Resources theories:
derivation of the foundational structure

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Abstract

For general processes theories described by symmetric monoidal categories, we introduce the notions of resource and resource theory. For any resource theory we construct a preordered monoid that compares resources, where the monoid structure accounts for compositionality. Any monotone function to the reals which also preserves the monoid structure then provides a quantitative measure of relative cost. For general resource theories we also define the notions of conversion rate and catalysis. We analyse the structure of resource theories, consider existing resource theories, some paradigmatic real world examples, and examples of a purely mathematical nature.

1 Introduction

The term resource theory has occurred at many occasions in the physics literature. For example, there are the resource theories of *entanglement* [17], *purity* [15], *asymmetry* [19] and *athermality* [4]. The purpose of these theories is to establish a quantitative measure of *relative cost* of the resources. Moreover, the notion of a resource is so generic that we can think of many kinds of resources in other areas of science, e.g. in chemistry:

$$2\mathrm{H_2O_2} + \mathrm{MnO_2} \to 2\mathrm{H_2O} + \mathrm{O_2} + \mathrm{MnO_2}$$

as well as in real life situations, for example, the required quantities of ingredients in any production process, ranging from food production to construction work.

But what is a resource theory? That is, is there a general mathematical definition that underpins all existing resource theories, and in particular, form which a natural quantitative notion of relative cost arises? Such a structure can then be studied as a mathematical entity in its own right, and provide results that can be fed back to all concrete resource theories. Certain new questions can then also be posed, e.g. are there other concepts, features and general principles that are common to all resource theories?

The general pattern of the example resource theories mentioned above, is that one has a theory about some kind of stuff, some of which is free, and some of which isn't [16]. The stuff usually consists of a family of states and/or processes, for example, in entanglement theory the non-free stuff would be entangled states and/or entangling operations, while local operations and classical communication which are the operations that do not increase entanglement, are freely available. Nielsen showed in [22] that bipartite entangled states give rise to a partially ordered set of equivalence classes of states, i.e. a *preordering* on the states, where the order expresses which states can be converted into other states by local operations and classical communication. Hence, this *LOCC-comparison relation* it captures the *relative value* of the resources: if a resource is above another one in the preordering, then one can obtain the later from the first one without any cost.

Here we start with a general theory of systems and processes, including states, and define the notion of a resource theory. The resulting general definition of a resource theory gives rise to a preordering, which moreover always comes together with a monotone composition operation. This composition operation captures an essential part of the structure of a resource theory, and any good measure of cost should respect it too, including in the case of LOCC-comparison. To prove this point we provide examples of fundamentally different resource theories which yield the same preorderings, and their difference is only revealed in terms of the composition operation. We also give examples of monotone cost functions that do not respect this composition operation, and expose their inadequacy.

This composition operation also allows to compare tuples of resources and yields the notion of a *conversion rate*, that is, the fraction of resources of one kind are required to produce resources of another kind. Composition dependent quantitative features of this kind are clearly also important for entanglement theory, e.g. entanglement distillation [2].

Within our conception of a resource theory, not only states but also processes can be resources. This accounts for situations where the use of a device that transforms states does not come for free, e.g. hiring a super computer to perform a computation, labour cost in construction work, energy cost of a cooker, etc. As a consequence, focussing back on quantum information theory, our framework also generates the resource inequalities of Devetak et al. in the context of quantum Shannon theory [12, 1]. Process transformations have also been considered elsewhere in quantum information, e.g. in [5].

2 Data of a resource theory

The definition of a resource theories builds on that of a process theory.

2.1 General process theories

Our starting point is a theory describing:

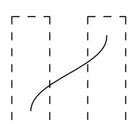
1. a collection of *systems* $A, B, \ldots$ *and processes* $f : A \to B, g : C \to D, \ldots$, where to $A \to B, C \to D, \ldots$ we refer as the *type* of the process.
2. which can be *composed in parallel* yielding $A \otimes B$ and $f \otimes g$,
3. and processes can also be *composed sequentially* when the input/output systems allow to do so, that is, in computer science terms, 'when the types match':
$$g \circ f := \Big(\mathrm{in}(f) \xrightarrow{f} \mathrm{out}(f) = \mathrm{in}(g) \xrightarrow{g} \mathrm{out}(g)\Big) \ ;$$
4. we moreover assume the existence of a *trivial system* I (cf. 'nothing'), and, for each system A, of an *identity process* $1_A : A \to A$ which doesn't alter the system.

This data is presented as a symmetric monoidal category (SMC) $(\mathbf{C}, \otimes, I)$ [10]. We will use the diagrammatic calculus of SMCs [24]. Processes of the type $I \to A$ are *states*.

