De-linearizing Linearity

bOB cOECKE @ QPL III

Oxford University

se10.comlab.ox.ac.uk:8080/BobCoecke/Home_en.html

(or Google ‘‘Bob Coecke’’)

THE GRAND CHALLENGE

Why did discovering quantum teleportation take 60 year?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\textbf{GOOD QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\textbf{HIGH-LEVEL language}}{\texttt{low-level language}}$$

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\mathbf{GOOD\ QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\textbf{GOOD QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\textbf{HIGH-LEVEL language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I'll present it to you.

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since ‘too low level’ cf.

$$\frac{\text{“}\mathbf{GOOD\ QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Wouldn’t it be nice to have a ‘good’ formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I’ll present it to you.

Isn’t it absurdly abstract coming from you guys?

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since ‘too low level’ cf.

$$\frac{\text{“}\mathbf{GOOD\ QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\mathbf{HIGH\text{-}LEVEL\ language}}{\texttt{low-level language}}$$

Wouldn’t it be nice to have a ‘good’ formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I’ll present it to you.

Isn’t it absurdly abstract coming from you guys?

Claim: It could be taught in **kindergarten**!

THE ACTS: QPL I, QPL II, QPL III

1. Analyse quantum compoundness.

$\Rightarrow$ A notion of **quantum information flow** emerges.

- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

1. Analyse quantum compoundness.

$\Rightarrow$ A notion of **quantum information flow** emerges.

- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. Axiomatize quantum compoundness.

$\Rightarrow$ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

1. **Analyse** quantum compoundness.

⇒ A notion of **quantum information flow** emerges.

- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. **Axiomatize** quantum compoundness.

⇒ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

3. **This act** (an unplanned duet), ...

⇒ ... **quantum logic** and **open systems/CPM's**!

- De-linearizing Linearity: Projective Quantum Axiomatics from SCC. Coecke (2005) QPL'05; `quant-ph/0506134`.
- †-CCC's and Completely Positive Maps. Selinger (2005) QPL'05.

THE ACTORS: $\otimes$ and $\oplus$

1. $\otimes$: the multiplicative fragment

$\Rightarrow$ Captures the bulk of the quantum structure, ...

1. $\otimes$: the multiplicative fragment

$\Rightarrow$ Captures the bulk of the quantum structure, ...

$\Rightarrow$ Never considered before in quantum logic, ...

1. $\otimes$: the multiplicative fragment

$\Rightarrow$ Captures the bulk of the quantum structure, ...

$\Rightarrow$ Never considered before in quantum logic, ...

$\Rightarrow$ Caused the failure of quantum logic, ...

1. $\otimes$: the multiplicative fragment

$\Rightarrow$ Captures the bulk of the quantum structure, ...

$\Rightarrow$ Never considered before in quantum logic, ...

$\Rightarrow$ Caused the failure of quantum logic, ...

2. $\oplus$: The additive fragment

$\Rightarrow$ Merely filling in the blanks, ...

1. $\otimes$: the multiplicative fragment

$\Rightarrow$ Captures the bulk of the quantum structure, ...

$\Rightarrow$ Never considered before in quantum logic, ...

$\Rightarrow$ Caused the failure of quantum logic, ...

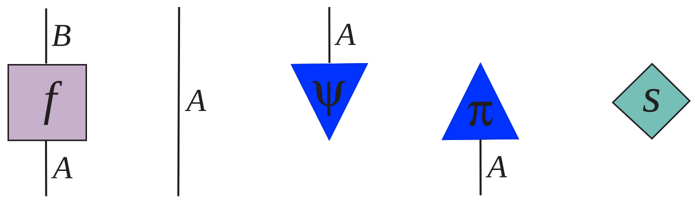
2. $\oplus$: The additive fragment

$\Rightarrow$ Merely filling in the blanks, ...

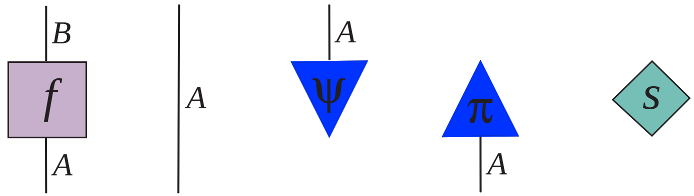
$\Rightarrow$ More or less what quantum logic has been about, ...

$\otimes$: the MULTIPLICATIVE fragment

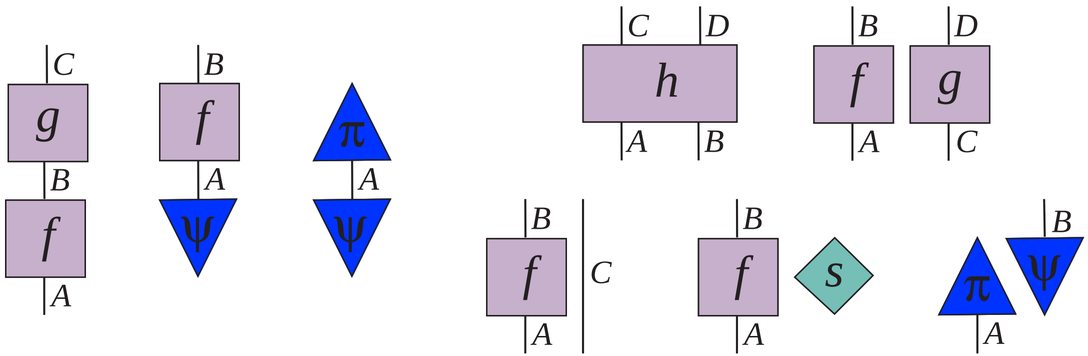
Primitive data:



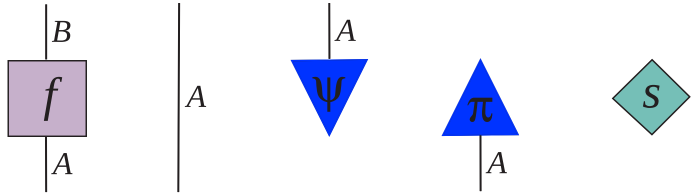
Primitive data:



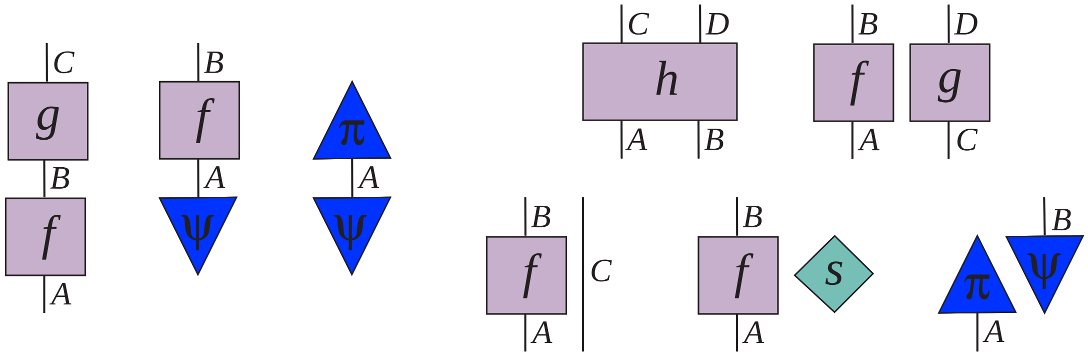
Sequential and parallel composition:



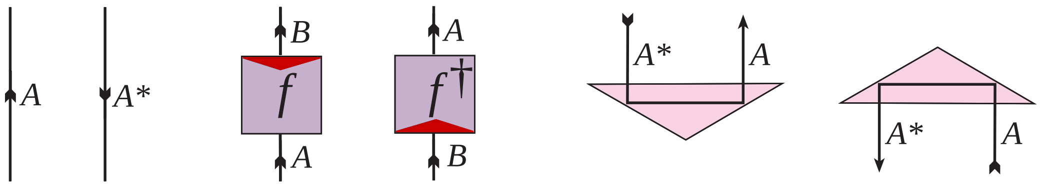
Primitive data:



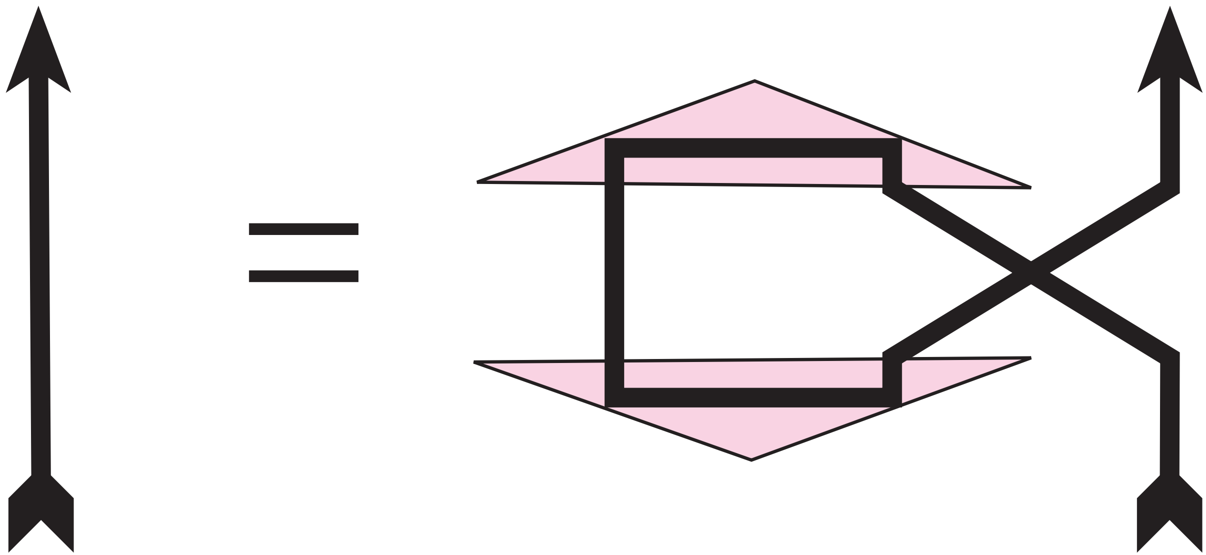
Sequential and parallel composition:



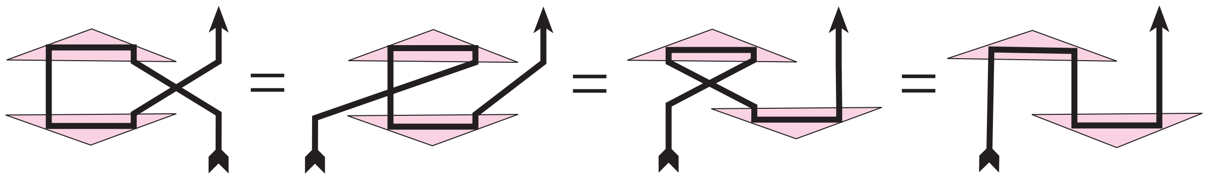
Duals, adjoints and EPR-states:



