Structure, Logicality and Sense in Quantum Theory

STILL DRAFT VERSION — FINAL VERSION WILL BE POSTED

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Abstract

In this course, rather than giving a brief standard textbook account on quantum theory, besides a historical account and the jargon that enables access to literature (references will be provided), we will provide "easy to grasp metaphors", have attention for order-theoretic, operational and logical issues, structural theorems and in particular aim towards a focus on structural similarities of current research in the foundations of quantum theory and the foundations of computer science. We stress that by the latter we do not (only) refer to quantum computation in its current incarnation as a research field.

Scope: This is not a standard quantum theory course, since it is pushed towards mathematical structure and even, logicality, and since it is also pushed towards "understanding", at least in the sense that "images are provided", either operational images, metaphorical images, classical mechanical representations or representations in different mathematical categories. The questions that are asked are about the particular objects emerging from quantum theory and the sense of these mathematical objects. The first section approaches a general audience. The second has a rather mathematical focus. The third section directs towards logic, and exposes current research. Selection of topics has been strongly motivated by possible connections with the foundations of computer science, what also justifies the epilogue "physically approaching computer science". In the second and third section we assume some knowledge of order-theory and logic. In the first section of basic linear algebra of vector spaces. Category theory is (in principle) not a preliminary. A particular emphasis has been made on discrete structures and classical representation. The role of physical space has been given a less primal position than in many textbooks. Thermodynamical notions including energy are absent and so are most considerations on unitary dynamics.

1 The orthodox quantum kitchen

In this section we discuss orthodox quantum mechanics, how it is supposed to be applied, and what, at a first glimpse, are its implications concerning "quantum physical reality". As a standard textbook, we propose Isham (1995). Have no doubt about the fact that von Neumann (1932), which fixed the mathematical and theoretical framework of quantum theory for once and for all, is still of great value.

1.1 Historical account

The historical development in itself tells the story on why quantum theory, compared to for example relativity theory (both special and general) is kind of a mess, in such a way that people teach it as a bunch of recipes — what is strange for a fundamental theory of physics. We briefly state this argument: It was not just one idea that carried the new theory but a lot of them, brought in by many different people, this during a relatively long period of time, leading to a bunch of rules not everybody was happy with — this is probably an understatement. Ultra brief historical development — for more details, see for example Kleppner and Jackiw (2000):

- M. Planck (1900): Radiation happens in quanta ; Planck himself considered this as totally absurd:

  "an act of desparation" ;

  what they tried to understand was the so-called 'ultra-violet catastrophe' in the spectrum of light emitted by hot bodies — one of the at that time "three remaining problems to solve in physics (and then the story of physics would have been finished)", one of the other two led to relativity theory.

- A. Einstein (1905): Light itself is quantified in a way depending on its frequency, moreover, light behaves like particles, and, has momentum.
- N. Bohr (1913): The quantum atom model.
- L. DeBroglie (1923): Particles behave like waves, i.e. they have a wave length.
- W. Pauli, E. Schrödinger, W. Heisenberg, J. von Neumann, e.a. (1925-1928): Full quantum theory.

We recall A. Einstein's comment:

> "God does not play dice!"

It is in the light of this uncomfortable feeling of Einstein with quantum theory that the famous so-called EPR-paper was written formulating the Einstein–Podolsky–Rosen paradox (Einstein, Podolsky and Rosen 1935 ; Bell 1964) , resulting in their slogan "Quantum theory is an incomplete theory" , what still is a point of disagreement — see also below. We now quote W. Heisenberg:

> "Even for the physicist is the fact that he is able to communicate his knowledge in an ordinary language to others the best criterion for the degree of understanding that he himself has obtained."

Unfortunately, it are exactly the contributions to science by Heisenberg himself that yielded some kind of crisis in physics in that the tendency of the scientific community to even try to understand the physical notions of quantum theory in any (not even deep) way seemed to vanish in terms of what could be called the "no-interpretation interpretation of quantum theory". We now situate quantum theory within the field of theoretical physics, naively put:

$$\begin{tikzcd}
& \mathrm{QG} & \\
\mathrm{GR} \arrow[ur, dash] & & \mathrm{QFT} \arrow[ul, dash] \\
\mathrm{SR} \arrow[u, dash] & & \mathrm{QT} \arrow[u, dash] \\
& * \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

QG:= Quantum gravity, GR:= General relativity, SR:= Special relativity, QFT:= Quantum field theory, QT:= Quantum theory. The structure of this diagram clearly shows that a true understanding of quantum theory should be an ingredient in all what follows, i.e. quantum field theory and the construction of a theory for quantum gravity — this is however in general not the case in current research programs in the sense that the "no-interpretation interpretation of quantum theory" is strongly adopted. In particular, even in absence of any philosophical considerations, the "no-interpretation interpretation of quantum theory" makes the theory difficult to understand in the sense that no "real models" are provided. The attitude of the "quantum structures community" is to look for models or equivalent representations in different mathematical categories.

1.2 Its recipes: Hilbert's cookery-book

Basic quantum theory is actually a bunch of instructions, with not a lot a priori significance attached to it. We give these rules. The original formulations took their form in either wave mechanics or matrix mechanics, both of them ending up to be an incarnation of the same thing, Hilbert space quantum mechanics. Although in many introductory textbooks only wave mechanics is presented, and in others first wave mechanics and later the abstract Hilbert space version, we will start with Hilbert space version and present wave mechanics as a spatial incarnation, i.e. the quantum theory of physical space — we stress here that physical space is not an a priori in the Hilbert space version. In the textbook Isham (1995) wave mechanics is only briefly discussed as an introduction to the Hilbert space version. In the monograph Piron (1976) a completely different attitude is adopted. Starting from primitive physical notions and down to earth principles, quantum theory is reconstructed, starting within lattice theory and appropriate axioms, going to Hilbert space via the representation theorem of projective geometry and obtaining the probabilistic structure via Gleason's theorem (Gleason 1957). In the monograph Jauch (1968) the attitude is somewhat in-between, but it could serve as a step-stone to Piron (1976), which is harder. Piron (1976) can be seen as the realisation of the program launched in the monograph Mackey (1963) which also still has some interesting features. In the context of these reconstructional programs we also refer to the introduction of the volume [?].

Let $\mathcal{H}$ be a (complex) Hilbert space, i.e.

- A (complex) vector space $(\mathcal{H}, \underline{o}, +, \lambda. - | \lambda \in \mathbb{C})$, that is additionally an inner product space, so there exists $\langle - | - \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that satisfies

$$\langle \psi | \lambda_1 \phi_1 + \lambda_2 \phi_2 \rangle = \lambda_1 \langle \psi | \phi_1 \rangle + \lambda_2 \langle \psi | \phi_2 \rangle \tag{1}$$

$$\langle \lambda_1 \phi_1 + \lambda_2 \phi_2 | \psi \rangle = \overline{\lambda}_1 \langle \phi_1 | \psi \rangle + \overline{\lambda}_2 \langle \phi_2 | \psi \rangle \tag{2}$$

$$\langle \psi | \psi \rangle \geq 0 \; with \; \langle \psi | \psi \rangle = 0 \Leftrightarrow \psi = \underline{o} \tag{3}$$

$$\langle \phi | \psi \rangle = \overline{\langle \psi | \phi \rangle}\,, \tag{4}$$

  which is complete with respect to strong convergence of Cauchy sequences and has a countable base.

Elements of $\mathcal{H}$ are in the context of quantum theory indeed frequently denoted by ψ and ϕ. For many purposes in terms of insights, particular quantum features and even structural results, we can and will forget about the infinite dimensional aspects in this definition, i.e. we will restrict ourselves to finite dimensional Hilbert spaces. In some cases we will even assume that the Hilbert space is real instead of complex.

We then define the following notions that constitute the core of quantum theory (of a quantum system):

- *States* are (represented by) 'rays' in $\mathcal{H}$, i.e. one dimensional subspaces, denoted by some $\psi \in ray(\psi)$.
- *Evolutions* are 'unitary (linear) operators' $U : \mathcal{H} \to \mathcal{H}$, i.e. $\exists U^{-1}$ and $\langle U\psi | U\phi \rangle = \langle \psi | \phi \rangle$.
- *Measurements* are 'self-adjoint (linear) operators' $H : \mathcal{H} \to \mathcal{H}$, that is $\forall \psi, \phi : \langle \psi | H\phi \rangle = \langle H\psi | \phi \rangle$.

These are the key ingredients of the initiating definition of Hilbert space quantum mechanics. Crucial is the fact that there are, separately described, *Measurements* and *Evolutions*. Recall here from linear algebra (in order to provide some initial intuition) that:

i. Unitary transformations are the structure preserving isomorphisms in an inner product space (preserve linearity and also the angles following from the inner product); They as such preserve the "structure(al description) of the system".

ii. Self-adjoint operators, or, in the particular case of a finite dimensional vector space over the reals, symmetric operators, "produce eigenvalues" $a \in \sigma(H)$ via the equation $H\psi_a = a\psi_a$ for eigenvectors ψ_a; The eigenvalues should be thought of as values of physical quantities, the eigenvectors as "eigenstates", i.e. those states who yield the corresponding value as outcome in a measurement.

We have the following extra rules with respect to these three notions, state, measurement and evolution:

- If two quantum systems are involved, we describe their states in the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ in which we would describe the individual systems. However obvious this might seem to be mathematically (from a categorical perspective — we work in the category of Hilbert spaces), this is the source of many of the as "mysterious" considered quantum phenomena.
- Given a (classical) Hamiltonian then evolution of the system is given by the Schrödinger equation

$$i\bar{h}\partial_t \psi = H\psi \tag{5}$$

  where H is a self-adjoint operator, the corresponding quantum Hamiltonian. It actually represents the 'measurement of the system's energy'. We obtain (Stone's theorem)

$$U(t) = e^{-\frac{i}{\hbar}Ht}\,. \tag{6}$$

  The construction of the quantum Hamiltonian is called quantisation. It is however just an instance of the general quantisation/quantification that happens when constructing the self-adjoint operator for any measurement, e.g. position or momentum. Unitary evolution will not play an important role in our further presentation! (since it just produces isomorphisms)

- When performing a measurement on the system in state ψ where the corresponding self-adjoint operator has $\sigma(H)$ as its spectrum of eigenvalues (which for formal simplicity we assume at this point to be discrete — for a general rigorous version, see spectral decomposition further in this text) then we obtain as outcome some $a \in \sigma(H)$ with probability:

$$Prob_\psi^H(a) = \langle\psi|P_a\psi\rangle \ (= \langle P_a\psi|P_a\psi\rangle = |P_a\psi|^2) \tag{7}$$

  where $P_a : \mathcal{H} \to \mathcal{H}$ is the (orthogonal) projector on the subspace of eigenvectors with eigenvalue a and ψ is normalised. Note here that $P_a\psi$ denotes an eigenstate with eigenvalue a. Note also that these probabilities indeed add up to one when considering all possible outcomes since $\sum_{a\in\sigma(H)} P_a\psi$ denotes the decomposition of ψ over the different eigenspaces corresponding to the eigenvalues, and, the $P_a\psi$ are mutually orthogonal, so $\sum_{a\in\sigma(H)} |P_a\psi|^2 = |\sum_{a\in\sigma(H)} P_a\psi|^2 = 1$.

