Completeness Results for the Graphical Language of Dagger Compact Closed Categories

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Thesis submitted for the degree of

MSc Mathematics and Foundations of Computer Science

3 September 2012

Abstract

The categorical reformulation of quantum mechanical principles by S. Abramsky and B. Coecke in 2004 [5] proved to be a suitable framework for modelling quantum protocols abstractly. It allows non-standard models for Quantum Informatics and provides a formally rigorous high-level diagrammatic language making very recently discovered phenomena such as quantum teleportation visually obvious.

We therefore aim to gain deeper insights into the underlying graphical calculus by generalizing Selinger's completeness theorem [1] stating that an equation in the graphical language holds if and only if it is valid in the category of finite-dimensional Hilbert spaces **FHilb** [1]. We will derive variations of this result for general free finite-dimensional R-semimodules if $\mathbb{N} \subset R$ in the dagger and non-dagger case as well as equations between trace expressions disproving completeness for various other semi-rings R.

Furthermore we will give a combinatorial criterion for the existence of non-trivial trace equations between matrices with bounded dimensions and draw multiple conclusions. Ultimately we will deduce how usual completeness and completeness for interpretations with bounded dimensions are related and obtain that in the latter case validity of many kinds of completeness are equivalent and strongly linked to solvability of this combinatorial problem.

Acknowledgements

First I would particularly like to thank my supervisor Bob Coecke for introducing me to the subject of completeness and for always having time for discussions. His suggestions and explanations helped me to gain a better understanding and were crucial for the completion of this thesis. I am also indebted to Peter Selinger whose detailed comments and given insights were very useful for finding new approaches and improving my results. Moreover I thank my course supervisor Mehrnoosh Sadrzadeh for help and advice during the first two terms.

Finally I would like to express my gratitude to my parents and my brother who supported me all the time and who made it possible for me to study in Oxford. As this dissertation marks the end of my studies here I thank all my friends from my College and Department who made this one year so enjoyable.

Notations

Throughout this work $\mathbb{N}$ denotes the set of natural numbers containing the 0 whereas $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ represents the natural numbers plus $+\infty$ endowed with the intuitive order. The letters i, j, k, l, m, n are reserved for natural numbers. We also use the Kronecker-delta δ_{ij} which is 1 if and only if $i = j$ and 0 otherwise. We will denote a disjoint union with $\dot\cup$. $B \,\dot\cup\, B'$ might also mean that we implicitly assume B and B' to be disjoint. If f, f' are functions defined on disjoint domains B, B', then we write $f \,\dot\cup\, f'$ for the function on $B \,\dot\cup\, B'$ mapping every $b \in B$ to $f(b)$ and every $b' \in B'$ to $f'(b')$.

For any given alphabet Σ we will refer to its set of words with $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$. We also write strings $\mathpzc{s}$ in an italic style indicating that we are working in the context of alphabets and words. If the strings consist of natural numbers then we will usually use the variables $\mathpzc{k}, \mathpzc{d}$. Besides $|\mathpzc{s}|$ as well as $\#\mathpzc{s}$ denote the length of the string $\mathpzc{s}$ while $\mathpzc{s}_i$ indicates its i-th character. The empty string is written as ϵ. Moreover we sometimes write $\oplus$ for the addition modulo a natural number that has to be derived form the context.

In the category theoretical chapters we will use the letters $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$ for categories, $A, B, C, \ldots$ for objects, $F, G, \ldots$ for functors and $\phi, \theta, \chi, \ldots$ for natural transformations. We indicate the type of a natural transformation by a double lined arrow $\Rightarrow$, e.g. $\phi : F \Rightarrow G$ denotes a natural transformation from F to G. Besides we will sometimes denote functors in a more concise way by tensor expressions with place holders, so e.g. $- \otimes B$ describes the functor acting as $A \mapsto A \otimes B$ and $f \mapsto f \otimes \mathrm{id}_B$. Moreover id_* might - depending on $*$ - represent an identity morphism, an identity functor, or an identity transformation. Also, the symbol $\simeq$ refers to an equivalence between categories.

Furthermore for a given category $\mathcal{C}$ we denote its set/class of objects with $\mathrm{Ob}\,\mathcal{C}$ and its set/class of morphisms with $\mathrm{Mor}\,\mathcal{C}$. For two objects $A, B \in \mathrm{Ob}\,\mathcal{C}$ we will use the notations $\mathcal{C}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A, B)$ interchangeably for the A, B-homset of $\mathcal{C}$, i.e. for the set/class of arrows from A to B. We presuppose familiarity with the categories **Set** resp. **Rel** of sets and functions resp. relations between them, the categories $\mathbf{Vect}_K$/**Hilb** of vectorspaces over the field K/Hilbert spaces and linear maps between them, and the categories $\mathbf{FVect}_K$/**FHilb** which are the restrictions of $\mathbf{Vect}_K$/**Hilb** to finite-dimensional vector-/Hilbert spaces. Besides we will use the term *semimodule*, refering to the obvious generalization of usual modules to arbitrary semi-rings.

The notations for (symmetric/braided) monoidal, for traced (dagger) and (dagger) compact closed categories are adopted from [7]. Also their diagrammatic language - in particular the habit to draw and read diagrams from south to north as well as the usage of wedged boxes - is motivated by [7]. Furthermore we will use the Bra-ket-notation of quantum mechanics in the more general setting of arbitrary R-semimodules..

Finally we make heavy use of *multisets*. The term *multiset* refers to a generalization of the usual concept of a set, allowing to contain the same element multiple times. We will use the brackets $\langle$ and $\rangle$ to indicate a multiset. $\langle 0,0,0,1,1\rangle$ e.g. represents the multiset containing 0 three times and 1 twice. We also write $\langle a_i\rangle_{i\in I}$ (where I is an arbitrary set of indices) for the multiset containing a as often as there are $i \in I$ with $a = a_i$ (which might be infinite; in this case 'as often' means equal cardinalities).

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Chapter 0

Introduction

0.1 Why categorifying Quantum Mechanics?

The study of Quantum Informatics revealed fundamental differences from the concepts of classical computation. The usage of *quantum bits* which - contrary to classical bits - do not only allow the states $|0\rangle$ and $|1\rangle$ but also any *superposition* $\alpha|0\rangle + \beta|1\rangle$, leads to highly counterintuitive results. The *no-cloning theorem* [16] and *no-deleting theorem* [29] for example make it impossible to copy qubits or to erase their informational content (without performing measurements), while non-orthogonal states are not even reliably distinguishable (cf. [13] p.87). From a quantum computational perspective *Shor's algorithm* [21] allowing integer factorization in polynomial time as well as *Grover's algorithm* [22] speeding up database searches quadratically, are the most striking achievements. An essential ingredience of these and further protocols are entangled states. Their *entanglement* - causing correlations between their simultaneous measurement outcomes even when spacially separated - and the hereby induced *non-locality* are unique for Quantum Physics (cf. Bell's Inequality [13] pp.111-117).

The mathematical formalism for quantum mechanics introduced by von Neumann [27] in 1932 generally suffices to verify the mentioned results. It regards qubit states as normalized elements in 2-dimensional Hilbert spaces and processes as applications of unitary matrices. Transitions to composite systems and consecutive applications of operations are modelled by the tensor product $\otimes$ resp. the composition $\circ$ of linear maps. However, this *low level* description does not provide any intuition for their correctness, making it - despite easy proofs - incredibly hard to develop new quantum protocols and algorithms.

In 2004 B. Coecke and S. Abramsky were able to reformulate the underlying quantum mechanical axioms in category theortical terms [5] laying the cornerstone for what is now called *Categorical Quantum Mechanics.* This more general setting does not only constitute an abstraction from Hilbert spaces as it takes place in the context of *dagger compact closed categories* with *biproducts* that matches non-standard models like the category of finite sets and relations as well. It also admits formally rigorous graphical visualizations of the involved algebraic structure making propositions obvious whose verifications require extensive calculations in the Hilbert space formalism. In fact, while the discovery of *quantum teleportation* [12] in 1993 took - as Shor's and Grover's algorithms - more than sixty years after having a mathematical foundation of quantum mechanics, the graphical language makes its correctness intuitively trivial (cf. [6], p.16 or [28], p.43). Similarly the *entanglement swapping protocol* [18] as well as the no-cloning/no-deleting theorems are diagrammatically evident (cf. e.g. [6] p.17) resp. rely on deeper category theoretical insights (cf. [17] pp.20-29). However, variations of compact closed categories turned out to be a useful tool in pure mathematics like e.g. knot theory as well [11].

