Categorical quantum mechanics meets the Pavia principles: towards a representation theorem for CQM constructions (position paper)

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This position paper serves two purposes:

(1) In the light of recent reconstructions of quantum theory, this paper shows the need for a representation theorem for categorical quantum mechanics (CQM), which establishes how conceptually-meaningful constructions on processes and their compositions lead to Hilbert space structure, without any reference of instrumentalist concepts such as measurement.

(2) As a first step towards this goal we show how several of the instrumentalist principles underpinning Pavia's reconstruction of finitary quantum theory, most notably purification and local discriminability, the latter up to an abstract counterpart to polar decomposability, are already subsumed by CQM in terms of compactness and Selinger's CPM-construction. We also reformulate the other Pavia principles without reference to measurement.

We expect that this position paper will generate interesting discussion between proponents of the information-theoretic stance towards reconstructions, proponents of the operational stance, proponents of compositionally and process ontology, and pure mathematicians.

1 Introduction

Recently there has been much activity in the area of reconstructing quantum theory, which involves reformulating it in terms of principles which are physically more reasonable. Several authors have produced a set of such principles from which they reconstruct the Hilbert space formalism of quantum mechanics [9, 20, 8, 25], starting with Hardy [23]. In particular, Hardy [24, 25] and Chiribella-D'Ariano-Perinotti (hereafter referred to as *Pavia*) [7, 8] relied in part on the *compositional process structure* of *categorical quantum mechanics* (CQM) [1, 12, 13] to formulate their principles, more specifically, on the diagrammatic incarnation of strict symmetric monoidal categories (sSMCs) [30, 27, 37]. In this paper we show how many of these principles, in particular those of Pavia, are already subsumed by the more sophisticated CQM-structures of *dagger-compactness* [1, 2], *Selinger's CPM-construction* [35], and the resulting notion of *environment* [11, 19]. In doing so, we make the first steps towards a representation theorem for CQM. This is desirable for a number of reasons:

- While all the existing reconstructions take the concept of a measurement device as a primitive, a measurement-free process-based reconstruction would be more appealing for most researchers in quantum gravity and cosmology. Measurement would then be a definable special kind of process, rather than a primitive, enabling extraction of information about systems. By focussing on processes and their (de-)composition one retains an operationalist stance without being instrumentalist. One could refer to this weaker form of operationalism as *process operationalism* (cf. [12, 13]

for a discussion thereof). The stance of elevating processes to a privileged role in quantum theory was already present in the work of Whitehead in the 1950s [40] and the work of Bohr in the early 1960s [6], and became more prominent in the work of Bohm and Hiley [3, 4, 5]. Of course, ideas on process ontology trace back as far as the pre-Socratics, most notably Heraclitus in the 6th century BC.

- Another desirable feature is to obtain as much as possible of the relevant structure by conceptually meaningful (in the process-operational sense) constructions. In this paper we make ample progress in this direction, by providing operationally meaningful constructions (see Section 2) that lead to Pavia's causality and purification postulates. In this light, the earlier ideas on decoherence and algebra-doubling in Prigogine [33] and Hiley [26] are also conceptually relevant.

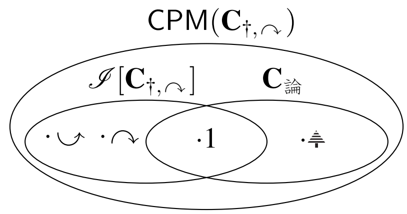
- The existing reconstructions are by no means mathematical axiomatizations. There are many additional assumptions (cf. Section 3 below), either justified operationally, or simply drawn from the practice of modern theoretical physics e.g. the dominant role of the continuum in describing state spaces and probabilities. For a case against this stance we refer to Isham [22]. It would be desirable to have an axiomatization which clearly states every single mathematical assumption, as a stepping-stone to a framework for generalized theories of physics, in which certain assumptions like the continuum can be dropped, possibly towards a theory of quantum gravity. For this reason we refer to our work as primarily a *representation theorem* as opposed to the *reconstruction* of the related work mentioned previously, since our work emphasises mathematical precision and explicitness in its formulation.

- In the same vein, elegant mathematical axioms which refer to well-understood 'clean' mathematical concepts would also be more appealing to mathematicians who would like to contribute to the *quantum axiomatization* research program. An example here is the Piron's 'almost' realization [31, 32] of Mackey's reconstruction program [28, 29], which led to a large area of quantum axiomatics research in terms of ordered structures in the '70s and '80s, for which we refer the reader to [16]. Of course, an existing genuinely mathematical reconstruction starting form compact SMCc is the Tannakian one, due to Deligne [21]. However, many assumptions that cannot be physically justified are made, e.g. the existence of biproduct structure at the pure state level against which a no-go theorem is in [10].

