangle \mapsto |0\rangle \otimes |0\rangle \\ |1\rangle \mapsto |1\rangle \otimes |1\rangle \end{array}} \\
\mathbb{C} \simeq \mathbb{C} \otimes \mathbb{C} & \xrightarrow[\;1 \otimes 1 \mapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\;]{} & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C})
\end{array}
$$

Entangled Bell-state vs. disentangled state

From coherence to neo-C(C)C

— with Saunders Mac Lane and Max Kelly —

coherence

Mac Lane in *Coherence in categories*, LNM **281** (1970):

> Lambek obtained a preliminary coherence result which recognizes a connection between [coherence for closed monoidal categories] and the cut-elimination theorem of Gentzen-style proof theory, and then Kelly-Mac Lane combined some of Lambek's ideas with the notion of "graph" of a generalized natural transformation to obtain a much more extensive coherence theorem covering many of the diagrams arising in closed categories.

coherence

Max Kelly in *Coherence in categories*, LNM **281** (1970):

> ***Moreover, such things appear in nature. Define a compact closed category as . . .***

Examples of Compact (Closed) Categories:

- ‘multi-linear algebra’ e.g. ‘catégories tannakiennes’
- ‘interaction categories’ in computational concurrency
- ‘cobordism categories’ for topological quantum fields
- ‘Lambek pregroup’ = non-sym. compact preorder
- ‘knots’ = morphism in non-sym. compact category
- ‘quantum entanglement’ = *strongly* compact category = ‘essence of full-blown quantum mechanics’

Compact (Closed) Categories

Def. A ***compact closed category*** is a *1-object bicategory* in which *each 1-cell has a left adjoint*.

Bénabou, J. (1967) ***Introduction to bicategories.*** LNM **47**.

Kelly, G. M. (1972) ***Many-variable functorial calculus.*** LNM **281**.

Kelly & Laplaza, M. L. (1980) ***Coherence for CCCs.*** JPAA **19**.

Compact (Closed) Categories

Symmetric Monoidal Tensor with for each object A

- **dual** A^*
- **unit** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ and **counit** $\epsilon_A : A \otimes A^* \longrightarrow \mathrm{I}$

$$
\begin{array}{ccccc}
A & \xleftarrow{\ \simeq\ } & \mathrm{I} \otimes A & \xleftarrow{\ \epsilon_A \otimes 1_A\ } & (A \otimes A^*) \otimes A \\
{\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\ \simeq\ ]{} & A \otimes \mathrm{I} & \xrightarrow[\ 1_A \otimes \eta_A\ ]{} & A \otimes (A^* \otimes A)
\end{array}
\qquad (1)
$$

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$$
\begin{array}{ccccc}
A^* & \xleftarrow{\;\simeq\;} & A^* \otimes \mathrm{I} & \xleftarrow{\;1_{A^*} \otimes \epsilon_A\;} & A^* \otimes (A \otimes A^*) \\
\uparrow{\scriptstyle 1_{A^*}} & & & & \uparrow{\scriptstyle \simeq} \\
A^* & \xrightarrow[\;\simeq\;]{} & \mathrm{I} \otimes A^* & \xrightarrow[\;\eta_A \otimes 1_{A^*}\;]{} & (A^* \otimes A) \otimes A^*
\end{array}
\qquad (2)
$$

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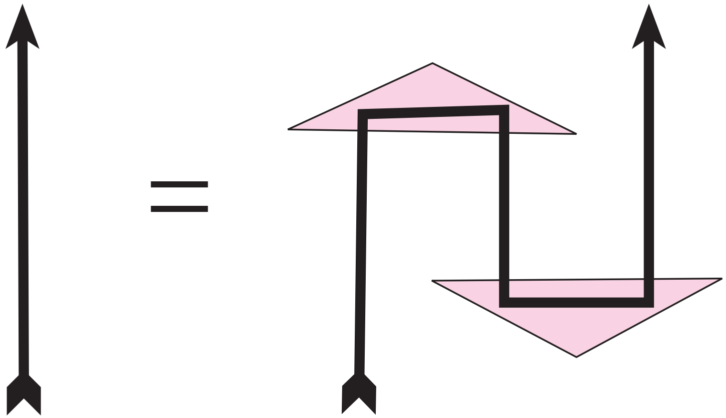

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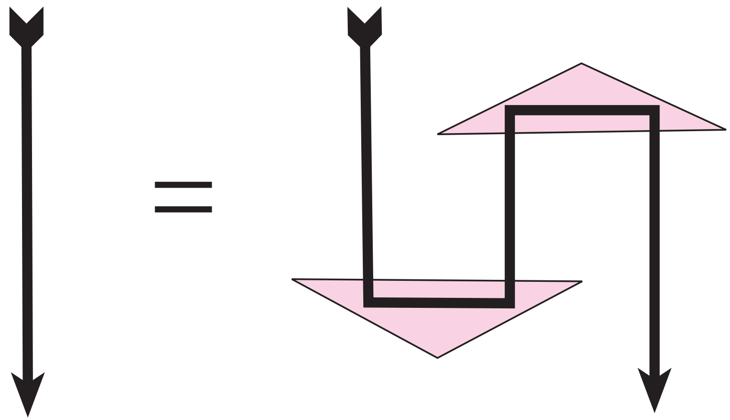

Compact (Closed) Categories

A contravariant $\otimes$-involution

$$f : A \to B \quad \mapsto \quad f^* : B^* \to A^*$$

arises as

$$\begin{array}{ccccc}
A^* & \xleftarrow{\;\simeq\;} & A^* \otimes \mathrm{I} & \xleftarrow{1_{A^*} \otimes \epsilon_B} & A^* \otimes B \otimes B^* \\
{\scriptstyle f^*}\uparrow & & & & \uparrow{\scriptstyle 1_{A^*} \otimes f \otimes 1_{B^*}} \\
B^* & \xrightarrow[\simeq]{} & \mathrm{I} \otimes B^* & \xrightarrow[\eta_A \otimes 1_{B^*}]{} & A^* \otimes A \otimes B^*
\end{array}$$

Compact (Closed) Categories

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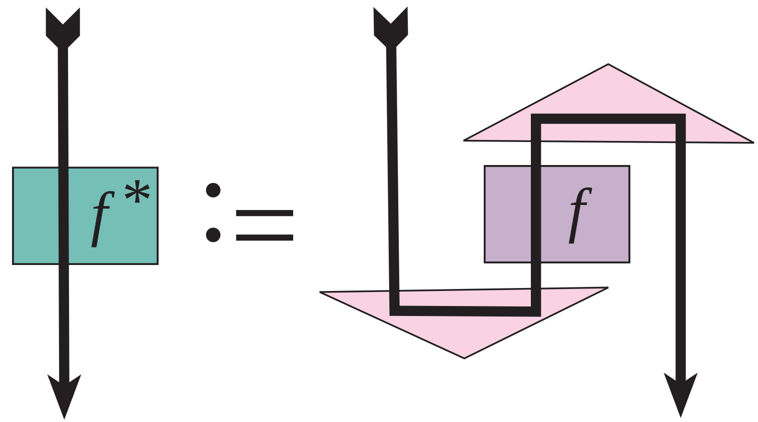

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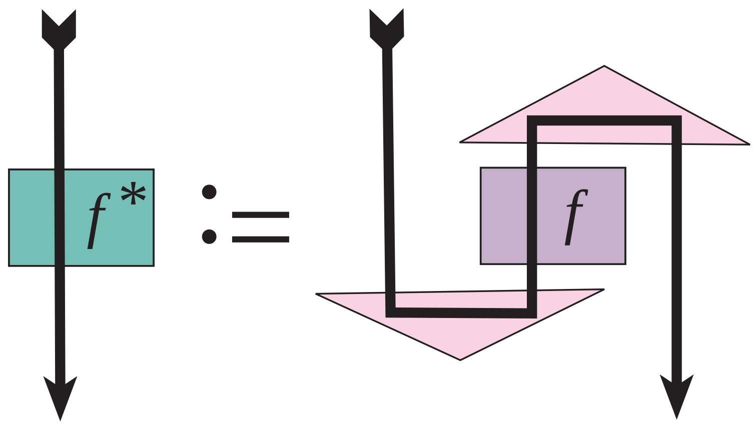

$\Rightarrow$ $*$-autonomy

$$\frac{\textbf{$*$-autonomy} \;+\; \otimes \equiv \otimes^*}{\textbf{compact closure}}$$

cf. $A \Rightarrow B := A^* \otimes^* B = (A \otimes B^*)^*$ via De Morgan

Closedness:

$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, B \Rightarrow C)$$

$\Rightarrow$ Multiplicative Logic

$*$-Autonomy:

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$\Rightarrow$ Multiplicative Logic with Negation

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$\Rightarrow$ Multiplicative Logic with Negation

Compactness:

$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, B^* \otimes C)$$

$\Rightarrow$ Weird Logic

By closedness

$$\mathbf{C}(B, C) \simeq \mathbf{C}(\mathrm{I}, B \Rightarrow C)$$

$$B \xrightarrow{f} C \quad \overset{\simeq}{\longleftrightarrow} \quad \mathrm{I} \longrightarrow B \Rightarrow C$$

$\Rightarrow$ internalization as elements

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$\Rightarrow$ internalization as elements

By compactness

$$\mathbf{C}(B \otimes C^*, \mathrm{I}) \simeq \mathbf{C}(B, C) \simeq \mathbf{C}(\mathrm{I}, B^* \otimes C)$$

$$B \otimes C^* \longrightarrow \mathrm{I} \quad \overset{\simeq}{\longleftrightarrow} \quad B \xrightarrow{f} C \quad \overset{\simeq}{\longleftrightarrow} \quad \mathrm{I} \longrightarrow B^* \otimes C$$

$\Rightarrow$ internalization as co-elements and as elements

Compact (Closed) Categories

A **partial trace**

$$f : C \otimes A \to C \otimes B \quad \mapsto \quad \mathrm{Tr}_C(f) : A \to B$$

arises as

$$\begin{CD} B @<{\simeq}<< \mathrm{I} \otimes B @<{(\sigma_{C,C^*} \circ \epsilon_C) \otimes 1_B}<< C^* \otimes C \otimes B \\ @A{\mathrm{Tr}_C(f)}AA @. @AA{1_{C^*} \otimes f}A \\ A @>>{\simeq}> \mathrm{I} \otimes A @>>{\eta_C \otimes 1_A}> C^* \otimes C \otimes A \end{CD}$$