This should now enable you to predict the behaviour of quantum systems. We do recall here the definition of projectors which will dominate the next section.

Definition 1. *A projector is a self-adjoint idempotent linear operator, i.e.* $P^2 = P = P^\dagger$.

The adjoint of a linear operator M is indeed in general denoted by $M^\dagger$ and is by definition such that $\forall\psi,\phi : \langle\psi|H\phi\rangle = \langle H^\dagger\psi|\phi\rangle$ — we ignore at this point issues to do with the domain where an operator is defined. There is a way to relate this adjoint to Galois adjoints for isotone maps between orthocomplemented posets (see further).

1.3 It's infant terrible: von Neumann's projection postulate

What does quantum theory say about sequential measurements? Here the disagreement between physicists starts (at a formal level). In the original formulation of the mathematical framework by John von Neumann, this matter was taken care of by his *projection postulate* (von Neumann 1932) — here the projectors start to play a role beyond a merely formal tool to calculate outcome probabilities, they describe "change of state". This section is entirely devoted to the role of projectors. This includes the spectral decomposition theorem that does some of the above (where we made restrictions on the spectrum of operators) in a mathematically rigorous way, in particular taking into account the existence of self-adjoint operators with continuous spectra.

1.3.1 The projection postulate, and its weakenings

So, von Neumann's projection postulate is both the cornerstone to behaviour of physical systems under measurement and the starting point of disagreement on quantum theory itself.[1] Let us first formulate a minimal (actually reduced) version:

- If a measurement of (a *physical observable* represented by) H yields $a \in \sigma(H)$ then an immediate second measurement of H would again yield a as outcome.

To give any sense to this phenomenon, here's *von Neumann's projection postulate*:

- If a measurement of (an observable represented by) H yields $a \in \sigma(H)$ then the state of the system changes from its initial state ψ into $P_a\psi$, i.e. in an eigenstate for a.

Note that there is not just an assumption of the fact that the system changes its state but there is also a very particular specification of how, more precisely, the system's state doesn't just change to any eigenstate of the measurement outcome but to $P_a\psi$. Any other choice would obviously cause different behaviour in a next measurement. As such, the minimal version yields a truly different theory (which is even less complete in Einstein's sense, and which, although some people remain attached to it, doesn't seem to coincide with experiments), so we won't consider it.

Due to the projection postulate, projections become maps describing transitions, namely

$$P_a^{(H)} : \mathcal{H} \to \mathcal{H} : \psi \mapsto P_a\psi \tag{8}$$

[1]You either accept it, or you don't, the latter either via an "after the measurement the system is destroyed, i.e. sequential measurements are to difficult to perform" mentality, or by aiming towards true modification of the whole of quantum theory without a projection postulate, possibly by assuming that quantum theory is non-sequential in its essence.

describes the transition of the system under measurement H when the outcome a is obtained (so we do have some form of conditioning here). Note that one might rather write this map as

$$P_a^{(H)} : \mathcal{H} \setminus A^\perp \to \mathcal{H} : \psi \mapsto P_a \psi \tag{9}$$

where A denotes the subspace of $\mathcal{H}$ spanned by the eigenvectors of H with eigenvalue a. In this way expressing that this transition is not possible for any $\psi \in A^\perp$ since we then have $Prob_\psi^H(a) = \langle\psi|P_a\psi\rangle = \langle\psi|\underline{o}\rangle = 0$. One could conclude that the von Neumann projection postulate introduces a *dynamic ingredient* in the description of a measurement. There are models in terms of a non-linear Schrödinger equation that describe this state transition in terms of continuous (but non-unitary) dynamics (Gisin and Piron 1981). And now for something bizarre: One should be aware of the fact that the above in some way presupposes that states as an ingredient of the theory have a counterpart in reality. It might sound very weird that I even mention this, but, the reality attributed to the states is an unfinished dispute in the physical community. In the "no-interpretation interpretation of quantum theory" one obviously doesn't even ask the question. In a radical instrumental perspective, the state is a purely formal construction that is part of the calculation of outcome probabilities in measurements. This is a rather popular attitude. We will attribute a counterpart in reality to the state, with as a minimal motivation that this is the only way to provide some image of what "happens" in terms of models.

Another formulation of the projection postulate (essentially motivated by the spatial incarnation of quantum theory that will be discussed below — it is in a sense a wave mechanics version of the above) is the one referred to as *collapse or reduction of the wave packet*:

- For sake of the argument, assume that H has a non-degenerated spectrum such that there exists an ortho(normal) base $\{\psi_a | a \in \sigma(H)\}$ of $\mathcal{H}$ consisting of H-eigenvectors. One can consider this as a preferred base with respect to H. Next, given an initial state $\phi \in \mathcal{H}$ on which we are going to perform a measurement, we envision ϕ in terms of its decomposition over $\{\psi_a | a \in \sigma(H)\}$, i.e. $\phi = \sum_{a\in\sigma(H)} \lambda_a \psi_a$ for $\lambda_a = \langle\psi_a|\phi\rangle$ the respective coefficients of projection. One refers to $\phi = \sum_{a\in\sigma(H)} \lambda_a \psi_a$ as a *superposition state with respect to H*. Note here that in many textbooks on a wave mechanics presentation one forgets the "with respect to H" and seems to talk about an absolute notion of superposition. (We discuss that below, physical space offers in a sense a preferred basis and set of measurements.) In this context, the projection postulate says that all terms in the sum $\sum_{a\in\sigma(H)} \lambda_a \psi_a$ that are not labelled by the outcome of the measurement disappear after the measurement. One calls this *collapse* or *reduction of the wave packet*. In case of a degenerated spectrum every outcome might have more terms in the linear combination, so there is no full reduction.

So there are two views here, the 'reduction view' and the 'change of state view'. One senses here that from a dynamical perspective the 'change of state view' propagates a more mechanistic sequential picture.

1.3.2 Spectral decomposition (von Neumann)

We formulate the *spectral decomposition theorem* that appeared in von Neumann (1932). This is the true mathematically well-founded core of Hilbert space quantum theory. Indeed, although we have (slightly abusively) interpreted the self-adjoint operators in terms of "producing (eigen)values for (eigen)states" general self-adjoint operators (possibly only partially defined) on an infinite dimensional Hilbert space might have not any eigenstates at all! We discuss such examples in the section on the Heisenberg uncertainty relation. The eigenstate picture does work well in the finite dimensional case and is for that reason valuable.

Theorem 1. *For any self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ there exists a spectral measure that is, a projection valued measure on its spectrum, say $P_-^H : \mathcal{B}(\sigma(H)) \to \mathcal{P}(\mathcal{H})$, where $\mathcal{B}(\sigma(H))$ denote the Borel sets in $\sigma(H)$ and $\mathcal{P}(\mathcal{H})$ the projectors on $\mathcal{H}$. This spectral measure reproduces the self-adjoint operator as $\int_{\sigma(H)} \mu \, dP_\mu^H$.*

Compared to the decomposition in the reduction view we rewrite the operator here in terms of a sum over projectors, with corresponding eigenvalues as weights. This theorem also allows to define probabilities for any measurable subset $B \in \sigma(H)$ in case of a non-discrete spectrum as $\langle\psi|P_B^H\psi\rangle$. In the discrete case we would have $H = \sum_{a\in\sigma(H)} a \, P_a$. This allows us, in view of eq.(**??**) that specifies probabilities in measurements, to define *expectation values of observables* as a weighted sum:

$$Exp_\psi^H = \sum_{a\in\sigma(H)} a \, Prob_\psi^H(a) = \sum_{a\in\sigma(H)} a \, \langle\psi|P_a\psi\rangle = \langle\psi|H\psi\rangle \tag{10}$$

At this point the crucial role of projectors in quantum theory becomes explicit. However, the values that any spectral measure takes are of course uniquely determined by the corresponding subspaces on which they project. Any Borel set $E \in \mathcal{B}(\sigma(H))$ in the spectrum of an observable H defines as such a subspace of the Hilbert space that we can consider as representative for a *physical property*, namely the property of having a value in E with respect to observable H. It will become clear later why it makes sense to distinguish between projectors as a special type of measurements and subspaces as physical properties. An initial intuition could be that a system has a certain property a corresponding with a subspace A when the measurement represented by the corresponding projector P_a yields answer $1\,(\in \sigma(P) = \{0,1\})$ with certainty, what in Hilbert space quantum mechanics incarnates as $\psi \in A \Rightarrow Prob_{\psi}^{P_a}(1) = \langle\psi|P_a\psi\rangle = 1$.

1.3.3 Bra's and Ket's

For jargon purposes we now also briefly describe Dirac's "Bra-ket"-formalism. This is motivated by the particular role projectors play. Call $|\psi\rangle$ a "ket", i.e. a normal vector, and $\langle\phi|$ a "bra", i.e. an abnormal vector that actually lives in the dual vector space $\overline{\mathcal{H}}$. The in-product is then a "braket" $\langle\phi|\psi\rangle$. Example of its use are writing $\phi = \sum_{a\in\sigma(H)}\langle\psi_a|\phi\rangle\langle\psi_a|$ for decomposition over a non-degenerated discrete spectrum and writing expectation values as a sandwich $\langle\psi|H|\psi\rangle$ between a bra-ket pair. This formalism truly embodies quantum 'cookery', in the sense that it is not entirely mathematically motivatable. Discrete decomposition of $|\phi\rangle$ yields $|\phi\rangle = \sum_{a\in\sigma(H)}|\psi_a\rangle\langle\psi_a|\phi\rangle$ and thus, since $\langle\phi|\phi\rangle = 1$ we obtain a class of equivalent representations for the identity as $\sum_{a\in\sigma(H)}|\psi_a\rangle\langle\psi_a| = id_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$. Analogously, $\sum_{a\in\sigma(H)}|\psi_a\rangle a\langle\psi_a| = H$ by $\langle\psi|H|\psi\rangle$ via spectral decomposition. Writing $\Psi \in \overline{\mathcal{H}}_1 \otimes \mathcal{H}_2$ as $\Psi = \sum_{i,j} c_{i,j}\langle\psi_i^1| \otimes |\psi_j^2\rangle$ for bases $\{\psi_i^1\}_i$ and $\{\psi_i^2\}_i$ we obtain an equivalence with expressions of type

$$\sum_{ij}|\psi_j^2\rangle c_{ij}\langle\psi_i^1| \,:\, \mathcal{H}_1 \to \mathcal{H}_2 \,:\, |\phi\rangle \mapsto \sum_{ij} c_{ij}\langle\psi_i^1|\phi\rangle|\psi_j^2\rangle\,, \tag{11}$$

i.e. linear maps. Actually we obtain Hilbert-Schmidt maps in case of infinite dimensional Hilbert spaces — see Dunford and Schwartz (1957) or Weidmann (1981).