Continuing on the subject of earlier axiomatizations, most notably the Mackey-Piron program which was completed by Solèr [38] in 1995, we want to emphasize the upshot of the more recent 'far less mathematical' reconstructions. While earlier programs were able to obtain Hilbert space from a 'tiny' set of axioms (in particular as compared to the amount of unstated mathematical axioms in the current reconstructions, e.g. see Section 3 below), they failed to obtain a conceptual account on how compound quantum systems behave. That is, to reconstruct tensor product like behavior, they first had to go all the way to Hilbert space structure. However, already as early as 1935, Schrödinger had emphasized the crucial role of the behavior of compound quantum systems for the conceptual understanding of quantum theory [34]. On the other hand, already in Hardy's paper [23] there was a key role for compound quantum systems. This became much more apparent in [24, 7, 8, 25] when the diagrammatic framework of CQM was taken as a background structure, hence introducing composition of systems as a primitive at the initial stage of the reconstruction. The power of doing so is what this paper aims to emphasize, as well as the important role that more sophisticated CQM structures can play in reconstructions and beyond.

2 Pure, mixed and deterministic morphisms in CQM

Let $\mathbf{C}_{\dagger,\cap}$ be a dagger-compact category, and let f_* denote the *conjugate* of f [1, 2]. Using Selinger's CPM-construction [35] we can construct another dagger-compact category $\mathsf{CPM}(\mathbf{C}_{\dagger,\cap})$, which has the same objects as $\mathbf{C}_{\dagger,\cap}$, and where morphisms of type $A \to B$ are those of the form $(1_B \otimes \cap \otimes 1_{B^*}) \circ (f \otimes f_*) : A \otimes A^* \to B \otimes B^*$ in $\mathbf{C}_{\dagger,\cap}$ where $f : A \to B \otimes C$. There is a functor $\mathscr{I} : \mathbf{C}_{\dagger,\cap} \to \mathsf{CPM}(\mathbf{C}_{\dagger,\cap}) : A \mapsto A; f \mapsto f \otimes f_*$ which is 'almost an embedding'. The *cap* $\cap : A \otimes A^* \to \mathrm{I}$ of the compact structure in $\mathbf{C}_{\dagger,\cap}$ will be denoted by $\text{⏚} : A \to \mathrm{I}$ in $\mathsf{CPM}(\mathbf{C}_{\dagger,\cap})$, and referred to as *ground*. We can now construct a third category $\mathbf{C}_{\text{論}}$ as a subcategory of $\mathsf{CPM}(\mathbf{C}_{\dagger,\cap})$, typically without a dagger or caps/cups, by restricting morphisms to those that preserve the ground, that is, $F : A \to B \in \mathbf{C}_{\text{⏚}}$ iff $\text{⏚}_B \circ F = \text{⏚}_A$.[1] We refer to $(\mathsf{CPM}(\mathbf{C}_{\dagger,\cap}), \mathbf{C}_{\dagger,\cap}, \mathbf{C}_{\text{論}})$ as a *CQM process structure of the constructive kind*, or in short, $\mathsf{CPM}(\mathrm{con})$. Since for these categories the objects are the same, we can represent the Hom-inclusions as follows:



(1)

We interpret objects A as physical systems and morphisms as physical processes which take a system of type A into a system of type B. While those in $\mathbf{C}_{\dagger,\cap}$ are interpreted as pure or closed processes, those in $\mathsf{CPM}(\mathbf{C}_{\dagger,\cap})$ are interpreted as mixed or open, and those in $\mathbf{C}_{\text{論}}$ are interpreted as deterministic.

Conceptualizing compactness. The conceptual idea behind compactness is simply that one can realize bipartite states and effects which enable post-selected teleportation. This has been discussed in great detail in earlier work [1, 12]. Note here in particular that this property is by no means specific to quantum theory; it can also be realized with classical probabilities. The key point is that in the quantum realm it can be achieved with pure states and effects, which is not possible classically. Within CQM, compactness is a strong assertion about the nature of the behavior of compound quantum systems, within the general framework of (s)SMCs (cf. the discussion in the introduction on earlier reconstructions).