Joyal, A, Street, R & Verity, D (1995) *Traced Monoidal Categories.*

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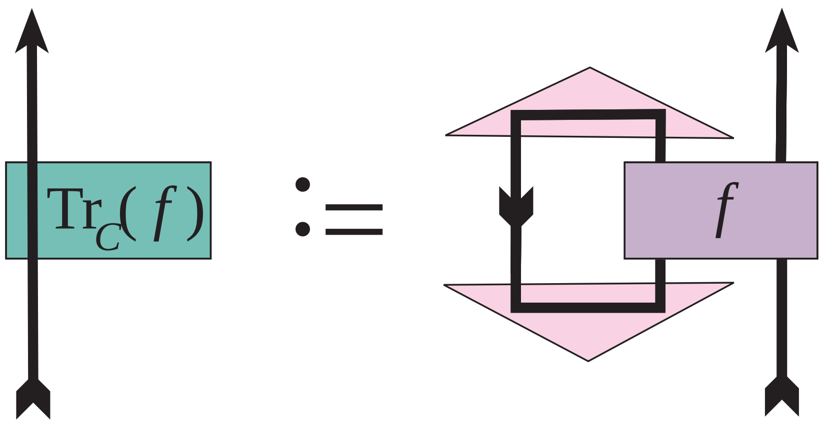


Joyal, A, Street, R & Verity, D (1995) ***Traced Monoidal Categories.***

Compact (Closed) Categories

A **full trace**

$$h : A \to A \quad \mapsto \quad \mathrm{Tr}(h) : \mathrm{I} \to \mathrm{I}$$

arises as

$$\begin{array}{ccc} \mathrm{I} & \xleftarrow{\sigma_{A,A^*} \circ \epsilon_A} & A^* \otimes A \\ {\scriptstyle \mathrm{Tr}(h)} \uparrow & & \uparrow {\scriptstyle 1_{A^*} \otimes h} \\ \mathrm{I} & \xrightarrow[\eta_A]{} & A^* \otimes A \end{array}$$

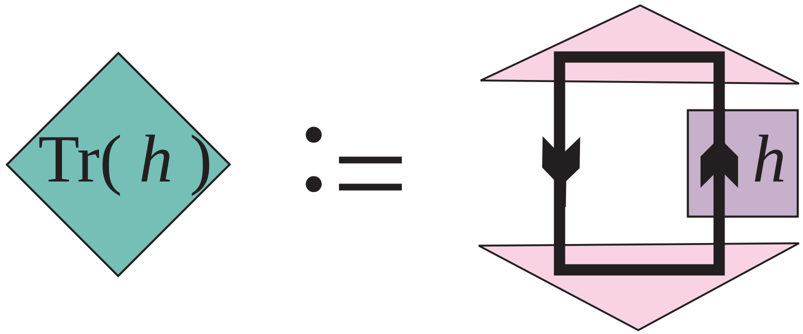
Joyal, A, Street, R & Verity, D (1995) *Traced Monoidal Categories.*

Compact (Closed) Categories

A **full trace**

$$h : A \to A \quad \mapsto \quad \mathrm{Tr}(h) : \mathrm{I} \to \mathrm{I}$$

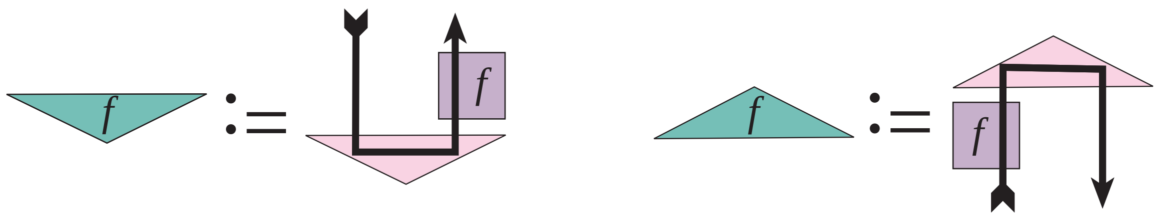
arises as



Joyal, A, Street, R & Verity, D (1995) ***Traced Monoidal Categories.***

Compositionality

When defining **names** and **conames**



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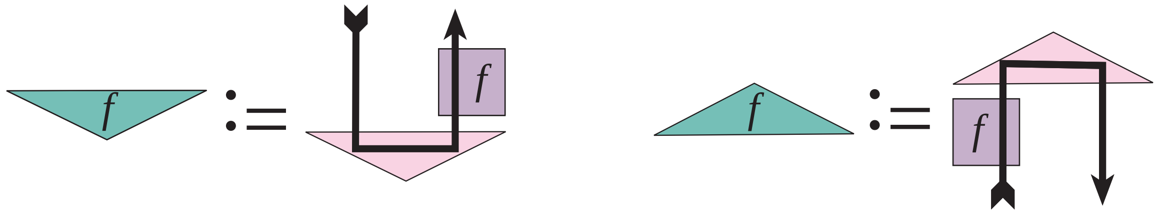
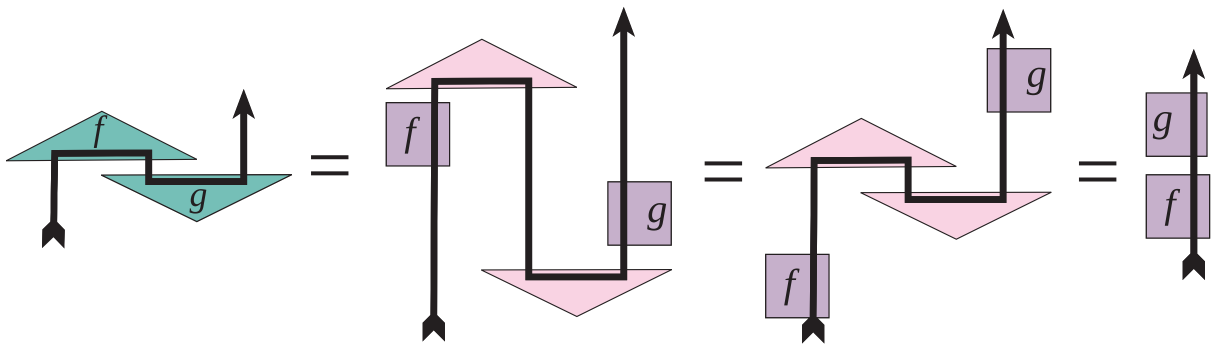
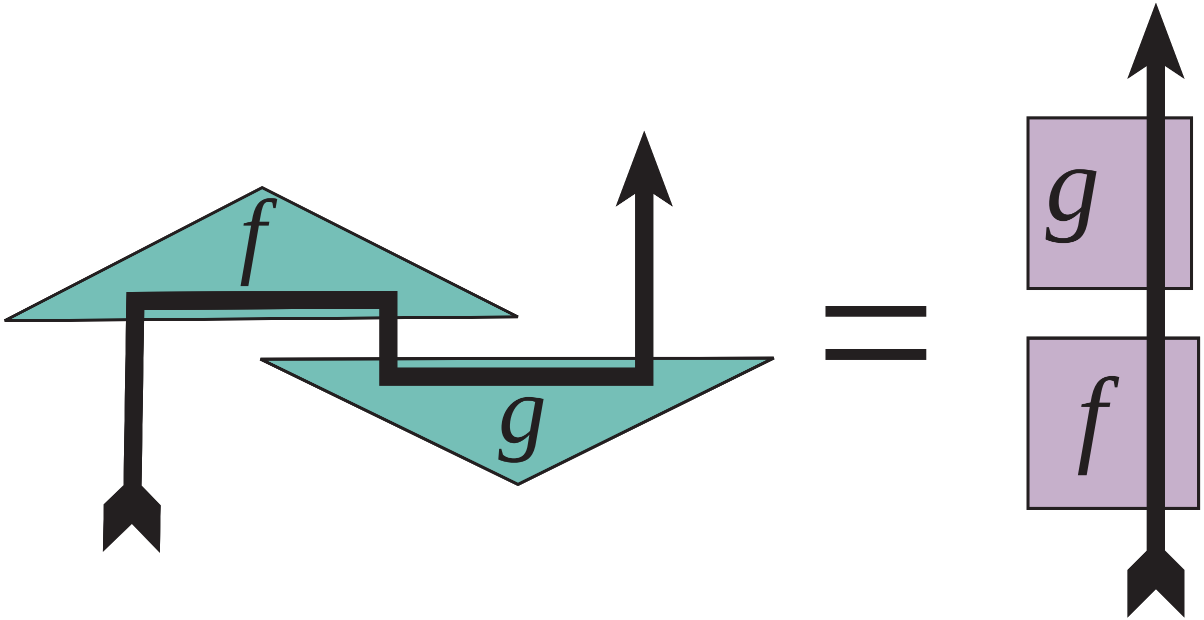
When defining **names** and **conames**



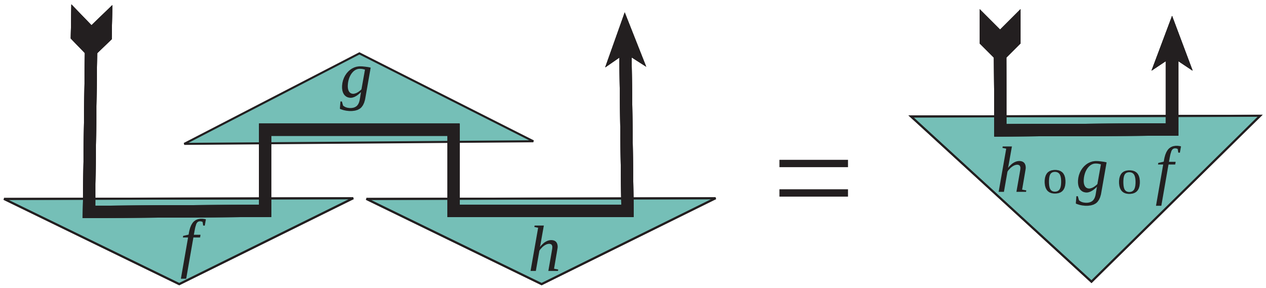
we obtain



Compositionality



Compositionality



If $\mathbf{C}$ has a 0-object, products and coproducts and if all morphisms with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are isos then $\mathbf{C}$ has **biproducts**.

If $\mathbf{C}$ is $\mathbf{Ab}$-enriched and if there are morphisms

$$A \underset{q_1}{\overset{p_1}{\leftrightarrows}} A \oplus B \underset{q_2}{\overset{p_2}{\rightleftarrows}} B$$

with

$$p_i \circ q_j = \delta_{ij} \qquad \sum_i q_i \circ p_i = 1_{A \oplus B}$$

then $\mathbf{C}$ has **biproducts**.

Biproduct categories admits matrix calculus

Categorics of matrix calculi

Let $\mathbf{BP}$ be a biproduct category with an object I such that $\mathbf{BP}(\mathrm{I},\mathrm{I})$ is commutative. Define full subcategory.