The calculus underpinning graphical reasoning allows us to deform diagrams when keeping their topologoy invariant and therewith matches our intuition well, in contrast to the underlying algebraic laws. This approach can also be considered as *high-level* since the graphical language suppresses some hidden structure, like the *tensor unit*, *associators*, *unitors* and inner brackets which - due highly non-trivial *Coherence Theorems* - behave like it is necessary to make diagrammatic reasoning work. Hence the graphical calculus is crucial for gaining a deeper understanding of the possibilities and constraints of Quantum Mechanics and might represent the foundation of more advanced algorithms in Quantum Computation.

0.2 The Idea of Completeness

Due to this importance we seek more profound insights into the diagrammatic language. Hasegawa, Hofman and Plotkin dealt in [3] with the question whether it is possible to retrace validity of equations in the graphical language of traced categories when working in the specific category of finite-dimensional vectorspaces $\mathbf{FVect}_K$ over fields with vanishing characteristic K. In other words, does every equation holding in $\mathbf{FVect}_K$ follow from the graphical calculus already and is therefore valid in every traced category? In this case we say $\mathbf{FVect}_K$ is *complete* for traced categories. The relevance of completeness originates on the one hand from its meaning that $\mathbf{FVect}_K$ has - informally speaking - the maximal "calculational expressiveness" provided by general traced categories.

However completeness of a category $\mathcal{C}$ can also be regarded as an indicator for whether $\mathcal{C}$ is a suitable model for the dagger compact closed fragment of Quantum Mechanics. Since a unified physical *Theory of Everything* could not be successfully developed yet, it might be a promising approach to drop the Hilbert space formalism and to consider alternatives instead. Thus completeness analyses might even lay the theoretical ground for discovering mathematical foundations of a *Quantum Gravitation* theory.

One can distinguish between several strengths of completeness, depending on e.g. whether a diagrammatic equation holds if it holds for all instantiations in $\mathbf{FVect}_K$ or if just one is already enough. Similarly completeness questions can be examined for other quantum models, especially those arising from $\mathbf{FVect}_K$ when replacing K by a general semi-ring R. Moreover both categories with and without dagger operations motivate separate analyses. Completeness for dagger compact closed categories was firstly considered by Selinger in [1]. Finally - as quantum computation operates with quantum *bits*, i.e. their state space is restricted to dimension 2 - it is worth to figure out in which cases completeness holds when interpreting objects appearing in diagrams as spaces with bounded dimensions. In this thesis we elaborate precise definitions of these different notions of completeness by formalizing diagrams and providing the category theoretical background first, and explore afterwards in what cases which kind of completeness holds.

0.3 Outline and Contributions

We assume familiarity with the basic category theoretical notions, especially with *categories*, *functors* and *natural transformations*. The first chapter introduces various structures culminating in the concepts of dagger compact closed and traced dagger categories as well as their graphical language. Compatibility with these structures also accounts for the consideration of monoidal functors and monoidal transformations. With this background traced (dagger) diagrams and their instantiations in specific categories will become subjects of mathematical inquiry themselves by formalizing them as *networks* and *interpretations* (cf. [3]). Next, the **Int**-construction will serve as a link between traced and compact closed categories, where we will extend the construction in [4] by a dagger component and provide more detailed proofs (cf. Theorems 2.9 and 2.14). Throughout Chapters 1 and 2 we will discuss the motivation behind definitions since usually the main work in Category Theory lies in the development of useful structure.

Chapters 3 and 4 will profit from a variety of - mainly non-categorical - mathematical approaches. There we make several contributions to the issue of completeness.

- We give formal completeness definitions comprehensively accommodating the variety of completeness types there are to consider and introduce a unified terminology for the results of Selinger and Hasegawa, Hofmann, Plotkin. (cf. Section 3.1)
- We generalize Selinger's result that any pair of diagrams can be separated in $\mathbf{FVect}_K$ by interpretations only depending on *one* of those diagrams if K provides transcendentals, to discrete semi-rings for both the dagger and non-dagger case. (cf. Sections 3.4, 3.5)
- We observe that in the presence of transcendentals the dagger and non-dagger case coincide. (cf. Section 3.5)
- We classify in how far the presence of trivial cycles affects completeness results. (cf. Section 3.7)
- We relate completeness of interpretations with bounded dimensions to the unbounded case and observe that for bounded dimensions the completeness questions essentially collapse to one central problem. (cf. Section 3.4, 3.5, 3.6, 3.7)
- We translate the search for equations between trace expressions in n dimensions into an equivalent combinatorial question. Consequences thereof will enable us to disprove a vast amount of potential trace equations. In this context we also examine in how far general diagrams can be reduced to diagrams consisting of traces. (cf. Sections 4.1, 4.2, 4.3)
- We derive various trace equations holding in some (semi-)rings R not containing $\mathbb{N}$ and therewith exemplify non-completeness for these R.

Chapter 1

Category Theoretical Prerequisites

Before analyzing to what extent graphical reasoning can be imitated in particular categories we will introduce the graphical calculus as well as the underpinning category theoretical concepts and shortly sketch the motivation behind the defined structure and its graphical visualization.

First we will deal with symmetric monoidal categories and present Mac Lane's coherence results. In the second section we extend them to traced and compact closed categories and explain in what way a dagger functor has to interact with these structures. Throughout these discussions we will illustrate why the involved axioms comply with the graphical intuition and therewith how the graphical language essentially captures the algebraic calculus of the category.

After that we discuss free finite-dimensional R-semimodules since they will serve as the major example of a (dagger) compact closed category we will include in our completeness analyses. The last section finally specifies what kind of functors and natural transformations preserve monoidal structures.

1.1 Symmetric monoidal categories and the graphical language

1.1.1 Definition and Coherence

The usual vectorspace tensor $\otimes$ mapping every two vectorspaces V, W to the vectorspace $V \otimes W$ freely generated by the pairs of basis vectors of V, W is crucial for quantum mechanics, as it describes how to pass from the state spaces of two quantum mechanical systems to the state space of the composite system. In order to model this categorically we consider categories whose classes of objects provide a monoidal structure (cf. [2] p.161ff, p.251ff).

Definition 1.1 *A* monoidal category M *is a tuple* $M = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *consisting of*

- *a category* $\mathcal{C}$*,*
- *a bifunctor* $-\otimes- : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, called the* tensor*,*
- *a specified object* I *of* $\mathcal{C}$ *called the* tensor unit*,*

- *natural isomorphisms*

$$\alpha : (- \otimes -) \otimes - \implies - \otimes (- \otimes -) \qquad \textit{called the } \text{associator},$$

$$\lambda : I \otimes - \implies - \qquad \textit{called the } \text{left unitor},$$

$$\rho : - \otimes I \implies - \qquad \textit{called the } \text{right unitor},$$

that make the pentagon

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[dl, "\alpha_{A \otimes B,C,D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B \otimes C,D}"'] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[drr, "\alpha_{A,B,C \otimes D}"] & & & & A \otimes ((B \otimes C) \otimes D) \arrow[dll, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] \\
& & A \otimes (B \otimes (C \otimes D)) & &
\end{tikzcd} \tag{1.1}$$

and the triangle diagram

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[d, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"] & \\
& A \otimes B \\
A \otimes (I \otimes B) \arrow[ur, "\mathrm{id}_A \otimes \lambda_B"'] &
\end{tikzcd} \tag{1.2}$$

commute. M *is called* strict *if all* α, λ, ρ *are the identity transformations.* M *together with a natural isomorphism* c *called the* swap *and consisting of a bunch of maps* $c_{A,B} : A \otimes B \to B \otimes A$ *is called* braided*, if the* hexagons

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "c_{A,B} \otimes \mathrm{id}_C"] \arrow[dl, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[dr, "\alpha_{B,A,C}"] & \\
A \otimes (B \otimes C) \arrow[dr, "c_{A,B \otimes C}"'] & & & B \otimes (A \otimes C) \arrow[dl, "\mathrm{id}_B \otimes c_{A,C}"] \\
& (B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"] & B \otimes (C \otimes A) &
\end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "c_{A \otimes B,C}"] \arrow[dl, "\alpha_{A,B,C}"'] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[dr, "\mathrm{id}_A \otimes c_{B,C}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"] & (A \otimes C) \otimes B \arrow[ur, "c_{A,C} \otimes \mathrm{id}_B"'] &
\end{tikzcd} \tag{1.4}$$

commute. (M, c) *is called* symmetric *if* $c^{-1}_{A,B} = c_{B,A}$ *for all objects* A, B.

