White paper: The logic of quantum mechanics - take II

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Abstract. We outline the motivation, discuss the conceptual significance, and point to available literature, on a research program which aims to equip quantum physics with a tensorial axiomatics, a genuine logic, an intuitive diagrammatic calculus, and where structural elements should represent empirical facts.

1 Yet another axiomatization of quantum theory?

A major difference between our approach to axiomatize quantum mechanics [1, 5, 6, 8, 10, 11, 13, 16, 17, 18, 19] and the other ones which have been around is our choice of primitive concepts:

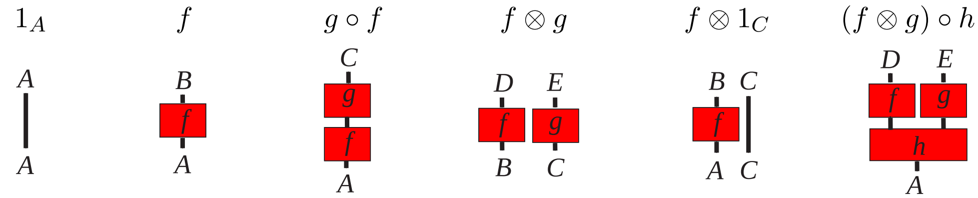
- We aim to axiomatize the *tensor product structure*, that is, how two quantum entities make up one whole. Most other approaches start from measurement-related concepts, be it either spaces of observables, spaces of probability measures, or collections of properties attributed to physical systems [72, 77, 79, 75, 84]. The fact that an axiomatics for quantum theory with 'conceiving two systems as one' as a primitive concept hasn't been proposed before is somewhat surprising: even in the early days of quantum theory Schrödinger was very aware of the key role which the Hilbert space tensor product played in the theory. Moreover, the inability to produce a canonical description for joint systems has been a major stumbling block for many of the approaches based on measurement-related concepts.
- At the same time we also axiomatize the *compositional structure of operations/processes* i.e. we aim at an a priori dynamical theory. The phenomena associated with the particular structure of the tensor product are of an essentially dynamical nature involving multiple processes. Also, recent progress in quantum informatics has exposed the need to conceive measurements as processes involving both classical and quantum information-flow e.g. measurement-based computational schemes [53, 71, 59, 66, 65, 62]. All of these point at the existence of an important structural ingredient of quantum theory involving both parallel ($\simeq$ tensorial) and sequential composition of processes, which has not yet been unveiled.

Basic structure. The mathematical structure capturing 'composing systems' and 'composing operations' actually already existed, but was not build for this purpose: *symmetric monoidal categories* [31]. The first people to conceive this mathematical structure as the canonical way to describe sequential and parallel composition of processes were computer scientists. Here are the structural ingredients from a physics perspective:

- We represent types of systems (or 'kinds' if you prefer), be it physical systems such as photons and electrons, or classical data types, or combinations thereof (e.g. the pair consisting of a quantum systems together with the observed data), by their names $A, B, C, \ldots$
- We represent operations/processes, be it evolution of a systems between time t_1 and time t_2, or the preparation of a system in a certain state, or a computation which takes data of type A as input and produces data of type B, or a measurement taking a quantum system A as input, destroys it, and produces an outcome of data type B, by arrows $A \to B$ where A is the input type and B is the output type.
- We represent the composite of two operations/processes $g : B \to C$ and $f : A \to B$, which are performed/happening one after the other, by $g \circ f : A \to C$. Doing nothing, or if you prefer, nothing happens, induces an 'identity' $1_A : A \to A$ for each system.
- The joint system of A and B is denoted by $A \otimes B$, 'nothing' is denoted by I, and the joint process/operation of processes/operations $f : A \to C$ and $g : B \to D$ is denoted $f \otimes g : A \otimes B \to C \otimes D$.

All these pieces of data, together with the obvious structural rules which mainly control how sequential and parallel composition interact, canonically make up a symmetric monoidal category [7, 12].