Conceptualizing the CPM-construction. How can we understand the CPM-construction? To a process of type $A \to B$ one adjoins an ancillary system C through which the process interacts with (the relevant part of) the environment other than the systems A and B that are already in its specification. So we consider a process of type $f : A \to B \otimes C$. Note here that by compactness it does indeed not matter whether we consider C either as an input or an output. We will now argue that $(1_B \otimes \cap \otimes 1_{B^*}) \circ (f \otimes f_*) : A \otimes A^* \to B \otimes B^*$ describes the the process that the system and the environment jointly undergo, f being the one that the system undergoes, f_* being the one the environment undergoes, and $\cap : C \otimes C^* \to \mathrm{I}$ being the flow of information between the processes f and f_*. That it makes sense to model the relevant process of the environment as the canonically related 'conjugate' (or mirror-image in the diagrammatic calculus) can be understood as follows. One can think of a system and its environment as 'complements',[2] that is, the environment is everything that is not the system. However, surely not the entire environment is relevant to the system, nor is every single detail of the system necessarily always relevant either. What is relevant is the 'surface' (or 'interface') through which the system interacts with other systems, which

[1] The symbol 論 is one of the three Japanese characters for determinism, and is pronounced 'ketteiron'.

[2] Evidently, here we don't use 'complement' in its strict set-theoretic sense.

of course are all part of its environment. Clearly, the relevant part of the environment is again that same surface. The difference between system and environment is that they are located at opposite sides of this surface, cf. the black areas in the following picture:

Conjugation (or mirroring) is what witnesses these opposite directions relative to the surface, similarly to how conjugation witnesses time-direction in quantum theory. Therefore, in the CPM-construction we can indeed interpret f and f_* respectively as the process a system undergoes and the process (the relevant part of) its environment undergoes.

Conceptualizing $\top$-preservation. A morphism $\top$ is interpreted as the process of *discarding* a system [13]. Hence the processes in $\mathbf{C}_{\top}$ are those that do not affect discarding. If a theory comes with a notion of probability, then the probability for a process f to take place (e.g. an effect) may depend on its input. In this case it would affect discarding. On the other hand, if it would happen with probability 1, no matter what its input is, then, either performing it and then discarding its output, or, not performing it and discarding its input, would 'have the same effect' i.e. they 'are' the same effect.[3] However, by defining $\mathbf{C}_{\top}$ in the manner that we did we avoided any a priori reference to probabilities.

One can define a triple of categories as in (1), in an axiomatic manner rather than as a construction, by relying on Coecke's axiomatization of Selinger's CPM-construction [11, 19]. From now on, in the statement of axioms, we will specify types of morphisms as subscripts.

Definition 1. By an *environment structure* for a dagger compact category $\mathbf{C}_{\dagger,\curvearrowright}$ we mean a dagger compact category $\mathbf{C}_{\top}$, which has the same objects as $\mathbf{C}_{\dagger,\curvearrowright}$ and in which $\mathbf{C}_{\dagger,\curvearrowright}$ is included, and which for every object A contains a special morphism $\top_A : A \to \mathrm{I}$ subject to the following axioms:

- <u>CPM(ax)1</u>. $\forall E_{A\to\mathrm{I}} \in \mathbf{C}_{\top}, \exists f_{A\to B} \in \mathbf{C}_{\dagger,\curvearrowright} : E = \top_B \circ f\,;$
- <u>CPM(ax)2</u>. $\forall f_{A\to B}, g_{A\to C} \in \mathbf{C}_{\dagger,\curvearrowright} : f^\dagger \circ f = g^\dagger \circ g \Longleftrightarrow \top_B \circ f = \top_C \circ g\,;$
- <u>CPM(ax)3</u>. $\top_{A\otimes B} = \top_A \otimes \top_B,\ \top_{\mathrm{I}} = 1_{\mathrm{I}}\,.$

Theorem 2 ([11]). *For an environment structure $\mathbf{C}_{\top}$ on $\mathbf{C}_{\dagger,\curvearrowright}$ we have $\mathsf{CPM}(\mathbf{C}_{\dagger,\curvearrowright}) \simeq \mathbf{C}_{\top}$ via the isomorphism $(1_B \otimes \curvearrowright \otimes 1_{B^*}) \circ (f \otimes f_*) \mapsto (1_B \otimes \top_C) \circ f$. Conversely, $\mathsf{CPM}(\mathbf{C}_{\dagger,\curvearrowright})$ provides an environment structure for $\mathscr{I}[\mathbf{C}_{\dagger,\curvearrowright}]$ with $\top = \curvearrowright$, provided that the following axiom holds:*[4]