Note that diagrams (1.3) and (1.4) are equivalent for symmetric monoidal categories when replacing A, B, C by C, A, B. It is easy to verify that if a category $\mathcal{C}$ has finite products or coproducts, the operations $\times$ resp. $+$ constitute a tensor operation with a terminal/initial object serving as tensor unit, making $(\mathcal{C}, \times)$ resp. $(\mathcal{C}, +)$ to a symmetric monoidal category. As described above the most important of a symmetric monoidal category for us is $\mathbf{Vect}_K$ with the vectorspace tensor which we will focus on throughout this work.

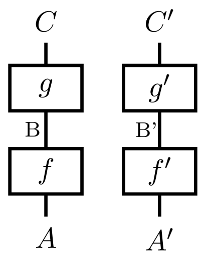
Unlike in proper monoids we in general do not equate e.g. $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ but provide only a systematic way of transforming one into the other (via the associators and unitors). This is due to the fact that requiring α, λ, ρ to be identites would make the previous examples lose their monoidal structure in an irreducible way. Indeed a transition to *skeletal* categories - these are categories arising from considering equivalence classes of isomorphic objects - does not work, as Mac Lane has shown for the skeleton $\mathbf{Set}_s$ of $\mathbf{Set}$ (cf. [2] p.164).

However the diagrams (1.1) and (1.2) ensure that there is a unique way of using the additional structure α, λ, ρ to create an isomorphism between every two objects, which would coincide in a monoid. In the case of symmetric monoidal categories this even holds for objects that would be equal in a commutative monoid (cf. [2] p.165-170, p.253ff).

Theorem 1.2 (Coherence Theorem, Mac Lane) *Let $M = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a (symmetric) monoidal category. Every well-typed diagram containing only morphisms consisting of* $\mathrm{id}, \circ, \otimes, \alpha, \lambda, \rho$ *(and c) commutes.*

1.1.2 Graphical visualization

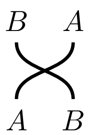
Mac Lane's Coherence Theorem suggests a way of depicting maps of a (symmetric) monoidal category graphically. We denote morphisms as boxes and their domain and codomain as labelled wires entering und leaving the box from south to north. Composition and the tensor operation will be depicted as vertical and horizontal concatenation. E.g. the left diagram visualizes the morphism



$$(g \circ f) \otimes (g' \circ f') = (g \otimes g') \circ (f \otimes f')$$

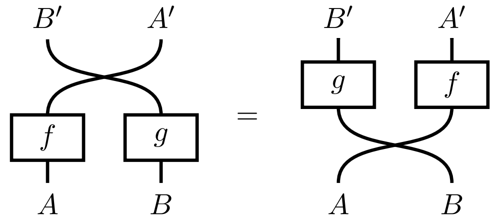
for given arrows $f : A \to B, g : B \to C, f' : A' \to B', g' : B' \to C'$. This example also shows how the diagrammatic language naturally encapsulates the above *interchange law.*

The tensor unit I as well as the morphisms $\alpha_{-,-,-}, \lambda_-, \rho_-$ will not be drawn at all. Likewise we omit bracketing between tensor products of maps. Due to naturality of α, λ, ρ two maps yielding the same diagram differ only by an isomorphism consisting of the α, λ, ρ. According to Mac Lane's Coherence Theorem this isomorphism is unique. We will say, the morphisms are equal *up to a unique isomorphism provided by the monoidal structure.* In the case of symmetric monoidal categories the isomorphisms $c_{A,B}$ are depicted as



The version of Theorem 1.2 for symmetric monoidal categories implies that two diagrams consisting only of swaps, describe the same maps up to a unique isomorphism provided by the monoidal structure if they are *topologically equivalent.*

As naturality of the swap diagrammatically means



for all $f : A \to A', g : B \to B'$ we see that all diagrams in the graphical language of symmetric monoidal categories that are topologically equivalent, represent the same arrow (up to a unique isomorphism provided by the monoidal structure). The precise meaning of topological equivalence will be formally captured by introducing the network terminology (cf. Chapter 2) and describing how to associate a diagram to a network. However it suffices for the intuition to regard yanking wires and pulling boxes along wires as these operations which do not affect the topological structure. Also, no axiom of symmetric monoidal categories states equality of morphisms with topologically different diagrams. Hence its graphical calculus is *sound*, i.e. (topological) equality in the graphical calculus implies equality (up to a unique homomorphism provided by the monoidal structure) of the depicted morphism, and *complete*, i.e. equal morphisms lead to topologically equal diagrams.

Theorem 1.3 *Two morphisms in symmetric monoidal categories are equal up to an isomorphism provided by the monoidal structure, i.e. consisting only of* $\mathrm{id}, \circ, \otimes, \alpha, \lambda, \rho$ *if and only if their diagrams in the graphical language of symmetric monoidal categories are topologically equivalent. In this case the relating isomorphism is unique.*

1.2 Traces, Duals and the Dagger

1.2.1 Traced Symmetric Monoidal Categories

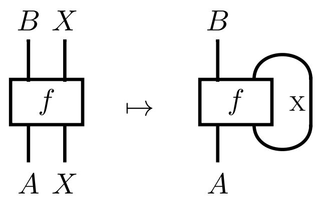
Joyal, Street and Verity were the first who endowed monoidal categories with a partial trace operation [4]. The resulting concept of *traced monoidal categories* can be seen as motivated by being a category theoretical abstraction of the partial trace used in qunatum computation to evaluate the state of a quantum system after measuring a subsystem (cf. [13] p.105ff). But here its strong interconnection with compact closed categories raises our interest. Unlike Joyal, Street and Verity who more generally defined a trace for balanced monoidal categories[1] we restrict ourselves to the symmetric case.

Definition 1.4 *A* traced symmetric monoidal category *is a symmetric monoidal cate-*

[1]This is a braided monoidal category together with a collection of *twists* $\theta_A : A \to A$ subject to certain coherence conditions (cf. [14] p.16).

gory $\mathcal{C}$ together with a collection of trace operators

$$\begin{aligned} \mathrm{Tr}^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) &\to \mathcal{C}(A,B) \\ f &\mapsto \mathrm{Tr}^X_{A,B}(f) \end{aligned}$$



satisfying

(i) naturality *in* A, B, *i.e.*

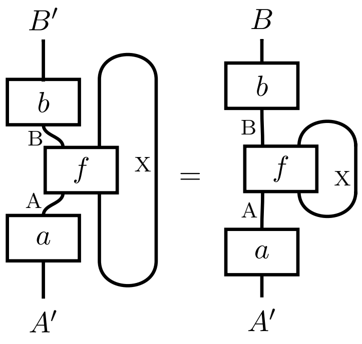
$$\mathrm{Tr}^X_{A,B}((b \otimes \mathrm{id}_X) \circ f \circ (a \otimes \mathrm{id}_X)) = b \circ \mathrm{Tr}^X_{A,B}(f) \circ a \quad \textit{for all} \quad a : A' \to A,\, b : B \to B'$$

(ii) yanking, *i.e.* $\mathrm{Tr}^X_{X,X}(c_{X,X}) = \mathrm{id}_X$.

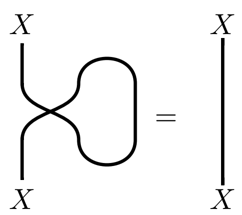
(iii) superposing, *i.e.* $\mathrm{Tr}^X_{C\otimes A, C\otimes B}(\mathrm{id}_C \otimes f) = \mathrm{id}_C \otimes \mathrm{Tr}^X_{A,B}(f)$.

(iv) *the* exchange rule, *i.e.* $\mathrm{Tr}^X_{A,B}(\mathrm{Tr}^Y_{A\otimes X, B\otimes X}(f)) = \mathrm{Tr}^{X\otimes Y}_{A,B}(f)$.

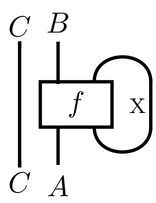
The names of these axioms refer to their diagrammatical expressions. Naturality and yanking mean



and



while superposing and the exchange law simply state that both ways of reading the diagrams



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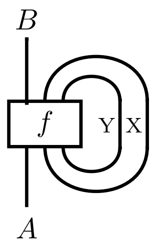


yield the same morphisms. Hence diagrams in the language of traced symmetric monoidal categories are well-defined (up to a unique isomorphism provided by the monoidal structure) and the trace axioms do not change their topology. Conversely it can be shown that all operations keeping the topological structure invariant, follow from rules (i) - (iv). In fact, the original definition in [4] stated every operation leaving the topology invariant, as an axiom (p.448-450) while Hasegawa - whose more concise axiomatization we followed here - has illustrated their equivalence to (i) - (iv) in [10] (p.237ff). We therefore gain the analagous result to Theorem 1.3.