1.3.4 Complementarity and non-commutativity

The complementarity principle (due to N. Bohr) in quantum theory in first order says that there exist incompatible observables (actually, that in general they are incompatible), that is, observables of which the value cannot be known at the same time, i.e. a system cannot be in an eigenstate for both observables. Note that the particular formulation of the complementarity principle might insinuate the perspective that the system "has values for an observable" and not "the system obtains values in a measurement". In his original formulation Bohr was however very careful and referred to measurement arrangements and that these arrangements are in general incompatible. Different formulations can be found in this sense. This complementarity principle is not much of a surprise in view of the above mathematical development. Take as self-adjoint operators on a two dimensional Hilbert space the projectors on the one-dimensional subspaces respectively spanned by ψ and $\psi+\psi^\perp$, canonically denoted as P_ψ and $P_{\psi+\psi^\perp}$. Then, non-complementarity would mean that the state of the system has to be in

$$\{\psi, \psi^\perp\} \cap \{\psi+\psi^\perp, \psi-\psi^\perp\}\,, \tag{12}$$

which is empty.

If subspaces can be spanned by vectors of a common base, then the corresponding projectors commute: Represent the vectors on which the projectors act in the decomposition over that common base and consider reduction with respect to the projections involved; we clearly have independence of the order of reducing with respect to the different projectors since we end up only with terms in the intersection. Explicitly we have for commuting projectors $P_A P_B = P_B P_A = P_{A\cap B}$. (Note here that this particular formula exhibits an advantage of working with subspaces compared to working with projectors: We can consider arbitrary intersections and even intersections of subspaces corresponding to non-commuting projectors, but there is no equivalent easily expressible operation on projectors — this will obviously be of importance when we will consider the lattice of properties/closed subspaces/projectors induced by taking arbitrary intersections of subspaces with closed linear span as resulting corresponding join) This as such links complementarity to non-commutativity. Explicitly (discrete in benefit of insight), consider $H_1 = \sum_{a_1\in\sigma(H_1)} a_1 P_{a_1}$ and $H_2 = \sum_{a_2\in\sigma(H_2)} a_2 P_{a_2}$ so

$$H_1 H_2 = \sum_{a_1\in\sigma(H_1)} a_1 P_{a_1} \sum_{a_2\in\sigma(H_2)} a_2 P_{a_2} = \sum_{a_1\in\sigma(H_1), a_2\in\sigma(H_2)} a_1 a_2 P_{a_1} P_{a_2}\,. \tag{13}$$

then, if all $P_{a_1}$and P_{a_2} commute then H_1 and H_2 commute. One however verifies that if there is any pair $\{P_{a_1}, P_{a_2}\}$ such that $P_{a_1}P_{a_2} \neq P_{a_2}P_{a_1}$ that then we have $H_1H_2 \neq H_2H_1$. (Formally, an important role is played in quantum theory by so-called commutators $[H_1, H_2] = H_1H_2 - H_2H_1$.) Conclusively, note that the essential correspondence here lies in the fact that

$$A \leftrightarrow B \ \Leftrightarrow \ P_AP_B = P_BP_A \,(= P_{A\cap B}) \tag{14}$$

where A and B are subspaces of $\mathcal{H}$ and $A \leftrightarrow B$ denotes that these subspaces can be spanned by vectors of a common base, and that this extends to the observables via spectral decomposition. This relation $\leftrightarrow$ is in Jauch (1968) and Piron (1976) referred to as *compatibility*. This (additionally to the spectral decomposition theorem) motivates the view that observables play a secondary role in the mathematical structure that is characteristic for quantum theory "that make mathematical considerations just more complicated". This was part of von Neumann's (and Birkhoff's) perspective when formulating his(/their) ideas towards quantum logic. This is not the perspective of many physicists since the "physical quantity", the "physical observable", the "energy", the "position", the "momentum" are the notions they want to see at first glance in their theory, and those are encoded as non-idempotent observables.

1.4 Indeterminedness and uncertainty relations

Historically, there is no doubt that Heisenberg's uncertainty relation(s) manifested an essential non-classicality of quantum theory (or weirdness if one wants). However, this relation on many occasions being read as "one cannot attribute both a precise position or momentum to a physical system", where we leave the interpretation of "one" and "attribute" vaguely, there is the fact that this aspect manifests already at a level of what one could call indeterminedness relations. As we will show, these indeterminedness relations do not apply to the particular observables position and momentum with respect to physical space, due to the fact that the latter are not definable on the whole Hilbert space and that in particular there does not really exist position states at all! All this are to be seen as consequences of "having to work in an infinite dimensional Hilbert space when encoding physical space".

1.4.1 "Indeterminedness relations"

Let us recall what a classical observable would be. If Σ is a classical space of states then any function $f : \Sigma \to \mathbb{R}^{(n)}$ represents a real valued observable. Saying that for a particular state p two observables f and g can "take a particular value" just means that $f(p)$ and $g(p)$ exist. So except for partially defined observables that do not apply to certain states we have no indeterminedness of observable values. This situation does not apply to quantum theory. Just consider the above considered case of incompatibility.

1.4.2 The spatial incarnation of quantum theory.

Mathematically what happens is that as Hilbert space one takes square Lebesque integrable functions on $\mathbb{R}^3$ on which one additionally puts an equivalence relation "being equal up to measure zero". It turns out that in this incarnation of Hilbert space quantum theory position X and momentum M , having a particular significance with respect to the underlying space $\mathbb{R}$ (we take one dimension for formal simplicity) that plays the role of arguments for the wave functions, "informally" (we don't specify a domain at this point) are respectively:

$$X : \psi(x) \mapsto (X\psi)(x) := x\psi(x) \tag{15}$$

$$M : \psi(x) \mapsto (M\psi)(x) := -i\bar{h}\tfrac{d\psi(x)}{dx} \tag{16}$$

We are not going to derive them here and refer to one of the cited textbooks for details. However, the problem here is that these are not actually bounded operators. Leaving that aside for a while (we are absolutely not going to be rigorous here, but outrageously intuitive!) that the following are necessary conditions on the respective eigenfuctions for these operators:

- In order that X produces only one value, $\psi(x)$ can only have one non-zero point ;
- $-i\bar{h}\frac{d\psi(x)}{dx} = a\psi(x)$ has $\psi(x) = \eta e^{\frac{iax}{\bar{h}}}$ as only (differentiable) solutions .

Both requirements are heavily contradictory and as such can be seen as an incarnation of an indeterminedness relation. This argument is however heavily abusive if only for there reason that both requirements characterise "objects" that are not at all in $\mathcal{H}(\mathbb{R})$. It however provides a feel on a manifestation of incompatibility of "having both position and momentum" that doesn't go via the statistical analysis of the next section.

1.4.3 Heisenberg's uncertainty relation

We proceed now with the derivation of the real Heisenberg uncertainty relations. What one does to consider the above position operator is restricting the domain $\mathcal{H}(\mathbb{R})$ to those functions that satisfy $\int_{\mathbb{R}} x^2|\psi(x)|^2 dx < \infty$. In case of the momentum operator one requires $\frac{d\psi(x)}{dx} \in \mathcal{H}(\mathbb{R})$ besides existence of the derivative. Obviously one cannot formulate the incompatibility of position and momentum in terms of an indeterminedness relation. The thing that does the trick is the fact that the expectation value of the commutator of the position and momentum operators (now obviously considered on the restricted domains) is a constant independent on the state, and, that the product of the *statistical notion uncertainty* is proportionally to this commutator. Explicitly, in classical statistics, an expectation value is

$$\langle H \rangle = \sum_{a\in\sigma(H)} a\, Prob(a) \tag{17}$$

and corresponding variance is

$$\Delta^2(H) = \sum_{a\in\sigma(H)} (a - \langle H\rangle)^2\, Prob(a)\,. \tag{18}$$

A little calculation then yields

$$\Delta^2(H) = \langle H^2\rangle - \langle H\rangle^2 \tag{19}$$

where $\langle H^2\rangle = \sum_{a\in\sigma(H)} a^2\, Prob(a)$. It is inspired by this that the notion of *uncertainty* in quantum theory is defined as, depending on the initial state:

$$\Delta^2_\psi(H) = \langle\psi|H^2\psi\rangle - \langle\psi|H\psi\rangle^2 = Exp_\psi^{H^2} - \left(Exp_\psi^H\right)^2 \tag{20}$$

or

$$\Delta_\psi(H) = \sqrt{Exp_\psi^{H^2} - \left(Exp_\psi^H\right)^2}\,. \tag{21}$$

one can then prove (using Schwarz inequality $|\langle\psi|\phi\rangle| \leq \langle\psi|\psi\rangle^{\frac{1}{2}}\langle\phi|\phi\rangle^{\frac{1}{2}}$ which is valid in Hilbert space) that

$$\Delta_\psi(H_1)\Delta_\psi(H_2) \geq \frac{1}{2} Exp_\psi^{[H_1,H_2]} \tag{22}$$

for any two self-adjoint operators H_1 and H_2. One has that

$$[X,M]\psi(x) = XM\psi(x) - MX\psi(x) = x(-i\bar{h})\frac{d\psi(x)}{dx} - (-i\bar{h})\frac{d(x\psi(x))}{dx} = i\bar{h}\psi(x) \tag{23}$$

so $[X,M] = i\bar{h}$ (i.e. multiplication with the constant $i\bar{h}$) such that the uncertainty relation becomes

$$\Delta_\psi(X)\Delta_\psi(M) \geq \frac{1}{2} = Exp_\psi^{[X,M]} = \langle\psi|i\bar{h}\psi\rangle = i\bar{h}\,. \tag{24}$$

i.e., state independent!

1.4.4 Non-locality

The feature that comes in the picture in the spatial, or equivalently, the wave mechanics representation of quantum theory is that of non-locality:

"Particles do not necessarily have localised spatial qualities".

Note that by this we do not mean that they are smeared out in space, one should in a sense look at them as indivisible. A space-like measurement is a measurement where any open subset of the spectrum of eigenvalues defines a region in physical space. The smaller the region, the more localised the spatial qualities of the system will be after a measurement with positive outcome. To get a flavour of non-locality, take a 20 pound bill (or 100, or 5, depending on the impression one wants to make) and tear it in two parts. Put one in your pocket, leave the other one where you were, and take a plane to the other side of the world. You just dislocalised 20 pound (not the bill, that is smeared out, but the financial content). Destruction of one part would indeed imply destruction of the full 20 pound. Non-locality has another incarnation within quantum theory besides the measurement of spatial qualities of particles, namely in terms of non-local interaction. This involves entanglement of multiple particles.

1.5 Entanglement

Entanglement is one of the major ingredients in quantum theory that causes confusion. There where in classical physics two systems are described by pairing states, say they are described in the Cartesian product, in quantum theory we also have to consider superpositions of such pairs. Let us note here that contra the somewhat artificial nature of superposition as it incarnates in the reduction of the wave packet picture of von Neumann's projection postulate (where it has only significance with respect to a particular measurement, i.e. the specification of a base) here we have an absolute aspect of superposition. The fact that we have (labelled) subsystems a priori restricts the bases one can consider for the Hilbert space description of the compound system, giving rise two families of states: pure tensors and the others, the superposed ones. We demonstrate this in detail in the first subsection below.