Categorics of matrix calculi

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- $Objects := \mathbb{N} \simeq \{\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n} \mid n \in \mathbb{N}\}$
- $\mathbf{D}(n,m)= n \times m$ matrices in $\mathbf{BP}(\mathrm{I},\mathrm{I})$
- $(\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n}) \otimes (\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{m}) := \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n\times m}$
- $n^* := n$ and $\eta_n = \epsilon_n^T := \Delta^{(n)} =: \mathrm{I} \to \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n\times n}$

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$\Rightarrow$ **we obtain a compact closed category**

Deligne, P. (1990) ***Catégories tannakiennes.*** *Grothendieck Festschrift*

distributivity natural transformation

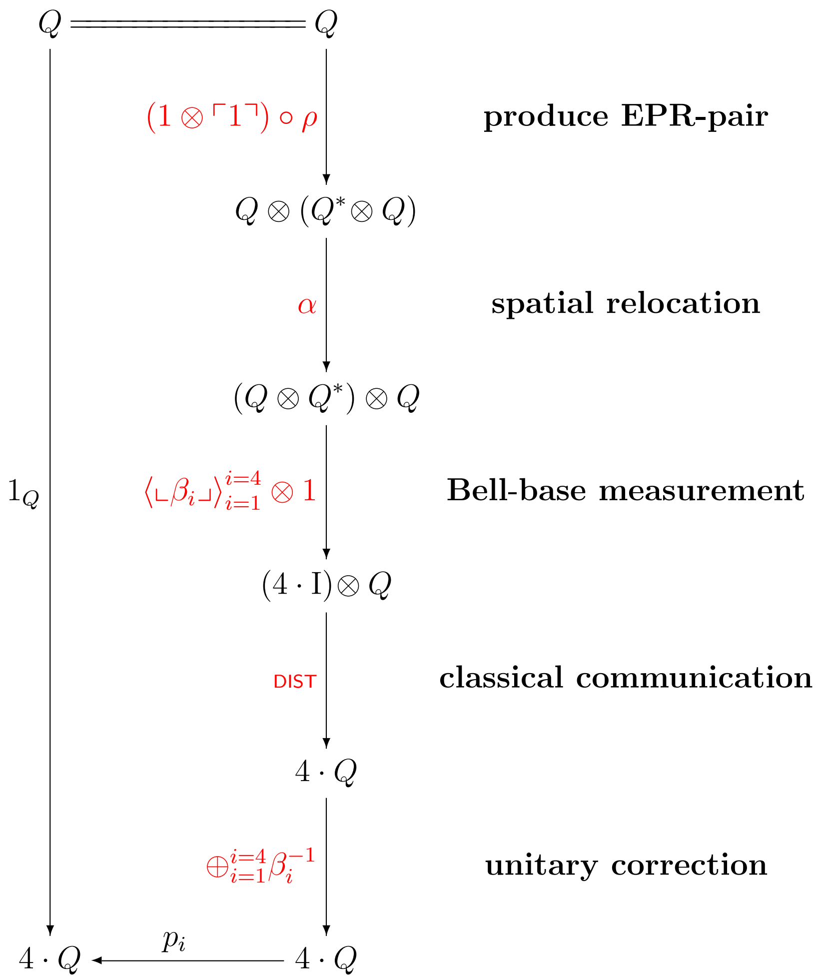
$$
\begin{array}{ccc}
(A_1 \oplus A_2) \otimes C & \xrightarrow{(f_1 \oplus f_2) \otimes g} & (B_1 \oplus B_2) \otimes D \\
\downarrow {\scriptstyle \mathsf{DIST}_{A_1, A_2, C}} & & \downarrow {\scriptstyle \mathsf{DIST}_{B_1, B_2, D}} \\
(A_1 \otimes C) \oplus (A_2 \otimes C) & \xrightarrow[(f_1 \otimes g) \oplus (f_2 \otimes g)]{} & (B_1 \otimes D) \oplus (B_2 \otimes D)
\end{array}
$$

distributivity natural transformation

$$\begin{array}{ccc}
(A_1 \oplus A_2) \otimes C & \xrightarrow{(f_1 \oplus f_2) \otimes g} & (B_1 \oplus B_2) \otimes D \\
\Big\downarrow \mathsf{DIST}_{A_1,A_2,C} & & \mathsf{DIST}_{B_1,B_2,D} \Big\downarrow \\
(A_1 \otimes C) \oplus (A_2 \otimes C) & \xrightarrow[(f_1 \otimes g) \oplus (f_2 \otimes g)]{} & (B_1 \otimes D) \oplus (B_2 \otimes D)
\end{array}$$

Distribution of data:

$$(\mathrm{I} \oplus \mathrm{I}) \otimes Agent \simeq (\mathrm{I} \otimes Agent) \oplus (\mathrm{I} \otimes Agent)$$


Q
Q
(1 ⊗ ⌜1⌝) ∘ ρ
produce EPR-pair
Q ⊗ (Q* ⊗ Q)
α
spatial relocation
(Q ⊗ Q*) ⊗ Q
1_Q
⟨⌞β_i⌟⟩_{i=1}^{i=4} ⊗ 1
Bell-base measurement
(4 · I) ⊗ Q
DIST
classical communication
4 · Q
⊕_{i=1}^{i=4} β_i^{-1}
unitary correction
4 · Q
p_i
4 · Q

$$\boxed{\frac{\dagger\text{-compactness}}{\text{compactness}} \simeq \frac{\text{inner-product space}}{\text{vector space}}}$$

Abramsky & Coecke (2004) ***A categorical semantics of quantum protocols.*** IEEE – Logic in Computer Science – `quant-ph/0402130`.

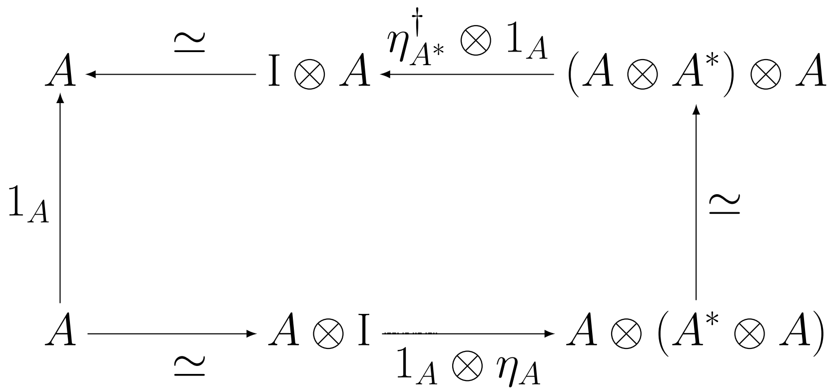
Abramsky & Coecke (2005) ***Abstract physical traces.*** TAC 14.

Selinger, P. (2005) **†-*CCC and completely positive maps.***

†-CCC

Symmetric Monoidal Tensor with

- **dual** A^*
- **unit** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$
- contravariant $\otimes$-involutive **adjoints** $f^\dagger : B \longrightarrow A$



†-CCC

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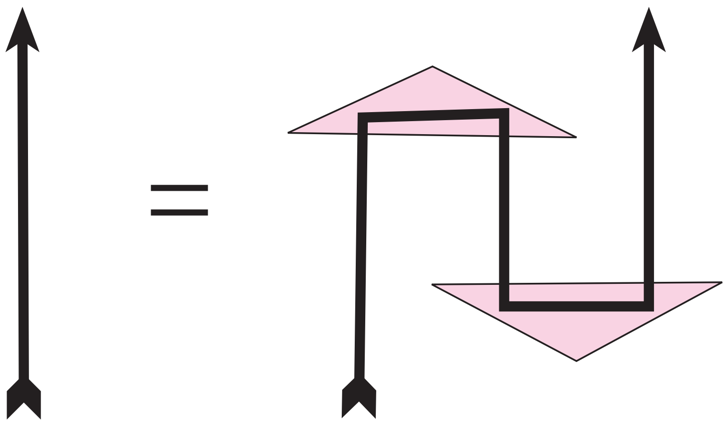

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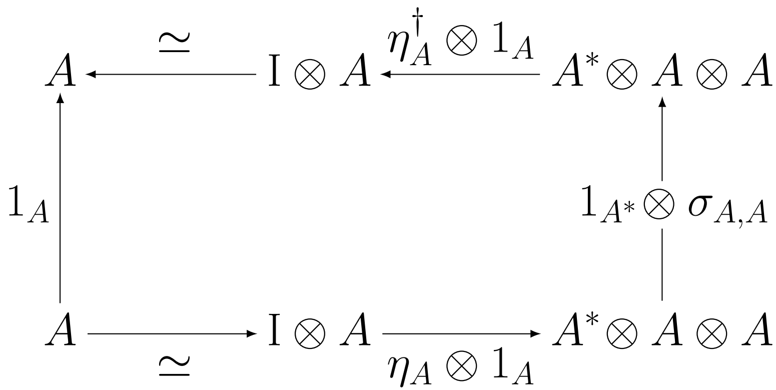

†-CCC

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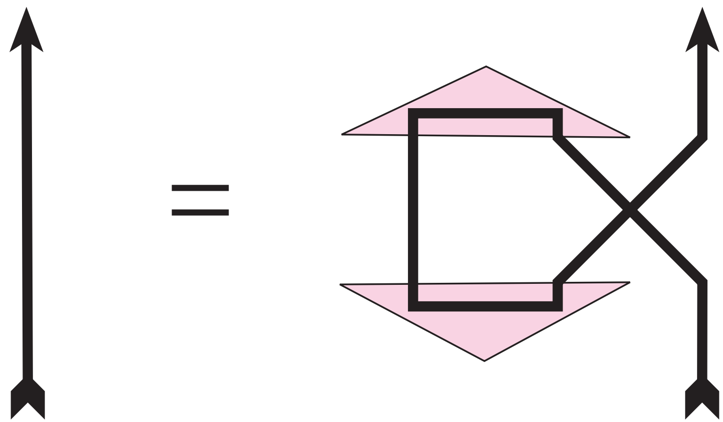

†-CCC

A covariant $\otimes$-involution

$$f : A \to B \quad \mapsto \quad f_* : A^* \to B^*$$

arises as

$$\begin{array}{ccccc}
B^* & \xleftarrow{\;\simeq\;} & \mathrm{I} \otimes B^* & \xleftarrow{\eta_A^\dagger \otimes 1_{B^*}} & A^* \otimes A \otimes B^* \\
\uparrow {\scriptstyle f_*} & & & & \uparrow {\scriptstyle 1_{B^*} \otimes f^\dagger \otimes 1_{A^*}} \\
A^* & \xrightarrow[\;\simeq\;]{} & A^* \otimes \mathrm{I} & \xrightarrow[1_{A^*} \otimes \eta_{B^*}]{} & A^* \otimes B \otimes B^*
\end{array}$$