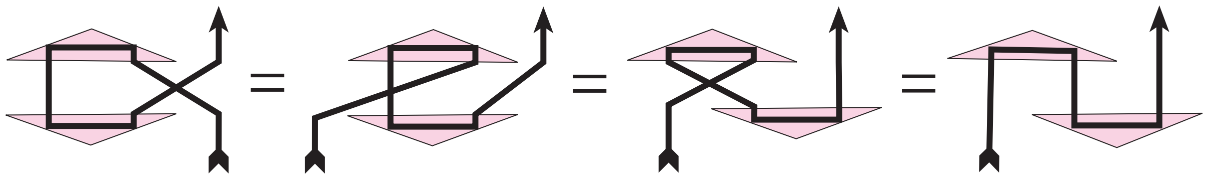
THE SOLE AXIOM



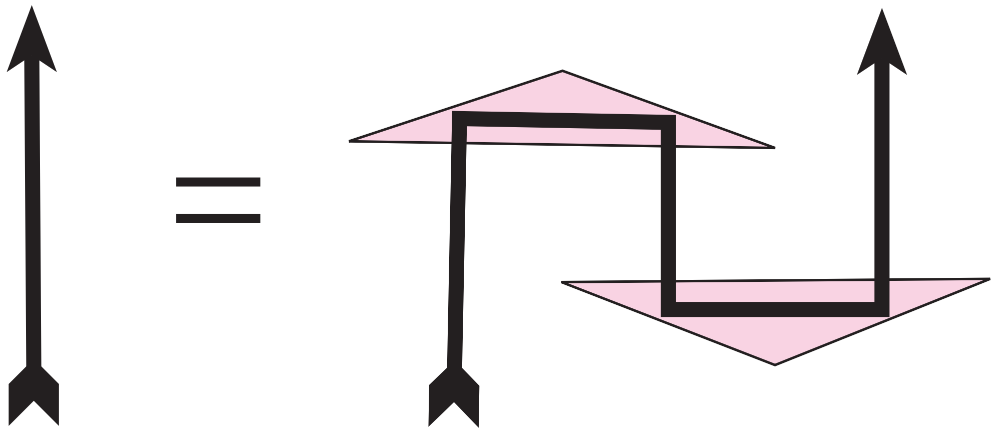
Since



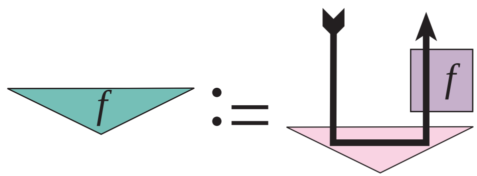
Since

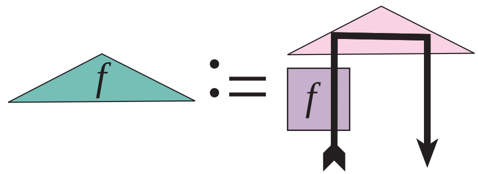


the axiom is equivalent to

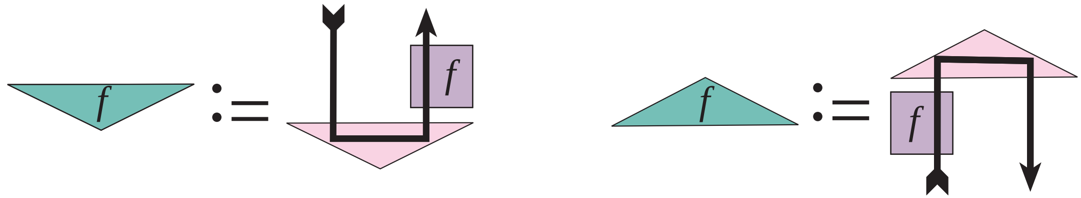


When setting

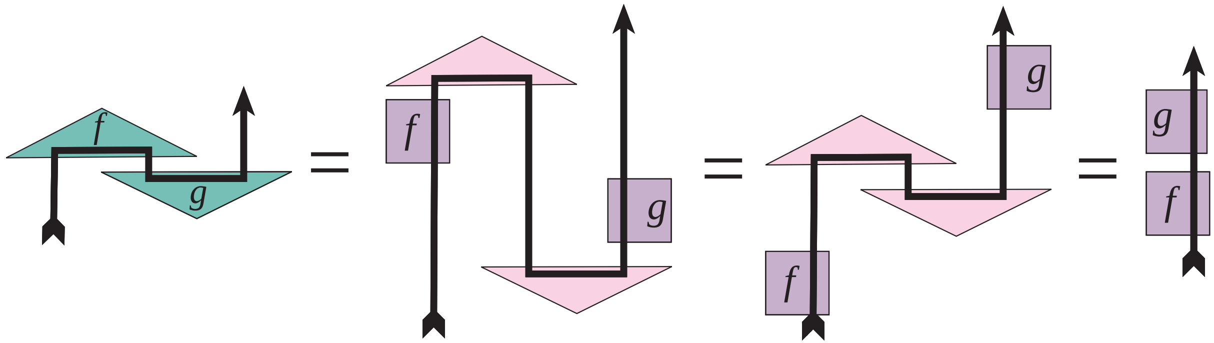




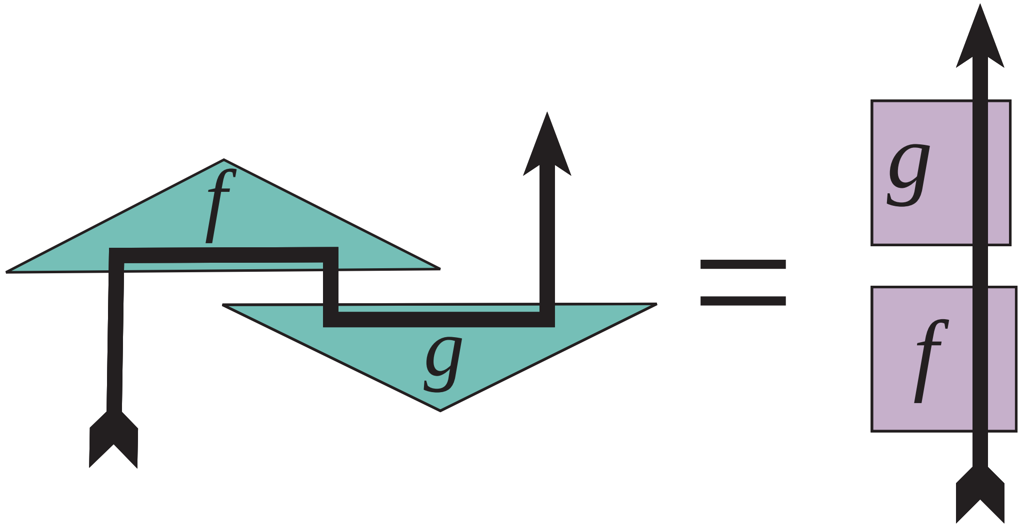
When setting



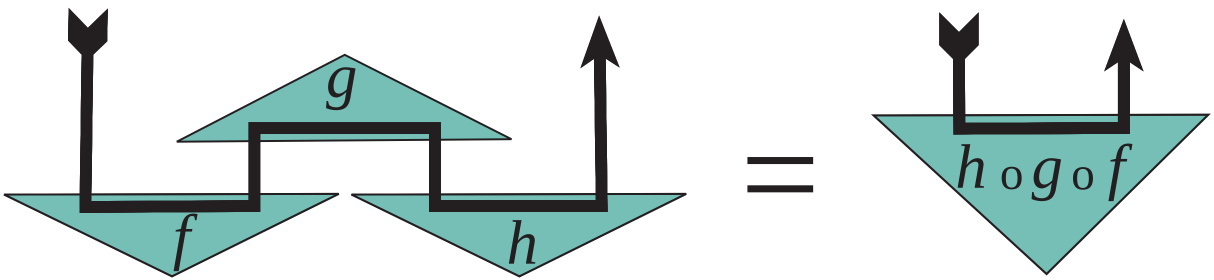
we obtain



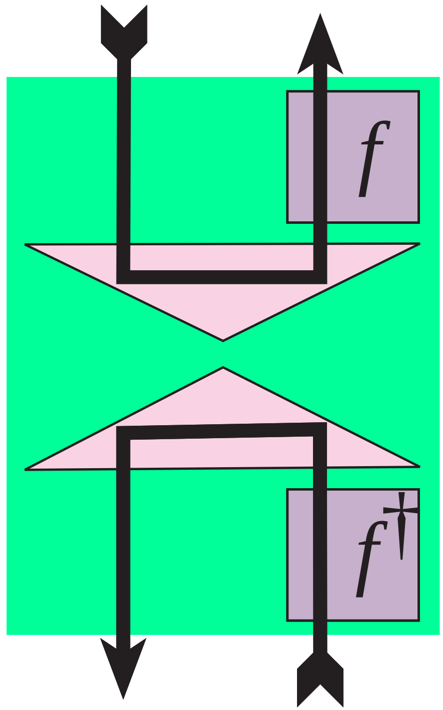
COMPOSITIONALITY



COMPOSITIONALITY bis

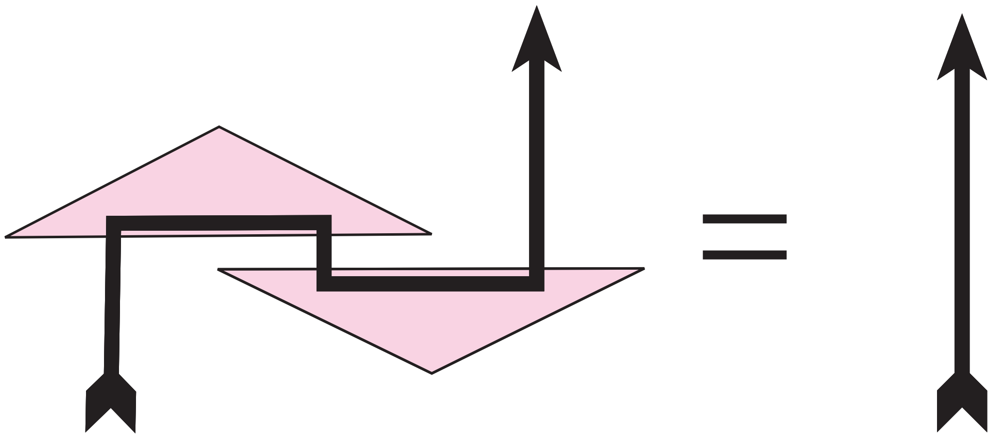


BIPARTITE PROJECTOR

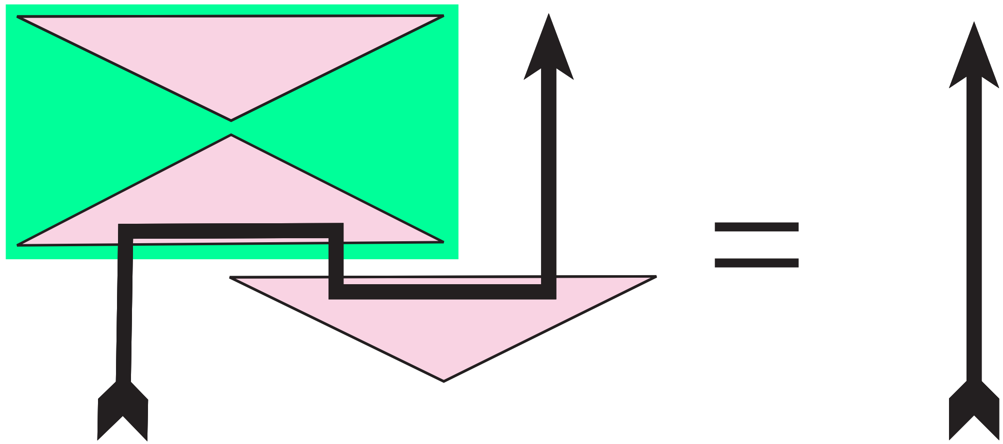


$$\mathrm{P}_f : A^* \otimes B \to A^* \otimes B$$

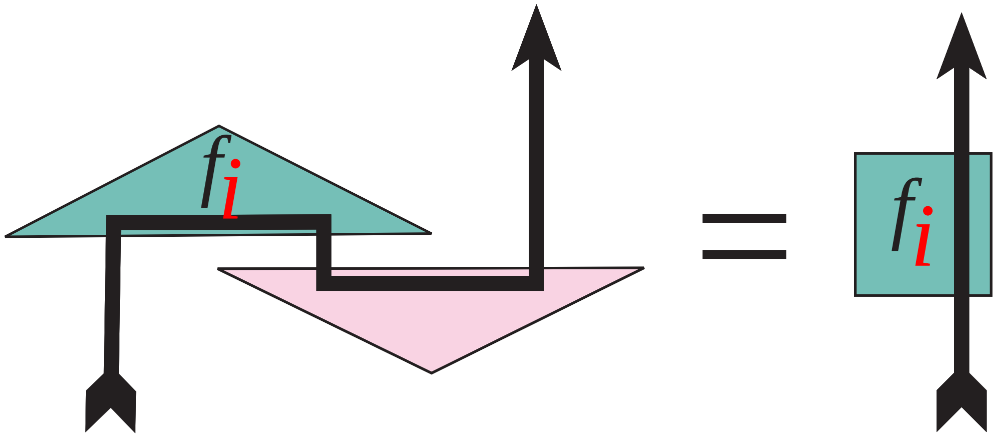
$\frac{1}{4}$th-TELEPORTATION



$\frac{1}{4}$th-TELEPORTATION

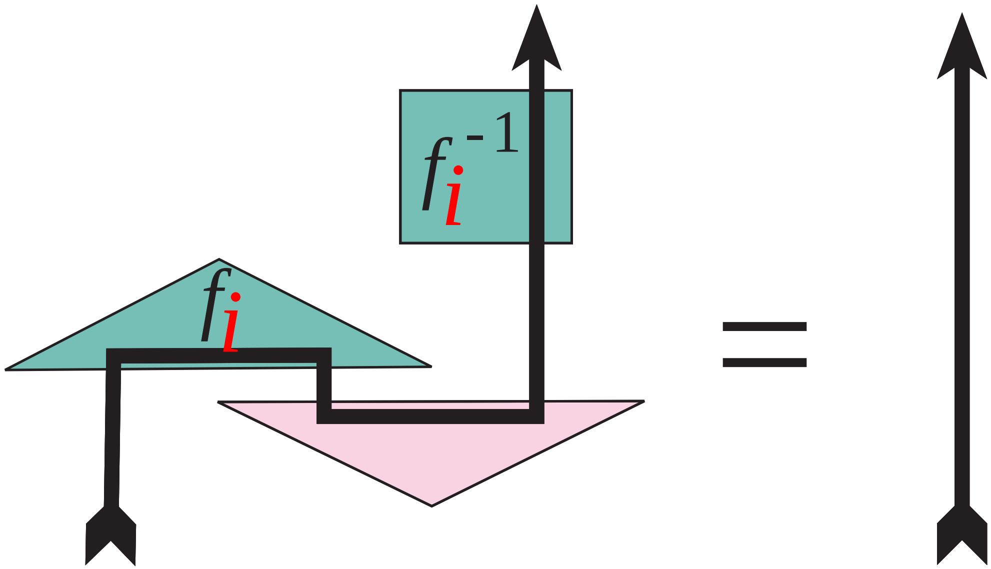


FULL TELEPORTATION



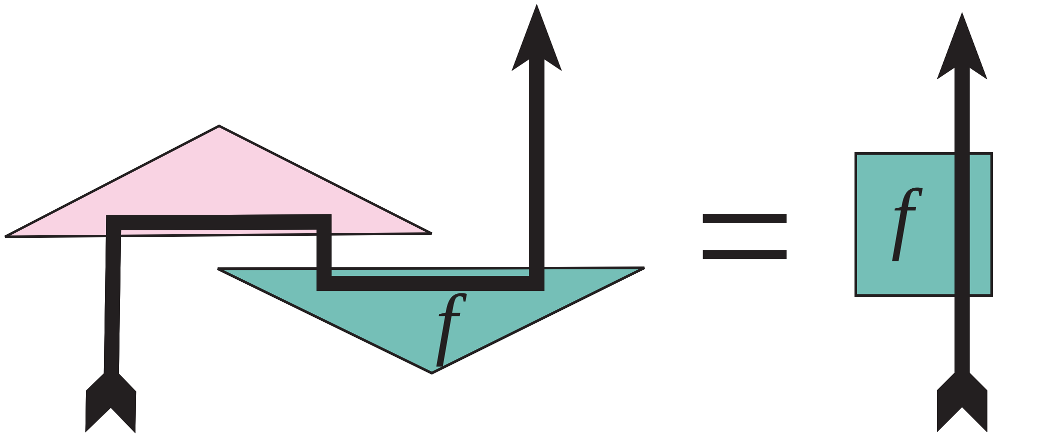
for $1 \leq i \leq 4$

FULL TELEPORTATION

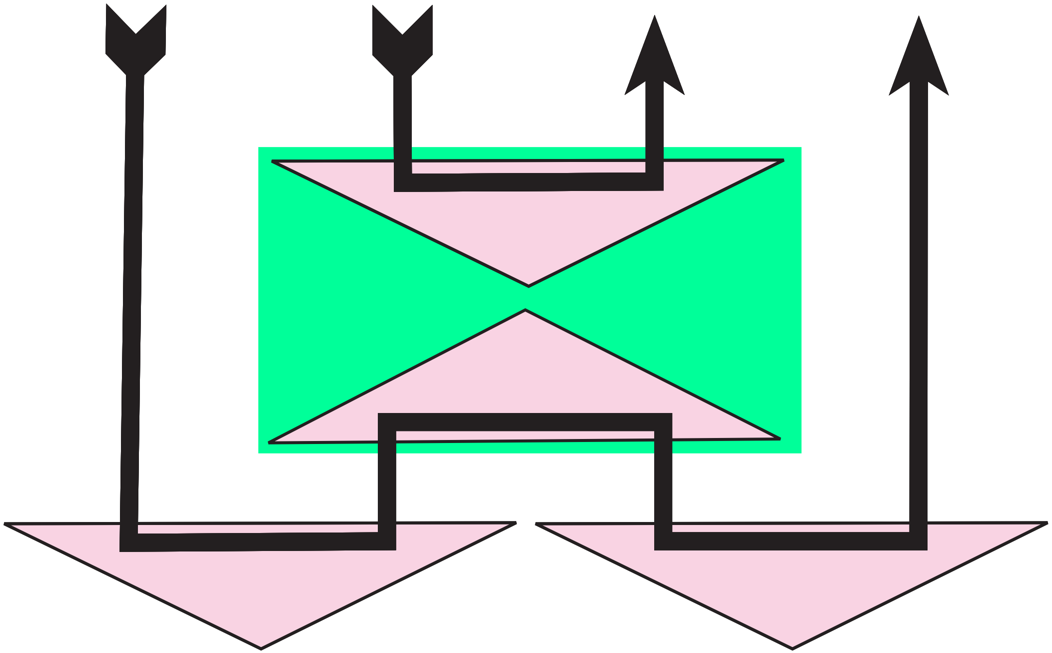


for $1 \leq i \leq 4$

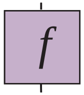
LOGIC GATE TELEPORTATION



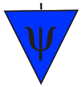
ENTANGLEMENT SWAPPING



HILBERT SPACE QM



$: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map

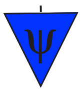


$: \mathbb{C} \longrightarrow \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

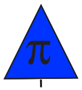
$: \mathbb{C} \longrightarrow \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

$\mathcal{H}^*$:= conjugate Hilbert space of $\mathcal{H}$

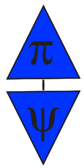
$f^\dagger$:= linear adjoint of f



$= |\psi\rangle$

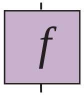


$= \langle\phi|$ for $\pi := \phi^\dagger$

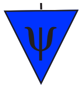


$= \langle\phi \mid \psi\rangle$

HILBERT SPACE QM



$: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map

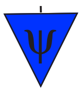


$: \mathbb{C} \longrightarrow \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

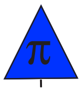
$: \mathbb{C} \longrightarrow \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

$\mathcal{H}^*$:= conjugate Hilbert space of $\mathcal{H}$

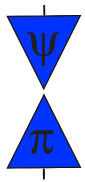
$f^\dagger$:= linear adjoint of f