1.5.1 Bases of pure tensors

As mentioned above, compound systems are described in quantum theory by the tensor product. The elements in this tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ consist of the Hilbert space spanned by a base of the form

$$\{\psi_i^1 \otimes \psi_j^2 \,|\, \psi_i^1 \in Base(\mathcal{H}_1)\,, \psi_j^2 \in Base(\mathcal{H}_2)\}\,, \tag{25}$$

by requiring linearity both at the left and the right of the tensor with respect to the two underlying Hilbert spaces, i.e.

$$(\sum_i c_i^1 \psi_i^1) \otimes (\sum_j c_j^2 \psi_j^2) = \sum_{i,j} c_i^1 c_j^2 \ \psi_i^1 \otimes \psi_j^2\,. \tag{26}$$

In general we as such obtain members of the form

$$\Psi = \sum_{i,j} c_{i,j}\, \psi_i^1 \otimes \psi_j^2\,. \tag{27}$$

Any state/vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ that can be written as $\psi \otimes \phi$ is called a product state/pure tensor. One shouldn't confuse the pure in 'pure tensors' with the utterance found in some textbooks of 'pure states': both pure and non-pure tensors are in fact so-called pure states since they both are represented by one-dimensional subspaces of the Hilbert space. The bases of $\mathcal{H}_1 \otimes \mathcal{H}_2$ that are meaningful to consider are now of the form of eq.(**??**) . This strongly restricts the number of representations of the system by choosing a particular base. However, there is absolutely no uniqueness of representation of a member of the tensor product sensu eq.(**??**) since for example

$$\psi \otimes \phi + \psi^\perp \otimes \phi + \psi \otimes \phi^\perp + \psi^\perp \otimes \phi^\perp \tag{28}$$

can be written as a product state/pure tensor

$$(\psi + \psi^\perp) \otimes (\phi + \phi^\perp)\,. \tag{29}$$

in the base

$$\big\{x \otimes y \ \big|\ x \in \{\psi + \psi^\perp, \psi - \psi^\perp\}, y \in \{\phi + \phi^\perp, \phi - \phi^\perp\}\big\}\,. \tag{30}$$

Contrary, the so-called singlet state (set $\mathcal{H}_1 = \mathcal{H}_2$)

$$\Psi_S = \sqrt{\frac{1}{2}}(\psi \otimes \psi^\perp - \psi^\perp \otimes \psi) \tag{31}$$

cannot be written as a product. Actually, this is in fact the only state a *physical entity* described in $\mathbb{C}^2 \otimes \mathbb{C}^2$ can have due to the so-called *Pauli exclusion principle* which requires anti-symmetry of the formal expression in eq.(**??**) .

Intermezzo: The Pauli exclusion principle requires anti-symmetry of the representation of multiple particles with spin in $\mathbb{N}+\frac{1}{2}$ *, and a* $\mathbb{C}^2$*-description corresponds exactly with spin-*$\frac{1}{2}$ *. Spin-*$\frac{1}{2}$ *is actually equivalent to what is understood nowadays as a qubit, the unit of quantum information theory in its present incarnation. Spin itself is a quality that particles have besides for example position and momentum qualities. The incompatibility arises here when one wants to measure spin as well along the x-, the y- and the z-axis of a Cartesian referential frame in space — this can easily be seen in the spherical model that we will present below for spin-*$\frac{1}{2}$ *. Particles with spin in* $\mathbb{N}+\frac{1}{2}$ *are called fermions, those with spin in* $\mathbb{N}$ *are called bosons, other don't exist: This is a so-called superselection rule, that is, a rule that excludes certain superpositions in a particular base. Another superselection rule is for example that the charge of an electron is either* $+e$ *or* $-e$ *. Thus, quantum theory is not a pure Hilbert space theory — see also later, Piron's theorem. Actually, bosons and fermions give rise to two completely different statistical theories for systems consisting of many particles, respectively called Bose-Einstein and Fermi-Dirac statistics — see for example Jauch (1968).*

Let us verify this claim of Ψ_S being the only anti-symmetric state. Consider $\Phi = \phi\otimes\phi' - \phi'\otimes\phi$ which is clearly anti-symmetric in the sense that exchange of whatever is at the left of a tensor with that what is on the right gives a vector with opposite sign. Assume $\phi = a\psi + b\psi^\perp$ and $\phi' = c\psi + d\psi^\perp$. Then we have

$$\begin{aligned}
\Phi &= \phi\otimes\phi' - \phi'\otimes\phi \\
&= (a\psi + b\psi^\perp)\otimes(c\psi + d\psi^\perp) - (c\psi + d\psi^\perp)\otimes(a\psi + b\psi^\perp) \\
&= ac\psi\otimes\psi + ad\psi\otimes\psi^\perp + bc\psi^\perp\otimes\psi + bd\psi^\perp\otimes\psi^\perp \\
&\quad -(ac\psi\otimes\psi + ad\psi^\perp\otimes\psi + bc\psi\otimes\psi^\perp + bd\psi^\perp\otimes\psi^\perp) \\
&= (ad - bc)(\psi\otimes\psi^\perp - \psi^\perp\otimes\psi) \\
&= \Psi_S\,.
\end{aligned}$$

We mention a theorem (Schmidt 1907) that to some extend characterises the non-uniqueness of representability of states in the tensor product. Given any state $\Psi \in \mathcal{H}_1\otimes\mathcal{H}_2$, there exists a base of $\mathcal{H}_1$ and one of $\mathcal{H}_2$ such that we have

$$\Psi = \sum_i c_i \psi_i^1 \otimes \psi_i^2\,. \tag{32}$$

The decomposition is essentially unique up to situations with coinciding coefficients c_i . This theorem seems in the context of quantum theory (to my knowledge) having only been used in the so-called modal interpretation of quantum theory, which at some point in the last decade was quite popular but now seems to loose all its followers.

The in-product on the tensor product, about which we didn't speak yet, is due to the construction of the latter via a chosen base of the underlying Hilbert spaces fully characterised by its values on product states/pure tensors

$$\langle\psi\otimes\phi|\psi'\otimes\phi'\rangle = \langle\psi|\psi'\rangle\langle\phi|\phi'\rangle \tag{33}$$

what then by eq.(**??**) and eq.(**??**) leads to

$$\langle\sum_{i,j} c_{i,j}\psi_i\otimes\phi_j|\sum_{k,l} c_{k,l}\psi'_k\otimes\phi'_l\rangle = \sum_{i,j,k,l}\bar{c}_{i,j}c_{k,l}\langle\psi_i|\psi'_k\rangle\langle\phi_j|\phi'_l\rangle\,. \tag{34}$$

The actual fact that we have two systems defines a special class of self-adjoint operators on $\mathcal{H}_1\otimes\mathcal{H}_2$, namely those of the form $H_1\otimes H_2$, defined by

$$(H_1\otimes H_2)(\psi\otimes\phi) = H_1(\psi)\otimes H_2(\phi)\,. \tag{35}$$

Produced values should be envisioned as pairs $(a_1, a_2)\in\sigma(H_1)\times\sigma(H_2)$.

$$\begin{aligned}
Prob_\Psi^{H_1\otimes H_2}(a_1,a_2) &= \langle\Psi|P_{a_1}\otimes P_{a_2}\Psi\rangle & (36)\\
&= \sum_{i,j,k,l}\bar{c}_{i,j}c_{k,l}\langle\psi_i^1|P_{a_1}\psi_j^1\rangle\langle\psi_k^2|P_{a_2}\psi_l^2\rangle & (37)
\end{aligned}$$

for $\Psi = \sum_{i,j} c_{i,j}\,\psi_i^1\otimes\psi_j^2$ — note that we have $(P_{a_1}\otimes I)(I\otimes P_{a_2}) = P_{a_1}\otimes P_{a_2}$. Comming back to the statement made in the introduction of this section, the spectral decomposition of these operators obviously goes over a base consisting only of pure tensors of projectors, and this justifies a true structural notion of superposition in a more instrumentalist fashion.

1.5.2 Models for compoundness

We refer to some models that produce the quantum probabilistic structure. A first model that was implicit in Gisin and Piron (1981) and made explicit in Aerts (1986) is one for spin-$\frac{1}{2}$, i.e. an entity described in a two-dimensional complex Hilbert space, or in more fashionable terms, a qubit. The idea is to represent the one-dimensional subspaces on a sphere (this works, slightly abusively we have $\mathbb{C}^2/\mathbb{C} \cong \mathbb{R}^3/\mathbb{R}_+$) where orthogonality becomes being antipodically located. So states are represented by points on this sphere and measurements by two antipodically located points (the two eigenstates of the self-adjoint operator which we assume neither to be the identity nor the nil operator) where we assign a value to each of them (the corresponding eigenvalues). We can generate quantum probabilistic behaviour in the following way. Consider a line that connects the two antipodically located points q_+ and q_- that characterise the measurement. Then project the state p of the system orthogonally on that line, giving location $p(q_+, q_-)$. Assume a stochastic variable λ uniformly distributed on that line. If the actual location of λ is in $[q_-, p(q_+, q_-)]$ then q_+ will be the outcome state. If the actual location of λ is in $[p(q_+, q_-), q_+]$ then q_- will be the outcome state. One verifies that as such we exactly obtain transitions according to the von Neumann projection postulate with probabilities according to standard quantum theory. For a spin-1 model, i.e. a model of an entity described in a three-dimensional complex Hilbert space, we refer to Coecke (1995) where a spin-1 system is modelled as a compound system consisting of two spin-$\frac{1}{2}$ systems, say two entangled qubits.[2] A singlet state model was proposed in (Aerts 1991). A model for arbitrary compound quantum systems can be found in (Coecke 1998).

1.6 The measurement problem, density (informally) and NoGo (informally)

Let us first introduce the notion of mixed state vs. pure state, i.e. everything what we up to now called state. Consider the states of a classical system, i.e. a set Σ that characterises position and momentum qualities of a system. For a point particle the state is exactly the pair $(position, momentum)$. A mixed state would be any representation that encodes a lack of knowledge on the precise position and momentum, say in probabilistic terms. So, this is a probability measure $\mu : \mathcal{B}(\Sigma) \to [0,1]$, with $\mathcal{B}(\Sigma)$ a σ-algebra of measurable Σ-subsets. Discretely, this comes down to assigning to each $p_i \in \Sigma$ a weight $\mu_i \in [0,1]$ such that $\sum_i \mu_i = 1$. Note that alternatively to a lack of knowledge perspective we can also consider this same mathematical object as representative for the relative distribution of states in statistical ensembles, or, for relative frequencies, or, for a probability of truth. We can do the same in quantum theory, consider a weight on pure states expressing a lack of knowledge. That's a mixed state. As an example, the mixed states for a set of two states constitute the unit interval and those for a set of three states can be represented in a triangle with corners the unit vectors of a base in three dimensional real space where the co-ordinates then stand for the respective probabilities (which add up to one). There is a powerful theorem, Gleason's theorem (1957), that provides a formal representation for such mixed states in terms of a particular kind of self-adjoint operators, so-called density operators (or matrices). It actually turns out that in view of "behaviour under measurements" many different weights that one can put on pure states turn out to be equivalent, and as such can be given the same formal representation. An example would be the pairs $\{\psi, \psi^\perp\}$ and $\{\psi + \psi^\perp, \psi - \psi^\perp\}$ where all weights are $\frac{1}{2}$. One for example easily verifies that in the above presented model that produces quantum probabilities one can actually represent the mixed states as points inside the sphere, and the two above would both be represented by the centre. More general, mixed states are constructable via convex combination of pure states by means of the weights, as such yielding a convex closed set, essentially of dimension $n^2 - 1$ in real space where n is the dimension of the corresponding Hilbert space (see Gleason's theorem for the representation of mixed states as density matrices what gives an indication of this fact). From that perspective, pure states are extremal elements in a convex structure. There is a very elaborated approach to the foundations of quantum theory that takes this convex structure as a starting point, namely Ludwig (1964, 1967, 1968) and many followers and in particular the (hard to read but) impressive treatise Ludwig (1985, 1987). We also mention the more accessible papers Cattaneo and Nisticò (1993) and Cattaneo and Laudisa (1994). We will present Gleason's theorem later. We note that in discourses on foundations of physics this way of describing mixed states in terms of lack of knowledge is in many cases not present and one sometimes even refers to them as states.