†-CCC

A covariant ⊗-involution

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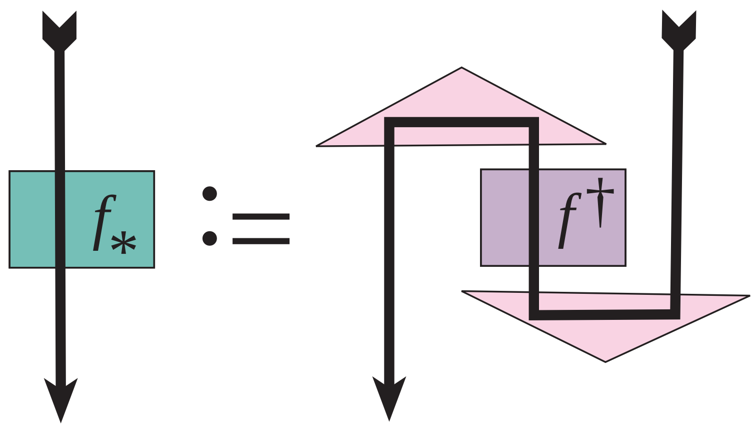

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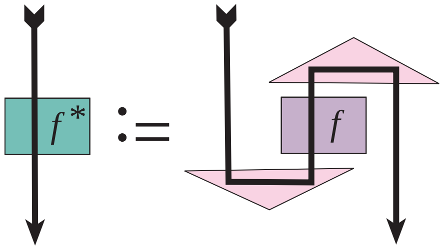
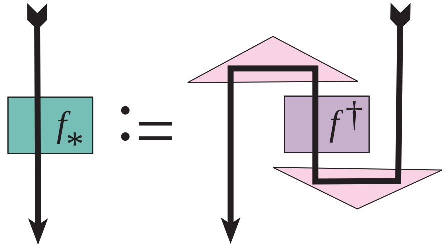


and



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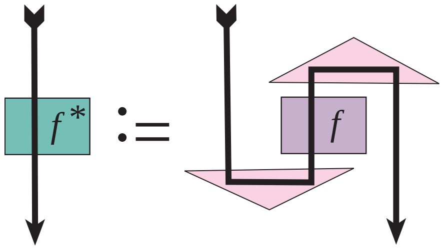
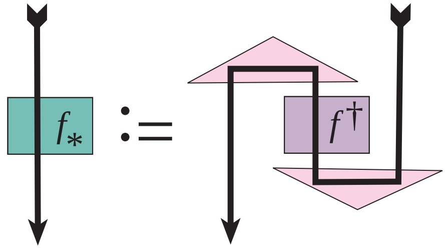
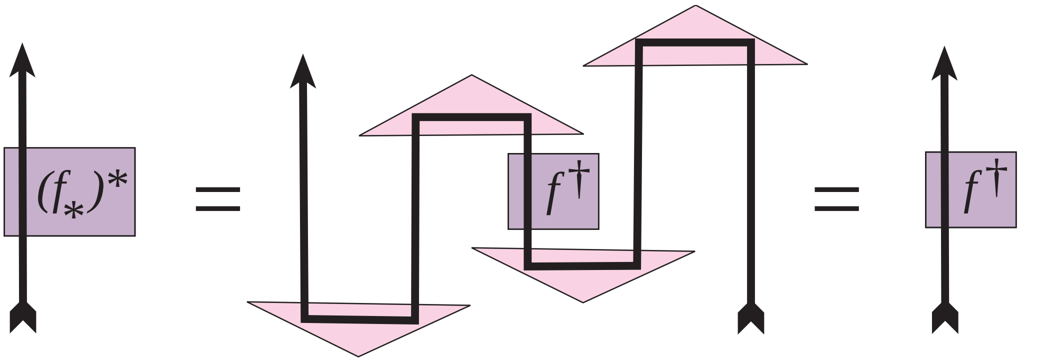


and



follows



†-CCC

The adjoint decomposes:

$$f^{\dagger} = (f^{*})_{*} = (f_{*})^{*}$$

†-CCC

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$$f^{\dagger} = (f^{*})_{*} = (f_{*})^{*}$$

E.g. in **FdHilb**:

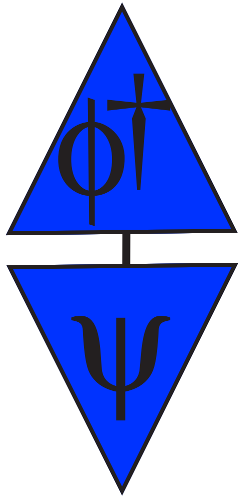
$(-)^{*}$ **:= transposition**

$(-)_{*}$ **:= complex conjugation**

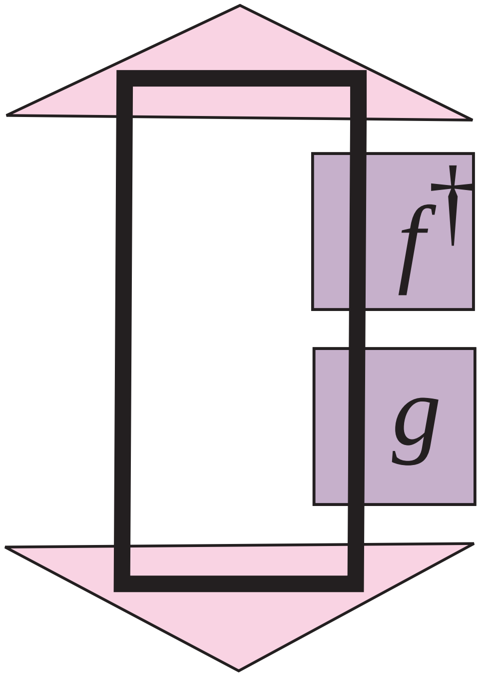
inner-product/norm

For $\phi, \psi : \mathrm{I} \to A$ we set:

$$\langle \phi \mid \psi \rangle := \phi^\dagger \circ \psi : \mathrm{I} \to \mathrm{I}$$

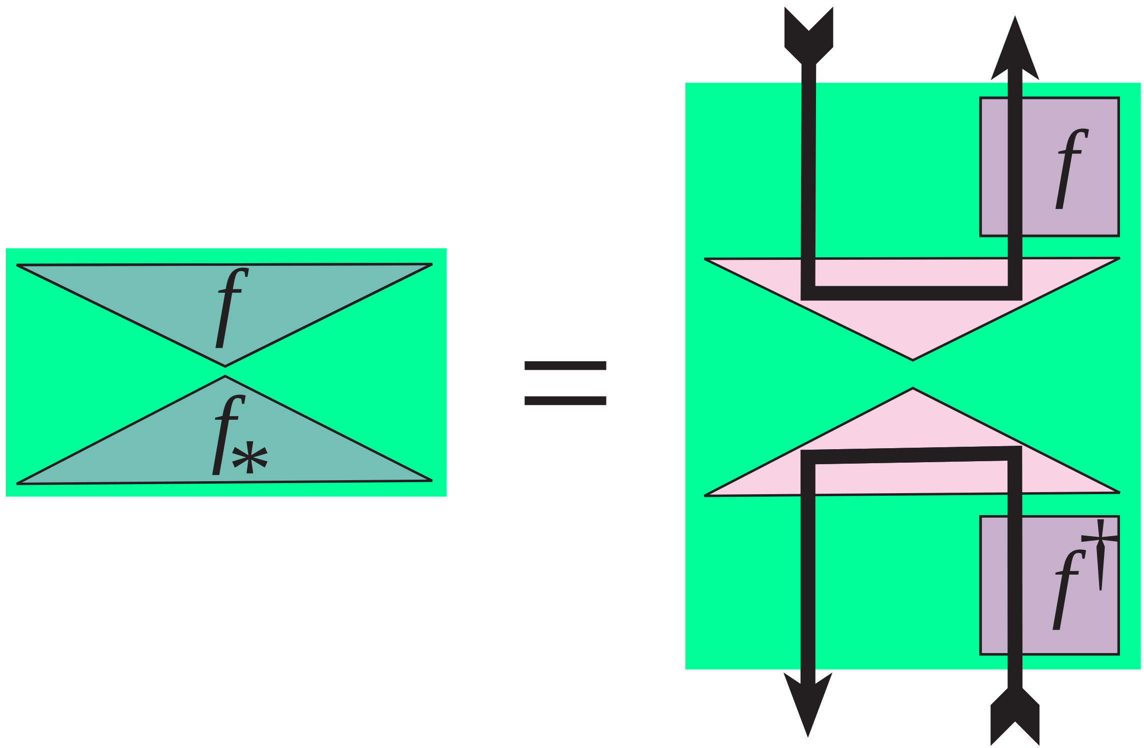


Hilbert-Schmidt inner-product/norm



$$\langle f \mid g \rangle := \mathrm{Tr}(f^\dagger \circ g)$$

complete bipartite projector

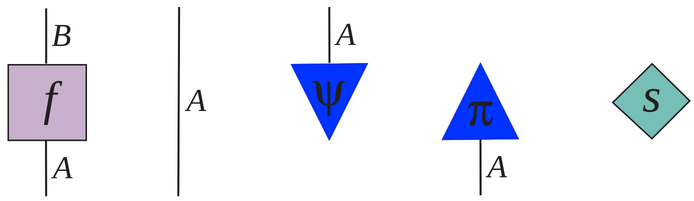


$$\mathrm{P}_f : A^* \otimes B \to A^* \otimes B$$

Application:
kindergarten quantum mechanics

Coecke (2005) ***Kindergarten quantum mechanics.*** `quant-ph/0510032`

Primitive data:



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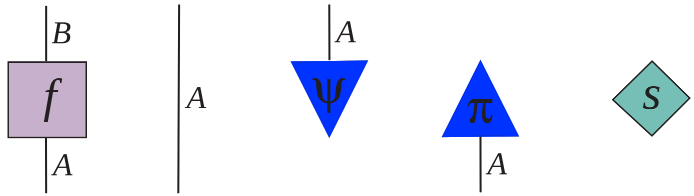
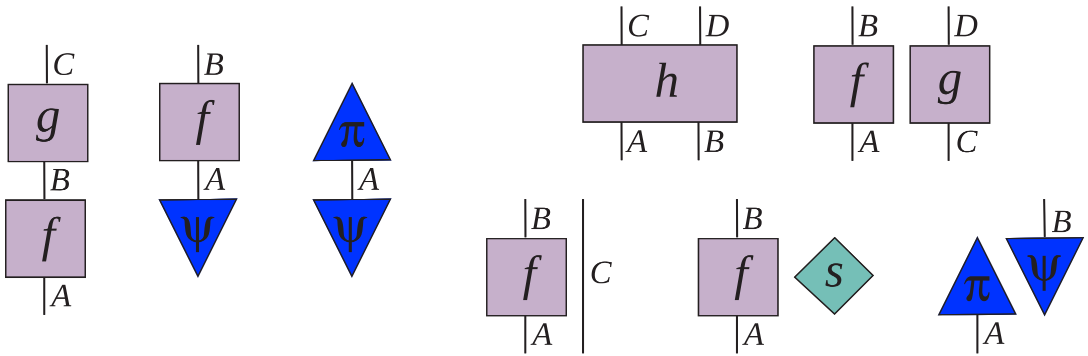
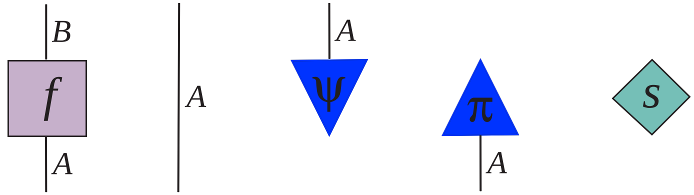