Theorem 1.5 *Two morphisms in traced symmetric monoidal categories are equal up to a unique ismomorphism provided by the monoidal structure if and only if the corresponding diagrams in the graphical language are topologically equivalent.*

Instead of speaking of traced symmetric monoidal categories we will often use the more concise term *traced category* in the following. Now we introduce the concept of compact closed categories which will turn out to be special instances of traced symmetric monoidal categories.

1.2.2 Compact Closed Categories

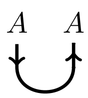
Definition 1.6 *A symmetric monoidal category $\mathcal{C}$ is a* compact closed category *if for every object A of $\mathcal{C}$ there is an object $A^* \in \mathrm{Ob}\,\mathcal{C}$ - called the* dual *of A - and morphisms*

$$\eta_A : I \to A^* \otimes A, \qquad \epsilon_A : A \otimes A^* \to I$$

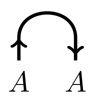
called the unit *and* counit *of A, satisfying the* snake equations

$$\lambda_A \circ (\epsilon_A \otimes \mathrm{id}_A) \circ \alpha^{-1}_{A,A^*,A} \circ (\mathrm{id}_A \otimes \eta_A) \circ \rho_A^{-1} = \mathrm{id}_A,$$
$$\rho_{A^*} \circ (\mathrm{id}_{A^*} \otimes \epsilon_A) \circ \alpha_{A^*,A,A^*} \circ (\eta_A \otimes \mathrm{id}_{A^*}) \circ \lambda^{-1}_{A^*} = \mathrm{id}_{A^*}.$$

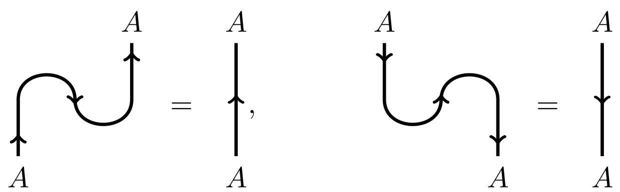
When drawing η_A and ϵ_A as



and



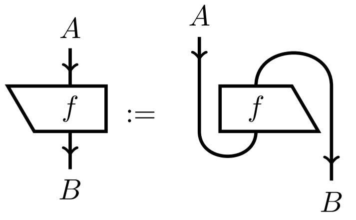
the snake equations become



(1.5)

explaining their name. Note that we never label wires with a dual A^* but indicate a * by endowing the wire with a direction pointing downwards. The quantum mechanical relevance of duals originates from the interpretation of units as preparing entangled qubits and of counits as projective measurements.

In order to state an analogous result to Theorem 1.5 we need to assign a meaning to upside-down boxes in order to let diagrams make sense where boxes were pulled around cups and caps. We therefore give a box belonging to $f : A \to B$ an orientation by cutting off the north east edge and define the upside-down box as the map $f^* : B^* \to A^*$ given by the diagrammatic equation



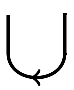
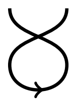
Obviously this definition is equivalent to the fact that pulling boxes around cups and caps does not change the represented morphism. Also, the snake equations express equality of two topologically equivalent diagrams.

We observe that this definition turns * into a contravariant functor from $\mathcal{C}$ to $\mathcal{C}$, the so-called *duality functor*. However the action of * on objects as well as the units and counits are not specified as compact closedness only states their existence. Hence we implicitly presuppose a certain choice for the *duality structure* A^*, η_A, ϵ_A for all $A \in \mathrm{Ob}\,\mathcal{C}$ when drawing or speaking of f^*. Again the axioms of a compact closed category are - as in the case of traced categories - sufficient to show that all topologically equal diagrams represent equal morphisms (cf. [8], p.10).

Theorem 1.7 *Two morphisms in compact closed categories (with chosen duality structure) are equal up to a unique ismomorphism provided by the monoidal structure if and only if the corresponding diagrams in the graphical language are topologically equivalent.*

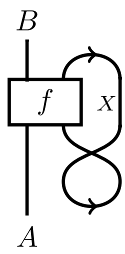
1.2.3 Compact Closed Structure induces a unique Trace

The appearance of cups and caps in the graphical language of both traced and compact closed categories is no coindicence. In fact, when thinking of

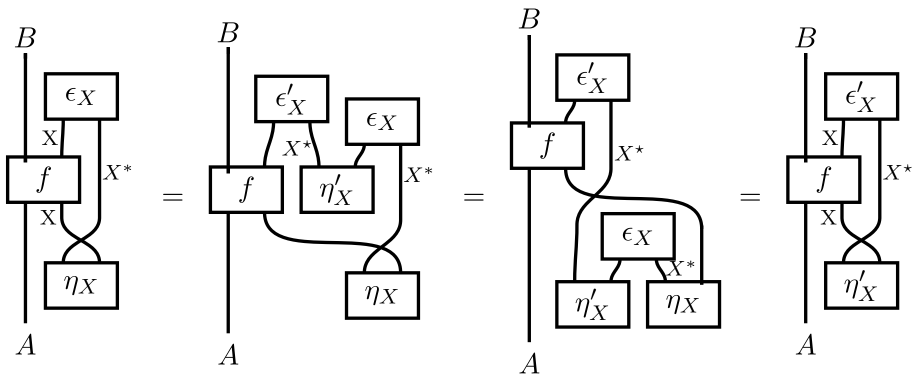
 as 

soundness and completeness of the graphical calculus make it evident that

$$\mathrm{Tr}^X_{A,B}(f) := (\mathrm{id}_B \otimes \epsilon_X) \circ (f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_A \otimes c_{X^*,X}) \circ (\mathrm{id}_A \otimes \eta_X) =$$



where $f : A \otimes X \to B \otimes X$, defines a trace operation in a compact closed category. Interestingly the trace operations do not depend on the chosen duality structure since for two different X^*, η_X, ϵ_X and $X^\star, \eta'_X, \epsilon'_X$ the snake equations reveal (cf. [7] p.38f)



Hence compact closed categories are special instances of traced symmetric monoidal categories.

Proposition 1.8 *Let $\mathcal{C}$ be a compact closed category. Then the compact closed structure induces unique trace operators turning $\mathcal{C}$ into a traced category.*

1.2.4 The dagger functor

Finally we have to capture categorically the fact that quantum mechanics makes heavy use of the scalar product of Hilbert spaces. Its existence can be encoded by means of a contravariant functor since we have $\langle u|v\rangle = (|u\rangle)^\dagger \cdot |v\rangle$ in finite dimensional Hilbert spaces, where here $\dagger$ denotes the self-inverse operation of conjugate transposing. Conversely we know from linear algebra that for every A of appropriate dimension we have $\langle u|Av\rangle = \langle A^\dagger u|v\rangle$ and $A^\dagger$ is unique with this property. The following definition specificies how a generalized $\dagger$-operation has to interact with the various structures we have defined so far.

Definition 1.9

(i) *A* dagger category $(\mathcal{C}, \dagger)$ *is a category $\mathcal{C}$ together with a contravariant functor $\dagger$ called the* dagger functor*, which acts as identity on the objects of $\mathcal{C}$ and which is self-inverse, i.e.* $\dagger \circ \dagger = \mathrm{id}_{\mathcal{C}}$.