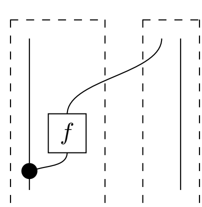
Here are some examples:

	systems	processes
Physics	physical systems	evolutions, measurements etc.
Programming	data types $\mathbb{B}$, $\mathbb{N}$, ...	programs
Chemistry	chemicals H_2, O_2, ...	chemical reaction
Cooking	ingredients carrot, potato, ...	boiling, spicing, mashing, etc.
Math. practice	propositions	proofs (cf. lemma, theorem)
Finance	e.g. currencies \$, €, ...	money transactions
Construction work	kinds of building materials, ...	acquiring materials, construction work

Example 1. In many cases one is interested in communication processes. One can take the objects of the category to be structured, where Alice and Bob now respectively refer to the 1st and the 2nd component of the tuple. Hence objects are of the form (A, B) with $A, B \in |\mathbf{C}|$, and their composition is componentwise, that is $(A, B) \otimes (A', B') := (A \otimes A', B \otimes B')$. Processes of type $(A, B) \to (A', B')$ can then be taken to be subsets of $\mathbf{C}(A \otimes B, A' \otimes B')$. For example, the process of Alice sending some data to Bob has type $(A, I) \to (I, A)$ and can be depicted as a morphism of $\mathbf{C}(A \otimes I, I \otimes A)$ as:



where the dotted boxes refer to the two agents. If Alice takes a copy of her data and then transforms one copy of her data by means of a process $A \to B$ before sending it to Bob, who already possesses data of type C, then the overall process has type $(A, C) \to (A, B \otimes C)$ and can be depicted in a diagram as:



where the dot depicts copying. More complex communication situations can be modelled similarly. If agents can posses two types of data, e.g. quantum and classical, then we can consider each component of the object again to consist of a pair, now referring to each of the two kinds of data. Operations could then transform data of one type into data of the other type, e.g. quantum measurement and quantum control operations.

2.2 Resource theories

Definition 1. A *resource theory* consists of an SMC $\mathbf{C}$ together with a proper all-objects-including sub-SMC of so-called *free processes*:

$$\mathbf{C}_{\text{free}} \overset{\neq}{\hookrightarrow} \mathbf{C}\,.$$

Any process which is not free is called a *resource*.

We write

$$\text{Sys} := |\mathbf{C}| \qquad \text{Proc} := \bigcup_{A,B\in|\mathbf{C}|} \mathbf{C}(A,B) \qquad \text{Proc}_{free} := \bigcup_{A,B\in|\mathbf{C}|} \mathbf{C}_{\text{free}}(A,B)$$

for the collection of all systems, processes, and free processes respectively.

Examples of already developed resource theories in quantum information are:

	free processes	resources
entanglement theory [17]	local operations, classical communication	entangled states, entangling processes
purity theory [15]	maximally mixed states, unitary operations	other states/operations
asymmetry theory [19]	symmetry-respecting operations	symmetry-breaking operations
athermality theory [4]	states in T-equilibrium, energy-conserving unitaries	athermal states, other operations

But one can consider new ones too:

	resources	free processes
mixedness theory	pure states, unitary operations	other states/operations

as well as examples outside physics:

	resources	free processes
DIY building	acquiring materials	construction work

What is considered as free depends on the situation. If one builds one's own house labor is for free and only materials have to be accounted for, and the same is true for cooking. But this is for example not the case for a building company and a restaurant.

We can now also consider examples presented in more direct mathematical terms:

	$\mathbf{C}$	$\mathbf{C}_{\text{free}}$
non-determinism	$\mathbf{Rel}_\emptyset$ (relations)	$\mathbf{Set}_\emptyset$ (functions)
irreversibility	$\mathbf{Set}_\emptyset$	$\mathbf{Perm}_\emptyset$ (permutations)
CQM quantum mixedness	$\text{CPM}(\mathbf{C})$ [23]	$\text{WP}(\mathbf{C})$ [8]
CQM classical mixedness	$\text{Stoch}(\mathbf{C})$ [11]	$\text{Func}(\mathbf{C})$ [11]

3 Parallel resources

We now initiate the study of the conversion of resources, that is, which resources can be 'produced' when others are 'consumed' in the presence of an unlimited supply of the free processes. In particular, we identify the foundational mathematical structure that captures conversion of resources, namely, the structure of a *non-negative preordered monoid*.

For reasons of clarity, we first consider a special kind of resource that can only be consumed in parallel, that is, resources $f_1, \ldots, f_n$ can only be consumed as the single (compound) resource $f' := f_1 \otimes \ldots \otimes f_n$. This assumption will provide a stepping stone to the general more complex case, which we discuss in Section 5. But parallel resources do already capture one very important example of resources, namely *resource states*, which due to the nature of their type can only be composed in parallel.