Sequential and parallel composition:



Primitive data:



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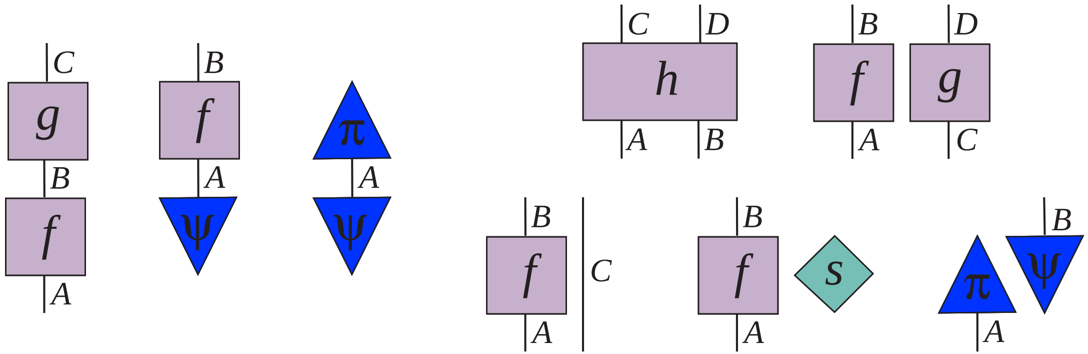
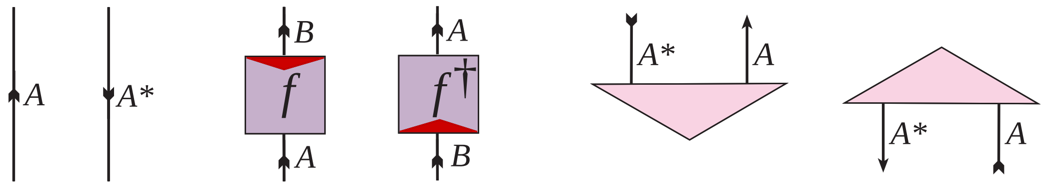


Duals, adjoints and EPR-states:



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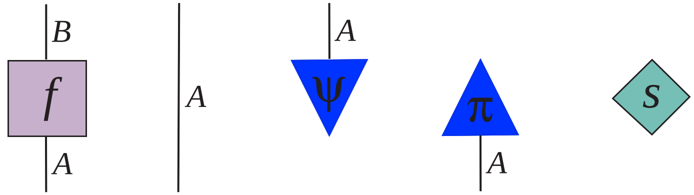

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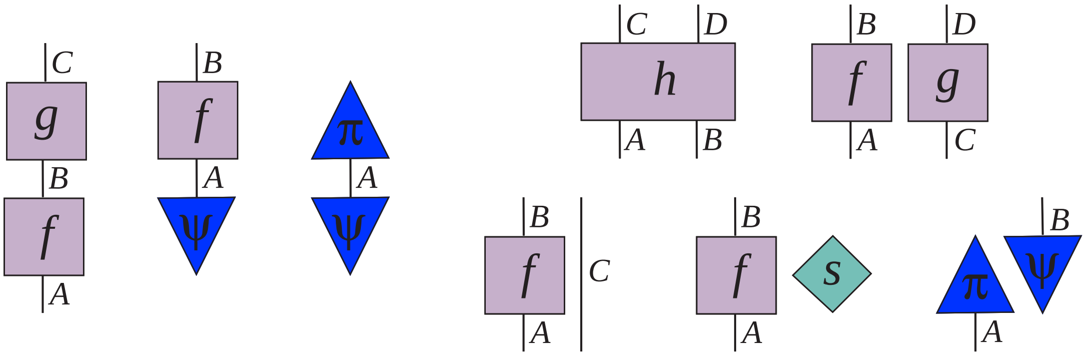

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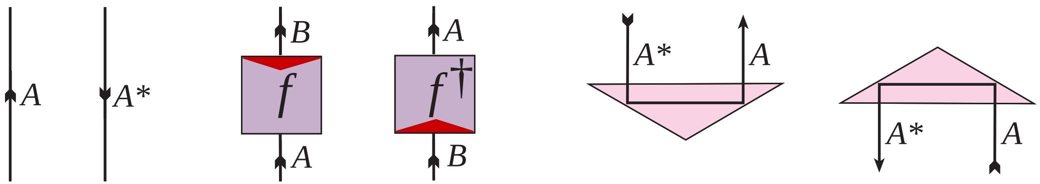
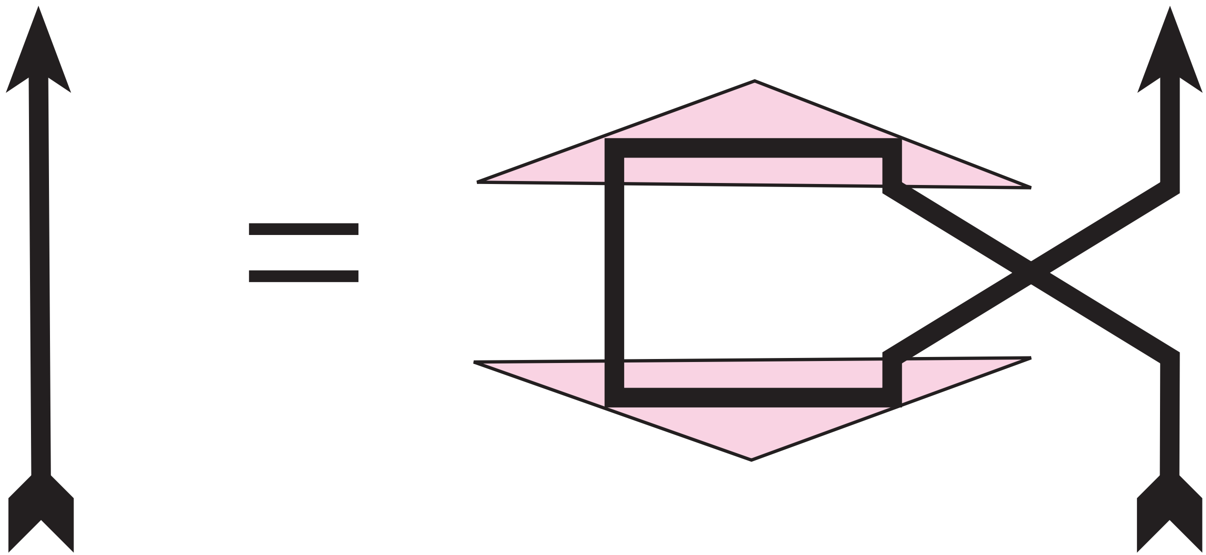
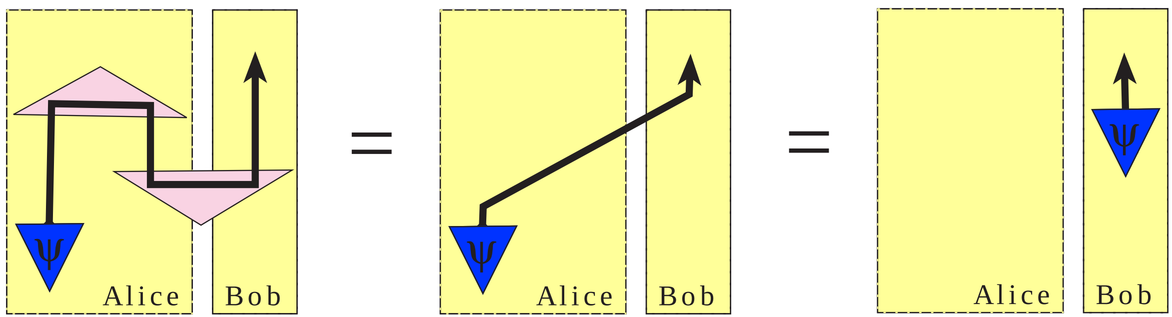


THE SOLE AXIOM

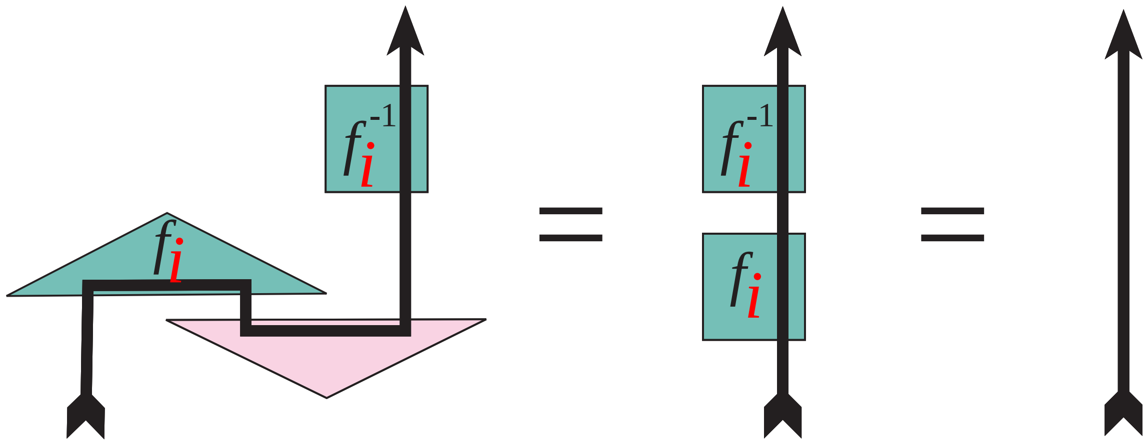


quantum teleportation



6 eminent researchers (1993) ... ***teleporting*** **....** 60 years after the actual birth of von Neumann's quantum formalism in PRL (of course).

quantum teleportation



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quantum teleportation: textbook version

Description. Alice has an 'unknown' qubit $|\phi\rangle$ and wants to send it to Bob. They have the ability to communicate classical bits, and they share an entangled pair in the EPR-state, that is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which Alice produced by first applying a Hadamard-gate $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the first qubit of a qubit pair in the ground state $|00\rangle$, and by then applying a CNOT-gate, that is

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |01\rangle \quad |10\rangle \mapsto |11\rangle \quad |11\rangle \mapsto |10\rangle,$$

then **she sends the first qubit of the pair to Bob**. To teleport her qubit, Alice first performs a bipartite measurement on the unknown qubit and her half of the entangled pair in the Bell-base, that is

$$\big\{|0x\rangle + (-1)^z|1(1-x)\rangle \mid x, z \in \{0,1\}\big\},$$

where we denote the four possible outcomes of the measurement by xz. Then **she sends the 2-bit outcome xz to Bob using the classical channel**. Then, if $x = 1$, Bob performs the unitary operation $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on its half of the shared entangled pair, and he also performs a unitary operation $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on it if $z = 1$. Now **Bob's half of the initially entangled pair is in state $|\phi\rangle$**.