(ii) *A* monoidal dagger category *is a monoidal category M equipped with a dagger functor $\dagger$ satisfying for all objects A, B, C and maps f, g*

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, \quad \alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C}, \quad \lambda^\dagger_A = \lambda^{-1}_A, \quad \rho^\dagger_A = \rho^{-1}_A.$$

(iii) *A* symmetric monoidal dagger category *is both a symmetric monoidal and a monoidal dagger category additionally satisfying*

$$c^\dagger_{A,B} = c^{-1}_{A,B} (= c_{B,A}).$$

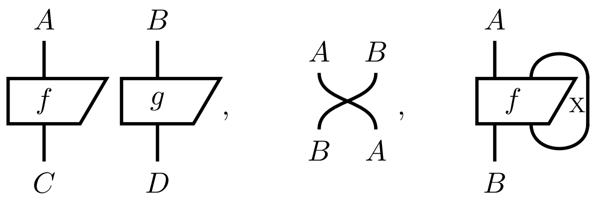
(iv) *A* traced (symmetric monoidal) dagger category *is both a traced symmetric monoidal and a symmetric monoidal dagger category additionally satisfying*

$$\mathrm{Tr}^X_{A,B}(f^\dagger) = \mathrm{Tr}^X_{B,A}(f)^\dagger.$$

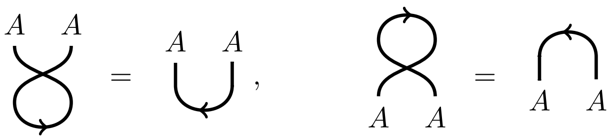
(v) *A* dagger compact closed category *is both a compact closed and a symmetric monoidal dagger category providing a duality structure that additionally satisfies*

$$\eta_A^\dagger = \epsilon_A \circ c_{A^*,A}, \qquad \epsilon_A^\dagger = c_{A^*,A} \circ \eta_A.$$

The motivation behind these requirements becomes clear when considering how to depict the dagger graphically. As it swaps domain and codomain of morphisms the most obvious way to do it is mirroring diagrams on the horizontal axis (but maintaining wire directions as the dagger does not invert duality). But soundness of the graphical languages requires that invisible maps α, λ, ρ stay invisible, explaining the axioms regarding the monoidal structure. The other axioms in (ii) to (iv) ensure that the diagrams



cannot be interpreted in different ways while the axioms in (v) equate topologically equal wires:



As before it can be proven that these axioms are sufficient to make the graphical languages of the different types of categories equipped with a dagger sound and complete.

Theorem 1.10 *Two morphisms in dagger compact closed/traced dagger/(symmetric) monoidal dagger categories are equal up to a unique ismomorphism provided by the monoidal structure if and only if the corresponding diagrams in the graphical language of dagger compact closed/traced dagger/(symmetric) monoidal dagger categories are topologically equivalent.*

Observe that also Theorems 1.3, 1.5, 1.7 and 1.10 express coherence of the respective structure, generalizing Mac Lane's Coherence Theorem. In fact, due to the invisibility of α, λ, ϕ in the graphical language, 1.3 can only hold if the monoidal structure provides at most one isomorphism between any pair of objects. Likewise 1.5, 1.7 and 1.10 generalize this statement for the additional invisible maps and operations $c_{I,I}, \mathrm{Tr}^I_{A,B}, \eta_I, \epsilon_I$. Hence we will often refer to these theorems as coherence for e.g. (dagger) compact closed categories.

1.3 Examples for (dagger) compact closed categories

Rel will serve as our first example of a dagger compact closed category.

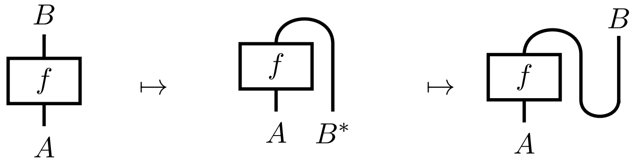
Example 1.11 Obviously **Rel** forms a symmetric monoidal category together with the cartesian product as tensor operation $\otimes$, a singleton as tensor unit $I = \{*\}$ and $c_{X,Y} \subset (X \times Y) \times (Y \times X)$ with $(x, y) \sim (y', x')$ iff $x = x', y = y'$ as the swap. When defining $X^* = X$ and η_X, ϵ_X as the relations given by

$$* \sim (x, x') \quad \text{iff } x = x' \qquad \text{and} \qquad (x, x') \sim * \quad \text{iff } x = x'$$

we obtain a duality structure turning **Rel** into a compact closed category. Finally we can endow **Rel** with a dagger by setting $R^\dagger = \{(y, x) \in Y \times X \mid (x, y) \in R\}$ for all relations $R \subset X \times Y$. Apparently the same constructions work when only allowing finite sets X. Hence **FRel**, i.e. the restriction of **Rel** to finite sets, forms a dagger compact closed structure as well.

Having a dagger compact closed structure is a quite strong requirement for a category. Unlike **Rel** the category **Set** together with the cartesian product as tensor e.g. cannot be equipped with cups and caps as otherwise the operations

$$\mathrm{Hom}(A, B) \;\to\; \mathrm{Hom}(A \otimes B^*, I) \;\to\; \mathrm{Hom}(A, B)$$



would be self-inverse and therefore bijective in contradiction to $\mathrm{Hom}(A \otimes B^*, I) = \{*\}$ since I is a singleton and hence terminal. The categories $\mathbf{FVect}_K$ and **FHilb** allow a (dagger) compact closed structure and are special instances of a more general class of categories.

Definition 1.12 *Let* $(R, +, \cdot)$ *be a* commutative semi-ring, *i.e.* $(R, +)$, $(R, \cdot)$ *are commutative monoids with* $1 \neq 0$ *and multiplication distributes over addition. We define* $\mathbf{FMod}_R$ *as the category whose objects are free finite-dimensional* R*-semimodules, i.e. semimodules isomorphic to* R^n *for some* $n \in \mathbb{N}$*, and whose morphisms are linear maps between them.* $\mathbf{FMod}_R$ *can be endowed with a compact closed structure. Let* A, B *be* R*-semimodules with bases* $(|e_i\rangle)_i$ *and* $(|f_j\rangle)_j$.

- *Define - as in the case of vectorspaces -* $A \otimes B$ *as the* R*-semimodule generated by* $(|e_i\rangle \otimes |f_j\rangle)_{ij}$*,* $I \cong R$ *and* $f \otimes g$ *by* $|e_i\rangle \otimes |f_j\rangle \mapsto f|e_i\rangle \otimes g|f_j\rangle$ *for linear maps* f, g *defined on* A, B.

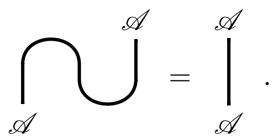
- *Define the swap as* $c_{A,B} : A \otimes B \to B \otimes A,\ |e_i\rangle \otimes |f_j\rangle \mapsto |f_j\rangle \otimes |e_i\rangle$

- *Define* $A^* = A$ *and*

$$\eta_A : I \to A \otimes A,\ 1 \mapsto \sum_i |e_i\rangle \otimes |e_i\rangle, \qquad \epsilon_A : A \otimes A \to I,\ |e_i\rangle \otimes |e_j\rangle \mapsto \delta_{ij}$$

If R *is a field we will use the notations* $\mathbf{FVect}_R$ *and* $\mathbf{FMod}_R$ *interchangeably.*

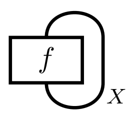
We know from linear algebra that $\mathbf{FMod}_R$ together with the above data forms a symmetric monoidal category and it is straightforward to verify the snake equations for η, ϵ. We need to require $(R, \cdot)$ to be commutative, as $(R, \cdot)$ can be categorically recovered by considering the collection of maps (*scalars*) of type $I \to I$. Together with the composition they form a commutative monoid (cf. [7] p.17f) that coincides with $(R, \cdot)$ in the case of $\mathbf{FMod}_R$. This is why we will just speak of semi-rings and implicitly include commutativity in the following. The additional requirement of having an addition is motivated by quantum mechanics having superpositions of states. However, $\mathbf{FMod}_R$ still comprises a large class of categories as the existence of finite products or coproducts already induces finite biproducts and a superposition rule (cf. [24]). The restriction to (free) finite-dimensional semimodules is necessary since if a duality structure for an infinite dimensional R-semimodule $\mathscr{A}$ existed - addressing meanings to cups and caps such that the snake equation



(1.6)

holds - we would derive a contradiction. In fact, every linear function $\epsilon : R \to \mathscr{A}^* \otimes \mathscr{A}$ maps only into a finite-dimensional subspace of $\mathscr{A}^* \otimes \mathscr{A}$. Hence so does the left hand side of (1.6) and therefore must be unequal to $\mathrm{id}_{\mathscr{A}}$. Finally when allowing R to be a semi-ring the restriction to free R-semimodules is necessary for obtaining a compact closed category (cf. [23] p.291).

We also observe that the trace $\mathrm{Tr}(f) = \mathrm{Tr}^X_{I,I}(\lambda_X^{-1} \circ f \circ \lambda_X)$ for maps $f : X \to X$ induced by the compact closed structure of $\mathbf{FMod}_R$ matches the notion of traces used in linear algebra, namely the sum of the diagonal entries of a square matrix.