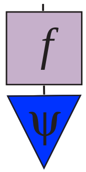
$= |\psi\rangle$



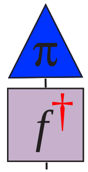
$= \langle\phi|$ for $\pi := \phi^\dagger$



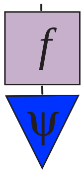
$= |\psi\rangle\langle\phi|$

$| f \circ \psi \rangle =$ 

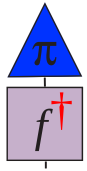
$= f \circ \psi$

$\langle f \circ \phi | =$ 

$= \phi^{\dagger} \circ f^{\dagger}$

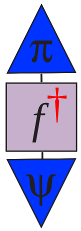
$| f \circ \psi \rangle =$ 

$= f \circ \psi$

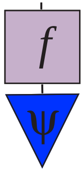
$\langle f \circ \phi | =$ 

$= \phi^\dagger \circ f^\dagger$

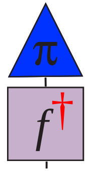
Adjointness implies

$\langle f \circ \phi | \psi \rangle =$ 

$= \langle \phi | f^\dagger \circ \psi \rangle$

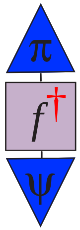
$$| f \circ \psi \rangle =$$ 

$$= f \circ \psi$$

$$\langle f \circ \phi | =$$ 

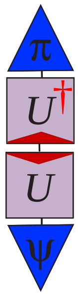
$$= \phi^\dagger \circ f^\dagger$$

Adjointness implies

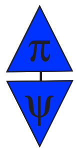
$$\langle f \circ \phi \mid \psi \rangle =$$ 

$$= \langle \phi \mid f^\dagger \circ \psi \rangle$$

Unitarity means $U^{-1} = U^\dagger$ so

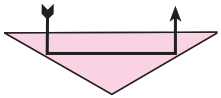
$$\langle U \circ \phi \mid U \circ \psi \rangle =$$ 

$=$ 

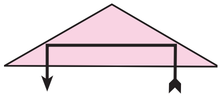
$=$ 

$$= \langle \phi \mid \psi \rangle$$

EPR-states and their adjoints:



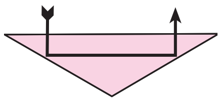
$$: \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \Big|\sum_i e_i \otimes e_i\Big\rangle$$

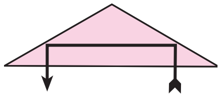


$$: \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C} :: \Phi \mapsto \Big\langle \sum_i e_i \otimes e_i \Big| \Phi \Big\rangle$$

$$:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 \mid \phi_2 \rangle$$

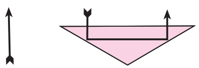
EPR-states and their adjoints:

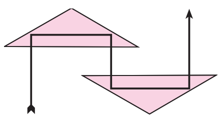
 $: \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \left| \sum_i e_i \otimes e_i \right\rangle$

 $: \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C} :: \Phi \mapsto \left\langle \sum_i e_i \otimes e_i \,\middle|\, \Phi \right\rangle$

$$:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 \mid \phi_2 \rangle$$

We verify the axiom:

 $= (-) \otimes \left(\sum_i e_i \otimes e_i \right) = \sum_i (- \otimes e_i) \otimes e_i$