Proof. In the case that the measurement outcome of the Bell-base measurement is xz, for

$$\mathrm{P}_{xz} := \langle 0x + (-1)^z 1(1-x)| - \rangle |0x + (-1)^z 1(1-x)\rangle$$

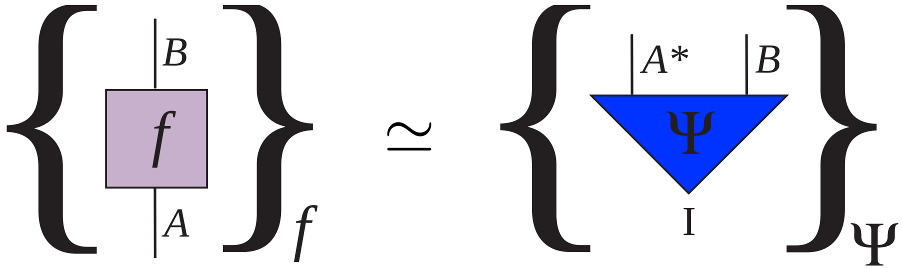
we have to apply $\mathrm{P}_{xz} \otimes \mathrm{id}$ to the input state $|\phi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Setting $|\phi\rangle = \phi_0|0\rangle + \phi_1|1\rangle$ we rewrite the input as

$$\frac{1}{\sqrt{2}}(\phi_0|000\rangle + \phi_0|011\rangle + \phi_1|100\rangle + \phi_1|111\rangle) = \frac{1}{\sqrt{2}}(\phi_0 \sum_{x=0,1} |0xx\rangle + \phi_1 \sum_{x=0,1} |1(1-x)(1-x)\rangle)$$

and application of $\mathrm{P}_{xz} \otimes \mathrm{id}$ then yields

$$\frac{1}{\sqrt{2}}|0x + (-1)^z 1(1-x)\rangle \otimes (\phi_0|x\rangle + (-1)^z \phi_1|1-x\rangle)\,.$$

There are four cases concerning the unitary corrections U_{xz} which have to be applied. For $x = z = 0$ the third qubit is $\phi_0|0\rangle + \phi_1|1\rangle = |\phi\rangle$. **If** $x = 0$ **and** $z = 1$ it is $\phi_0|0\rangle - \phi_1|1\rangle$ which after applying $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ becomes $|\phi\rangle$. **If** $x = 1$ it is $\phi_0|1\rangle + (-1)^z\phi_1|0\rangle$ which after applying $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ brings us back to the previous two cases, what completes this proof. □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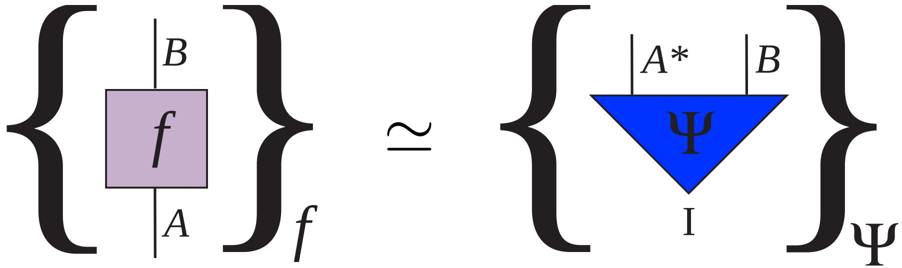
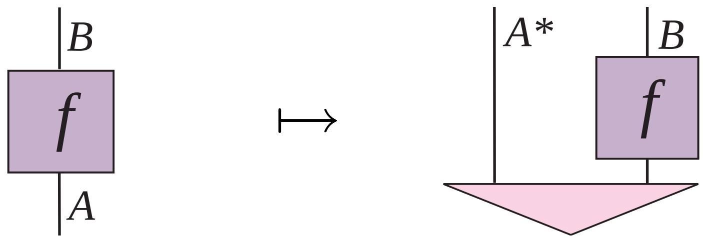
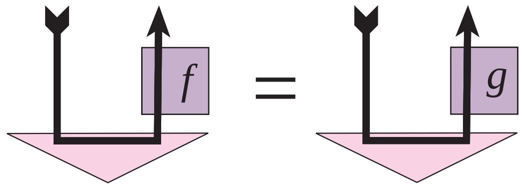


for the bijection i.e.



Proof of injectivity.



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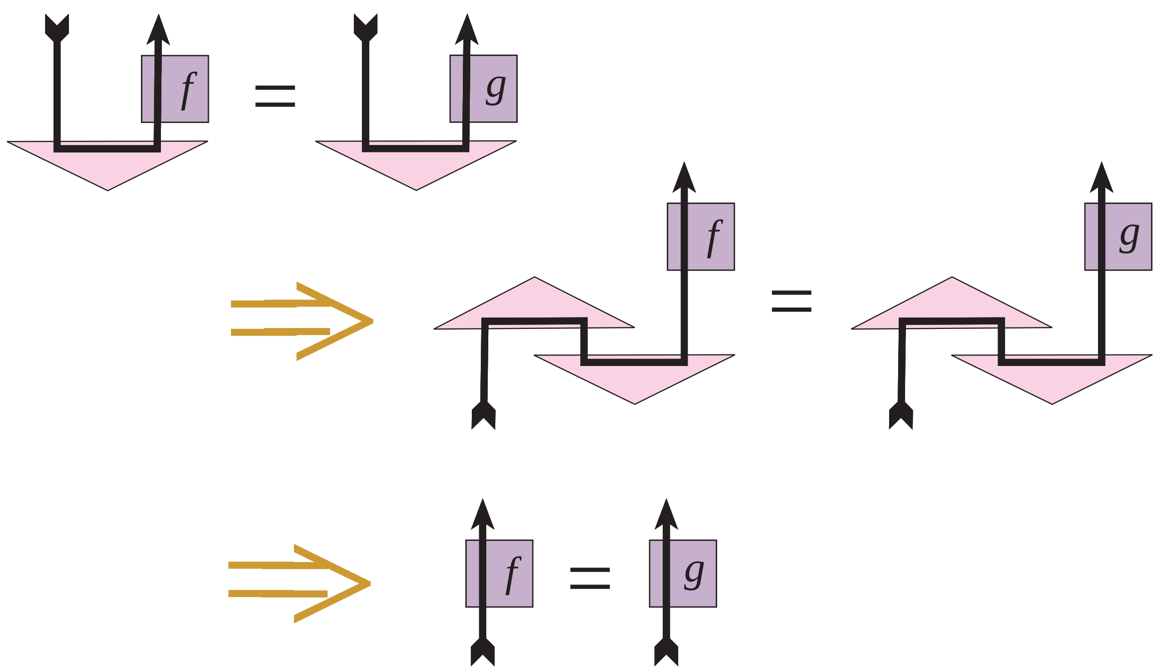

Proof of surjectivity.

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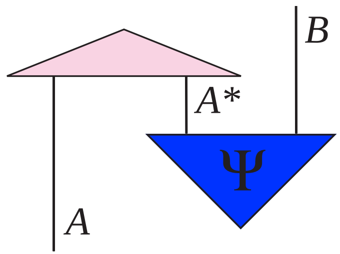

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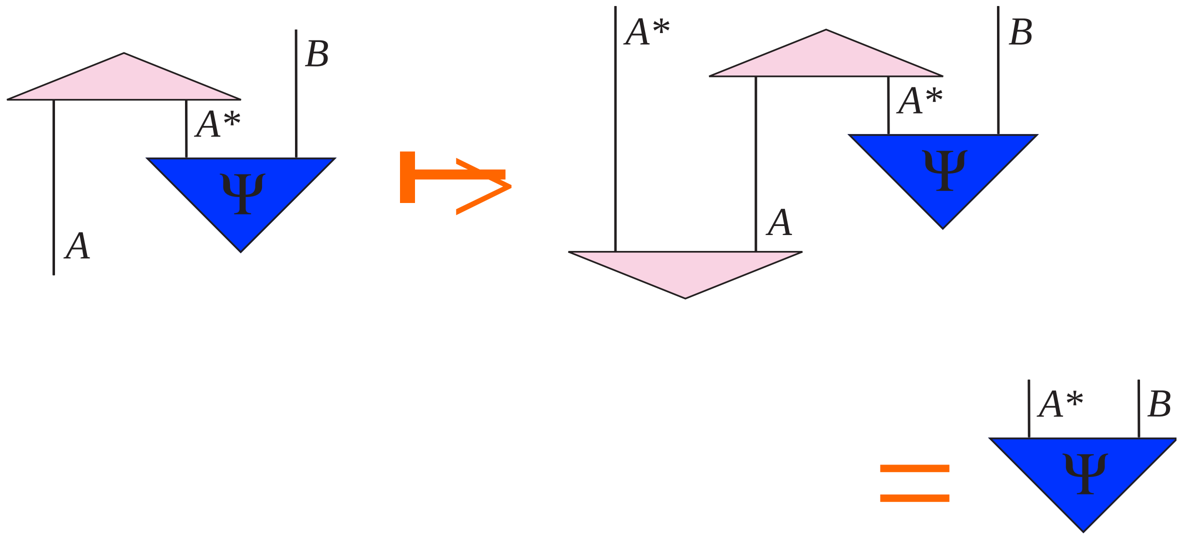


Reconsideration: quantum logic which works

Coecke (2005) ***De-linearizing linearity: Projective quantum axiomatics from strong compact closure.*** `quant-ph/0506134`

Prehistory

quantum mechanical formalism [von Neumann 1932]

Birkhoff, G. (1958) ***von Neumann and lattice theory.*** Bull. AMS **64**.

Rédei, M. (1997) ***Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead).*** Stud. Hist. Phil. Mod. Phys. **27**.