Furthermore there are different ways of endowing $\mathbf{FMod}_R$ with a †-functor. The scalar product of a Hilbert space leads to transpose conjugation as †-operation since $\langle u|A|v\rangle = \langle \overline{A}^t u|v\rangle$. When trying to generalize the complex conjugation to R we need to keep compatibility with $+$ and $\cdot$ due to $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ and we have to consider an involution of R as $\dagger \circ \dagger = \mathrm{id}$. Conversely it is obvious that a †-functor transposing matrices over R and conjugating its entries, will then satisfy the conditions (i) - (v) of Definition 1.8.

Definition 1.13 *A* conjugation $^-$: $R \to R$ *of a semi-ring* R *is an involution satisfying* $\overline{x+y} = \overline{x} + \overline{y}$ *and* $\overline{xy} = \overline{x}\,\overline{y}$ *for all* $x, y \in R$. *The identity conjugation is called* trivial conjugation*, all other conjugations are called* non-trivial*. The* $\dagger$*-functor associated to* $^-$ *is the transpose conjugation, i.e.* $A^\dagger = (\overline{a_{ji}})_{ij}$ *for all* $A = (a_{ij})_{ij} \in R^{m\times n}$*. We write* $\mathbf{FProd}_R$[2] *for the dagger compact closed category arising from* $\mathbf{FMod}_R$ *by endowing it with the dagger functor induced by* $^-$*. In the case* $R = \mathbb{C}$ *we write* **FHilb** *instead of* $\mathbf{FProd}_{\mathbb{C}}$*.*

Apparently the Boolean Algebra $B_0 = (\{0,1\}, \vee, \wedge)$ is a semi-ring and a comparison of Definition 1.12 with 1.11 reveals $\mathbf{FRel} = \mathbf{FProd}_{B_0}$ with respect to the trivial conjugation of B_0. However **Rel** cannot be interpreted as the category of all B_0-semimodules since the relation $\eta_X : * \sim (x,x)$ for all $x \in X$ cannot even be regarded as a linear map from B_0 to $B_0^{|X|}$ for infinite sets X as free semimodules do not contain infinite sums of basis elements. Nevertheless **Rel** carries a dagger compact closed structure as B_0 has the special property that every infinite sum converges.

Remark 1.14 *Let* $R \subset \mathbb{Q}$ *be a sub-semiring of* $\mathbb{Q}$*.*

(i) *The only conjugation on* R *is the identity.*

(ii) *The only non-trivial conjugation of* $R[X]$ *is given by* $\overline{x} = x$ *for all* $x \in R$ *and* $\overline{X} = -X$*.*

We will make use of this remark by speaking of the conjugation of R or $R[X]$ without describing it explicitly. The only non-trivial claim is the uniqueness of the non-trivial conjugation of $R[X]$ characterized by $\overline{X} = -X$. But for a general conjugation $^-$ there is a polynomial p with $\overline{X} = p(X)$ implying $X = p(\overline{X}) = p(p(X))$ and therewith $p = \pm X$.

1.4 Monoidal functors and transformations

1.4.1 Monoidal functors

After having introduced various kinds of additional structures for categories we have to discuss what kinds of functors preserve those structures. This will lead to the concept of traced (symmetric monoidal) dagger functors and weaker notions preserving only parts of the listed structures.

Definition 1.15 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be monoidal categories (in* (i)-(iii)*) with tensor units* $I_{\mathcal{C}}$ *and* $I_{\mathcal{D}}$ *and tensor product* $\otimes$*, associator* α *and unitors* λ, ρ[3]*.*

(i) *A* (strong) monoidal functor[4] *is a triple* (F, ϕ, ϕ_0) *consisting of a functor* $F : \mathcal{C} \to \mathcal{D}$*,*

[2] This notations shall remind of the term *Inner Product Space*.

[3] Although the tensor products of $\mathcal{C}$ and $\mathcal{D}$ are different functors we denote them with the same symbol $\otimes$ for notational convenience. We will assume the same convention for α, λ, ρ and (when we work with braidings, traces and daggers) also for c, Tr and $\dagger$. It has to be derived from the surrounding objects what tensor product (or associator, unitor, swap, trace or dagger) is meant.

[4] In this work we will usually not mention strongness explicitly. However this convention must not be mixed up with the notion of a monoidal functor in other papers where the requirement on $\phi_{-,-}, \phi_0$ to be isomorphisms is dropped.

a natural isomorphism $\phi : F - \otimes F- \Rightarrow F(- \otimes -)$ *and an isomorphism* $\phi_0 : I_{\mathcal{D}} \to FI_{\mathcal{C}}$ *making the diagrams*

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[r, "\alpha_{FA,FB,FC}"] \arrow[d, "\phi_{A,B} \otimes \mathrm{id}_{FC}"'] & FA \otimes (FB \otimes FC) \arrow[d, "\mathrm{id}_{FA} \otimes \phi_{B,C}"] \\
F(A \otimes B) \otimes FC \arrow[d, "\phi_{A \otimes B, C}"'] & FA \otimes F(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{A,B,C}"] & F(A \otimes (B \otimes C))
\end{tikzcd} \tag{1.7}$$

as well as

$$\begin{tikzcd}
I_{\mathcal{D}} \otimes FA \arrow[r, "\lambda_{FA}"] \arrow[d, "\phi_0 \otimes \mathrm{id}_{FA}"] & FA \\
FI_{\mathcal{C}} \otimes FA \arrow[r, "\phi_{I_{\mathcal{C}},A}"] & F(I_{\mathcal{C}} \otimes A) \arrow[u, "F\lambda_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes I_{\mathcal{D}} \arrow[r, "\rho_{FA}"] \arrow[d, "\mathrm{id}_{FA} \otimes \phi_0"] & FA \\
FA \otimes FI_{\mathcal{C}} \arrow[r, "\phi_{A,I_{\mathcal{C}}}"] & F(A \otimes I_{\mathcal{C}}) \arrow[u, "F\rho_A"']
\end{tikzcd} \tag{1.8}$$

commute for all objects A, B, C *of* $\mathcal{C}$ *(cf. [2] p.255f).*

(ii) *If* $\mathcal{C}, \mathcal{D}$ *are braided, then* F *is called a* braided monoidal functor *if* F *is a monoidal functor and additionally the diagrams*

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "c_{FA,FB}"] \arrow[d, "\phi_{A,B}"] & FB \otimes FA \arrow[d, "\phi_{B,A}"] \\
F(A \otimes B) \arrow[r, "Fc_{A,B}"] & F(B \otimes A)
\end{tikzcd} \tag{1.9}$$

commute for all objects A, B *of* $\mathcal{C}$*. If in this case* $\mathcal{C}$ *and* $\mathcal{D}$ *are symmetric* F *is called a* symmetric monoidal functor *(cf. [14] p.15).*

(iii) *If* $\mathcal{C}, \mathcal{D}$ *are traced symmetric monoidal categories, then* F *is a* traced symmetric monoidal functor *if it is a symmetric monoidal functor and additionally satisfies*

$$F\mathrm{Tr}^X_{A,B}(f) = \mathrm{Tr}^{FX}_{FA,FB}(\phi^{-1}_{B,X} \circ Ff \circ \phi_{A,X}) \tag{1.10}$$

for all objects A, B, X *and arrows* $f : A \otimes X \to B \otimes X$ *of* $\mathcal{C}$*.* (cf. [4], p.452)

(iv) *Now suppose that* $\mathcal{C}, \mathcal{D}$ *are dagger categories. A* dagger functor *from* $\mathcal{C}$ *to* $\mathcal{D}$ *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *satisfying* $F(f^\dagger) = (Ff)^\dagger$ *for all arrows* f *of* $\mathcal{C}$ *(cf. [14] p.49). If* $\mathcal{C}, \mathcal{D}$ *are (traced symmetric/symmetric/braided) monoidal dagger categories, then a* (traced symmetric/symmetric/braided) dagger functor *is a (traced symmetric/symmetric/braided) functor that is also a dagger functor additionally satisfying*

$$\phi^\dagger_{A,B} = \phi^{-1}_{A,B}$$

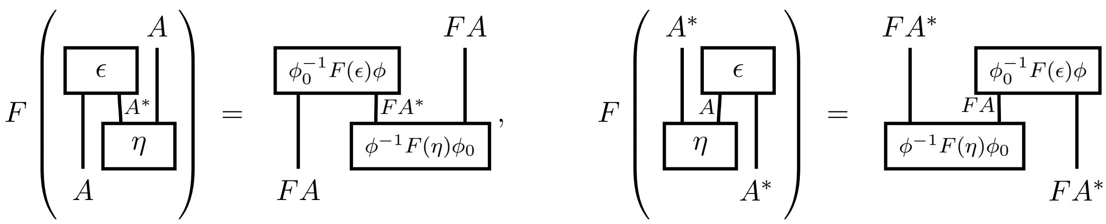
for all objects A, B *of* $\mathcal{C}$*.*

We notice that these axioms resemble those we would intuitively expect if we tried to define a homomorphism (namely $FA \otimes FB = F(A \otimes B), FI_{\mathcal{C}} = I_{\mathcal{D}}$ etc.) but instead of identifying two objects with each other, we again only require a systematic reversible way of converting the one into the other by means of ϕ and ϕ_0.