<u>PS</u>. $\forall \rho_{\mathrm{I}\to A}, \sigma_{\mathrm{I}\to A} \in \mathsf{CPM}(\mathbf{C}_{\dagger,\curvearrowright}) : \rho \circ \rho^\dagger = \sigma \circ \sigma^\dagger \Rightarrow \rho = \sigma\,.$

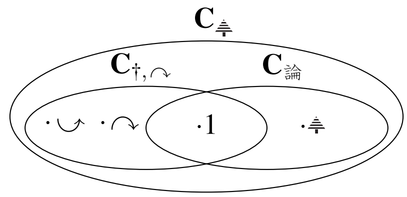
The axiom PS was called *preparation-state agreement* in [10], since it states that if two ket-bras agree (cf. projection as preparation procedure), then the kets agree too (cf. the state resulting from the preparation procedure). In [11] it is shown that PS does not always hold in $\mathsf{CPM}(\mathbf{C}_{\dagger,\curvearrowright})$, and moreover that $\mathsf{CPM}(\mathbf{C}_{\dagger,\curvearrowright})$ is axiomatizable by an environment structure *if and only if* PS holds. So by Thm. 2, up to PS, the CPM-construction and Defn. 1 are essentially one and the same thing. Hence, a third category $\mathbf{C}_{\text{論}}$ can be obtained as before, by restricting $\mathbf{C}_{\top}$. We refer to a triple of categories $(\mathbf{C}_{\top}, \mathbf{C}_{\dagger,\curvearrowright}, \mathbf{C}_{\text{論}})$ as described above as a *CQM process structure of the axiomatic kind*, or in short, $\mathsf{CPM}(\mathrm{ax})$. Hence:

[3] There is of course an issue here of what is understood by equality in the theory, which we ignore for now.

[4] A sufficient condition for this to hold is that the functor $\mathscr{I}$ is faithful.

Corollary 3. $\mathsf{CPM}(\mathrm{ax}) \Longleftrightarrow (\mathsf{CPM}(\mathrm{con}) \wedge \mathsf{PS})$.

Hom-inclusions now look as follows:



The power of CPM(ax)2 was recently demonstrated in [19] where in combination with complementary basis structures [14] it enabled to derive many protocols including the classical control structure.

3 Comparison with the Pavia principles

In [8] Pavia proposed a set of principles, organized as six statements referred to as (i) *causality* (caus), (ii) *perfect distinguishability* (dist), (iii) *ideal compression* (comp), (iv) *local distinguishability* (loc), (v) *pure conditioning* (cond), and (vi) *purification* (pur), on the basis of which they reconstructed the Hilbert space formalism, that is to say, the finite-dimensional version without explicit Schrödinger evolution. Pavia referred to these principles as axioms, but some qualifications should be made:

- When converting these statements to mathematical language, as quantified equational statements, some of these principles may involve more than one independent statements, for example pur, as we discuss below. These principles are defined relative to a background framework, built on top of (s)SMC-structure, which also comes with equational content.
- Pavia moreover assumes additional 'branching'-structure on operations. Concretely, operations are specified in terms of components, which then via coarse-graining lead to composites. In a dual persective, this means that non-deterministic operations (cf. $\mathbf{C}_{\triangleq}$) arise from a decomposition structure on deterministic ones (cf. $\mathbf{C}_{\text{論}}$). We discuss this structure from the CQM perspective in Section 3.5; it boils down to categorical enrichment in commutative monoids.
- It is implicit in the definitions of state, effect and operation, that behavioral-equivalence style principles are imposed by construction, or in other words, well-pointedness principles, which is also discussed below.
- Several other 'concreteness'-assumptions are made, which include:
  - The monoid of scalars is $([0,1],\times)$. The Pavia notion of probabilistic framework moreover assumes from the start a strong probabilistic constraint, namely that any sum of scalars (cf. enrichment in commutative monoids) is one.
  - Continuity assumptions are made for the spaces of states and effects.

The above should not be seen as a criticism. While from an axiomatic perspective these implicit assumptions are definitely undesirable, many physicists, and quantum information scientists in particular, would take them for granted. This is what justifies the Pavia approach, which in its current form is aimed at that community. Most recent reconstructions are in the spirit of Einstein's derivation of special relativity from the principle of relativity and the finite speed of light which also invokes additional assumptions of the same kind (e.g. see Hardy's recent 'book' for a discussion [25]).