Prehistory

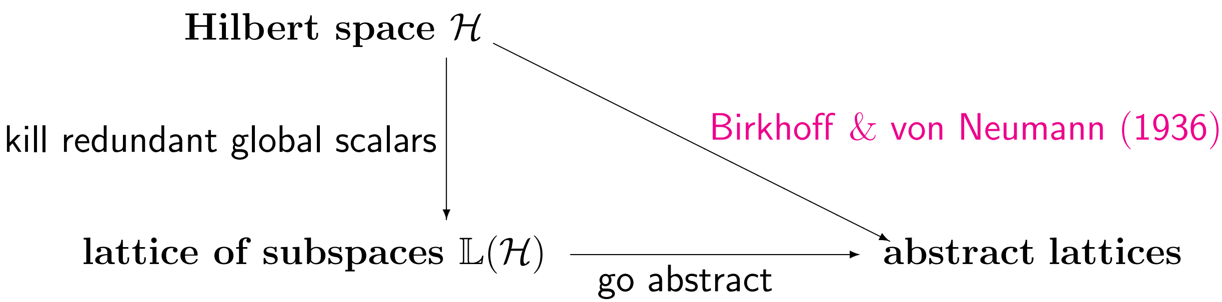
quantum mechanical formalism [von Neumann 1932]

"I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic.) [von Neumann 1935]

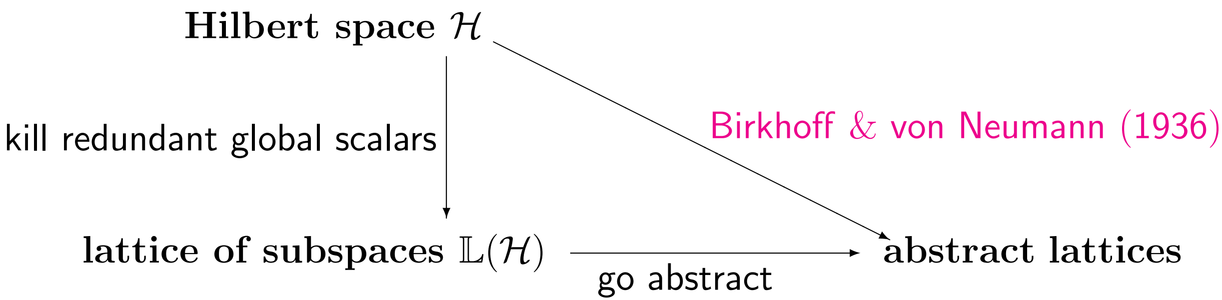
Birkhoff, G. (1958) ***von Neumann and lattice theory.*** Bull. AMS **64**.

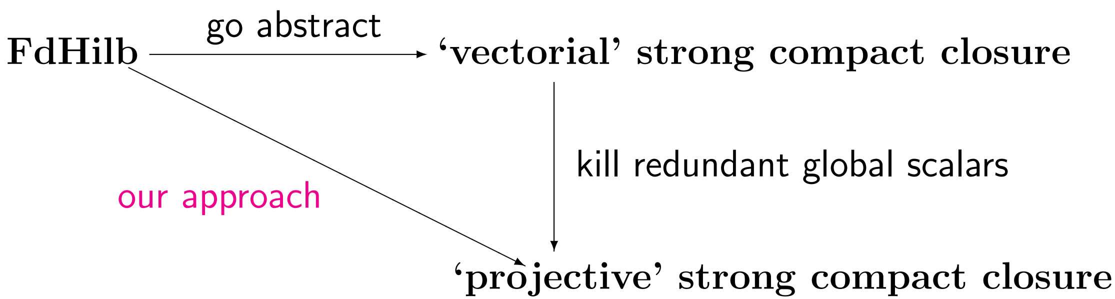
Rédei, M. (1997) ***Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead).*** Stud. Hist. Phil. Mod. Phys. **27**.

vectorial vs. projective

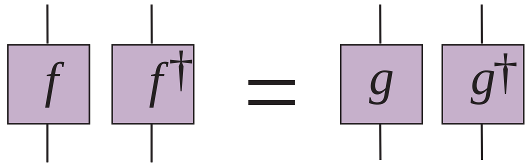


vectorial vs. projective

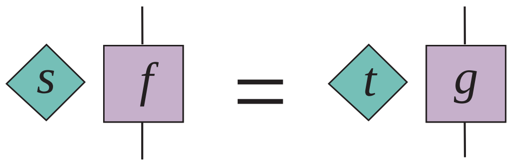


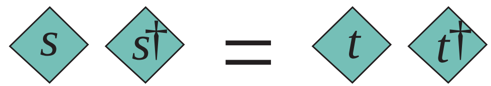


If



then there exist diamonds s, t such that:

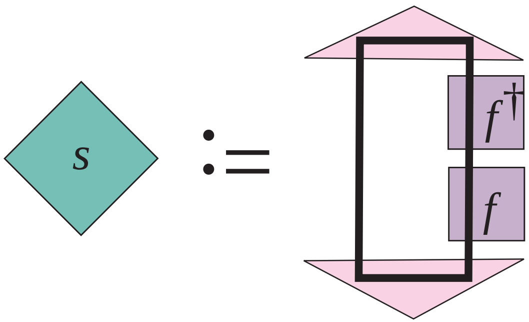


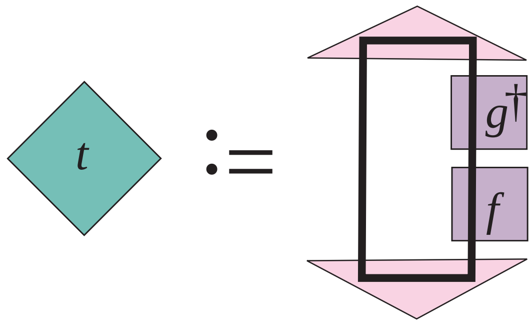


Formally:

$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t : \; s \bullet f = t \bullet g \; , \; s \circ s^\dagger = t \circ t^\dagger$$

Proof.





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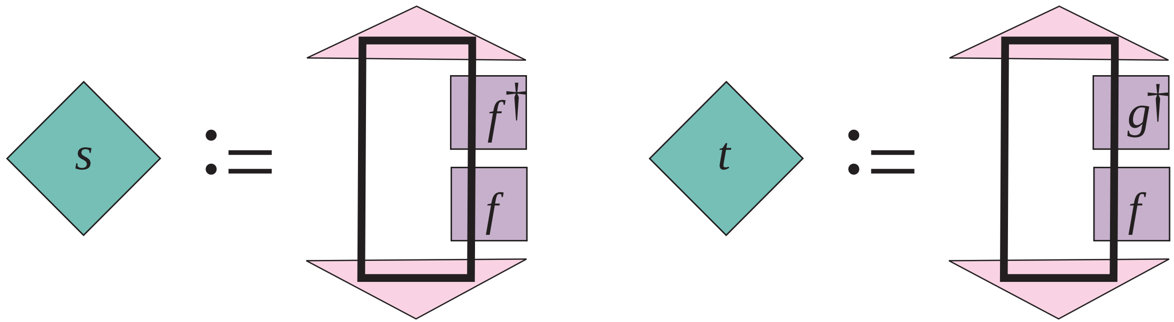

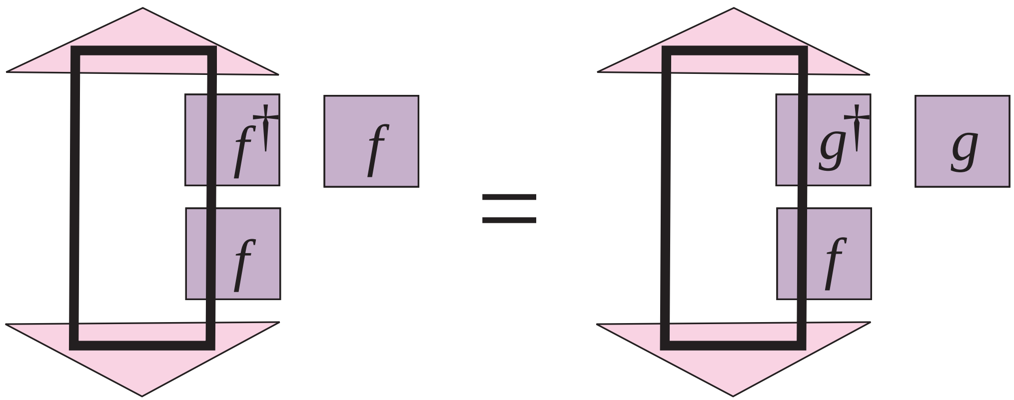

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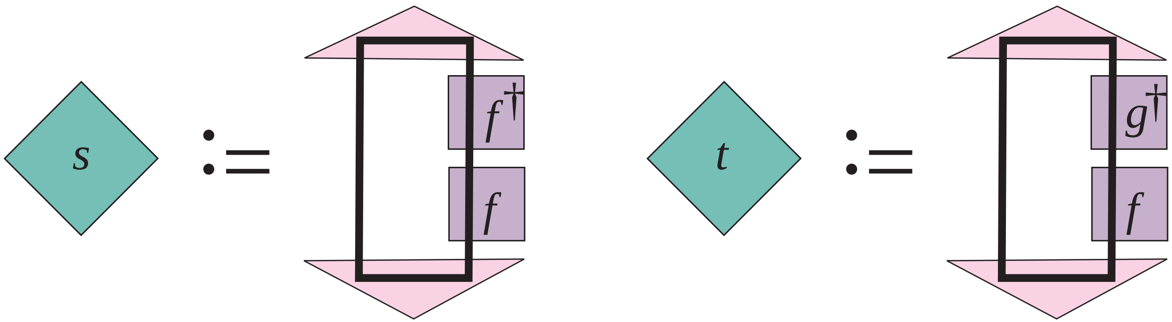