3.1 Comparing parallel resources

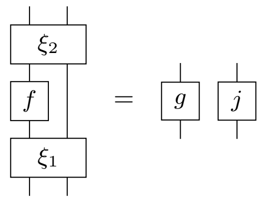
Definition 2. For $f, g \in \mathrm{Proc}$ we set:

$$f \succeq g$$

whenever

$$\exists Z \in \mathrm{Sys}, \xi_1, \xi_2 \in \mathrm{Proc}_{free}, j \in \mathrm{Proc} : \xi_2 \circ (f \otimes 1_Z) \circ \xi_1 = g \otimes j \ . \tag{1}$$

In string diagrams, the equality in (1) becomes:



(2)

We refer to Z as the *ancilla*, that is, the additional system space required while processing the resource, to ξ_1 as *pre-processing*, to ξ_2 as *post-processing*, and to j as *junk*. Given a resource f, when composing it with free processes, either in parallel or sequentially, then we always obtain an expression of the form of the LHS of (2).

Lemma 3. *Any expression in the language of symmetric monoidal categories that includes a single occurrence of some morphism f can be put in the form of the LHS of (2).*

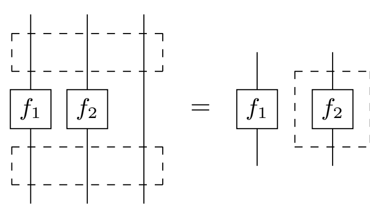
The presence of the junk process j allows for some morphism f not to be entirely consumed in the production of g. For example, for $f = f_1 \otimes f_2$ we may have:

$$\xi_2 \circ ((f_1 \otimes f_2) \otimes 1_Z) \circ \xi_1 = g \otimes (f_2 \otimes j') \ , \tag{3}$$

where f_2 is not consumed and reappears as part of the junk $j = f_2 \otimes j'$. More importantly, when taking $\xi_1 = 1$ and $\xi_2 = 1$ we have:

$$f_1 \otimes f_2 \succeq f_1 \tag{4}$$

i.e. 'the whole is more costly than its parts'. In a diagram (2) is realised as follows:



(5)

where the two dotted boxes in the LHS represent ξ_1 and ξ_2 and the one in the RHS represents j. We will use this manner of establishing that (2) is realised throughout this paper.

The presence of a junk process in RHS implicitly assumes that resources are *disposable*, that is, not consuming a resource doesn't come at any cost. This immediately excludes 'negative resources' such as financial debt, as well as 'undesirable resources', e.g. nuclear waste. One can deal with financial debt or nuclear waste in the present formalism by treating the processes of disposing of them as a proper resource.

In Section 3.6 we show that if certain conditions are met the junk process becomes redundant in Definition 2. One of these conditions has a physical origin while the other one puts a non-emptiness constraint on the of free states of any type.

Remark 1. If the SMC of free processes contains zero-processes, that is, processes $0 \in \mathrm{Proc}$ which are such that $f \otimes 0 = g \otimes 0$ for all pairs of processes $f, g \in \mathrm{Proc}$ of the same type, then (2) can always be trivially satisfied, by including such a zero-process within ξ_- and in j. Therefore, when constructing mathematical examples based on standard categories such as $\mathbf{Rel}$, one may have to exclude these zero-processes from the candidate junk processes. On the other hand, for real life resource theories zero-processes can never take place, as these represent the 'the impossible'.

For resource theories, it should be intuitively clear that, for example, if $f_1 \succeq g_1$ and $f_2 \succeq g_2$, then $f_1 \otimes f_2 \succeq g_1 \otimes g_2$. The following definition captures the nature of the $\succeq$-relationships for general resource theories.

Definition 4. A *preordered monoid* is a structure

$$(M , \succeq , \cdot , 1)$$

where $\succeq \subseteq M \times M$ is a reflexive transitive relation, $\cdot : M \times M \to M$ is associative and has unit 1 up to the equivalence relation induced by the preordering, that is,

$$\forall x, y, z \in M : (x \cdot y) \cdot z \simeq x \cdot (y \cdot z) \quad \text{and} \quad \forall x \in M : 1 \cdot x \simeq x \simeq x \cdot 1 \tag{6}$$

where $x \simeq y$ means $x \succeq y$ and $y \succeq x$. We require these structures to interact as follows:

$$a \succeq b \ , \ x \succeq y \ \Rightarrow \ a \cdot x \succeq b \cdot y \ . \tag{7}$$

Note that the unit of a preordered monoid need not be unique. Each preordered monoid induces a partially ordered monoid by quotienting relative to the equivalence relation:

$$(M/\simeq \ , \ \geq \ , \ \cdot \ , [1] \ )$$

where for equivalence classes $[x]$ and $[y]$ we have:

$$[x] \geq [y] \ \Leftrightarrow \ x \succeq y \ .$$

Now the unit is of course unique. We call this the *induced partial ordering*.