 $= \sum_i \langle - \mid e_i \rangle \cdot e_i \ = \ \mathsf{id}$

A key role is played by

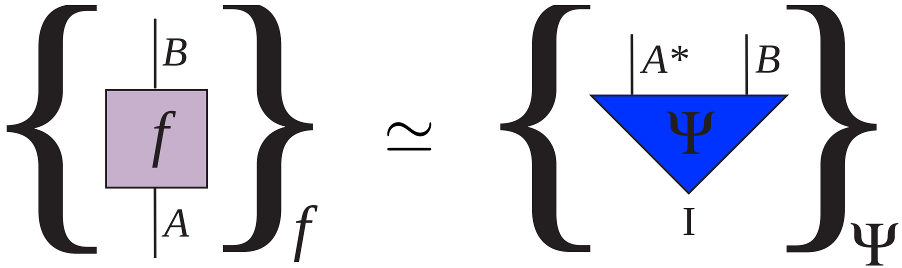
$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \quad \simeq \quad \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

i.e. bipartite states $\Psi \in \mathcal{H}_1^* \otimes \mathcal{H}_2$ are representable by linear functions $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and vice versa. Indeed

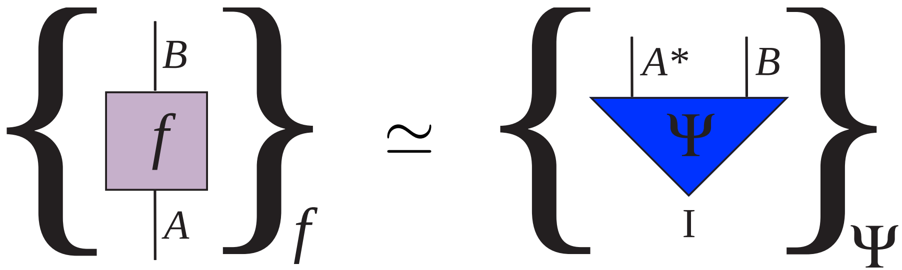
$$\Psi = \sum_{ij} m_{ij} \, |ij\rangle \quad \stackrel{\simeq}{\longleftrightarrow} \quad \begin{pmatrix} m_{11} \cdots m_{1n} \\ \vdots \quad \ddots \quad \vdots \\ m_{k1} \cdots m_{kn} \end{pmatrix}$$

$$\stackrel{\simeq}{\longleftrightarrow} \quad f = \sum_{ij} m_{ij} \, |j\rangle\langle i|$$

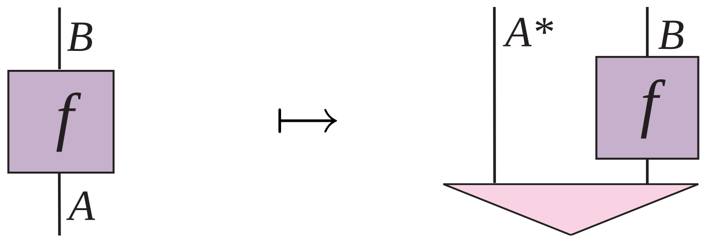
PROCESSES $\simeq$ 2-STATES



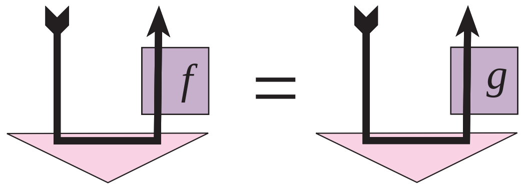
PROCESSES $\simeq$ 2-STATES



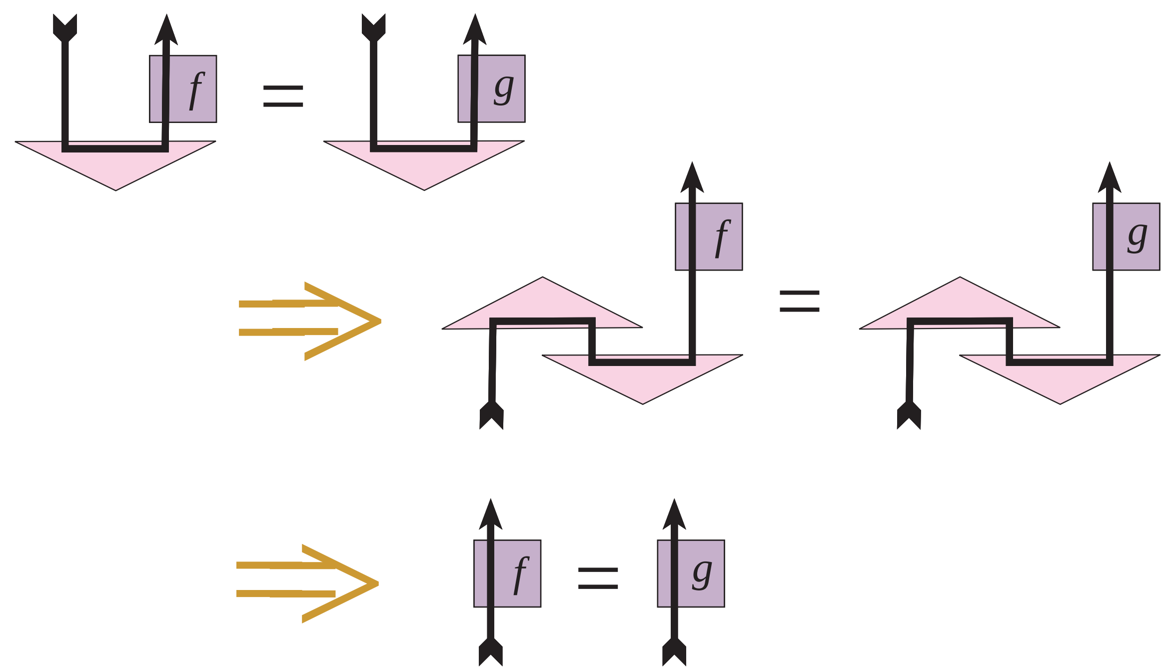
for the bijection $f \mapsto \ulcorner f \urcorner$ i.e.



Proof of injectivity.

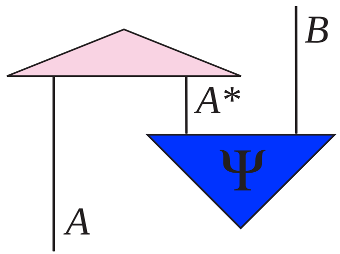


Proof of injectivity.

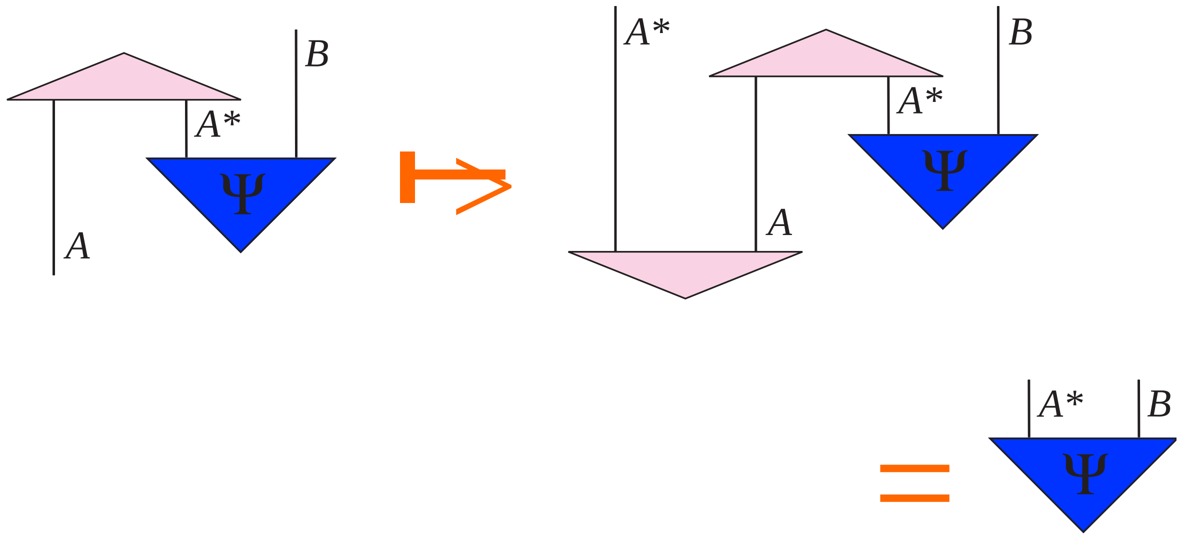


Proof of surjectivity.

Proof of surjectivity.



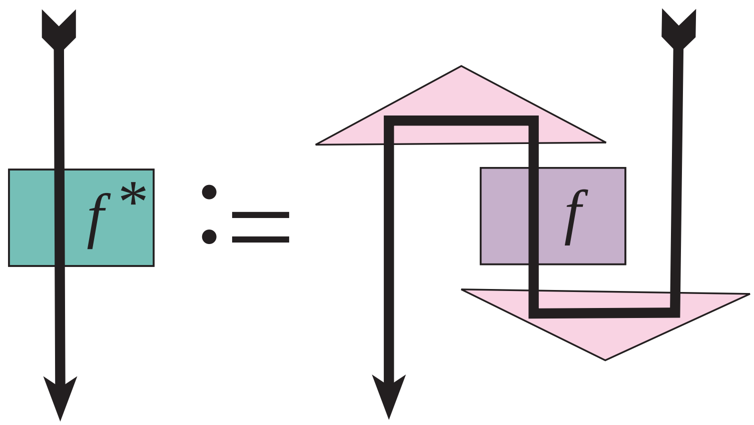
Proof of surjectivity.



UPPER STAR

A "contravariant" involution

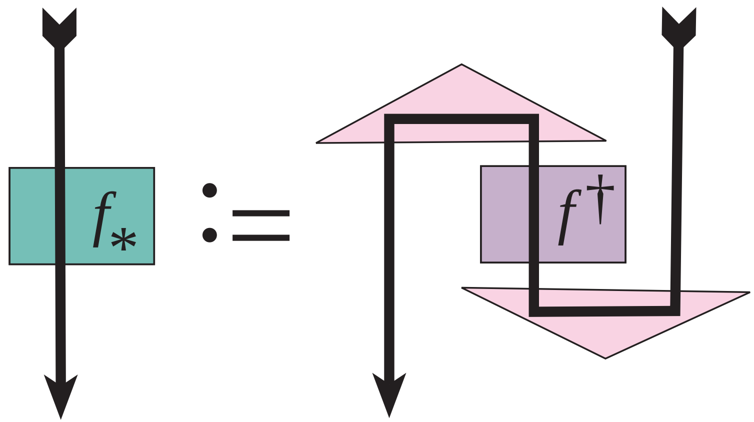
$$f : A \to B \quad \mapsto \quad f^{*} : B^{*} \to A^{*}$$



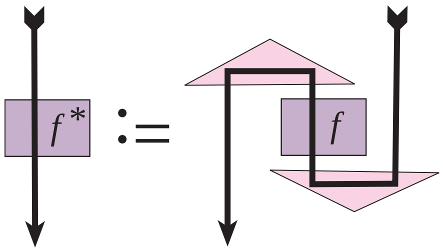
LOWER STAR

A "covariant" involution

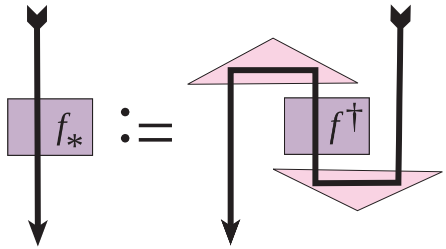
$$f : A \longrightarrow B \quad \mapsto \quad f_{*} : A^{*} \longrightarrow B^{*}$$