However:

- As already mentioned in the Introduction, it would be desirable to have a clearer view on what the precise equational content is that leads to the Hilbert space formalism. Also, in formulating the principles, much structure beyond purely compositional one is assumed, while it seems to us that one could go a long way without it.
- Secondly, one would also like to have an insightful direct proof, which is crucial when one wants to take the representation theorem as a starting point for a framework for general quantum-like physical theories that stretch well beyond quantum theory; verifying Pavia's current proof requires traveling a web of many papers via a plethora of Lemmas and Propositions.
- One physically inspired critique concerns the particular task Pavia embarked upon: to derive quantum theory from operational (cf. measurement) and information-theoretic principles, a task which traces back to Wheeler's 'it from bit' [39] and which has been more explicitly put forward as a concrete challenge by Brassard and Fuchs around 2000. While we wholeheartedly agree that structures and axioms should be conceptually (in the physical sense) justified, it is not clear that instrumentalist and information-theoretic concepts would really give us new insights into the nature of quantum reality. Concepts like measurement, local tomography, bit commitment (cf. [9]), compression, arguably are by no means naturally occurring physical phenomena. Indeed, they require lots of engineering effort for those that implement them, and typically involve working against the natural entropic arrow of time encoded in the 2nd law of thermodynamics, e.g. a computation is a process where from no knowledge about the answer one strives towards perfect knowledge.

3.1 Directly implicit axioms: causality and pure conditioning

Pavia's caus-principle is realized by construction in terms of the category $\mathbf{C}_{論}$. They relied on 'no-signalling from the future via measurements' to achieve this fact, and then derived [7, Lemma 3]):[5]

Proposition 4. *The tensor unit* I *is terminal in* $\mathbf{C}_{論}$.

Pavia's cond-principle follows directly by the fact that $\mathbf{C}_{\dagger,\curvearrowright}$ forms a (compact) category: the axiom states that any pure effect performed on a pure bipartite system results in a pure state.

3.2 Local discriminability from compactness and behavioral equivalence

Pavia's loc-principle, in its earlier incarnation of *local tomography* [41], has in some form or another been part of the new wave of reconstructions from the start [23]. It states that the state of a compound system can be obtained by only performing local operations. Formally we have:

$$\underline{\mathsf{Loc1}}.\ \big(\forall e_{A\to \mathrm{I}}, e'_{B\to \mathrm{I}} : (e\otimes e')\circ \Psi_{\mathrm{I}\to A\otimes B} = (e\otimes e')\circ \Phi_{\mathrm{I}\to A\otimes B}\big) \Longrightarrow \Psi = \Phi\,.$$

In this form it refers directly to compound systems. Remarkably, in the presence of compactness, it is actually nothing but a statement of behavioral equivalence for processes involving single systems:

$$\underline{\mathsf{Loc2}}.\ \big(\forall \psi_{\mathrm{I}\to A}, e_{B\to \mathrm{I}} : e\circ f\circ \psi = e\circ g\circ \psi\big) \Longrightarrow f = g\,.$$

Recall that principles of this kind are part of the background framework in Pavia. What this particular statement asserts is that no context is required to verify equivalent behaviors, and therefore one could refer to Loc2 as *context-free behavioral equivalence*. We conclude with the following.

Theorem 5. *In either* $\mathbf{C}_{\dagger,\curvearrowright}$ *or* $\mathbf{C}_{⏚}$, Loc2 *implies* Loc1.

[5] A derivation of this result from the existence of correlations between space-like separated regions is in [15].

Example. The strength of local discriminability can be seen by considering real-vector space quantum mechanics. It is well known that there exist mixed states in real quantum mechanics which do not satisfy $\mathsf{Loc2}$: i.e. there exist bipartite real density matrices $\rho \neq \rho'$ but for which all local measurements yield the same results. Given that $\mathsf{Loc1}$ fails in this manner we would expect that a category of all real density matrices contradicts one of the above locality assumptions. This indeed the case: we find that the category $\mathsf{CPM}(\mathbf{FdVect}_{\mathbb{R}})$ does not contain all real density matrices. In other words, because the CPM construction preserves compactness (and therefore has locality built in), it excludes the density matrices for which local discriminability fails.

Question. A matrix calculus in any semiring (e.g. over Booleans) gives rise to a compact closed category. What characterizes those semirings for which context-free behavioral equivalence holds? It is known that it holds for the field of complex numbers, but not for reals nor quaternions.