A preordered monoid can equivalently be defined as a small thin monoidal category (MC) $\mathbf{C}$, where $(\otimes, I)$ provides $(\cdot, 1)$ and 'non-emptyness of hom-sets' provides $\succeq$, that is,

$$A \succeq B \ \ \Leftrightarrow \ \ \mathbf{C}(A, B) \neq \emptyset \ . \tag{8}$$

Bifunctoriality then exactly yields (7). In fact, to any MC $\mathbf{C}$ we can associate a preordered monoid, namely, the so-called 'decategorification' of $\mathbf{C}$. If this MC is moreover symmetric then the monoid is commutative up to $\simeq$, that is,

$$\forall x, y \in M : x \cdot y \simeq y \cdot x \ . \tag{9}$$

We call such a preordered monoid also *symmetric*.

Definition 5. By the *decategorification* $\mathrm{DEC}(\mathbf{D})$ of a symmetric monoidal category $\mathbf{D}$ we mean the symmetric preordered monoid that has $|\mathbf{D}|$ as its elements, and such that for $A, B \in |\mathbf{D}|$ we have $A \succeq B$ if and only if $\mathbf{D}(A, B)$ is non-empty.

A preorder $(M, \succeq)$ can have several *bottom* elements, i.e. $\perp \in M$ such that

$$\forall x \in M : x \succeq \perp \ . \tag{10}$$

We call a preordered monoid $(M, \succeq, \cdot, 1)$ *non-negative* if 1 is a bottom. Then we have:

$$\forall x, a \in M : x \cdot a \succeq a \ , \tag{11}$$

that is, the operation $x \cdot - : M \to M$ is order increasing for any x.

By f_{free} we will mean 'any free process'.

Theorem 6. $(\mathrm{Proc}, \succeq, \otimes, f_{\mathrm{free}})$ *is a non-negative symmetric preordered monoid.*

The fact that 'the whole is more costly than the parts' (cf. (4)) yields:

Corollary 7. *In* $(\mathrm{Proc}, \succeq, \otimes, f_{\mathrm{free}})$ *the monoid multiplication is an upper bound, that is:*

$$\forall f, g \in \mathrm{Proc} : f \otimes g \succeq f \ .$$

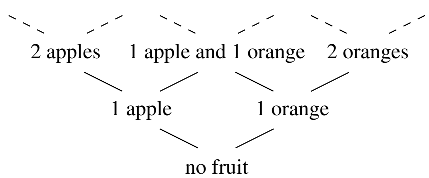
Example 2. Two important examples in the mathematics literature which involve an ordering that interacts with a monoid structure, where in both cases the ordering is a (complete) lattice, are lattice ordered groups [3], where the monoid admits inverses, and the commutative case of quantales [21], where there is a distributive law between the monoid multiplication and the suprema of the ordering. The inverses of lattice ordered groups contradict Corollary 7, since as already mentioned in Section 3.1, the presence of the junk process in (2) implicitly assumes resources to be 'non-negative'. Also the distributivity law for quantales does not seem to admit an obvious interpretation from the point of view of resource theories. Hence, the study of preordered monoids arising from resource theories seem to warrant a mathematical study in their own right.

Example 3. The preordering of [22] which underpins entanglement theory is an instance of the preordering of Theorem 6. One considers bipartite communication protocols as discussed in Example 1. The resources are entangled bipartite (pure) states. As composition was not accounted for in [22], one considers $\succeq$ only for single resources, that is, single entangled states. The free operations are generated by disentangled states, local operations, and classical communication between the parties, and then closed under sequential composition. Given a resource state Ψ, let $(r_1, \ldots, r_n)$ be the squares of the Schmidt coefficients in descending order. The main result established in [22] is that $\Psi \succeq \Psi'$ if and only iff $(r_1, \ldots, r_n)$ and $(r'_1, \ldots, r'_n)$ compare with respect to majorization order [20], that is, $r_1 \leq r'_1, r_1 + r_2 \leq r'_1 + r'_2, \ldots, \sum_i r_i \leq \sum_i r'_i$. This ordering does not account for converting a tuple of n entangled states into m entangled states although clearly this is very relevant to entanglement theory (e.g. entanglement distillation [2]).

3.2 Essence of the monoid structure

We provide an example of two fundamentally different resource theories with the same preorders. Their difference becomes apparent in terms of the difference of the corresponding monoid structures. This proves that the monoid structure exposes crucial features of resource theories that are invisible for the preordering.