When discussing statistical features in quantum theory we automatically end up with questions on the existence of so-called hidden-variable theories. Essentially this comes down to: "Can we replace the states in quantum theory by a larger

[2]The shift from two-dimensional complex Hilbert space to three-dimensional complex Hilbert space is essential in the sense that all structural theorems for quantum theory, including NoGo theorems only apply to entities described in an at least three-dimensional complex Hilbert space (see also subsection **??**).

class of states, on which we allow us to introduce a lack of knowledge such that the resulting mixtures are identified with the states of ordinary quantum theory, and that actually produce the same probabilistic structure?" The answer is NO, due to so-called NoGo-theorems, e.g. Kochen and Specker (1967). We will present a lattice version of a NoGo-theorem later (which is much simpler). Why were we able to produce a model that yields quantum probabilistic structures? The answer is "contextuality", i.e. dependence on the performed measurement. We indeed do not introduce an uncertainty on the states but on the particular interaction of that state with the performed measurement, i.e. we perform a mixed measurement. Another answer of a clever person with respect to the spin-$\frac{1}{2}$ model would be that it concerns a Hilbert space of dimension 2. Indeed, most NoGo-theorems (and also Gleason's) require a Hilbert space of dimension strictly greater than two. Therefore we provide a model that produces quantum probabilities of a three dimensional real Hilbert space (Aerts 1986). First note that the symmetry of a (finite dimensional) Hilbert space $\mathcal{H}$ is such that saying

- a state as a definite value for every measurement, or,
- every state has a definite value for a particular measurement with a $\mathcal{H}$-base as eigenvectors,

are the same statements. We will consider one measurement with eigenvalues $\{1,2,3\}$ and corresponding unit eigenvectors $\{\psi_1,\psi_2,\psi_3\}$. The matrix of the measurement is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{38}$$

Consider the three dimensional Euclidean space with co-ordinate axis spanned by $\{\psi_1,\psi_2,\psi_3\}$ and in it the triangle by convexly closing these vectors which we denote by $\triangle$. Then represent the state ϕ by the point

$$(\langle\phi|P_{\psi_1}\phi\rangle\,,\langle\phi|P_{\psi_2}\phi\rangle\,,\langle\phi|P_{\psi_3}\phi\rangle)\in\triangle \tag{39}$$

We indeed have $\langle\phi|P_{\psi_1}\phi\rangle+\langle\phi|P_{\psi_2}\phi\rangle+\langle\phi|P_{\psi_3}\phi\rangle=1$. This point now generates three subtriangles with respective corners

$$\{\phi,\psi_2,\psi_3\}\qquad\qquad\{\psi_1,\phi,\psi_3\}\qquad\qquad\{\psi_1,\psi_2,\phi\} \tag{40}$$

Consider a stochastic variable $(\lambda_1,\lambda_2,\lambda_3)\in\triangle$ that is uniformly distributed in the triangle. If $(\lambda_1,\lambda_2,\lambda_3)\in\{\phi,\psi_i,\psi_j\}$ then ψ_k for $k\neq i,j$ is the outcome. This produces quantum probabilities. It suffices to see that the surface of the subtriangle $\{\phi,\psi_i,\psi_j\}$ is proportional to $\langle\phi|P_{\psi_k}\phi\rangle$, and as such the probability of $(\lambda_1,\lambda_2,\lambda_3)\in\{\phi,\psi_i,\psi_j\}$, i.e. of obtaining ψ_k, is $\langle\phi|P_{\psi_k}\phi\rangle$.

Another example of a contextual hidden variable model was provided by Bohm (1952), as a theorem one could say, but that nowadays is elevated to a very popular interpretation of quantum theory. So here we reach the domain of the "non-(no interpretation interpretations)", e.g. the Bohm hidden variable interpretation, consistent histories interpretation, GRW interpretation etc. We won't discuss them. The motivation of all these comes from being not at ease with what Einstein referred to as the incompleteness of quantum theory. To questions are usually posed as:

1. What causes collapse/projection/reduction? — no acceptance obviously requires no answer!
2. What lies at the origin of the particular probabilistic nature of quantum theory?

The second one is the one that motivated hidden variable theories. The first one motivates crazy views on the world like the many-worlds interpretation which actually has a lot of followers (many here in Oxford) and with respect to which some even go so far that they claim that the actual realisation of quantum computation provides a proof of this perspective. Another (not popular anymore) perspective is that the collapse is created by our mind (this causes however a paradox referred to as Wigner's friend paradox). Also Schrödinger's cat paradox comes in here by envisioning the cat as an intermediate player between the quantum entity and the observer, hidden in a closed box and the cat's life depends on the state of an atom by means of some funny set-up. When the state is in a superposition, then so that cat should in terms of dead and alive, until the box is opened by the observer who causes the collapse. All this is captured by the utterance "the measurement problem".

2 Quantum structures: Some theorems and perspectives

The description 'quantum structures' is actually quite recent. It was adopted by a scientific community at foundation, the 'International Quantum Structures Association', in order to avoid the (negative) connotation that 'Quantum Logic' bared, i.e. its metaphysical connotation (see further). We apply the name for this section in which we discuss some 'structural' theorems in the context of quantum mechanics, and more general orthomodular lattices, that have no a priori interpretational assumptions but are purely mathematical results. However, they do have major implications with respect to approaching, understanding, situating or characterising the above defined framework of quantum theory.

2.1 The initial Birkhoff-von Neumann setting

Let us first investigate the kinematical structure encoded within quantum theory, i.e. we will abstract over evolution (classically envisioned as "the dynamics of the system"). In classical physics the algebra of properties that can be attributed to a system consists of the subsets of the state space, i.e. it is the powerset $\mathcal{P}(\Sigma)$. Any proposition on truth of an observable can be expressed in terms of such a subset. Indeed, let $f:\sigma\to\mathbb{R}$ be an observable, then $f^{-1}[E]\in\mathcal{P}(\Sigma)$ expresses the property "the value of f is in $E\in\sigma(f)$". The basic idea in Birkhoff and von Neumann (1936) is the same thing. All statements of the form "the value of H is in $E\subseteq\sigma(H)$" for a self-adjoint operator H can be represented by the projector P_E^H in the spectral decomposition, which at its turn defines some closed subspace A_E such that $P_{A_E}=P_E^H$. Therefore it makes sense to give a special status to $\mathcal{L}(\mathcal{H})$, the set of closed subspaces of a Hilbert space, or equivalently, the corresponding projectors $\mathbb{P}(\mathcal{H})$. In view $\mathcal{L}(\mathcal{H})\subseteq\mathcal{P}(\mathcal{H})$ we inherit ordering via inclusion and meets as intersection, i.e. we obtain a complete lattice — recall that a when for a poset L all greatest lower bounds (meets) exist then also all smallest upper bounds exist via $\bigvee_i a_i=\bigwedge\{a\in L|\forall i: a\geq a_i\}$. In $\mathcal{L}(\mathcal{H})$ this corresponds with taking the closed linear span. We also have an orthocomplementation defined by $A^\perp=\{\psi\in\mathcal{H}|\forall\phi\in A:\langle\psi|\phi\rangle=0\}$. Note that in general an orthocomplementation is defined as a map $(-)':L\to L$ that satisfies

$$a\leq b\Rightarrow b'\leq a' \tag{41}$$
$$a\wedge a'=0 \tag{42}$$
$$a\vee a'=1 \tag{43}$$
$$a''=a\,. \tag{44}$$

In principle there is not too much harm when thinking of meets as conjunction, however, due to superposition one shouldn't think of joins as disjunction and of the orthocomplement as negation. We discuss this further. However, having these lattice operations motivated the title "The Logic of Quantum Mechanics" of the Birkhoff-von Neumann paper. The manifest difference with classical logic is the non-distributivity of this lattice. Indeed, taking ψ, $\psi^\perp$ and $\phi=\psi+\psi^\perp$ in a two dimensional Hilbert space we obtain

$$\phi\wedge(\psi\vee\psi^\perp)=\phi\wedge\mathcal{H}=\phi \tag{45}$$
$$(\phi\wedge\psi)\vee(\phi\wedge\psi^\perp)=\underline{o}\vee\underline{o}=\underline{o}\,. \tag{46}$$

Some prefer to think in terms of the isomorphic lattice of projectors with the connectives arising as operations on corresponding matrices. The order then arises as

$$P_A\leq P_B\Leftrightarrow P_B\circ P_A=P_A\,(=P_A\circ P_B)\,. \tag{47}$$

For meets we have that whenever P_A and P_B commute

$$P_A\wedge P_B=P_A\circ P_B\,, \tag{48}$$

and for joins whenever P_A and P_B commute

$$P_A\vee P_B=P_A+P_B-P_A\circ P_B \tag{49}$$

what reduces for orthogonal joins, i.e. we additionally to commutation have $P_A\circ P_B=P_\emptyset$, to

$$P_A\oplus P_B=P_A+P_B \tag{50}$$

and for the orthocomplement of P_A we obtain

$$1 - P_A \,. \tag{51}$$

Given for example states $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we obtain then for the join of the corresponding projectors:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{52}$$

In terms of a subspace we have to write join in a parametrised way:

$$\left\{ c \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c' \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,\Big|\, |c|^2 + |c'|^2 = 1 \right\} . \tag{53}$$

Following more traditional quantum structural notation we will from now on adopt $a, b, c \in \mathcal{L}(\mathcal{H})$ where atoms, i.e. minimal elements of $\mathcal{L}(\mathcal{H}) \setminus \emptyset$ are denoted as $p, q \in \Sigma(\mathcal{H})$, which in quantum theory are exactly the states. The top and bottom will respectively be denoted by 1 and 0. To introduce some dialectics we will say that a property a is actual in a state p iff $p \leq a$, i.e. when a verification of it via a measurement, say test, yields a positive result with certainty. In more traditional terminology, the value of H is in $E \in \sigma(H)$ with certainty for any pair (H, E) that is represented by the subspace $A\,(\sim a)$ where $P_E^H = P_A$ when the state of the system is p.

2.2 Gleason's theorem (1957): Probability is implicit in the lattice!

We return to mixed states, but now seriously.