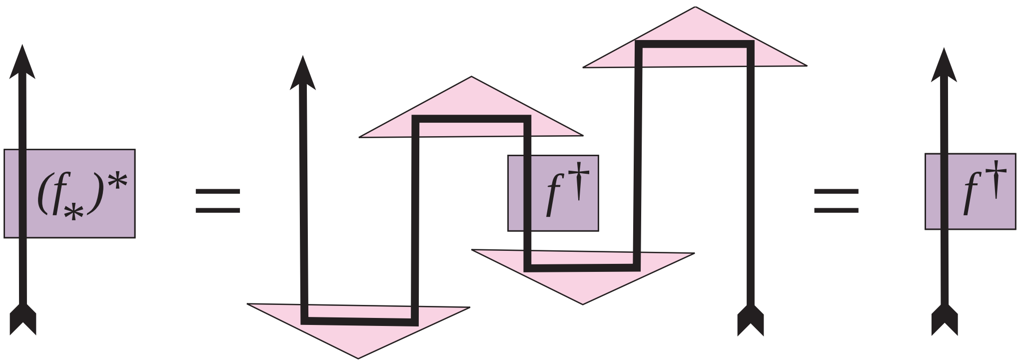
From



and

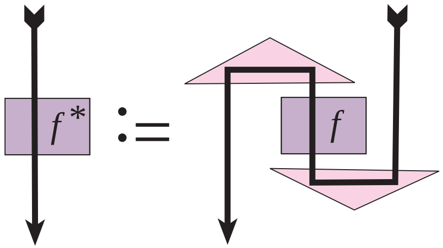


follows

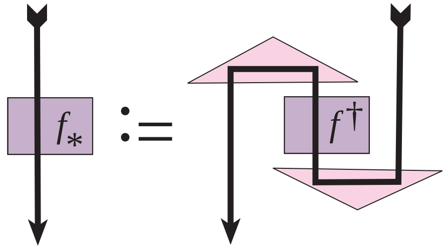


and analogous we can prove that $(f^*)_* = f^\dagger$

Hence the star operations



and



provide a decomposition of the adjoint:

$$f^{\dagger} = (f^{*})_{*} = (f_{*})^{*}$$

In particular, for the Hilbert space model we have

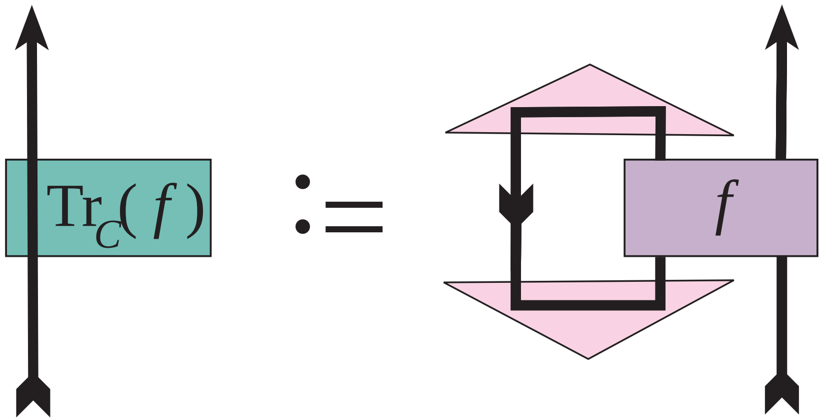
$$(-)^{*} := \textbf{transposition}$$

$$(-)_{*} := \textbf{complex conjugation}$$

TRACE

A Joyal-Street-Verity **trace**

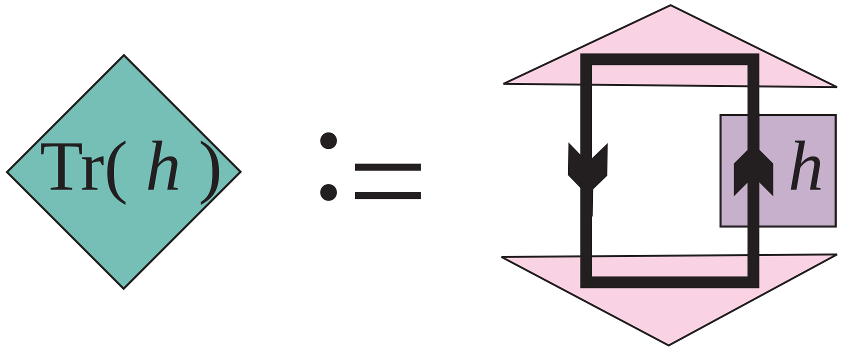
$$f : C \otimes A \to C \otimes B \quad \mapsto \quad \mathrm{Tr}_C(f) : A \to B$$



FULL TRACE

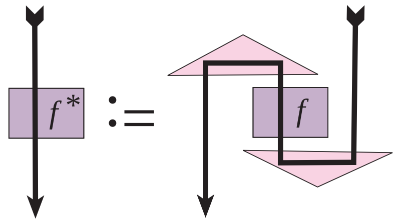
A corresponding **full trace**

$$h : A \to A \quad \mapsto \quad \mathrm{Tr}(h) : \mathrm{I} \to \mathrm{I}$$

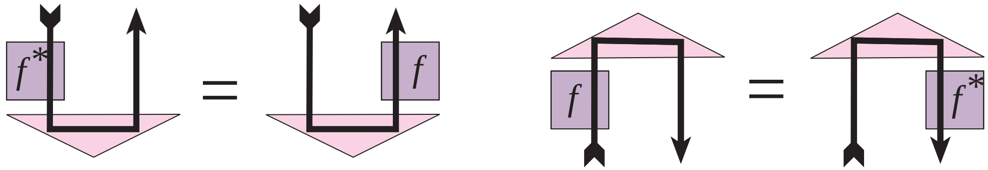


$\Rightarrow h$ "carries a diamond" cf. **probabilistic weight**

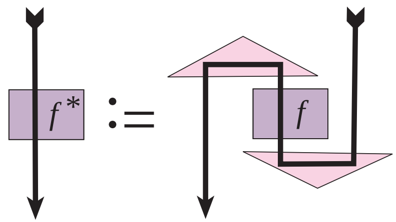
From



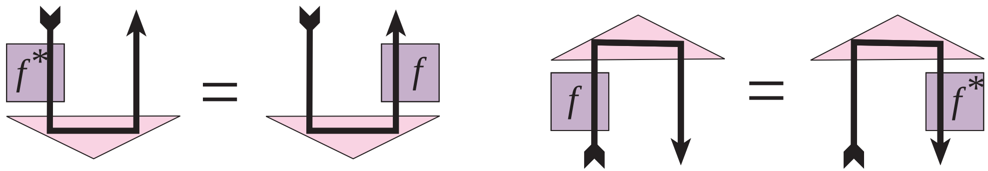
follows



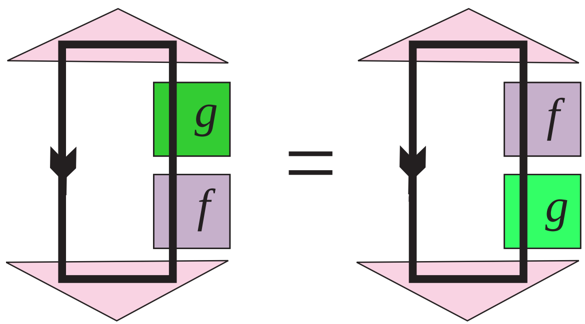
From



follows



and hence

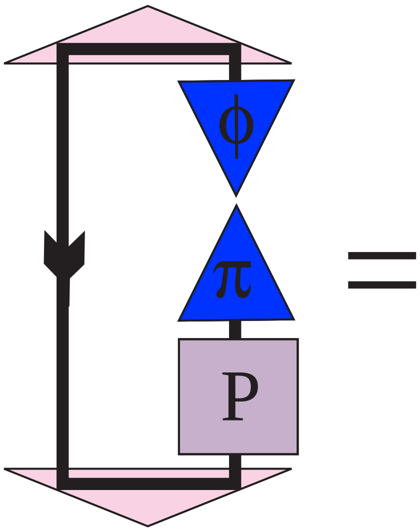


EQUIVALENT BORN RULES

$$\mathsf{Tr}(\rho_\phi \circ \mathrm{P}) \overset{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$

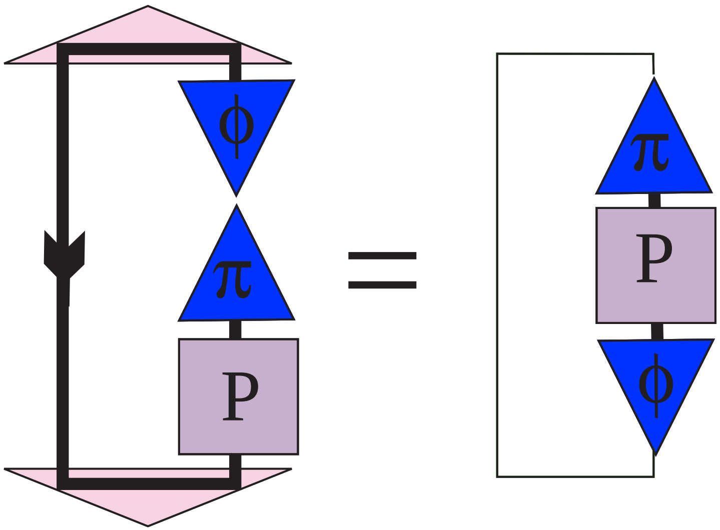
EQUIVALENT BORN RULES

$$\mathsf{Tr}(\rho_\phi \circ \mathrm{P}) \stackrel{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



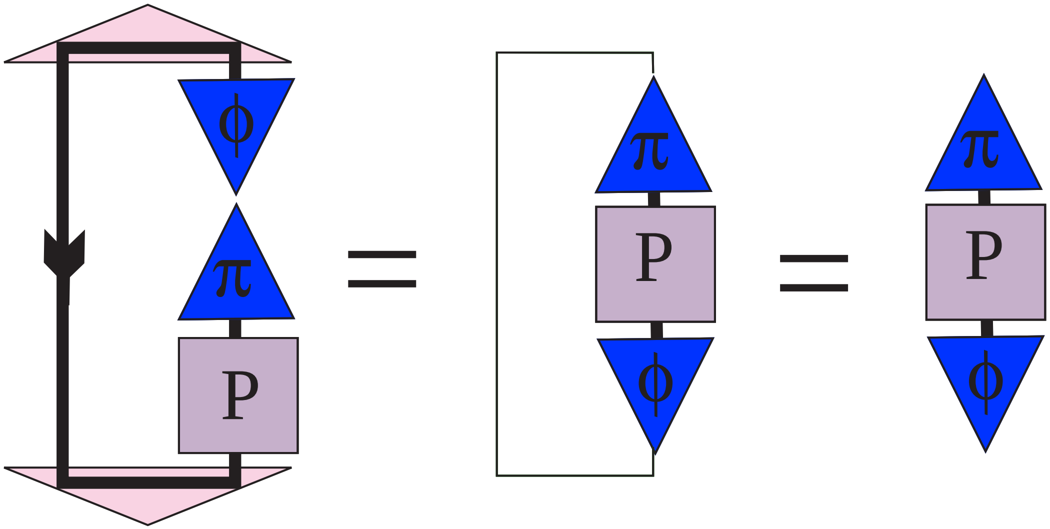
EQUIVALENT BORN RULES

$$\mathrm{Tr}(\rho_\phi \circ \mathrm{P}) \stackrel{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



EQUIVALENT BORN RULES

$$\text{Tr}(\rho_\phi \circ \text{P}) \stackrel{???}{=} \langle \phi \mid \text{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



$$\mathbb{C}^* \otimes \mathbb{C} \simeq \mathbb{C}$$

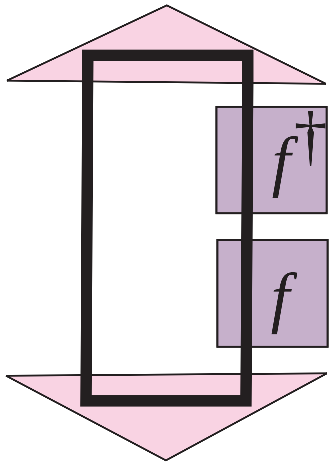
The **squared Hilbert-Schmidt norm**

$$||f|| = \sum_i \langle f(e_i) \mid f(e_i) \rangle$$

exists in the picture formalism as

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

i.e.