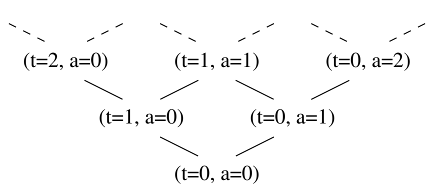
Example 4. Consider the resource theory $\mathbf{Fruit}$ which has multisets of fruit as its objects, e.g. '3 apples and 4 oranges', each of which only admits one state which we can interpret as obtaining that fruit e.g. 'buying 3 apples and 4 oranges'. The only morphisms of any other type are the identities, which are also the only free processes, and which we interpret as 'doing nothing to the fruit'. Hence, there are no non-trivial sequential compositions. Parallel composition of systems and states is multiset union, e.g. composing '3 apples and 4 oranges' and '2 apples and 3 pears' yields '5 apples, 4 oranges and 3 pears'. By Theorem 6 we know that the bottom elements of the preordering are the free processes, that is, the identities. The resources include the non-empty multisets of fruits, as well as compositions of these with identities. Since the only free processes are identities, $\succeq$-relationships can only be established by equations of the form (5). Each equivalence class contains exactly one state, and for these $\succeq$ is given by multiset reverse inclusion (cf. Corollary 7), that is, 'cost is proportional to the amount of fruit acquired' subject to 'apples and oranges don't compare'. Restricting to only apples and oranges, the Hasse diagram of the induced partial order looks as follows:



The monoid multiplication is componentwise addition.

Remark 2. If we would also consider the processes 'eating fruit' as a free processes, then this would still yield the same preordered monoid. When using these free processes in (2) this would simply lead to fruit 'being eaten in LHS', rather than being considered as junk in RHS, with the same resulting $\succeq$-relationships.

Example 5. Consider the resource theory $\mathbf{Quality}$ which only has one object I. There are two kinds of processes of type $I \to I$, namely, (i) the resources, which improve the quality of a certain product (e.g. maturing of whiskey) and we assume these to be totally ordered, and, (ii) the free processes, which cause decay in quality (e.g. deluting whiskey). The tensor 'aims at maximal quality', that is, $p \otimes q$ for p, q resources will be either p or q depending on which one improves the quality the most, $f \otimes g$ for f, g free processes will be either f or g depending on which one reduces the quality the least, and $p \otimes f$ will be p. Since the only manner in which non-equivalent $\succeq$-relationships arise is by reducing quality, i.e. by taking ξ_2 in (2) to be a quality decay process, the partial ordering induced by $\succeq$ is the total ordering representing quality levels that we started with. The monoid multiplication is the order-theoretic supremum. When considering a product that has two quality features, each of which being modelled as above, with the corresponding decay processes now specifying a decay level for each of these features, then the Hasse diagram of the induced partial order looks as follows (t := taste, a:= alcohol percentage):



The monoid multiplication is still the order-theoretic supremum.

So in both examples we obtain the same induced partial ordering! The radically different nature of these resource theories is captured by the distinct composition structures. While for $\mathbf{Quality}$ the composition is the supremum, this is not the case for $\mathbf{Fruit}$:

$$1 \text{ apple} \vee 1 \text{ apple} = 1 \text{ apple} \neq 2 \text{ apples} = 1 \text{ apple} + 1 \text{ apple}$$

where the composition is the sum. More precisely, the induced partial ordering is:

$$(\mathbb{N}, \geq) \times (\mathbb{N}, \geq)$$

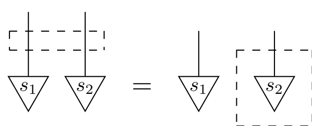
both for $\mathbf{Fruit}$ and $\mathbf{Quality}$, but the monoids respectively are:

$$(\mathbb{N}, +, 0) \times (\mathbb{N}, +, 0) \qquad \text{vs.} \qquad (\mathbb{N}, \vee, 0) \times (\mathbb{N}, \vee, 0)\ .$$

3.3 Origin of comparison relationships

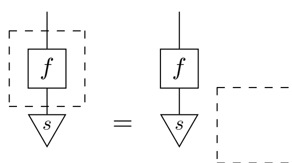
The examples $\mathbf{Fruit}$ and $\mathbf{Quality}$ also provide insights in the kinds of $\succeq$-relationships that may occur. The SMC of the resource theory $\mathbf{Fruit}$ is a commutative monoid where the composition is to be interpreted as parallel, to which a trivial sequential composition operation is freely adjoined. The SMC of the resource theory $\mathbf{Quality}$ is also a commutative monoid but where the composition is now to be interpreted as sequential, to which a trivial idempotent parallel composition relation is adjoined. In order-theoretic terms these are suprema, i.e. a property rather than a genuine structure. So these examples can be seen as 'mutually orthogonal 1D projections' of the 2D composition structure of an SMC.

As a consequence, these two monoids play a very different role with respect to (2), i.e. in establishing $\succeq$-relationships. As already mentioned above, for $\mathbf{Fruit}$ the $\succeq$-relationships arise only via (5), which since all resources are states becomes:



(12)

For $\mathbf{Quality}$ on the other hand, $\succeq$-relationships arise only via:



(13)

Note also that while for $\mathbf{Fruit}$ the operation $s \otimes -$ causes increase in the ordering, for $\mathbf{Quality}$ the operation $f \circ -$ causes decrease in the ordering.