1.4.2 Compatibility with compact closed structure

Another way of understanding the motivation behind these definitions is to observe that in the presence of a functor $F : \mathcal{C} \to \mathcal{D}$ the category $\mathcal{D}$ holds two different structures arising from the traced monoidal structure in $\mathcal{D}$, i.e. $\alpha^{(\mathcal{D})}, \lambda^{(\mathcal{D})}, \rho^{(\mathcal{D})}, c^{(\mathcal{D})}, \mathrm{Tr}^{(\mathcal{D})}, \dagger^{(\mathcal{D})}$, and arising from the image of the structure of $\mathcal{C}$ under F, i.e. $F\alpha^{(\mathcal{C})}, F\lambda^{(\mathcal{C})}, F\rho^{(\mathcal{C})}, Fc^{(\mathcal{C})}, F\mathrm{Tr}^{(\mathcal{C})}$, $F\dagger^{(\mathcal{C})}$. The previous axioms now ensure that both structures essentially coincide, i.e. after equalizing involved types by means of ϕ, ϕ_0, they yield the same morphisms. Thus when applying F on a term $M \in \mathrm{Mor}\,\mathcal{C}$ we may interchange F with the appearing structure in M when inserting the $\phi_{-,-}, \phi_0$ to avoid type errors. If an equation can be derived by using this fact we will say it follows from *the compatibility of F with the (traced symmetric/symmetric/braided) monoidal (dagger) structure of $\mathcal{C}$ and $\mathcal{D}$*. As this will simplify arguments in the next chapter a lot, we illustrate it for the following example, showing at the same time why we did not define a compact closed functor.

Example 1.16 Let $\mathcal{C}$ be a compact closed category and $A \in \mathrm{Ob}\,\mathcal{C}$ with unit η and counit ϵ. Moreover let $\mathcal{D}$ be a symmetric monoidal category and $(F, \phi, \phi_0) : \mathcal{C} \to \mathcal{D}$ a symmetric monoidal functor. Then the equations



follow from compatibility of F with the monoidal structures of $\mathcal{C}$, $\mathcal{D}$. Due to $F\mathrm{id}_A = \mathrm{id}_{FA}$ these equations prove that FA has a dual object in $\mathcal{D}$ namely FA^* witnessed by

$$\begin{array}{rll} \phi_{A^*,A}^{-1} \circ F\eta \circ \phi_0 : & I_{\mathcal{D}} \to FA^* \otimes FA & \text{as unit and} \\ \phi_0^{-1} \circ F\epsilon \circ \phi_{A,A^*} : & FA \otimes FA^* \to I_{\mathcal{D}} & \text{as counit.} \end{array}$$

Hence symmetric monoidal functors already preserve dualities, making the notion of a compact closed functor redundant (cf. [20] p.86). Compatibility of F with the symmetric monoidal structure of $\mathcal{C}, \mathcal{D}$ also shows its compatibility with the trace operations induced by the compact closed structures of $\mathcal{C}, \mathcal{D}$.

Finally an easy calculation shows that if $(F, \phi, \phi_0) : \mathcal{C} \to \mathcal{D}$ and $(G, \phi', \phi'_0) : \mathcal{D} \to \mathcal{E}$ are (traced symmetric/symmetric/braided) monoidal (dagger) functors then so is $(G \circ$

$F, G\phi_{-,-} \circ \phi'_{F-,F-}, G\phi_0 \circ \phi'_0)$. Hence the next definition makes sense.

Definition 1.17 *Let* **TrSMCat** / **CompCCat** / **TrSMDCat** / **DCompCCat** *denote the category whose objects are (small) traced / compact closed / traced dagger / dagger compact closed categories and whose morphisms are traced / symmetric monoidal / traced dagger / symmetric monoidal dagger functors between them. Due to Proposition 1.8 and the last example there are forgetful functors*

$$\mathscr{U} : \mathbf{CompCCat} \to \mathbf{TrSMCat} \qquad \text{and} \qquad \mathscr{U} : \mathbf{DCompCCat} \to \mathbf{TrSMDCat}$$

which we will both denote with the same symbol $\mathscr{U}$.

1.4.3 Monoidal natural transformations and 2-categories

Later results will also require to have a notion for natural transformations preserving monoidal structure.

Definition 1.18 *A* monoidal natural transformation *is a natural transformation $\theta : F \Rightarrow G$ between monoidal functors (F, ϕ, ϕ_0) and (G, ϕ', ϕ'_0) of the same type $(\mathcal{C}, \otimes, I_\mathcal{C}) \to (\mathcal{D}, \otimes, I_\mathcal{D})$, making the diagrams*

$$\begin{tikzcd} FA \otimes FB \arrow[r, "\theta_A \otimes \theta_B"] \arrow[d, "\phi_{A,B}"] & GA \otimes GB \arrow[d, "\phi'_{A,B}"] \\ F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"] & G(A \otimes B) \end{tikzcd} \qquad \begin{tikzcd} FI_\mathcal{C} \arrow[rr, "\theta_{I_\mathcal{C}}"] \arrow[dr, "\phi_0"] & & GI_\mathcal{C} \arrow[dl, "\phi'_0"] \\ & I_\mathcal{D} & \end{tikzcd} \tag{1.11}$$

commute for all objects A, B of $\mathcal{C}$. An equivalence of categories that is witnessed by monoidal natural transformations is called a monoidal equivalence. *If there is a monoidal natural isomorphism between (F, ϕ, ϕ_0) and (G, ϕ', ϕ'_0), then we shortly write $(F, \phi, \phi_0) \cong (G, \phi', \phi'_0)$ or just $F \cong G$.*

The following example will be useful in the second chapter and demonstrates how our various Coherence Theorems enable us to deal with structures without explicitly constructing them.

Example 1.19 Let $(F, \phi, \phi_0) : \mathcal{C} \to \mathcal{D}$ be a traced (dagger) functor between the traced (dagger) category $\mathcal{C}$ and the compact closed category $\mathcal{D}$ with chosen duality structure $({}^*, \eta, \epsilon)$. Consider the functor

$$\begin{array}{cccc} F - \otimes I_\mathcal{D} : & \mathcal{C} & \to & \mathcal{D} \\ & C & \mapsto & FC \otimes I_\mathcal{D} \\ & f : C \to C' & \mapsto & Ff \otimes \mathrm{id}_{I_\mathcal{D}} : FC \otimes I_\mathcal{D} \to FC' \otimes I_\mathcal{D} \end{array}$$

The compact closed structure of $\mathcal{D}$ together with monoidality of F provides isomorphisms

$$\phi'_{C,C'} : (FA \otimes I_D) \otimes (FB \otimes I_D) \to F(A \otimes B) \otimes I_{\mathcal{D}} \quad \text{and} \phi'_0 : I_{\mathcal{D}} \to FI_{\mathcal{C}} \otimes I_{\mathcal{D}}$$

turning $F - \otimes I_{\mathcal{D}}$ into a traced (dagger) functor. They likewise provide isomorphisms

$$\theta_C : FC \to FC \otimes I_{\mathcal{D}}$$

demonstrating $F \cong F - \otimes I_{\mathcal{D}}$. We do not need to write down the explicit algebraic expressions as coherence for (dagger) compact closed categories (Theorem 1.10) as well as compatibility of F with the traced (dagger) structure of $\mathcal{C}, \mathcal{D}$ with the above type requirements induce unique isomorphisms of the above types consisting only of the structure F and $\mathcal{D}$ provide. Coherence and compatibility make it also obvious that the diagrams (1.7), (1.8), (1.9) and (1.11) commute. Ultimately naturality of ϕ and θ as well as (1.10) and the requirements of Definition 1.14 (iv) are all trivial in the graphical language[5]. Likewise we have

$$F \cong F - \otimes FI_{\mathcal{C}}, \quad F \cong I_{\mathcal{D}} \otimes F-, \quad F \cong FI_{\mathcal{C}} \otimes F - .$$

In the following we claim that a natural transformation is monoidal without further explanation if it follows from an analogous application of coherence and compatibility.