Graphical calculus. Rather than stating these rules we will rely on a really nice feature about this particular mathematical structure: it can be equivalently presented as a purely graphical calculus. This calculus traces back to Penrose's work in the 1970's [44] but it took 20 more years to settle its precise algebraic and topological significance [35, 38, 47]. Currently it is significantly used in mathematical physics [30, 42], knot theory [48] and quantum group theory [46] — see in particular Baez' weekly finds [29]. Our variant emphasises the connection with Dirac notation. Processes/operations are represented by *boxes*, types of systems by *wires*, composition by connecting outputs and inputs by wires, and tensor by locating wires or boxes side by side e.g.



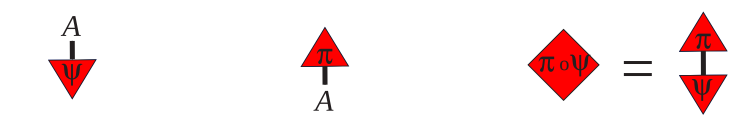
Typical axioms of the symmetric monoidal structure such as commutation of

$$\begin{tikzcd} A \otimes C \arrow[r, "1_A \otimes g"] \arrow[d, "f \otimes 1_C"'] & A \otimes D \arrow[d, "f \otimes 1_D"] \\ B \otimes C \arrow[r, "1_B \otimes g"] & B \otimes D \end{tikzcd} \qquad \begin{tikzcd} A \otimes C \arrow[r, "\sigma_{A,C}"] \arrow[d, "f \otimes g"'] & C \otimes A \arrow[d, "g \otimes f"] \\ B \otimes D \arrow[r, "\sigma_{B,D}"] & D \otimes B \end{tikzcd}$$

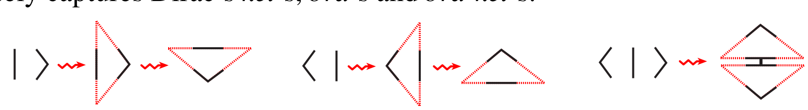
depict as

i.e. we are allowed to 'slide' boxes along the wires, and also along crossings. The void type I, i.e. 'nothing', is represented by 'no wire', which gives rise to triangles and diamonds:

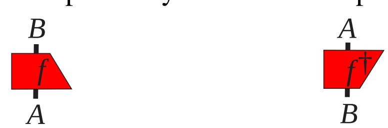
$\psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$



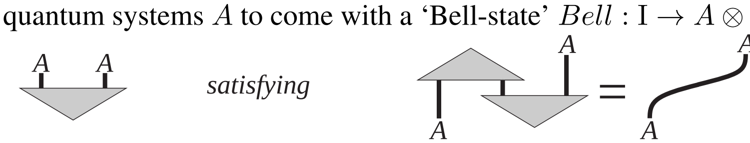
Note that this precisely captures Dirac's *ket*'s, *bra*'s and *bra-ket*'s:



The adjoint, which turns kets into bras is captured by reversal of the picture:

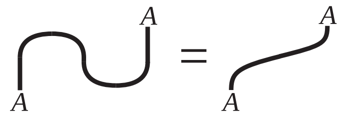


Quantum structure. We will now adjoin additional structure to this basic setting in order to build more specific theories. In conceptual terms this means that want will assert the existence of particular operations/processes which are characteristic for the specific theory we aim to capture axiomatically. The game to play is to add as little as possible, both formally and in terms of conceptual compromise, in order to express and prove as much as possible about that theory. For quantum theory, we will assert the existence of Bell-states (or the ability to prepare them if you wish) and their ability to realise phenomena such as teleportation. Explicitly we require quantum systems A to come with a 'Bell-state' $Bell : \mathrm{I} \to A \otimes A$, that is, a 'triangle'

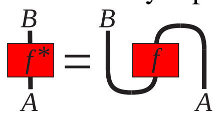


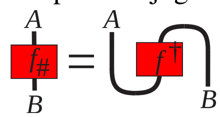
However, when rather than as a triangle we represent this quantum structure as a wire

satisfying

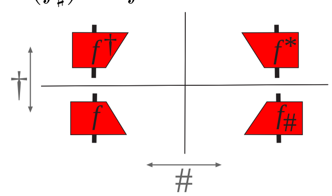


the axiom takes a more lucid form which boils down to 'yanking a piece of rope'. This seemingly tiny bit of structure is already enough to abstractly capture transposition and complex conjugation respectively as [1, 28]:

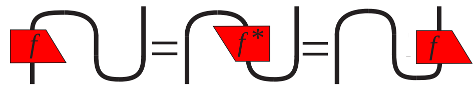




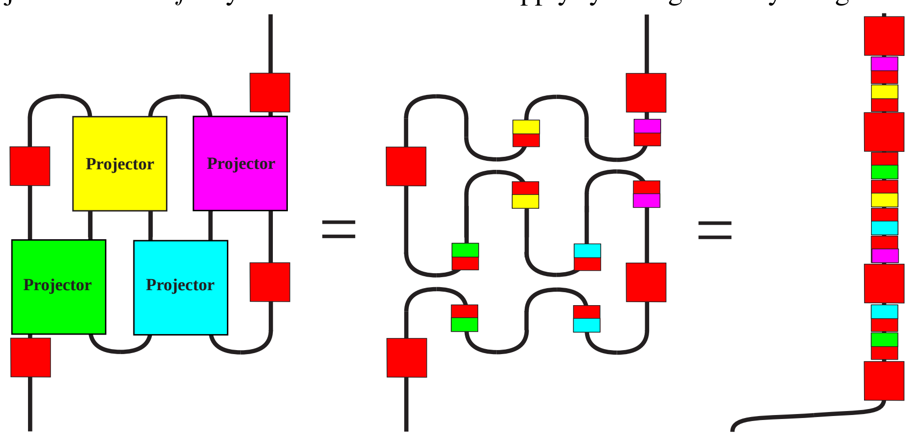
for which, in particular, we have $(f^*)_\sharp = (f_\sharp)^\sharp = f^\dagger$. We can use asymmetry to depict all of these [17]:



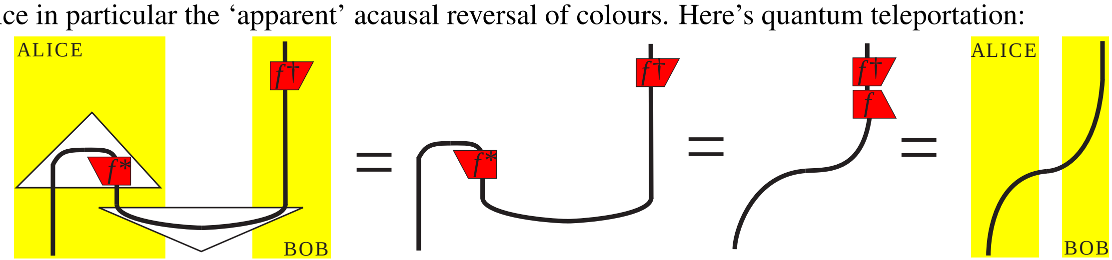
as reflections of f, and it immediately also follows that we can now 'slide' boxes along wires:



— to prove this just substitute f^* by its definition and then apply 'yanking'. Funny things also follow [4, 5]:



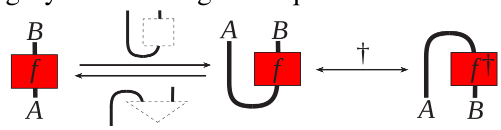
— notice in particular the 'apparent' acausal reversal of colours. Here's quantum teleportation:



where we require f to be unitary, that is, $f^\dagger \circ f = 1_A$ and $f \circ f^\dagger = 1_B$. The required classical information flow is implicit in the dependency of the correction $f^\dagger$ on the effect $Bell^\dagger \circ (1_A \otimes f^*)$. How we can make this classical information flow explicit in the picture will be discussed in Section 3.