3.3 Purification from the CPM-construction and polar decomposability

Pavia's pur-principle states that every system admits a purification, and, if a system admits two purifications over the same system then these purifications can be interconverted via a reversible transformation. They consider this particular principle to be particularly powerful and have an entire paper dedicated to its consequences, which include many quantum informatic results [7].

Rather than requiring a reversible transformation for purification over identical systems, we will require the existence of an isometry between purifications of arbitrary systems, in order to allow for an on-the-nose comparison with $\mathsf{CPM(ax)}$. For the same reason, we consider some dagger compact category $\mathbf{C}_{\top}$ and a sub dagger compact category $\mathbf{C}_{\dagger,\frown}$, without any further constraints initially. It is easy to see that this axiom decomposes in the following two statements:

$\underline{\mathsf{PPur1} = \mathsf{CPM(ax)1}}$. $\forall E_{A\to \mathrm{I}} \in \mathbf{C}_{\top}, \exists f_{A\to B} \in \mathbf{C}_{\dagger,\frown} : E = \top_B \circ f$;

$\underline{\mathsf{PPur2}}$. $\forall f_{A\to B}, g_{A\to C} \in \mathbf{C}_{\dagger,\frown} : \top_B \circ f = \top_C \circ g \Longrightarrow \exists_{isom} U_{B/C\to C/B} : g/f = U \circ f/g$;

where $\exists_{isom} U_{B\to C}$ is a shorthand for $\exists U_{B\to C} : U^\dagger \circ U = 1_B$ (i.e. U is an *isometry*), and the use of X/Y indicates that either of two statements holds in the obvious manner.

Remark 6. Strictly speaking, $\mathsf{PPur1} \wedge \mathsf{PPur2}$ only implies Pavia's pur-principle provided that isometries of type $A \to A$ are always isos; this issue will be addressed below in Section 3.4.

The polar decomposability of an arbitrary linear map $f : \mathscr{H} \to \mathscr{K}$ between Hilbert spaces can be written either as $U \circ \sqrt{f^\dagger \circ f}$ or $U^\dagger \circ \sqrt{f^\dagger \circ f}$ where U is an isometry of appropriate type. A variation involving two linear maps is the following. For $f : \mathscr{H} \to \mathscr{K}$ and $g : \mathscr{H} \to \mathscr{K}'$ with $f^\dagger \circ f = g^\dagger \circ g$ we have $g/f = U \circ f/g$ for some isometry $U : \mathscr{K}/\mathscr{K}' \to \mathscr{K}'/\mathscr{K}$. The following axiom abstracts this:

$\underline{\mathsf{PD}}$. $f^\dagger \circ f = g^\dagger \circ g \Longleftrightarrow \exists_{isom} U_{B/C\to C/B} : g/f = U \circ f/g$;

that is, the self-adjoint components coincide ($f^\dagger \circ f = g^\dagger \circ g$) iff the morphisms defer by an isometry.

Lemma 7. *Under the assumption that:*

$\underline{\mathsf{U}\top}$. $U^\dagger \circ U = 1_A \Rightarrow \top_B \circ U = \top_A$ *;*

then $\mathsf{PPur2}$ *becomes an equivalence:*

$\underline{\mathsf{PPur2}'}$. $\forall f_{A\to B}, g_{A\to C} \in \mathbf{C}_{\dagger,\frown} : \top_B \circ f = \top_C \circ g \Longleftrightarrow \exists_{isom} U_{B/C\to C/B} : g/f = U \circ f/g$.

Hence $\mathsf{PPur2} \wedge \mathsf{U}\top \Longleftrightarrow \mathsf{PPur2}'$.

Theorem 8. *We have the following triangle of equivalences (we omit quantifiers):*

$$\top_B \circ f = \top_C \circ g$$

$$\text{CPM(ax)2} \;\Updownarrow \qquad\qquad \Updownarrow\; \text{PPur2}'$$

$$f^\dagger \circ f = g^\dagger \circ g \quad \overset{\text{PD}}{\Longleftrightarrow} \quad \exists_{isom} U_{B/C \to C/B} : g/f = U \circ f/g$$

We conclude that there is a very close relationship between pur and CPM(ax): (i) PPur1 coincides with CPM(ax)1, and (ii) PPur2′ coincides with CPM(ax)2 up to polar decomposability.