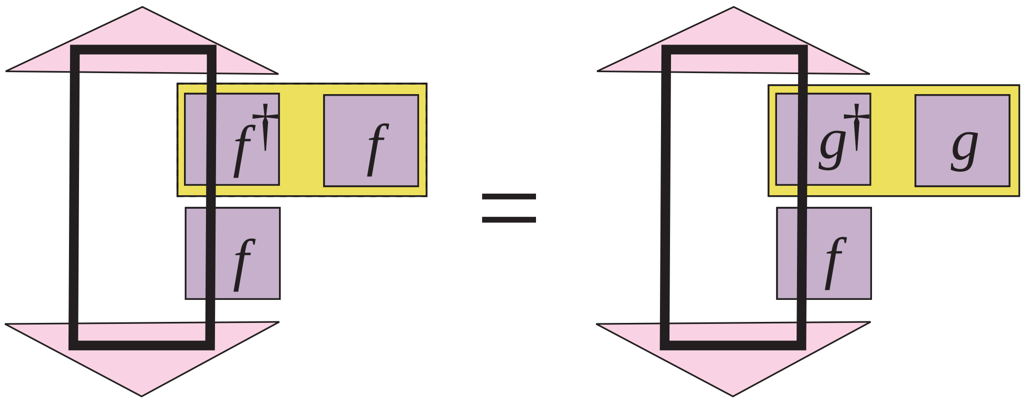

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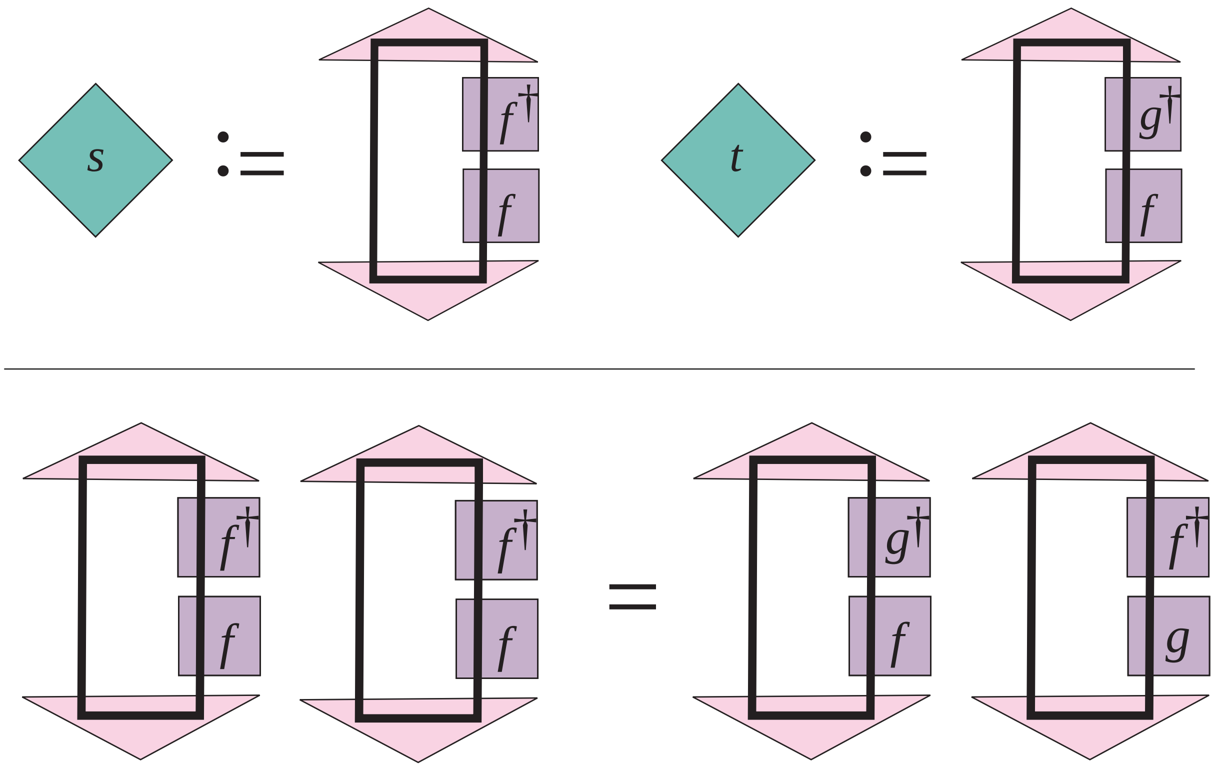

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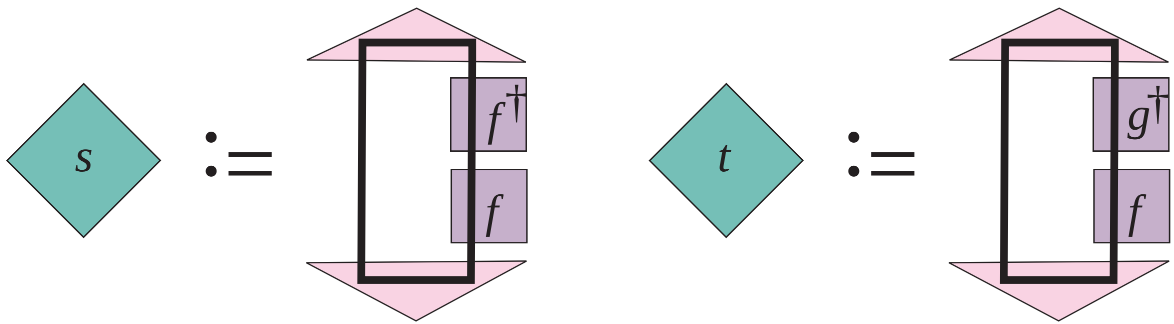
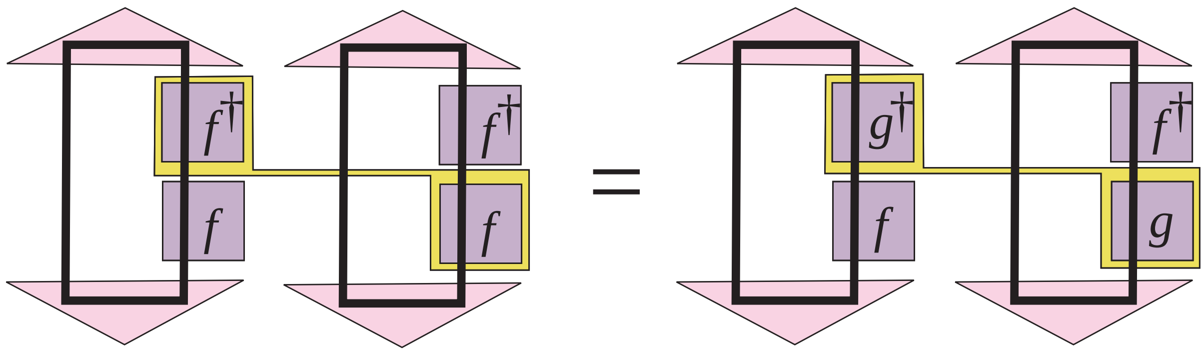
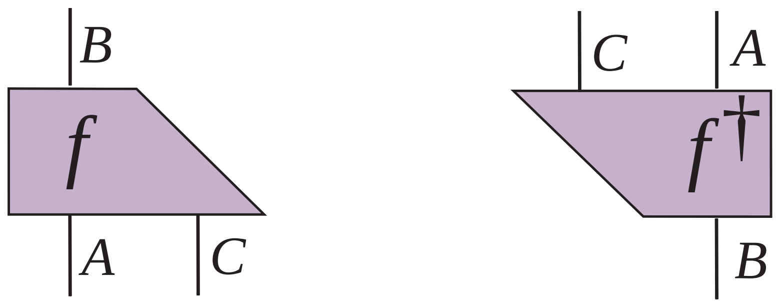


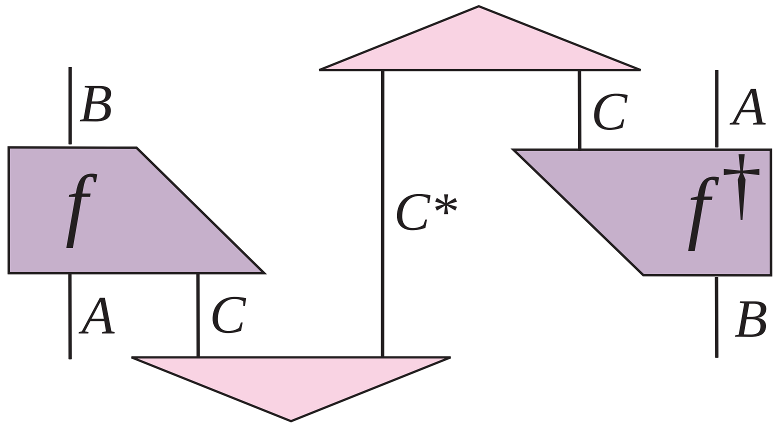


density matrices

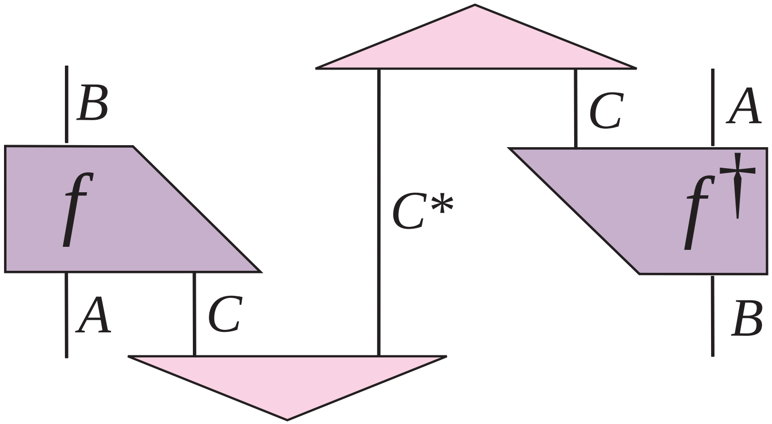
density matrices



density matrices



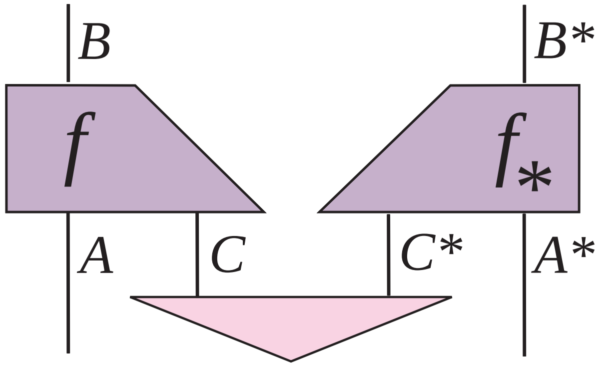
density matrices



In **FdHilb** we get **completely positive maps** as morphisms and **density matrices** as elements!

Selinger, P. (2005) ***†-CCC and completely positive maps.***

density matrices



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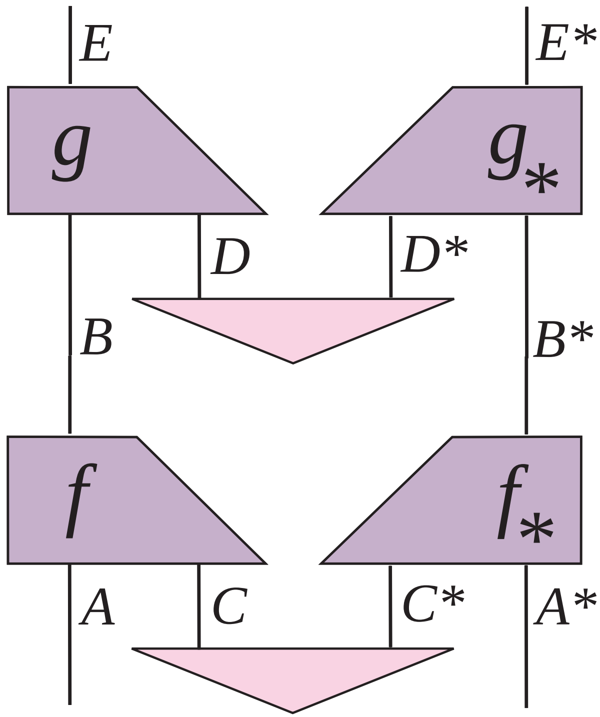
density matrices



Selinger, P. (2005) †-*CCC and completely positive maps.*

special thanx to

Michael Wright

for his initiative and enthusiasm