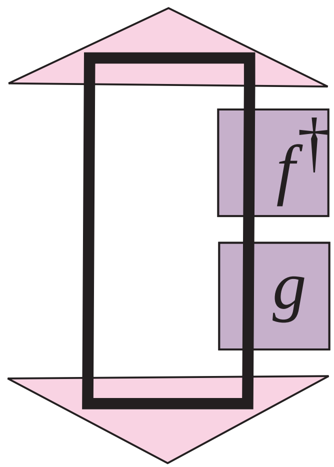
The **Hilbert-Schmidt inner-product**

$$\langle f, g\rangle = \sum_i \langle f(e_i) \mid g(e_i)\rangle$$

exists in the picture formalism as

$$\langle f, g\rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$$

i.e.



and generalizes bra-ket inner-product

ABSTRACT GLOBAL PHASES

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger$$

Let $f = e^{i\theta}\cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathbf{FdHilb}$.

$$f \otimes f^\dagger = e^{i\theta}\cdot g \otimes (e^{i\theta}\cdot g)^\dagger = e^{i\theta}\cdot g \otimes e^{-i\theta}\cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

$$f \otimes f^\dagger = g \otimes g^\dagger \quad \Longrightarrow \quad \exists s, t : s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger$$

Let $f = e^{i\theta} \cdot g : \mathcal{H}_1 \to \mathcal{H}_2$ in **FdHilb**.

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger = 1_\mathrm{I} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

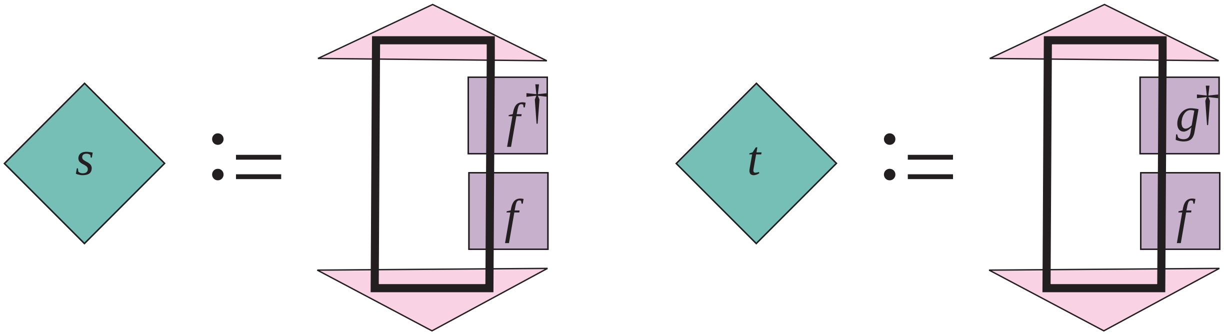
$$f \otimes f^\dagger = g \otimes g^\dagger \quad \Longrightarrow \quad \exists s, t : \; s \bullet f = t \bullet g \,,\, s \circ s^\dagger = t \circ t^\dagger$$

e.g.

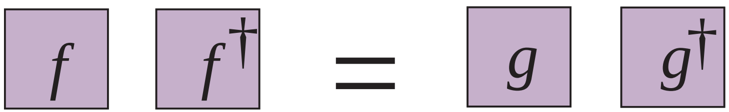
$$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \qquad \text{and} \qquad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

♯1 $s := (\ulcorner f \urcorner)^{\dagger} \circ \ulcorner f \urcorner$ and $t := (\ulcorner g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$

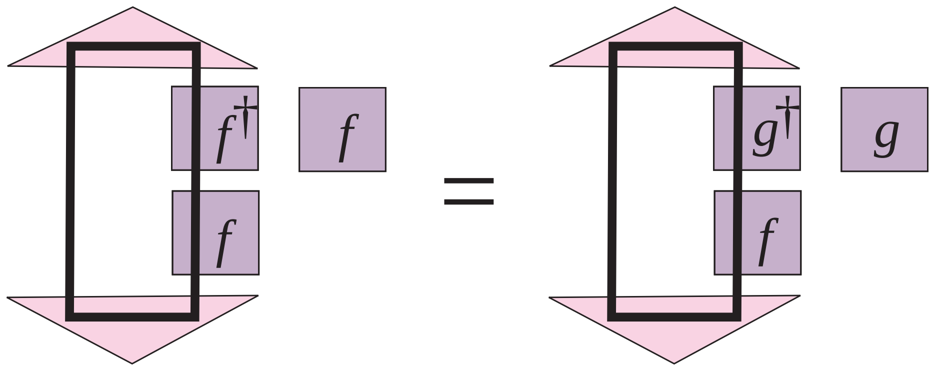


♯2 $f \otimes f^{\dagger} = g \otimes g^{\dagger}$



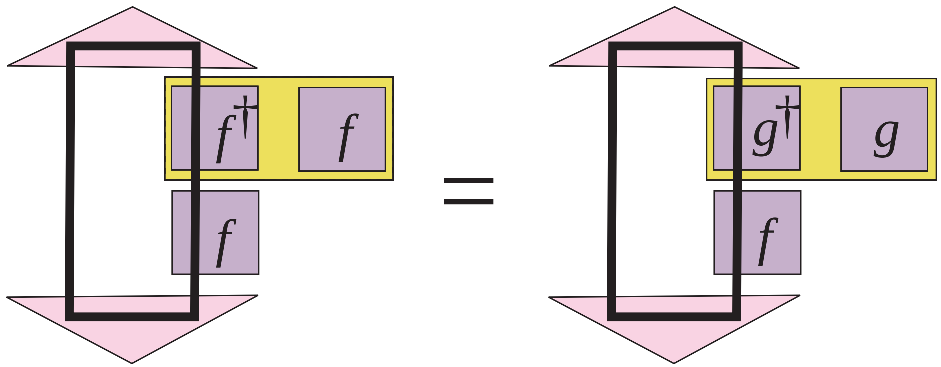
Proof.

$\sharp 3 \quad s \bullet f = t \bullet g$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



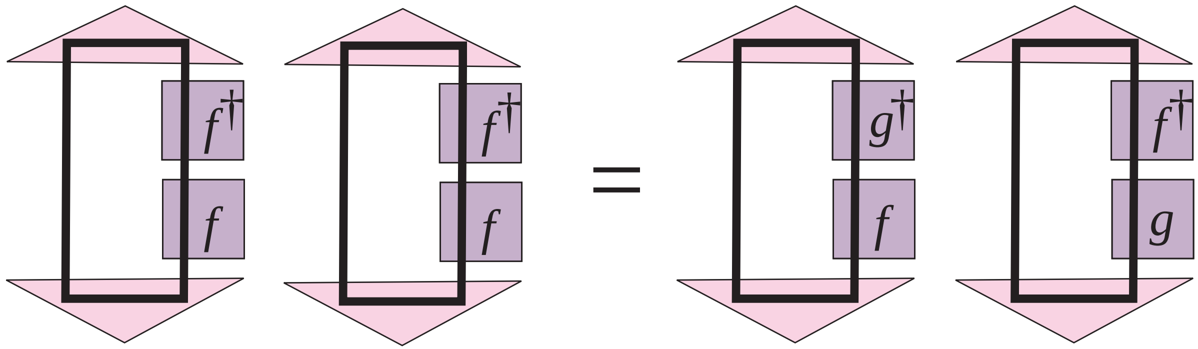
Proof.

$\sharp 3 \quad s \bullet f = t \bullet g$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$



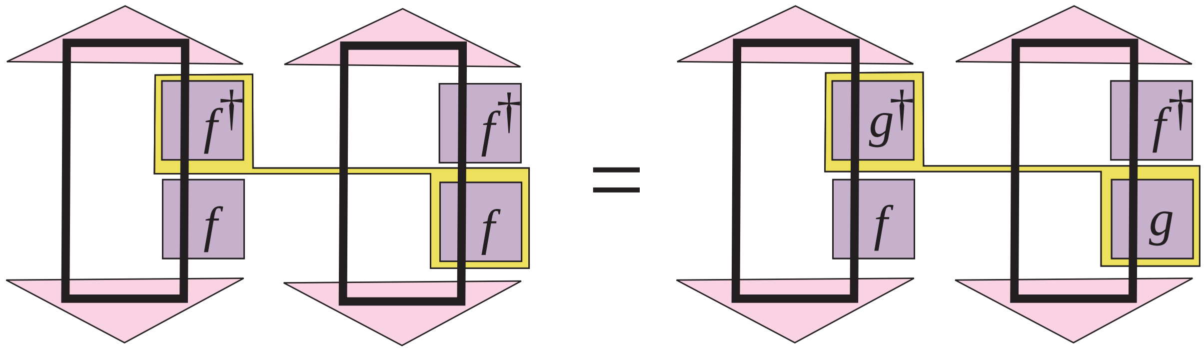
Proof.

$\sharp 4 \quad s \circ s^\dagger = t \circ t^\dagger$ with $s/t := (\ulcorner f/g \urcorner)^\dagger \circ \ulcorner f \urcorner$

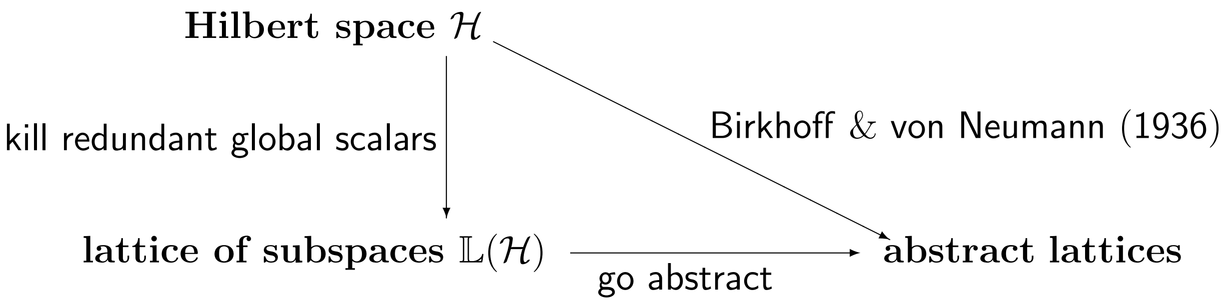


Proof.