Definition 8. The $\succeq$-relation is called *purely parallel* when:

$$\forall f \in \mathrm{Proc}, Z \in \mathrm{Sys}, \xi_1, \xi_2 \in \mathrm{Proc}_{free} : \xi_2 \circ (f \otimes 1_Z) \circ \xi_1 \succeq f\ , \tag{14}$$

and it is called *purely sequential* when:

$$\forall f, g \in \mathrm{Proc} : f \succeq f \otimes g \tag{15}$$

In a purely sequential resource theory one can think of resources as all being *zero*. The complementary notion is that of a resource theory where all the resources are positive.

Definition 9. We call a resource theory *positive* whenever for $f, g \in \mathrm{Proc}$:

$$f \succeq f \otimes g \Rightarrow g \in \mathrm{Proc}_{free}\ .$$

Positive resource theories typically exhibit the following features:

- **Extensivity.** This features was already clear in the contrast between the above two examples. While 'quantity' (cf. $\mathbf{Fruit}$) is an extensive attribute like for example 'volume' and 'weight', 'taste' and 'alcohol percentage' (cf. (t) and (a) in $\mathbf{Quality}$) are intensive ones like for example 'temperature'.

- **Finitarity.** If a resource is infinite, for example the number of rooms available in Hilbert's hotel, then one can produce clones of it without any cost, something which is forbidden by the following result.

3.4 No-cloning for positive resource theories

Theorem 10. *In positive resources theories, resources cannot be 'cloned':*

$$\forall f \in \mathrm{Proc} \setminus \mathrm{Proc}_{free} : f \not\succeq f \otimes f\ .$$

Remark 3. Note here the resemblance to *linear logic* [14] where premises also cannot be cloned. Linear logic is often referred to as a 'resource sensitive logic', or in Girard's own words: "While classical logic is about truth, linear logic is about food".

3.5 Resources as objects and resource conversion as morphisms

The ordering in $(\mathrm{Proc}, \succeq, \otimes, f_{\mathrm{free}})$ represents 'increase in resource' while the monoid multiplication represents 'combining resource', and as already indicated in the previous section, these relate to the 2D composition structure of the underlying SMCs. But this analogy is by no means a perfect match, since, as demonstrated in the previous section, the $\succeq$-relationships can arise both from sequential and parallel composition (cf. Definition 8).

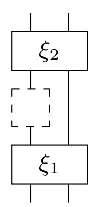
Obtaining a perfect match would mean that the preordered monoid $(\mathrm{Proc}, \succeq, \otimes, f_{\mathrm{free}})$ is the *decategorification* of the SMC (see Definition 5). We can realise such a perfect match by constructing a new SMC from the pair $(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$ with resources and free processes as objects, and with conversions thereof as morphisms.

This 'alternative presentation' of a resource theory will also be interesting in its own right. For example, for $\mathbf{Quality}$ it would be a natural choice to take the products as objects and their decay as morphisms. Also for $\mathbf{Fruit}$ we could have taken fruit as objects and eating fruit as morphisms. A category $\mathbf{Cooking}$ would have culinary ingredients as its objects and cooking receipts as its morphisms. In these examples the resources are states but the alternative presentation can also be achieved when resources are proper processes.

Given a resource theory $(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$ we construct a new SMC $\mathrm{RC}(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$ with Proc as its objects and the morphisms of type $f \to g$ being triples:

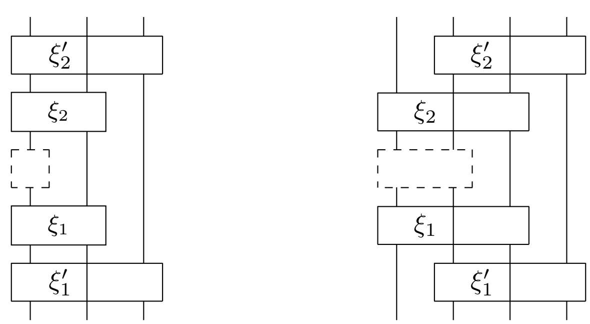
$$\left\{(Z, \xi_1, \xi_2) \in \mathrm{Sys} \times \mathrm{Proc}_{free} \times \mathrm{Proc}_{free} \;\middle|\; (2) \text{ holds for some } j\right\} \tag{16}$$

When representing these the triples (Z, ξ_1, ξ_2) in the following manner:

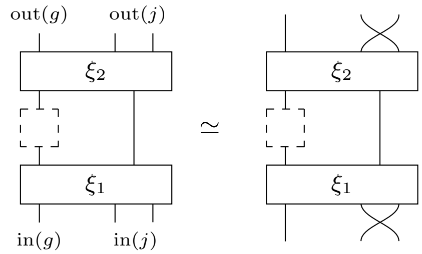


(17)

where one should think of the 'hole' indicated by the dotted line as allowing to insert a process of the appropriate type, sequential and parallel composition are as follows:



Finally, we impose a congruence on morphisms, reflecting the fact that junk is invariant under symmetry. This is a technical requirement that guaranties naturality of symmetry for $\mathrm{RC}(\mathbf{C}, \mathbf{C}_{\mathrm{free}})$. Concretely, we will treat the following morphisms as equivalent:



(18)

Theorem 11. *For all resource theories* $\mathrm{RC}(\mathbf{C}, \mathbf{C}_{\text{free}})$ *is an SMC and:*

$$\mathrm{DEC}(\mathrm{RC}(\mathbf{C}, \mathbf{C}_{\text{free}})) = (\mathrm{Proc}, \succeq, \otimes, f_{\text{free}}) \; . \tag{19}$$

So we established that a resource theory formulated as an inclusion of process theories admits an alternative presentation as a single SMC where processes are the objects and resource conversions are the morphisms. One may now also ask which additional properties an arbitrary SMC $\mathbf{D}$ must have for its objects to be interpreted as resources and its morphisms to be interpreted as resource conversions.

Lemma 12. $\mathrm{DEC}(\mathbf{D})$ *is non-negative whenever* $\mathbf{D}(A, I) \neq \emptyset$ *for all* $A \in |\mathbf{D}|$.

The morphisms of type $A \to I$ can be interpreted as disposing of A. On the other hand, $A \in |\mathbf{D}|$ is to be interpreted as free if there also exist a morphism of type $I \to A$. So dually, if for $A \in |\mathbf{D}|$ we have that $\mathbf{D}(I, A) = \emptyset$, then A is a resource.

3.6 Causality and free execution of processes

Definition 13. A process $f : A \to B$ in a resource theory is *freely executable* if there exists a free state $x : I \to A$. A a resource theory $(\mathbf{C}, \mathbf{C}_{\text{free}})$ admits *free execution of processes* if for every $A \in |\mathbf{C}|$ there exists a free state $x : I \to A$.

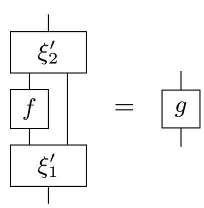
While a process may be free, its execution may not be whenever providing an input necessarily comes at a cost. Therefore, in (2) we may want to restrict the junk processes to freely executable ones. It may indeed be the case that the execution of the simulation of f as in (2) requires an input state at both inputs of ξ_1, and hence, since there should be no cost associated whatsoever with the junk process, that j must be freely executable.

Definition 14. A process theory $\mathbf{C}$ is *causal* if I is terminal, that is, if for $A \in |\mathbf{C}|$ there is a uniques process d of type $A \to I$ that we call *universal disposing*. A resource theory $(\mathbf{C}, \mathbf{C}_{\text{free}})$ is *freely causal* if $\mathbf{C}$ is causal and the universal disposing processes are free.

Causality in this form was derived as a principle in [6] and studied in the context of process theories as SMCs in [9]. Within a spatial context it is a principle that prevents signalling [9], and plays a crucial role in the reconstruction of quantum theory [7].

Under the reasonably mild assumptions of free execution of processes and free causality we can drop the presence of the junk processes in (2).

Lemma 15. *If a resource theory admits free execution of processes and is freely causal, then obeying (2) for some* $\xi_1, \xi_2 \in \mathrm{Proc}_{free}$ *and* $j \in \mathrm{Proc}$ *is equivalent to obeying:*



(20)

for some $\xi_1', \xi_2' \in \mathrm{Proc}_{free}$.

4 Quantitative concepts for resource theories

We present three notions that are common to general resource theories and which all crucially rely on the composition structure of the preordered monoid.

4.1 Cost measures

The non-negative reals $\mathbb{R}^+$ are ordered and admit several commutative monoid structures that make it into a preordered monoid, e.g. $\vee$ and $+$. Cost measures for resource theories may want to respect either of these, depending on the nature of the resource theory.

For example, for **Fruit** any natural measure would be a homomorphism of preordered monoids into $(\mathbb{R}^+, \geq, +, 0)$. Ignoring the monoid structure and simply requiring monotonicity would result in inadequate measures. For example, any $\vee$-preserving map into $(\mathbb{R}^+, \geq)$ would mean that the cost for n apples is the same as the cost for a single apple. On the other hand, for a homomorphism of preordered monoids into $(\mathbb{R}^+, \geq, +, 0)$ it suffices to assign a price for each individual piece of fruit since then preservation of the monoid multiplication fixes the cost of any multiset of fruit.

Definition 16. An *additive cost measure* is a homomorphism of preordered monoids:

$$\mu : (\mathrm{Proc}, \succeq, \otimes, f_{\mathrm{free}}) \to (\mathbb{R}^+, \geq, +, 0)$$

4.2 Conversion rate

Por a process $f \in \mathrm{Proc}$ let $n \cdot f := \overbrace{f \otimes \ldots \otimes f}^{n}$.