2.2.1 The theorem

Theorem 2. *There exists only one map $\omega_p : \mathcal{L}(\mathcal{H}) \to [0,1]$ with $dim(\mathcal{H}) \geq 3$ for $p \in \Sigma(\mathcal{H})$ such that*

$$\omega_p(a) = 1 \Leftrightarrow p \leq a \tag{54}$$

$$\textstyle\sum_i \omega_p(a_i) = \omega_p(\oplus_i a_i) \tag{55}$$

and this map is given by assigning the quantum probability for obtaining a positive answer in the verification of the corresponding property.

We should think of $\omega_p(a)$ as the probability to obtain a positive outcome a verification of the property $a \in \mathcal{L}(\mathcal{H})$ when the system in state p. In terms of self-adjoint operators, $\omega_p(a)$ stands for the value $\langle \psi_p | P^{H_i}_{E_{i,k_i}} \psi_p \rangle$ (the same for every i) of a family of projectors $\{P^{H_i}_{E_{i,k_i}}\}_{i,k_i}$ that exactly have the states for which a is actual as corresponding eigenstates. Then, eq.(**??**) expresses that if and only if the property a is actual for the state p then we obtain a certain positive outcome, and eq.(**??**) expresses that probabilities on mutually orthogonal projectors add, what traces back to the spectral decomposition theorem where different possible outcomes are identified by mutually orthogonal projectors. Indeed, if $E_k \cap E_l = \emptyset$ we clearly have

$$\langle \psi_p | P^H_{E_k} \psi_p \rangle + \langle \psi_p | P^H_{E_l} \psi_p \rangle = \langle \psi_p | P^H_{E_k \cup E_l} \psi_p \rangle \ . \tag{56}$$

Note here that the orthogonal join exactly expresses that we consider properties encoding disjoint eigenstate sets that can be attributed to one self-adjoint operator. We can derive the following corollary concerning mixed states.

Corollary 1. *For all maps $\omega : \mathcal{L}(\mathcal{H}) \to [0,1]$ that satisfy*

$$\omega(0) = 0 \tag{57}$$

$$\omega(1) = 1 \tag{58}$$

$$\textstyle\sum_i \omega(a_i) = \omega(\oplus_i a_i) \,. \tag{59}$$

we have that there exists a density operator ρ_ω such that $\omega(a) = tr(\rho_\omega P_a)$, i.e. they are of the form $tr(\rho P_{(-)})$ and thus bijectively representable by density operators.

Definition 2. *A density operator is a self-adjoint positive (i.e. $\forall\psi : \langle\psi|\rho\psi\rangle \geq 1$) linear operator with trace 1 (i.e. $tr(\rho) = \sum_i \langle\psi_i|\rho\psi_i\rangle = 1$) for some (and as such any) base $\{\psi_i\}_i$ of $\mathcal{H}$.*

In general we have $\rho^2 \leq \rho$ and pure states turn out to be those that saturate the inequality into an equality. The upper bound of the square of the density operator as such turns out to be a measure of "pureness". The moral of corollary **??** is that many different statistical mixtures of (pure) states end up having the same behaviour under measurements. Mixed states transform under verification of a property A by the so-called Lüders' rule

$$\rho \mapsto \frac{P_A \rho P_A}{tr(\rho P_A)}\,, \tag{60}$$

a generalisation of the von Neumann projection postulate. Note that the transition involves renormalization, i.e. it ignores the probability of transition. For the general issue of quantum probability theory we refer to Accardi (1982), Accardi and Fedullo (1982) and Gudder (1985). To conclude, we mention that one has to make a clear distinction between the following three sums:

- The (weighted) sum of two (unit) vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ that creates a so-called superposition with respect to a measurement that has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as eigenstates.
- The sum of two projectors $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ that creates the projector on the subspaces spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — see eq.(**??**).
- the (weighted) sum of the two density matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ that results in a lack of knowledge on having either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2.2.2 A NoGo-theorem

As an example we consider the Jauch-Piron which was the most compact version that captures the structural content of NoGo-theorems, and actually relies on the content of Gleason's theorem to provide a full picture. Two properties a and b are compatible, denoted $a \leftrightarrow b$, if the sublattice generated by $\{a, a', b, b'\}$ is distributive. Actually, this relations points to the connection between distributivity and commutativity: two projectors P_A and P_B in $\mathbb{P}(\mathcal{H})$ commute if the sublattice of $\mathcal{L}(\mathcal{H})$ generated by $\{A, A^\perp, B, B^\perp\}$ is distributive. A state p is represented by the unique Gleason quantum probability $\omega_p : \mathcal{L} \to [0,1]$. In Jauch and Piron (1963) it is shown that ω_p fulfils

$$\omega_p(0) = 0 \tag{61}$$

$$\omega_p(1) = 1 \tag{62}$$

$$a \leftrightarrow b \Rightarrow \omega_p(a) + \omega_p(b) = \omega_p(a \wedge b) + \omega_p(a \vee b) \tag{63}$$

$$\omega_p(a) = \omega_p(b) = 1 \Rightarrow \omega_p(a \wedge b) = 1 \tag{64}$$

These will be considered as requirements on everything that one might consider as a hidden state. Note that eq.(**??**) extends eq.(**??**) in a natural way.

Definition 3. *A theory is said to admit hidden variables if we can add extra variables Λ_p to every state such that there exist maps $(p,\lambda) : \mathcal{L} \to \{0,1\}$ (a binary set, not an interval!) and a probability measure $\mu_p : \mathcal{B}(\Lambda_p) \to [0,1]$ ($\mathcal{B}(\Lambda_p)$ is a σ-field of subsets of Λ_p) such that:*

$$\forall p \in \Sigma, \forall a \in \mathcal{L}, \exists \Lambda_p : \omega_p(a) = \int_{\Lambda_p} (p,\lambda)(a)\, d\mu_p(\lambda) \tag{65}$$

*and such that all (p,λ) fulfill eq.(**??**), eq.(**??**), eq.(**??**) and eq.(**??**).*

This cannot be realized (Jauch and Piron 1963)! At this point we mention a modest Go-theorem that exhibits at a lattice theoretic level why we were able to produce classical models that produce quantum probabilities. It will also give some insight in what is ment by contextuality and that it is not that much of an unnatural assumption, as many physicists claim it is. However, I have the feeling that in many cases this ends up being purely for methodological reasons in terms of "retaining control over the system under study" that is assumed to be sufficiently isolated from its surroundings and as such these surroundings shouldn't influence the outcomes of observations we make on it. Clearly, also acceptance of non-locality is problematic from this perspective. To that extend philosophy of physics at some point went that far to attribute the title "realist" to those who believe in non-locality and non-contextuality (in some sense quite an insult if I may say so). The analogue of eq.(**??**) would in that case be the following: we relate with every property a a measurement e_a, i.e. a test of the property a which we allow to decompose over a set Λ_a on which there exists a probability measure $\mu_a : \mathcal{B}(\Lambda_a) \to [0,1]$. This gives us:

$$\forall p \in \Sigma, \forall a \in \mathcal{L}, \exists \Lambda_a : \omega_p(a) = \int_{\Lambda_a} (p,\lambda)(a) d\mu_a(\lambda) \tag{66}$$

where again the maps $(p,\lambda) : \mathcal{L} \to \{0,1\}$ are $\{0,1\}$-valued. However, the very different nature of $(p,\lambda)(a)$ here, i.e. a component of a measurement and not a of state or in terms of more suggestive notations $(p)(a_\lambda)$ vs. $(p_\lambda)(a)$, doesn't even justify eq.(**??**) anymore what takes us completely out of the hidden variable setting. To that extend it is even possible to construct a model that produces quantum probabilities where we can omit the label a in Λ_a and μ_a, in the sense that the reference to the measurement as argument of the maps $(p,\lambda) : \mathcal{L} \to \{0,1\}$ suffices. Details, representational theorems and classification concerning can be found in Coecke (1997) and Coecke and Valckenborgh (1998).

2.3 Piron's theorem (1964)

This is the representation theorem for Hilbert space quantum theory in lattice terms. Also (wrongly) referred to as the Amemiya-Araki theorem due to the elaboration in Amemiya and Araki (1967) where under prompting by M.H. Stone, a geometric proof was obtained. The theorem was published in Piron (1964) and besides in Piron (1976) we also refer to Varadarajan (1968) for a proof. At this point we mention that Varadarajan (1968), two volumes with title "The Geometry of Quantum Theory", besides providing an extensive elaboration on geometrical aspects of quantum theory, deals with subjects like Mackey's systems of imprimitivity, and is as such still a very valuable mathematical account.

2.3.1 The theorem

A lattice $\mathcal{L}$ is a 'generalised Hilbert lattice' if complete, orthomodular, i.e. orthocomplemented and

$$a \leq b \implies a \vee (a' \wedge b) = b\,, \tag{67}$$

atomistic, i.e. every element can be written as a join of atoms, and satisfies the covering law, i.e. if p is an atom of $\mathcal{L}$ and $a \in \mathcal{L}$ then there are no elements between a and $p \vee a$. It is irreducible if not decomposable as direct union $\bigvee_\alpha \mathcal{L}_\alpha$ with elements $\{a_\alpha\}_\alpha$ and

$$\{a_\alpha\}_\alpha \leq \{b_\alpha\}_\alpha \Leftrightarrow \forall \alpha : a_\alpha \leq b_\alpha\,. \tag{68}$$

It is easily seen that the lattice of closed subspaces of a Hilbert space is an irreducible Hilbert lattice.

Theorem 3. (Piron) *Every generalised Hilbert lattice canonically induces a projective geometry, where irreducible components correspond to irreducible components of the projective geometry.*

The construction proceeds as

- 'Points':= Σ,
- 'Lines':= $\big\{\{p \in \Sigma | p \leq q \vee r\} | q, r \in \Sigma\,, q \neq r\big\}$.

Via the representation theorem for projective geometries one then obtains the following.

Corollary 2. (Piron; Amemiya and Araki) *An irreducible generalised Hilbert lattice (rank ≥ 4) can be realized as lattice of closed linear varieties of a 'generalised Hilbert space' $(\mathcal{V}, \phi, \mathbf{K})$, which is actually an ordinary Hilbert space if $\mathbf{K} = \mathbf{C}$. Every generalised Hilbert lattice can be realized by a family of them.*

So we have full characterisation up to specification of the division ring. In other words one could say that there is still an axiom missing. This task was essentially completed in Solèr (1995) where it was proved that the vector space admitting an infinite orthonormal sequence was a sufficient condition for the division ring to be standard, i.e. $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We give some additional details now. For an arbitrary inner product space $(\mathcal{V}, \phi, \mathbf{K})$, the complete atomistic ortholattice $\mathcal{L}(\mathcal{V})$ of biorthogonal subspaces need not be orthomodular. When it is, $(\mathcal{V}, \phi, \mathbf{K})$ is termed a generalised Hilbert space. This terminology is motivated by the remarkable result that if $\mathcal{V}$ is an inner product space over one of the standard division rings $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ then $\mathcal{L}(\mathcal{V})$ is orthomodular if and only if $\mathcal{V}$ is complete.