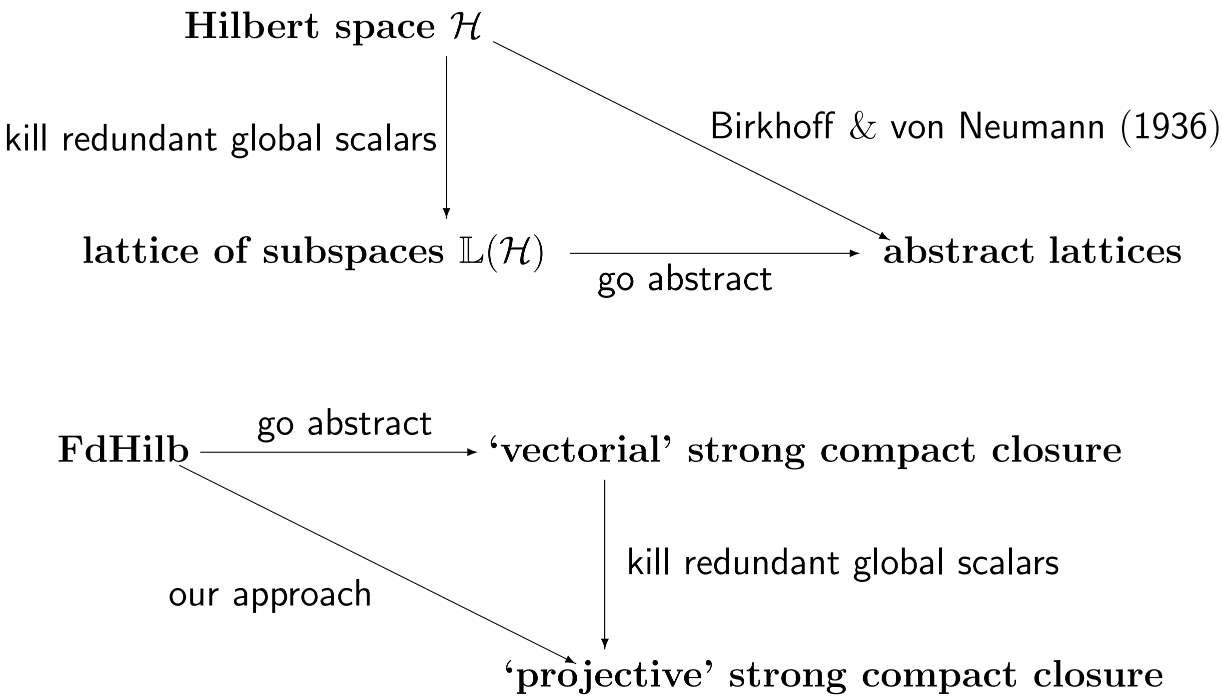
$\sharp 4 \quad s \circ s^{\dagger} = t \circ t^{\dagger}$ with $s/t := (\ulcorner f/g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$



PROJECTIVE vs VECTORIAL



PROJECTIVE vs VECTORIAL



ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^{\dagger} = g \otimes g^{\dagger} \implies f = g$$

ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

iff

$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^{\dagger} = g \otimes g^{\dagger} \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

iff

$$\psi \circ \psi^{\dagger} = \phi \circ \phi^{\dagger} \implies \psi = \phi$$

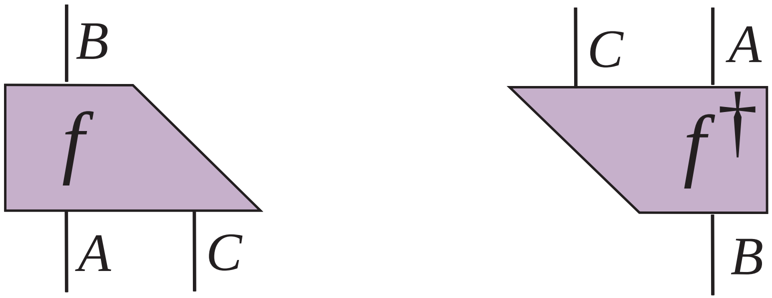
iff

Equal Preparations Produce Equal States

OPEN SYSTEMS AND CPMs

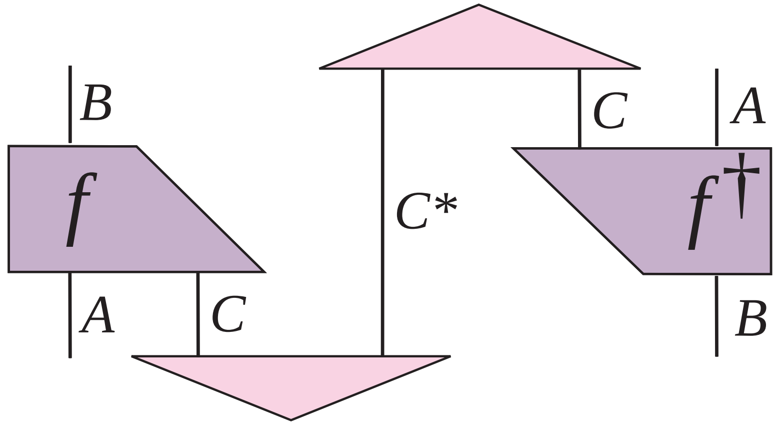
$\Rightarrow$ projective process

OPEN SYSTEMS AND CPMs



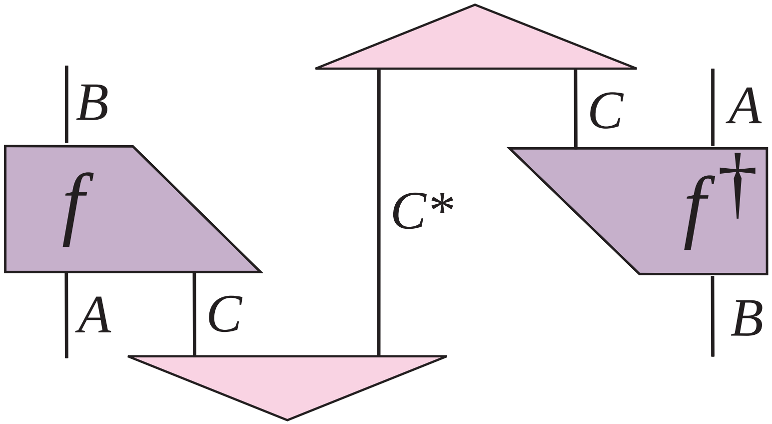
$\Rightarrow$ projective process with ancila

OPEN SYSTEMS AND CPMs



⇒ projective process with hidden ancila

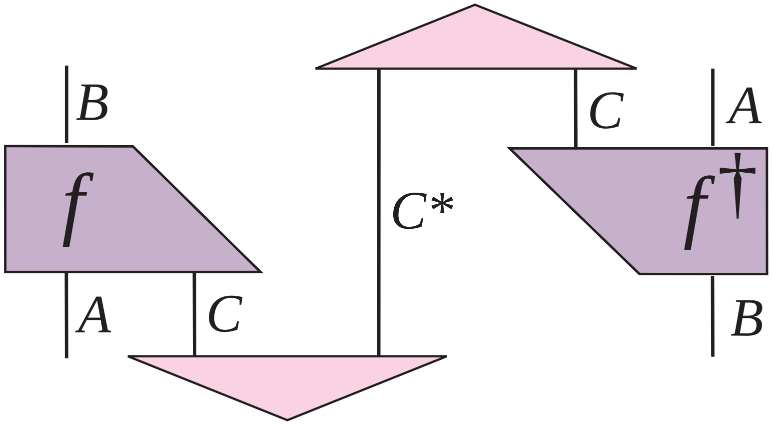
OPEN SYSTEMS AND CPMs



$\Rightarrow$ projective process with hidden ancila

= open process on open system

OPEN SYSTEMS AND CPMs



In the case of Hilbert spaces and linear maps we exactly obtain **completely positive maps** (Selinger 2005)!

ABSTRACT QM

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \to B$:= Morphism $f : A \to B$

ABSTRACT QM

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \rightarrow B$:= Morphism $f : A \rightarrow B$

State of A := Element $\psi : \mathrm{I} \rightarrow A$

Evolution of A := Unitary $U : A \rightarrow A$

Measurement on A := "Projectors" $\{\mathrm{P}_i : A \rightarrow A\}_i$

ABSTRACT QM

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \to B$:= Morphism $f : A \to B$

State of A := Element $\psi : \mathrm{I} \to A$

Evolution of A := Unitary $U : A \to A$

Measurement on A := "Projectors" $\{\mathrm{P}_i : A \to A\}_i$

- Data := $\nu \in \{i\}_i$
- Dynamics := $\psi \mapsto \mathrm{P}_\nu \circ \psi$
- Probability := $\psi^\dagger \circ \mathrm{P}_\nu \circ \psi = \mathrm{Tr}(\mathrm{P}_\nu \circ \rho_\psi) : \mathrm{I} \to \mathrm{I}$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \longrightarrow A\}_i$
- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \longrightarrow \ldots$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \longrightarrow A\}_i$

- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \longrightarrow \ldots$

E.g. for $U : A \longrightarrow \bigoplus_i A_i$ unitary let $\pi_j := p_j \circ U$ and

$$\{\mathrm{P}_i := \pi_i^\dagger \circ \pi_i\}_j \qquad M := \langle \mathrm{P}_i \rangle_i : A \longrightarrow \bigoplus_i A\,.$$

Extra additive structure is required for:

- **Specification** of the good families $\{\mathrm{P}_i : A \longrightarrow A\}_i$
- **Combining** families $\{\mathrm{P}_i\}_i$ in a single $M : A \longrightarrow \ldots$

E.g. for $U : A \longrightarrow \bigoplus_i A_i$ unitary let $\pi_j := p_j \circ U$ and

$$\{\mathrm{P}_i := \pi_i^\dagger \circ \pi_i\}_j \qquad M := \langle \mathrm{P}_i \rangle_i : A \longrightarrow \bigoplus_i A\,.$$

Cf. in **FdHilb** **biproducts** provide $\bigoplus_i$, p_j and $\langle\ \ \rangle_i$.

$\oplus$: the **ADDITIVE** fragment

$\Rightarrow$ **Strong Compact Closure:** **The (very) good!**

$\Rightarrow$ Strong Compact Closure: The (very) good!

Thm. No biproduct SCCC satisfies the preparation-state agreement and has non-trivial negative scalars.

$\Rightarrow$ **Strong Compact Closure:** **The (very) good!**

Thm. No biproduct SCCC satisfies the preparation-state agreement and has non-trivial negative scalars.

Col. Matrix calculi which satisfy preparation-state agreement cannot describe interference phenomena.

$\Rightarrow$ **Strong Compact Closure:** **The (very) good!**

Thm. No biproduct SCCC satisfies the preparation-state agreement and has non-trivial negative scalars.

Col. Matrix calculi which satisfy preparation-state agreement cannot describe interference phenomena.

$\Rightarrow$ **Biproducts:** **The Bad!**

$\Rightarrow$ **Strong Compact Closure:** **The (very) good!**

Thm. No biproduct SCCC satisfies the preparation-state agreement and has non-trivial negative scalars.

Col. Matrix calculi which satisfy preparation-state agreement cannot describe interference phenomena.

$\Rightarrow$ **Biproducts:** **The Bad!**

Thm. The passage $f \mapsto f^\dagger \otimes f$ kills products, coproducts and the symmetric monoidal structure.

$\Rightarrow$ **Strong Compact Closure:** **The (very) good!**

Thm. No biproduct SCCC satisfies the preparation-state agreement and has non-trivial negative scalars.

Col. Matrix calculi which satisfy preparation-state agreement cannot describe interference phenomena.

$\Rightarrow$ **Biproducts:** **The Bad!**

Thm. The passage $f \mapsto f^\dagger \otimes f$ kills products, coproducts and the symmetric monoidal structure.