Definition 17. For two processes $f, g \in \mathrm{Proc}$ of a resource theory the *conversion rate* is:

$$R(f \to g) := \bigvee \left\{ \frac{m}{n} \,\middle|\, n \cdot f \succeq m \cdot g, n, m \in \mathbb{N} \right\} . \tag{21}$$

4.3 Catalysis

Definition 18. For processes $f, g \in \mathrm{Proc}$ of a resource theory we set:

$$f \succeq_{cat} g$$

if there exist a *catalyst* $c \in \mathrm{Proc}$ such that:

$$f \otimes c \succeq g \otimes c$$

Theorem 19. $(\mathrm{Proc}, \succeq_{cat}, \otimes, f_{\mathrm{free}})$ *is a non-negative symmetric preordered monoid.*

We can also consider cost measures and conversion rates for catalysed processes when replacing $\succeq$ by $\succeq_{cat}$ in Definition 16 and Definition 17 above.

In particular, if μ is an additive cost measure, then it will also be an additive cost measure with respect to the catalytic ordering since $f \otimes c \succeq g \otimes c$ implies $\mu(f) + \mu(c) \succeq \mu(g) + \mu(c)$, and hence that $\mu(f) \succeq \mu(g)$, and additivity is trivialy inherited.

5 Sequentially composable resources

For a pair of resources (f_1, f_2) we may have:

$$f_1 \circ f_2 \succeq g \tag{22}$$

while

$$f_1 \otimes f_2 \not\succeq g , \tag{23}$$

for example, if $f_1 = f_2$ are $\pi/2$ rotations and g is a π rotation.

On the other hand, a pair of resources that arises in parallel as $f_1 \otimes f_2$ can in general not be consumed as in (22). This will force us to distinguish between resources that can only be used in parallel, which were thus far denoted as the monolithic process:

$$f_1 \otimes \ldots \otimes f_n$$

versus those that may also be used sequentially. We denote these as a list:

$$(f_1, \ldots, f_n) ,$$

although in fact, they form a multiset, so the order in the sequence doesn't matter.

We use $+$ to denote list concatenation, ϵ to denote the empty list and Proc^* to denote all lists (representing multisets) of processes contained in Proc, including the empty list ϵ. By $S(n)$ we denote the symmetric group of permutations on n elements.

Definition 20. For $F = (f_1, \ldots, f_n) \in \mathrm{Proc}^*$ and $g \in \mathrm{Proc}$ we set:

$$F \succeq g$$

whenever

$$\exists \sigma \in S(n), Z_1, \ldots, Z_n \in \mathrm{Sys}, \xi_0, \ldots, \xi_n \in \mathrm{Proc}_{free}, j \in \mathrm{Proc} : \\ \xi_n \circ (f_{\sigma(n)} \otimes 1_{Z_n}) \circ \xi_{n-1} \circ \ldots \circ \xi_1 \circ (f_{\sigma(1)} \otimes 1_{Z_1}) \circ \xi_0 = g \otimes j \ . \tag{24}$$

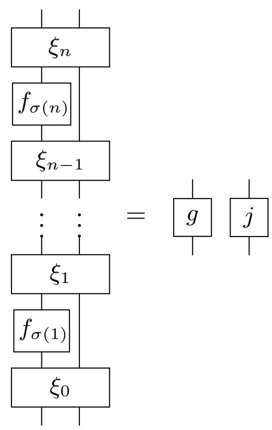
For $F = (f_1, \ldots, f_n), G = (g_1, \ldots, g_m) \in \mathrm{Proc}^*$ we set

$$F \succeq G$$

whenever there exists a partition of F into lists $F_1, \ldots, F_m$ such that:

$$F_1 \succeq g_1 \quad \ldots \quad F_m \succeq g_m \ .$$

In string diagrams the equality in (24) becomes:



(25)

Theorem 21. $(\mathrm{Proc}^*, \succeq, +, \epsilon)$ *is a non-negative symmetric preordered monoid.*

6 Closing

Having established non-negative preordered monoids as a fundamental structure for resource theories, from which notions such as cost, rate and catalysis can be derived, one can now initiate a study of resource theories starting from this structure. We already made substantial progress in this direction, e.g. [13] where several kinds of resources are classified in terms of the preordered monoid structure, and connections between these properties are established. Also a notion of approximate conversion of resources has been introduced, based on *epsilonification* within the context of topological vector spaces.

By no means do we consider Definitions 1 and 2 to be the final word on what a ‘general resource theory’ should be, but the generality of these definitions is surely sufficient to establish the study of preordered monoids within the context of resources as a worthwhile endeavour, and encompasses the existing examples of resource theories as well as many new ones.

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