Actually, this theorem covers both classical and quantum physics, or in general, quantum theory with superselection rules. Let us explain this. For this general version of the theorem, we omit the irreducability requirement such that we obtain a labelled family $\{\mathcal{H}_\alpha\}_\alpha$ of Hilbert spaces for any generalised Hilbert lattice (rank ≥ 4) where any arbitrary two subspaces either:

- allow superpositions, i.e. belong to the same Hilbert space
- are submitted to a superselection rule, i.e. belong to different Hilbert spaces.

A classical state space should be envisioned as a family $\Sigma = \{p_\alpha\}_\alpha$ of one-dimensional Hilbert spaces p_α (i.e. containing only one state) and is as such saturated with superselection rules. Note that the covering law and atomisticity are axioms that refer to points such that a (complete) orthomodular lattice can be envisioned as embodying 'pointless' quantum theory. Let us mention some theorems on orthomodularity. A first theorem where in some way dynamics is sneaking in is Foulis's theorem (1960) that identifies orthomodular lattices with so-called Baer *-semigroups (see also section **??**). Explicit steps towards using it as a starting point for a more process oriented version of quantum logic can be found in Pool (1968). At that point, some of the axioms remained unexplained. We note however that orthomodularity came in in terms of assuring the existence of an adjunction — see section **??** of this text. We also mention a theorem stating that decompositions of virtually any algebraic or topological structure constitute an orthomodular poset (Harding 1996). For a recent survey on orthomodular lattices we refer to Bruns and Harding (2000).

3 Quantum logic: A dead-end history?

In the first two subsections we give a historical account of the development of quantum logic. The last two sections focus on personal current research which indicates structural correspondences with developments in computer science.

3.1 Quantum implication via Sasaki hook: The great fiasco!

For an arbitrary orthocomplemented lattice $\mathcal{L}$, set

$$\varphi_a^*(-) : \mathcal{L} \to \mathcal{L} : b \mapsto a \wedge (a' \vee b) \qquad \text{and} \qquad \varphi_{a,*}(-) = (a \xrightarrow{S} -) : \mathcal{L} \to \mathcal{L} : b \mapsto a' \vee (a \wedge b)\,. \tag{69}$$

The double notation $\varphi_{a,*}(-) = (a \xrightarrow{S} -)$ will become clear below. We will in particular refer to $(- \xrightarrow{S} -)$ as the *Sasaki hook*. For the particular case of quantum theory the *Sasaki projections* φ_a^* actually exactly encode projectors. Indeed, for the lattice of closed subspaces of a Hilbert space we have that

$$\varphi_A^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) : B \mapsto A \cap (A^\perp \vee B) \tag{70}$$

encodes the action of the orthogonal projector P_A that projects on the subspace A, so in particular we have

$$\varphi_A^*(\psi) = A \cap \left(A^\perp \vee \psi\right) = P_A(\psi)\,. \tag{71}$$

We note that this perspective actually can be generalised to arbitrary orthomodular lattices where the Sasaki projections turn out to be the projectors in the so-called Baer *-semigroup of hemimorphisms on the underlying orthomodular lattice (Foulis 1960). We have the following equivalent characterisations of orthomodularity.

Proposition 1. *The following are equivalent for an ortholattice $\mathcal{L}$:*

i. $\mathcal{L}$ is orthomodular, i.e. $a \leq b$ implies $a \vee (a' \wedge b) = b$;

ii. For all $a \in \mathcal{L}$ we have $(a \xrightarrow{S} x) = 1 \Rightarrow$ (or $\Leftrightarrow$) $a \leq x$;

iii. For all $a \in \mathcal{L}$ we have $\varphi_a^*(-) \dashv (a \xrightarrow{S} -)$.

Statement (ii) indicates the minimal implicative nature of the Sasaki hook. Statement (iii) in some way "tries to add to that (but fails)". Let us explain this. In constructive logic the adjointness between $(a \wedge -)$ and the action of the hook $(a \to -)$ expresses both modus ponens and deduction, justifying the utterance 'logic'. Now, envisioning $(a \wedge -)$ as classical projections, what is even more motivated by the fact that for a distributive lattice we have

$$\varphi_a^*(-) = a \wedge (a' \vee -) = (a \wedge a') \vee (a \wedge a) = (a \wedge -)\,, \tag{72}$$

we obtain an analogy, or better, a situation of generalisation, with respect to

$$(a \wedge -) \dashv (a \to -) \qquad \text{and} \qquad a \wedge (a' \vee -) \dashv (a \stackrel{S}{\to} -) \tag{73}$$

in terms of adjointness of projection and action of hook. Nothing however a priori justifies the utterance 'logic' for the second situation in "any operational way" and its historical failure confirms this claim. For literature concerning, as well pro as contra we mention Blok, Köler and Pigozzi (1984), Dalla-Chiara (1986), Hardegree (1979), Herman, Marsden and Piziak (1975), Kalmbach (1974, 1983) and Malinowski (1990). For a recent survey and other references we refer to Smets (2001). Our perspective is that (iii) does not carry the logical but the physical content of orthomodularity but in a manifestly different (more fundamental) way than mere analogy (and from which logical constructs will emerge). We show this in sections **??** and **??**.

3.2 The operational motivation: Ontology vs. empiricism

There exist two constructive approaches towards quantum logicality, one deals with properties of a system, the other with observed events in a measurement. Roughly one could say that in the Jauch-Piron approach the most primitive mathematical structure is a complete lattice, i.e. the operation $\bigvee$ caries the 'conjunctive' physical content, there where in the Foulis-Randall approach it is the orthoalgebra, i.e. the partial operation (and implicitly, relation) $\oplus$ caries the physical content of 'macroscopical distinguishability. In (for example) Foulis and Randall (1972), Randall and Foulis (1973), Foulis, Piron and Randall (1983) and Wilce (2000) one considers a notion to which we prefer to refer to as "observed events that reflect something about the system's qualities", where in (for example) Emch and Jauch (1965), Jauch and Piron (1969), Piron (1976), Foulis and Randall (1984), Moore (1999) and Smets (2001) one considers "qualities of the system that cause certain events to occur", this depending on the particular environment (e.g., presence of a measurement device) — note in particular the cross-contributions for both approaches. As we know from quantum mechanics, the state of the system in general doesn't determine the outcome of a measurement, and, an event provoked by a measurement actually changes the system's qualities/properties. As such, it comes as no surprise that these perspectives yield different mathematical structures. To a certain extend one could say that both in the Jauch-Piron and Foulis-Randall perspective we are interested in how the system interacts with its environment, although in the first case from the 'system's perspective' where in the second case we rather consider the 'environments perspective', in other words, an endo- versus an exo-perspective. Obviously, since the Foulis-Randall perspective is an exo-perspective, the measurements are made explicit within the formalism. In the Jauch-Piron perspective where we focus on the system's behaviour itself this is a somewhat more subtle matter. However, this explicit consideration of the environment (or measurement context), even in the system's endo-perspective, is what gives the operational flavour to this approach. For a general survey on these operational approaches we refer to Coecke, Moore and Wilce (2000).

3.3 The quantum extension of classical logic

The attitude in pushing quantum logic towards a full logic was always one of "what do we loose?". However, the richness of quantum theory insinuates that an attitude of "what should we add?" might be the appropriate way. Let us go back to the description of the algebra of properties of a classical physical system, i.e. taking the powerset $\mathcal{P}(\Sigma)$ of its states in view of the fact that they represent propositional statements on the observables obtainable in a measurement, encoding explicitly "the value of f is in $E \in \sigma(f)$". In quantum theory this is taken care of by the lattice of closed subspaces, say the image under the linear closure, i.e. the operation that assigns to any set the smallest closed subspace that contains it. Indeed, we can envision this collection $T \in \mathcal{P}(\mathcal{H})$ of states as a disjunction in the sense "the true state is in T". Then, given any self-adjoint operator H we have that given a state in T is the true state of the system that "the value of H is in $E \in \sigma(H)$" couldn't be distinguished for any $T' \in \mathcal{P}(\mathcal{H})$ that has the same linear closure. Let's denote the closure by $\mathcal{R} : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$. It identifies a particular kind of subsets of the states space, which for classical physics should be envisioned as being $id_{\mathcal{P}(\Sigma)}$. In brief, the "static" algebra of properties of a classical system is $(\mathcal{P}(\Sigma), \bigcap, \bigcup, \neg)$:

i. $\mathcal{P}(\Sigma)$ *represents the logic of preparable states, and*

ii. $\mathcal{P}(\Sigma)$ *represents propositional statements on observables sensu "the value of f is in $E \in \sigma(f)$",*

there where the static algebra of properties of a quantum system is $(\mathcal{P}(\mathcal{H}), \bigcap, \bigcup, \neg, \mathcal{R})$:

i. $\mathcal{P}(\mathcal{H})$ *represents the logic of preparable states, and*

ii. $(\mathcal{P}(\mathcal{H}), \mathcal{R})$ *represents propositional statements on observables sensu "the (eigen)value of H is in $E \in \sigma(H)$",*

and reading $T \in \mathcal{P}(\mathcal{H})$ as "the state is in T", T, T' are indistinguishable sensu "the value of H is in $E \in \sigma(H)$" when $\mathcal{R}(T) = \mathcal{R}(T')$, i.e. the linear closure $\mathcal{R}$ identifies particular subsets of state space ($id_{\mathcal{P}(\Sigma)}$ classically).

Reading a classical implication $T \to T'$ as "if we proved $T \subseteq \mathcal{P}(\Sigma)$, for example by verifying $Q \subseteq T'$ or even $p \in T'$ in an experiment, then we know that T' can be derived", it is clear the Q plays a hidden role here. Doing the same for quantum theory this hidden role vanishes: Actual verification changes the state of the system. In other words, effective measurement goes with a "change of state" given by the von Neumann projection postulate. We can encode this as an hooks, where $T_1, T_2 \in \mathcal{P}(\mathcal{H})$ and $A \in \mathcal{R}[\mathcal{P}(\mathcal{H})]$,

$$(T_1 \stackrel{A}{\to} T_2) := {}^{c}T_1 \cup (A^{\perp} \vee (A \cap T_2)), \tag{74}$$

what results from setting

$$(T_1 \stackrel{A}{\to} T_2) := \{\psi \in \mathcal{H} |\, (\psi \in T_1) \Rightarrow (P_A(\psi) \in T_2)\} \tag{75}$$

Note here the analogy with constructive logic and functional programming where terms should here be envisioned as preparations of properties and hooks as (in/trans)ductions/external manipulations (Amira, Coecke and Stubbe 1998) of the system that produce actuality of another property starting from a given one. Since verifications correspond to the image of $\mathcal{R}$, we define a family of hooks labelled by $\mathcal{R}[\mathcal{P}(\mathcal{H})]$. The resulting picture becomes:

$$\Big(\mathcal{P}(\mathcal{H}), \mathcal{R}, \{(- \stackrel{A}{\to} -) | A \in \mathcal{R}[\mathcal{P}(\mathcal{H})]\}\Big) \tag{76}$$

We can derive this picture in an operational way and for arbitrary orthomodular lattices. The orthomodularity will then be the incarnation of causal duality. We will derive this below in a more general setting. The moral is, we have an additional operation $\mathcal{R}$, to which we refer as operational resolution, and a family of hooks that are "between kinematics and dynamics" in some way. They live on a level that has no classical counterpart. Let us calculate a classical 'limit' in benefit of insight. Classically means $(-)^{\perp} := {}^{c}(-)$, $(- \vee -) := (- \cup -)$ and in particular $A := \mathcal{H}$, i.e. the verification is trivial. We obtain

$$\begin{aligned}
(T_1 \stackrel{\mathcal{H}}{\to} T_2) &= {}^{c}T_1 \cup ({}^{c}\mathcal{H} \cup (\mathcal{H} \cap T_2)) && (77)\\
&= {}^{c}T_1 \cup (\emptyset \cup T_2) && (78)\\
&= {}^{c}T_1 \cup T_2 . && (79)
\end{aligned}$$

Fascinating here is that the Sasaki hook actually comes in play since we can write

$$(T_1 \stackrel{A}{\to} T_2) = (T_1 \to (A \stackrel{\text{Sasaki}}{\to} [T_2])) \tag{80}$$

where the first hook is classical implication and the second the pointwise extension of the Sasaki hook. This is formally due to adjointness of $\varphi^*_A[-]$ and $(A \stackrel{\text{Sasaki}}{\to} [-])$, extending adjointness of $P_A(-)$ and $(A \stackrel{\text{Sasaki}}{\to} -)$, which are both manifestations of so-called causal duality where the latter exactly encodes orthomodularity for an ortholattice. This provides an alternative interpretation of orthomodularity. We discuss this in subsection **??**.