$\Rightarrow$ **... we need something else:** **The Ugly?**

SCCC follows from BP ($\simeq$ abstract linear algebra) i.e.

SCCC is a component of linearity

SCCC follows from BP ($\simeq$ abstract linear algebra) i.e.

SCCC is a component of linearity

But

$$\nabla_A^{(n)} \circ \left(\bigoplus_i (\pi_i^\dagger \circ \pi) \right) \circ \Delta_A^{(n)} \neq 1_A$$

so BP is wrong!

SCCC follows from BP ($\simeq$ abstract linear algebra) i.e.

SCCC is a component of linearity

But

$$\nabla_A^{(n)} \circ \left(\bigoplus_i (\pi_i^\dagger \circ \pi) \right) \circ \Delta_A^{(n)} \neq 1_A$$

so BP is wrong! Hence

BP is too much but **SCCC is not enough**

So how much do we need to

de-linearize linearity, ...

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$
- projections $p_i : \bigoplus_i A_i \longrightarrow A_j$ for **measurement dynamics** $\mathrm{P}_i := (U \circ p_i)^\dagger \circ (p_i \circ U) : A \longrightarrow A$

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$
- projections $p_i : \bigoplus_i A_i \longrightarrow A_j$ for **measurement dynamics** $\mathrm{P}_i := (U \circ p_i)^\dagger \circ (p_i \circ U) : A \longrightarrow A$

What I get for **free**:

- $\prod$-types for **branches** $\prod_i f_i : \prod_i A_i \rightarrow \prod_i B_i$
- pairing $\langle\ \ \rangle_i$ for **branching** $\langle \mathrm{P}_i \rangle_i : A \rightarrow \prod_i A$

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$
- projections $p_i : \bigoplus_i A_i \longrightarrow A_j$ for **measurement dynamics** $\mathrm{P}_i := (U \circ p_i)^\dagger \circ (p_i \circ U) : A \to A$

What I get for **free**:

- $\prod$-types for **branches** $\prod_i f_i : \prod_i A_i \to \prod_i B_i$
- pairing $\langle\ \ \rangle_i$ for **branching** $\langle \mathrm{P}_i \rangle_i : A \to \prod_i A$

What I need to be **satisfied**:

- 'fancy' distributivity of $\otimes$ over $\oplus$

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$
- projections $p_i : \bigoplus_i A_i \longrightarrow A_j$ for **measurement dynamics** $\mathrm{P}_i := (U \circ p_i)^\dagger \circ (p_i \circ U) : A \longrightarrow A$

What I get for **free**:

- $\prod$-types for **branches** $\prod_i f_i : \prod_i A_i \rightarrow \prod_i B_i$
- pairing $\langle\ \ \rangle_i$ for **branching** $\langle \mathrm{P}_i \rangle_i : A \rightarrow \prod_i A$

What I need to be **satisfied**:

- 'fancy' distributivity of $\otimes$ over $\oplus$
- Born rule

What I need as **data**:

- $\bigoplus$-types for **orthogonality** $\bigoplus_i A_i$
- projections $p_i : \bigoplus_i A_i \longrightarrow A_j$ for **measurement dynamics** $\mathrm{P}_i := (U \circ p_i)^\dagger \circ (p_i \circ U) : A \longrightarrow A$

What I get for **free**:

- $\prod$-types for **branches** $\prod_i f_i : \prod_i A_i \to \prod_i B_i$
- pairing $\langle\ \ \rangle_i$ for **branching** $\langle \mathrm{P}_i \rangle_i : A \to \prod_i A$

What I need to be **satisfied**:

- 'fancy' distributivity of $\otimes$ over $\oplus$
- Born rule ... but what is a Born rule?

Born rule

$|-|_\xi$:= valuation which extracts probabilistic weight

$$
\begin{array}{ccc}
\mathbf{C}(A, \oplus_i B_i) & \xrightarrow{(p_1 \circ - \,,\; \ldots \;,\, p_n \circ -)} & \times_i \mathbf{C}(A, B_i) \\
\big\downarrow {\scriptstyle |-|_\xi} & & \big\downarrow {\scriptstyle |-|_\xi \times \;\ldots\; \times |-|_\xi} \\
\mathbf{C}(\mathrm{I}, \mathrm{I}) & \xleftarrow[\sum_i]{} & \times_i \mathbf{C}(\mathrm{I}, \mathrm{I})\,.
\end{array}
$$

Born rule

$|-|_\xi :=$ valuation which extracts probabilistic weight

$$
\begin{array}{ccc}
\mathbf{C}(A, \oplus_i B_i) & \xrightarrow{(p_1 \circ - \, , \; \ldots \; , \, p_n \circ -)} & \times_i \mathbf{C}(A, B_i) \\
\Big\downarrow {\scriptstyle |-|_\xi} & & \Big\downarrow {\scriptstyle |-|_\xi \times \; \ldots \; \times |-|_\xi} \\
\mathbf{C}(\mathrm{I}, \mathrm{I}) & \xleftarrow[\sum_i]{} & \times_i \mathbf{C}(\mathrm{I}, \mathrm{I}) \, .
\end{array}
$$

e.g.

$$\langle \psi | \psi \rangle = \sum_i \underline{\underline{\langle \psi | \mathrm{P}_i \circ \psi \rangle}} = \sum_i \langle p_i \circ \psi | p_i \circ \psi \rangle = \sum_i \langle \psi_i | \psi_i \rangle$$

Let $\mathbf{C}$ be a BP-SCCC (e.g. $\mathbf{FdHilb}$).

Prop. For $f \in \mathbf{C}(A, B_1 \oplus \ldots \oplus B_n)$

$$||f|| = ||f_1|| + \ldots + ||f_n||$$

Prop. For $f \in WProj(\mathbf{C})(A, B_1 \oplus \ldots \oplus B_n)$

$$\sqrt{||f||} = \sqrt{||f_1||} + \ldots + \sqrt{||f_n||}$$

Does any of these two valuations as a preferred status?

THM. If $\mathbf{C}$ is an SCCC with a symmetric monoidal $\oplus$-structure on positive maps and $|-|_\xi := ||-||^\nu$ then

$$||f||^\nu = ||f_1||^\nu + \ldots + ||f_n||^\nu$$

if and only if

$$\mathrm{Tr}(\mathrm{Tr}(h) \oplus \mathrm{Tr}(h')) = \mathrm{Tr}(h \oplus h')$$

$$\mathrm{Tr}(h) = \mathrm{Tr}(h_{11} \oplus h_{22}).$$

THM. If $\mathbf{C}$ is an SCCC with a symmetric monoidal $\oplus$-structure on positive maps and $|-|_\xi := ||-||^\nu$ then

$$||f||^\nu = ||f_1||^\nu + \ldots + ||f_n||^\nu$$

if and only if

$$\mathrm{Tr}(\mathrm{Tr}(h) \oplus \mathrm{Tr}(h')) = \mathrm{Tr}(h \oplus h')$$

$$\mathrm{Tr}(h) = \mathrm{Tr}(h_{11} \oplus h_{22}).$$

In particular is the sum on scalars

$$s + t := \left(\mathrm{Tr}(s^{\frac{1}{\nu}} \oplus t^{\frac{1}{\nu}})\right)^\nu$$

THM. If $\mathbf{C}$ is an SCCC with a symmetric monoidal $\oplus$-structure on positive maps and $|-|_\xi := ||-||^\nu$ then

$$||f||^\nu = ||f_1||^\nu + \ldots + ||f_n||^\nu$$

if and only if

$$\mathrm{Tr}(\mathrm{Tr}(h) \oplus \mathrm{Tr}(h')) = \mathrm{Tr}(h \oplus h')$$

$$\mathrm{Tr}(h) = \mathrm{Tr}(h_{11} \oplus h_{22}).$$

In particular is the sum on scalars

$$s + t := (\mathrm{Tr}(s^{\frac{1}{\nu}} \oplus t^{\frac{1}{\nu}}))^\nu$$

e.g. for $||-||$ and $\sqrt{||-||}$ we have

$$s + t := \mathrm{Tr}(s \oplus t) \quad \text{and} \quad s + t = \sqrt{||s \oplus t||}$$

0-maps (without 0-object)

$$\begin{array}{ccccccc}
B & \xleftarrow{\ \simeq\ } & \mathrm{I}\otimes B & \xleftarrow{\ \eta_0^\dagger\otimes 1_B\ } & (0^*\otimes 0)\otimes B & \xleftarrow{\ \simeq\ } & 0 \\
\uparrow{\scriptstyle 0_{A,B}} & & & & & & \uparrow{\scriptstyle 1_0} \\
A & \xrightarrow[\simeq]{} & \mathrm{I}\otimes A & \xrightarrow[\eta_0\otimes 1_A]{} & (0^*\otimes 0)\otimes A & \xrightarrow[\simeq]{} & 0
\end{array}$$

Pseudo-projections

$$p_{\underline{A},B} := r_A \circ (1_A \oplus 0_{B,O})$$

+-enrichment

$$\begin{array}{ccccccc}
B & \xleftarrow{\simeq} & \mathrm{I}\otimes B & \xleftarrow{\eta_2^\dagger\otimes 1_B} & (2^*\otimes 2)\otimes B & \xleftarrow{\simeq} & 2^*\otimes(B\oplus B) \\
\uparrow{\scriptstyle h+h'} & & & & & & \uparrow{\scriptstyle 1_{2^*}\otimes(h\oplus h')} \\
A & \xrightarrow[\simeq]{} & \mathrm{I}\otimes A & \xrightarrow[\eta_2\otimes 1_A]{} & (2^*\otimes 2)\otimes A & \xrightarrow[\simeq]{} & 2^*\otimes(A\oplus A)
\end{array}$$

for which we have

$$f\circ(h+h') = f\circ h + f\circ h'$$

Interpretation of axioms

Using the sum $\mathrm{Tr}(\mathrm{Tr}(h)\oplus\mathrm{Tr}(h')) = \mathrm{Tr}(h\oplus h')$ becomes

$$\mathrm{Tr}(h) + \mathrm{Tr}(h') = \mathrm{Tr}(h + h')$$

and $\mathrm{Tr}(h) = \mathrm{Tr}(h_{11} \oplus h_{22})$ becomes

$$\mathrm{Tr}(h) = \mathrm{Tr}(h_{11}) + \mathrm{Tr}(h_{22})$$

i.e.

- Linearity of classical mixing
- Relative phases carry no probabilistic weight

A calculation

$$
\begin{aligned}
\langle f, g\rangle &= \sum_i \langle f(e_i) \mid g(e_i)\rangle \\
&= \sum_i \langle f \circ p_i^\dagger \mid g \circ p_i^\dagger\rangle \\
&= \sum_i p_i \circ f^\dagger \circ g \circ p_i^\dagger \\
&= \sum_i (f^\dagger \circ g)_i \\
&= \mathrm{Tr}(f^\dagger \circ g)
\end{aligned}
$$

ERATA

We proudly point to some bugs in the proceedings:

- no terminal object needed for ortho-Bornian SCCC
- $h \star h' := \mathrm{Tr}(h \oplus h') \quad \rightsquigarrow \quad h + h'$
- pseudo-linearity of trace $\quad \rightsquigarrow \quad$ linearity of trace

BIG PICTURE

Non-idempotence of Selinger's construction on **FdHilb** indicates that open system is a relative concept. Hence quantum axiomatics come in pairs:

(**closed axiomatics** , **open axiomatics**)

where

closed axiomatics := **ortho-Bornian SCCC**

open axiomatics := **CPM(closed axiomatics)**