3.3.1 An aside: characterizing purity in terms of connectedness or operator-squaring

In CPM(ax) an environment adjoins mixed (or open) operations to a given collection of pure (or closed) operations $\mathbf{C}_{\dagger,\curvearrowright}$. But what does it mean to be pure? A natural requirement would be:

Pure. $e_{A\to \mathrm{I}} \in \mathbf{C}_{\dagger,\curvearrowright} \Longleftrightarrow \big((\forall f_{A\to B} \in \mathbf{C}_{\dagger,\curvearrowright} : e = \top \circ f) \Longrightarrow (\exists e'_{A\to \mathrm{I}}, \psi_{\mathrm{I}\to B} : f = \psi \circ e')\big)$;

The reason this is natural is because of its conceptual clarity: the righthand-side of the equivalence expresses that, if some manner of producing the effect $e : A \to \mathrm{I}$ involves $\top$, then e's input must be 'disconnected' from this involvement of the environment, that is, no essential part of the process (i.e. not of type $\mathrm{I} \to \mathrm{I}$) properly extends into the environment. Denoting the 'only if' and 'if' implication statements of Pure by $\Longrightarrow_{\mathsf{Pure}}$ and $\Longleftarrow_{\mathsf{Pure}}$ respectively, we have the following:

Proposition 9. PPur2 *implies* $\Longrightarrow_{\mathsf{Pure}}$ *and* $\big(\mathsf{PPur1} + \mathbf{C}_{\top}(\mathrm{I},\mathrm{I}) \subseteq \mathbf{C}_{\dagger,\curvearrowright}\big)$ *implies* $\Longleftarrow_{\mathsf{Pure}}$.

Remark 10. On the other hand, we cannot get $\Longrightarrow_{\mathsf{Pure}}$ from CPM(ax) alone; this requires the additional assumption that $f^\dagger \circ f$ being disconnected implies that f itself is disconnected.

Another characterization of purity of states that one typically finds in the quantum information literature is $tr(\rho^2) = tr(\rho)^2$. Here we extend this characterization from states to operations:

Pure′. $|\top_B \circ f \circ \top_A^\dagger|^2 = \curvearrowleft_A^\dagger \circ (1_{A^*} \otimes (f^\dagger \circ f)) \circ \curvearrowleft_A$;

Proposition 11. *Assuming* $\mathbf{C}_{\top}(\mathrm{I},\mathrm{I}) \subseteq \mathbf{C}_{\dagger,\curvearrowright}$*, both* CPM(ax) *and* PPur2 *imply* Pure′.

3.4 Ideal compression from universality of polar decomposition

So far we have accounted for four of the six Pavia axiom; we now consider the fifth axiom, comp. The content of this is that for every state $\rho_{A\to A} = \Sigma_i \sigma_i^\dagger \circ \sigma_i$ there exists a *maximally efficient and lossless compression scheme*: this consists of a pair of maps $(\mathscr{E}_{A\to C}, \mathscr{D}_{C\to A})$, which are respectively the *encoding* and *decoding* maps, such that $\forall i : \mathscr{D} \circ \mathscr{E} \circ \sigma_i = \sigma_i$, and for which $\dim(C) \le \dim(A)$. Concretely, for a density matrix ρ on an n-dimensional Hilbert space, this corresponds to representing it on an r-dimensional Hilbert space, where r is the rank of ρ.

We generalize this from states ρ to arbitrary morphisms f as follows.

IC. $\forall f_{A\to B} \exists_{isom} \mathscr{E}_{B\to C}, , \mathscr{D}_{C\to B}$:

a) $(\mathscr{E} := \mathscr{D}^\dagger) \wedge$

b) $(\mathscr{D} \circ \mathscr{E} \circ f = f) \wedge$

c) $(\forall \mathscr{D}'$ satisfying (a) and (b) $\exists!_{isom} U : \mathscr{D}' \circ U = \mathscr{D})$.

IU. All endomorphic isometries in $\mathbf{C}_{\text{⏚}}$ are unitaries.

Since endomorphic isometries are automatically unitaries only for finite-dimensional Hilbert spaces, IU can be seen as our statement of finite-dimensionality. Hence this imposes a restriction on models of our axioms, but one which matches the Pavia reconstruction, and other recent reconstructions such as [9, 20, 8, 25].