> *Intermezzo:* *Anticipating on a question raised by Prakash Panangaden during these lectures we mention that one could think of a probabilistic version of the above. The motivation is the fact that when pushing the constructive/opertational/functional perspective on implication one might argue that one true produces statistical mixtures and not disjunction. Although in some way i disagree at this last point it definitely makes sense to produce a probabilistic version. The motivation for not a priori incorporating probability at a primitive level is a personal suspicion about the mathematical concept of probability, and in particular any ontological connotation attributed to it within physics. An example of producing disjunction operationally would be the following. Imagine a factory that produces boxes that emit either states ψ or $\psi^{\perp}$ in some random way with probabilities x and $1 - x$, i.e. a mixed state, and they produce it for any non-zero x on request. Unfortunately, the production installation went berserk and ignores instructions with respect to the value of x. This produces an equivalence class of mixed states identifiable as disjunction.*

3.4 The operational pointless version of dynamic quantum logic

The pointless extension of the above essentially proceeds in two steps.

3.4.1 Disjunctive extension via Bruns-Lakser injective hulls

Reversing the reasoning of section **??** in the sense of starting of with a property lattice $\mathcal{L}$, then, replacing the old-style quantum 'logic' essentially comes down to embedding it in a disjunctive extension $\mathcal{DI}(\mathcal{L})$ where the inclusion $\mathcal{L} \hookrightarrow \mathcal{DI}(\mathcal{L})$ itself will provide the operator $\mathcal{R} : \mathcal{DI}(\mathcal{L}) \to \mathcal{DI}(\mathcal{L})$. How does one proceed to do this? A first candidate for encoding disjunctions would be the powerset $\mathcal{P}(\mathcal{L})$, where one read for example reads $\{a, b\}$ as "either a or b is actual". However, if $a \leq b$ we don't have $\{a\} \subseteq \{b\}$ so we do not preserve the initial order. We can clearly overcome this problem by restricting to order ideals

$$\mathcal{I}(\mathcal{L}) := \{\downarrow[\mathcal{A}] | \mathcal{A} \subseteq \mathcal{L}\} \subset \mathcal{P}(\mathcal{L})\,. \tag{81}$$

However, we encounter a second problem. In case the property lattice would be a complete Heyting algebra in which all joins encode disjunctions, then $\mathcal{A}$ and $\{\bigvee \mathcal{A}\}$ again mean the same thing. This redundancy can be eliminated by considering so-called distributive ideals $\mathcal{DI}(\mathcal{L})$ (Bruns and Lakser 1970), that is, order ideals, that are closed under 'joins of distributive sets' (abbreviated as 'distributive joins'), i.e. if $\mathcal{A} \subseteq I \in \mathcal{DI}(\mathcal{L})$ then $\bigvee \mathcal{A} \in I$ whenever we have

$$\forall b \in \mathcal{L} : b \wedge \bigvee \mathcal{A} = \bigvee \{b \wedge a \mid a \in \mathcal{A}\}\,. \tag{82}$$

For $\mathcal{L}$ atomistic and $\Sigma \subseteq \mathcal{L}$, $\mathcal{DI}(\mathcal{L}) \cong \mathcal{P}(\Sigma)$ which implies that $\mathcal{DI}(\mathcal{L})$ is a complete atomistic Boolean algebra. In general we obtain a (complete) Heyting algebra — the construction indeed factors over MacNeille completion. We can moreover provide a more rigorous reasoning which exhibits the canonical nature of this construction, and its full preservation of the physically given in the initial lattice of properties. Contrary to the general attitude in orthodox quantum logic the construction shows as well that for purposes of full generality one cannot abstract over states. In particular does the distributive hull via Bruns-Lakser distributive ideals construction require the fact that superposition states are reflected in the lattice of properties as so-called superposition properties. In the case the given lattice is orthocomplemented we can refine the construction in the sense that we can equip the disjunctive extension with a so-called 'operational complementation' that has the operational resolution $\mathcal{R}$ as its square. The fact that we embed a lattice of properties in a Heyting algebra provides it with an implication. Details can be found in Coecke (2001). All this sheds a new light on the following quote:

> "... whereas for logicians the orthocomplementation properties of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities as the weakest link in the algebra of logic." (Birkhoff and von Neumann 1936)

3.4.2 Causal duality: General dynamic hooks

We mentioned above that we would provide an alternative interpretation of orthomodularity. We will derive it for any general environment, the particular case of it for a verificational measurement will embody orthomodularity. Rather than giving a full derivation, we sketch a more intuitive way of looking at the obtained results. Assume (so we don't give a full proof here) for a system placed in an environment e during a time interval $[t_1, t_2]$ (which can be envisioned as being infinitesimal) that there exist the maps:

- 'Propagation of properties' $e^* : \mathcal{L}_1 \to \mathcal{L}_2$ that assigns to any property $a_1 \in \mathcal{L}_1$ the strongest property $e^*(a_1) \in \mathcal{L}_2$ of which actuality is implied at time t_2 due to actuality of a_1 at time t_1 ;
- 'Causal assignment of properties' $e_* : \mathcal{L}_2 \to \mathcal{L}_1$ that assigns to any property $a_2 \in \mathcal{L}_2$ the weakest property $e_*(a_2) \in \mathcal{L}_1$ whose actuality at time t_1 guarantees actuality of a_2 at time t_2 .

Since, given $a_2 \in \mathcal{L}_2$, $e_*(a_2) \in \mathcal{L}_1$ guarantees actuality of a_2 at time t_2 , $e_*(a_2)$ has to propagate to a property that is stronger (or equal) than a_2 and as such $e^*(e_*(a_2)) \leq a_2$. Analogously, given $a_1 \in \mathcal{L}_1$, since it propagates into $e^*(a_1)$, actuality of a_1 at t_1 guarantees actuality of $e^*(a_1)$ at t_2 and as such $a_1 \leq e_*(e^*(a_1))$. Thus, from $e^*(e_*(a_2)) \leq a_2$ and $a_1 \leq e_*(e^*(a_1))$ we obtain $e^* \dashv e_*$, and this adjunction is what we refer to as causal duality.

Intermezzo: *An interesting question raised by Peter Jevons during this lectures was one on the significance of the two closure operators e^*e_* and e_*e^* emerging from this adjunction. The answer to this could be that they are a measure for the irreversibility of the process since in the reversible limit they become identities. This point definitely deserves in the sense of further research. Aspects of computational efficiency might come in in this way. As some examples, the closure and anti-closure resulting from the Sasaki adjunction are $(a \wedge -)$ and $(a' \vee -)$ with respective fixpoints $\downarrow a$ and $\uparrow a'$. For unitary evolution this would twice be the identity with fixpoints $\downarrow 1$ and $\uparrow 1'$ so both $\mathcal{L}$.*

Note that the same derivation can be made for the powerset of the state space resulting in a condition that expresses forward preservation of disjunction and backward preservation of conjunctions. The causal duality for the properties expresses forward preservation of the join and backward preservation of the meet, which is also conjunctive. The generality of the principle lies in the fact that besides applying to temporal processes it also applies to compoundness (Coecke 2000). It actuality allows us to prove things and is such is not just a fancy way of writing something down. For a proof of linearity of Schrödinger flows, given that the property lattice of the corresponding system is $\mathcal{L}(\mathcal{H})$, see Faure, Piron and Moore (1995). For a proof that the tensor product of Hilbert spaces is appropriate to describe compoundness for systems with as property lattice $\mathcal{L}(\mathcal{H})$ (modulo Abramsky, Blute and Panangaden (1999) for the infinite dimensional case) see Coecke (2000). Conclusively, if the space in which we describe the system is linear, then causal duality forces temporal propagation and compoundness to be described by linear maps. These results essentially exploit the work done Faure and Frölicher (1993, 1994). For details on causal duality we refer to Coecke, Moore and Stubbe (2001).

We will consider a quantum measurement as an 'infinitesimal' version of placing a system in an environment e during a time interval $[t_1, t_2]$. By the von Neumann projection postulate we now that under a measurement 'propagation of properties' is given by the Sasaki projections which exactly encode the Hilbert space projectors. Causal duality becomes in this case the fact that the Sasaki projections admit a right adjoint what boils down to the requirement of orthomodularity. Note that all this indeed extends to pointless quantum theory, i.e. complete orthomodular lattices, and so does the development in section **??**. Actually we can do more. The hooks (with $\mathcal{A}$ a fixpoint of $\mathcal{R}$)

$$(- \stackrel{\mathcal{A}}{\to} -) = (- \to (\bigvee \mathcal{A} \stackrel{\text{Sasaki}}{\to} [-])) : \mathcal{DI}(\mathcal{L}) \times \mathcal{DI}(\mathcal{L}) \to \mathcal{DI}(\mathcal{L}) \tag{83}$$

admit an adjoint tensor $(- \otimes_{\mathcal{A}} -)$ that turns $\mathcal{DI}(\mathcal{L})$ into a commutative quantale. We can also define a backward hook with corresponding tensor that turns $\mathcal{DI}(\mathcal{L})$ into an in general non-commutative dual quantale. We can recover Sasaki projection and hook as

$$(\mathcal{L} \stackrel{\mathcal{A}}{\to} -) = (\mathcal{L} \to (\bigvee \mathcal{A} \stackrel{S}{\to} [-])) = (\bigvee \mathcal{A} \stackrel{S}{\to} [-]) \quad \text{and} \quad (\mathcal{L} \otimes_{\mathcal{A}} -) = \varphi^*_{\bigvee \mathcal{A}}[\mathcal{L} \cap -] = \varphi^*_{\bigvee \mathcal{A}}[-]\,. \tag{84}$$

For details on all this we refer to Coecke and Smets (2001).

4 Epilogue: Physically approaching computer science

Further current elaborations:

- Higher level quantum computation and derivates;
- Quantum process networks and types;
- Labelled quantum transition systems;
- Quantum process calculi.

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