2 Yet another quantum 'non-logic'?

By 'true' logic we mean a symbolic systems which support automated reasoning. This requires some sort of deductive mechanism which enables to 'resolve' logical expressions. It is well-known that Birkhoff-von Neumann style quantum logic did not have such a mechanism. Even more so, the identification 'quantum vs. classical' as 'non-distributive vs. distributive' can equally be stated as 'no deduction vs. deduction', since the semantic counterpart to a deduction mechanism is always some sort of distributive law. It seems to me that capturing the spirit of quantum theory should be in terms of 'true quantum features' and not in terms of 'classical features which fail to be true in the quantum realm'. In categorical logic a deductive mechanism typically means that 'morphisms' $f : A \to B$ internalise as 'elements' $\psi_f : \mathrm{I} \to A \Rightarrow B$. This is usually referred to as closedness of a category. In our categorical quantum structures we in fact have even more:



That is, morphisms internalise both as elements and as co-elements. In a sense we have a 'hyper-logic' [2, 16]. Physically this closedness boils down to the Hilbert-Schmidt and Jamiolkowsi dualities between states and operations [1, 17]. Passing from kets to bras then also involves the effects in a similar manner.

3 What about the classical world?

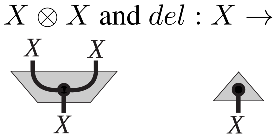
Resource-sensitivity has been an important topic in logic, proof theory and computer science during the past few decennia. A new kind of logic named *linear logic* arises when one drops the structural rules which allow one to freely *copy* and *delete* premisses [36]. In practice this means that every time one relies on a premise one consumes it, and hence repeated use requires several copies of that premise. Linear logic is subject to a beautiful purely diagrammatic proof theory for which there is no counterpart in traditional logic. We obtain traditional logic from linear logic by *adjoining structure which witnesses the ability to copy and delete data*:

Traditional Logic = Linear Logic + (copying, deleting)

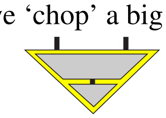
In physics, resource-sensitivity has also become a key paradigm within quantum information science due to the observation that no physical operation can *clone a* pure quantum state [70], that is, produce two copies of that state. Also, while classical probabilistic data is *broadcast-able*, that is, the information it comprises can be made available to two (or more) parties, this is not the case for quantum data [52]. Therefore:

Classicality = Quantumness + (copying/broadcasting, deleting)

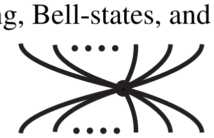
In fact, the distinction between copying and broadcasting will enable us to both extract classical deterministic and classical probabilistic structure from the quantum structure [13]. Hence rather than *quantizing a classical theory*, we do the dual, namely *classicizing a quantum theory*. If one has a quantum system represented by a Hilbert space $\mathcal{H}$ then specifying a non-degenerate classical context means choosing a base $\{e_i\}_i$. Hence the resulting classical structure is the pair $(\mathcal{H}, \{e_i\}_i)$ i.e. a Hilbert space with additional structure. One should note that this perspective is not that unfamiliar to quantum structures research. Indeed, given the lattice $\mathbb{L}(\mathcal{H})$ of subspaces of $\mathcal{H}$, specifying a classical context means picking some Boolean algebra B and a monomorphism $\xi : B \to \mathbb{L}(\mathcal{H})$ of ortholattices, resulting in a pair $(\mathbb{L}(\mathcal{H}), \xi)$. For us a classical context for a quantum structure will consist of two operations, *copy* $: X \to X \otimes X$ and *del* $: X \to \mathrm{I}$, respectively depicted as: [10, 13]:



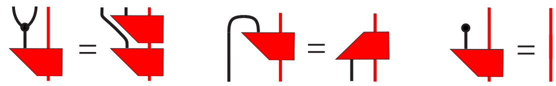
which 'refine' the Bell-state in the sense that we 'chop' a big triangle into a small one and a trapezoid:



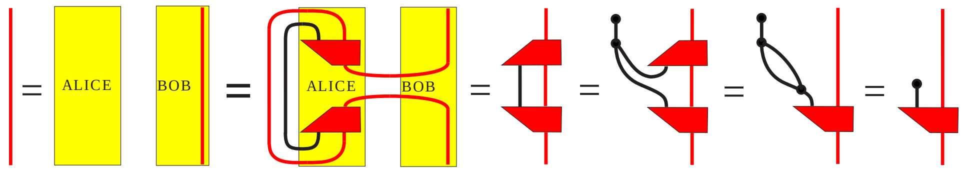
The structural requirements we impose on them are all conceptually very reasonable and translate in mathematical terms as a †-Frobenius algebra [10]. In diagrammatic terms these rules boil down to the fact that any *connected* network involving copying, deleting, Bell-states, and their adjoints is equal to a spider-shape [11]:



which is obtained by fusing dots together. This structure is powerful enough to extract categories of functions, relations, stochastic maps and permutations from any given category which we conceive to consist of quantum processes which come equipped with a classical context [13]. We can now 'define' measurements [10]:



where the reader might recognise the first requirement to be von Neumann's projection postulate. The second one relates eigenstates to resulting states, and slightly more surprising, the last one asserts in the presence of the two other ones that measurements cannot be used for faster-than-light-communication:



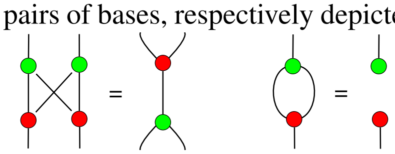
All this together provides a diagrammatic language for interacting quantum and classical information flows.

4 What more can we do? Some examples, ...

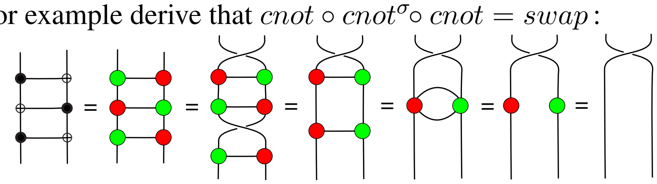
CPMs, decoherence, POVMs & Naimark's dilatation theorem. Please consult [17, 9], [13] and [11].

Structural resources for quantum informatic tasks. By diagrammatic 'reverse engineering' we discover which structural components are required for certain tasks. Please consult [10, 14]. E.g., for teleportation and dense coding, with very little work, we recover matching abstract counterparts to Werner's general teleportation and dense coding schemes [69]. It also follows that Perdrix' measurement based computational scheme [65] requires the presence of classical structure in order to describe the quantum information flow [14].

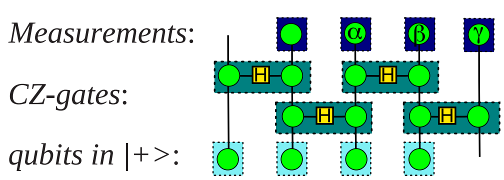
Mutually unbiased observables. We define mutually unbiased observables such as spin X and spin Z or position and momentum as particular pairs of bases, respectively depicted in green and red, which satisfy [15]:



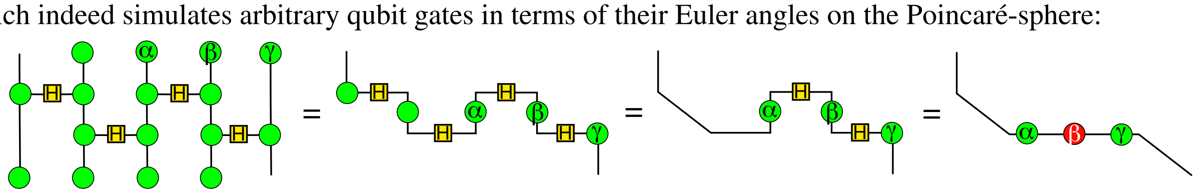
From these laws we can for example derive that $cnot \circ cnot^{\sigma} \circ cnot = swap$:



Measurement-based quantum Computing. To achieve more expressiveness we can decorate these dots with phases [15]. One can show that the spider-theorem still holds provided we add up all phases occurring in the network. This provides us with enough power to show that the one-way quantum computational model [66] enables to simulate arbitrary unitary operations on multiple qubits. The network to simulate an arbitrary qubit unitary is:



which indeed simulates arbitrary qubit gates in terms of their Euler angles on the Poincaré-sphere:



Quantum algorithms. With the same structure we can also compute the quantum Fourier transform [15] which is the key ingredient of Shor's factoring algorithm. Whether this calculus is complete in the model-theoretic sense, that is, whether every statement that can be proved in Hilbert space terms can be proved in our graphical calculus, remains an open question, but we have reasons to belief that this might actually be true.

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