3.5 Perfect distinguishability from CMon-enrichment

Pavia's dist-principle is the only one that requires making explicit reference to additive structure. Therefore we will assume that $\mathbf{C}_{\text{⏚}}$ is enriched in commutative monoids, **CMon**, that is, for each family of morphisms $\{f_i\}_{A\to B}$ of the same type, there exists a morphism $\bigsqcup_i f_i$, and for each type there exists a unique morphism $0_{A\to B}$, such that for any morphism f of appropriate type:

$$f\circ\left(\bigsqcup_i f_i\right)=\bigsqcup_i(f\circ f_i)\qquad\left(\bigsqcup_i f_i\right)\circ f=\bigsqcup_i(f_i\circ f)\qquad 0\circ f=0\qquad f\circ 0=0$$

$$f\otimes\left(\bigsqcup_i f_i\right)=\bigsqcup_i(f\otimes f_i)\qquad\left(\bigsqcup_i f_i\right)\otimes f=\bigsqcup_i(f_i\otimes f)\qquad 0\otimes f=0\qquad f\otimes 0=0$$

Conceptualizing CMon-enrichment. A 'sum' $\bigsqcup_i$ represents possibilities, i.e. one of the f_i in $\bigsqcup_i f_i$ is the process that actually happens. Hence it accounts for 'branching'-structure.

Remark 1. One should distinguish the above sum from non-determinism in computer science, which doesn't involve weighting, while here it is the morphisms f_i in $\bigsqcup_i f_i$ which carry the weights. Neither should it be confused with equally weighted mixing, e.g. $\frac{1}{2}f+\frac{1}{2}g$ which isn't associative; here every 'branch' is added with weight 1, but the branch itself may have lower (or higher) weight.

3.5.1 An interlude: equating mixedness and openness

In quantum mechanics the concepts of mixing and a system extending into the environment are interchangeable. Here, we have two clearly distinct concepts. The following axiom equates them:

Mix. $\exists\{e_i\}\subseteq\mathbf{C}_{\text{論}}:\text{⏚}=\bigsqcup_i e_i\,;$

Proposition 12. *Let* $e_{A\to\mathrm{I}}\in\mathbf{C}_{\text{⏚}}$ *and assume that* Mix *holds.* $(\exists f_{A\to B}\in\mathbf{C}_{\dagger,\curvearrowright}:e=\text{⏚}_B\circ f)\Longleftrightarrow(\exists\{e_i\}\subseteq\mathbf{C}_{\dagger,\curvearrowright}:e=\bigsqcup_i e_i)$.

3.5.2 Measurements, refinement and mutual exclusion

Definition 13. A *measurement* is any morphism $\bigsqcup_i f_i\otimes p_i$ with $\bigsqcup_i p_i=\text{⏚}^\dagger$ with $\{p_i\}\subseteq\mathbf{C}_{\text{論}}\bigcap\mathbf{C}_{\dagger,\curvearrowright}$ being the *measurement outcomes*. The corresponding *copying operation* is $\bigsqcup_i(p_i\otimes p_i)\circ p_i^\dagger$.

Central in Pavia's language is the notion of *refinement*. Here, it means that $\{f_i\}\subseteq\{g_i\}$ for two measurements $\bigsqcup_i f_i\otimes p_i$ and $\bigsqcup_i g_i\otimes p_i$. *Coarse-graining* is the obvious dual to refinement.

To obtain *mutual exclusiveness* of measurements with respect to the outcomes, as required in the notion of 'perfect distinguishability' it suffices for $\mathbf{C}(\mathrm{I},\mathrm{I})$ to be ordered, with 0_{I} as the bottom, and $s+-:\mathbf{C}(\mathrm{I},\mathrm{I})\to\mathbf{C}(\mathrm{I},\mathrm{I})$ to be monotone.

4 Conclusion

In principle, if one compiles the above reformulated Pavia principles together, one should be able to extract a reconstruction of quantum theory from [8] and papers referenced therein. In this context, a direct proof constrained by elegance, verifiability, and insightfulness would be highly desirable.

Even more importantly, there is no indication whatsoever that the assumptions stated above are by any means minimal. Crucially here, a lot of work in the Pavia proof is towards establishing ideal post-selected teleportation, which comes for free when assuming compactness. Also the contribution of the dagger structure has not been exploited yet. One may hope to get rid from brute force tricks such as stating the concrete nature of the scalar monoid, as is achieved in the Mackey-Piron reconstruction.

Once this has been done, one may ask the question about the relation between a representation theorem of this kind and a completeness theorem such as Selinger's [36]. In other words, to which extent can all equational statements be established in the language and by the axioms of the representation theorem, that is, diagrams and **CMon**-enrichment, or, maybe even diagrams alone, by relying on the implicit encoding of additive structure in terms of certain commutative Frobenius algebras [18, 17, 19].

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