We observe that for monoidal transformations $\theta : F \Rightarrow G$, $\chi : G \Rightarrow H$ between monoidal dagger functors F, G, H of type $\mathcal{C} \Rightarrow \mathcal{D}$ the pointwise composition $\chi \circ \theta : F \Rightarrow H$ satisfies again (1.11). Hence together with $\{\mathrm{id}_{FC} \mid C \in \mathrm{Ob}\,\mathcal{C}\}$ as identity transformation we obtain a category with monoidal (dagger) functors of type $\mathcal{C} \Rightarrow \mathcal{D}$ as objects and monoidal transformations as morphisms between them. Moreover monoidal transformations can be composed *horizontally*, i.e. in compliance with the composition of functors. For $F, G : \mathcal{C} \Rightarrow \mathcal{D}$, $F', G' : \mathcal{D} \Rightarrow \mathcal{E}$ and $\theta : F \Rightarrow G$, $\theta' : F' \Rightarrow G'$ the definition

$$(\theta' \bullet \theta)_C := \theta'_{GC} \circ F'\theta_C = G'\theta_C \circ \theta'_{FC} : F'FC \to G'GC$$

yields a new transformation $\theta' \bullet \theta : F' \circ F \Rightarrow G' \circ G$ which can easily be seen to be monoidal natural if so are its components. Thus (symmetric) monoidal/compact closed/traced (dagger) categories together with (symmetric) monoidal/traced (dagger) functors and monoidal transformations between them form a so-called 2-*category* which is essentially a category with "morphisms between morphisms".

Definition 1.20 *A* 2-category $\mathscr{C}$ *consists of* 0-cells *or objects* $A, B, C, \ldots,$ 1-cells *or arrows* $f, g, h, \ldots : A \to B$ *between objects and* 2-cells $\theta, \chi, \ldots : f \Rightarrow g$ *between arrows of the same type such that*

(i) *objects and arrows form together a category* $\mathcal{C}$*,*

[5] cf. Corollary 2.10 and proof of Theorem 2.14 to see why diagrams do not have to be drawn explicitly for this argument.

(ii) *for all objects A, B the hom-set $\mathcal{C}(A, B)$ together with the* 2*-cells between its elements form a category and*

(iii) *for all objects A, B, C there is a strictly associative functor*

$$\bullet : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

acting as the usual composition on 1*-cells that satisfies*

- $\theta \bullet \mathrm{id}_{\mathrm{id}_A} = \mathrm{id}_{\mathrm{id}_B} \bullet \theta$ *for all* 2*-cells $\theta : f \Rightarrow g$ between* 1*-cells $f, g : A \to B$ and*
- *the* interchange law *holds for $\circ, \bullet$, i.e. for all suitably typed* 2*-cells $\theta, \theta', \chi, \chi'$ we have*
$$(\chi \circ \theta) \bullet (\chi' \circ \theta') = (\chi \bullet \chi') \circ (\theta \bullet \theta').$$

2-categories are special instances of *enriched categories.* These are - slightly informally speaking - categories whose homsets are not sets or classes containing arrows as elements but objects of a monoidal category themselves, allowing us to endow categories with additional structure and e.g. define categories, that also contain 2-cells between their arrows. A detailed treatment of Enriched Category Theory can be found in [15].

Example 1.21 As discussed above, the categories **TrSMCat**, **CompCCat**, **TrSMDCat**, and **DCompCCat** are 2-categories with monoidal transformations serving as 2-cells.

Remark 1.22 If $\mathscr{C}$ is a 2-category then we gain a further category $\mathcal{C}^-$, consisting of the objects of $\mathscr{C}$ and equivalence classes of arrows of $\mathscr{C}$ identifying two arrows if there is an invertible 2-cell between them (w.r.t. vertical composition). Indeed for $f \cong g$, $f' \cong g'$ witnessed by θ, θ' the horizontal composition $\theta \bullet \theta'$ witnesses $f' \circ f \cong g' \circ g$ (if types match) due to the interchange law for $\circ, \bullet$ ensuring that $(\theta \bullet \theta')^{-1} = \theta^{-1} \bullet \theta'^{-1}$. Thus the composition is well-defined for equivalence classes.

Chapter 2

Traced Networks and the Int-construction

As we have seen in the last chapter, calculations in certain kinds of categories can essentially be done in a graphical language allowing deformations that do not affect its topological structure. Therefore we formalize in the first section what a (traced dagger) diagram resp. its topological structure is by endowing abstract diagrams (*networks*) which in turn form new (traced dagger) categories $\mathbf{Net}\,\mathscr{S}$.

After that the transition from networks to diagrams of a specific category will lead to the concept of *interpretations* which can be reduced to some key information, as we will see in the second section. In the last two sections we describe how to extend a traced (dagger) category $\mathcal{C}$ to a (dagger) compact closed category $\mathbf{Int}\,\mathcal{C}$. This construction will turn out to be the inverse (in the sense of adjoints) to the forgetful functor $\mathscr{U} : (\mathbf{D})\mathbf{CompCCat} \to \mathbf{TrSM}(\mathbf{D})\mathbf{Cat}$ and will therefore allow us to restrict ourselves to traced (dagger) daigrams. Here the coherence theorems of the first chapter as well as compatibility of monoidal functors will play a crucial role for verifying the required axioms.

2.1 The categories $\mathbf{Net}\,\mathscr{S}$

2.1.1 Signatures

First we have to specify what labels we allow for the boxes and wires appearing in diagrams. This leads to the notion of networks over a given *signature*[1].

Definition 2.1 *A* signature *is a tuple* $\mathscr{S} = (S, F, \mathrm{dom}, \mathrm{cod})$ *where* S *is a set of* sorts *or* object labels*,* F *is a set of* function labels *and functions*

$$\mathrm{dom} : F \to S^*, \qquad \mathrm{cod} : F \to S^*$$

[1]The terminology and definitions we will give here will be a combination of those of Selinger (cf. [1] p.6) and Hasegawa, Hofmann and Plotkin (cf. [3] p.5f) to achieve both highest generality and maximal conciseness.

assigning to every $f \in F$ *a* domain *or* source $\mathrm{dom}(f)$ *and a* codomain *or* target $\mathrm{cod}(f)$. *A* dagger signature *is a signature* $\mathscr{S} = (S, F, \mathrm{dom}, \mathrm{cod})$ *together with a fixpoint-free involution* $\dagger : F \to F$, *i.e.* $f^\dagger \neq f$ *and* $f^{\dagger\dagger} = f$, *satisfying* $\mathrm{dom}(f^\dagger) = \mathrm{cod}(f)$ *and* $\mathrm{cod}(f^\dagger) = \mathrm{dom}(f)$ *for all* $f \in F$. *In this case we regard* F *as splitted into a set of* non-dagger labels $f_1, f_2, \dots$ *and* dagger-labels $f_1^\dagger, f_2^\dagger, \dots$. *In both cases we define* $F_\bullet = \{\bullet_A | A \in S\}$.

Note that we will also write $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ for the domain and codomain of a map f in a particular category. Clearly words $\mathcal{A} = A_1 \dots A_l \in S^*$ correspnd to tensor products $A_1 \otimes \cdots \otimes A_l$ and $\bullet_A$ matches trivial cycles labelled with A in usual diagrams.

We will particularly pay attention to the *universal signature* $\mathscr{S}_\infty = (S_\infty, F_\infty, \mathrm{dom}, \mathrm{cod})$ consisting of countable sets S_∞, F_∞ of object and arrow labels and typing functions $\mathrm{dom}, \mathrm{cod}$ ensuring that for all $\mathcal{A}, \mathcal{B} \in S_\infty^*$ there are countably many $f \in F_\infty$ of type $\mathcal{A} \to \mathcal{B}$. The infinite amount of arrows of every type allows us to build any possible traced network (up to a change of names) and is therefore *universal* for traced networks. Endowing $\mathscr{S}_\infty$ with the dagger $\dagger$ yields the *universal dagger signature* if there are countably many non-dagger and countably many dagger labels of type $\mathcal{A} \to \mathcal{B}$ for all $\mathcal{A}, \mathcal{B}$. We will also write it as $\mathscr{S}_\infty$ since the presence of a dagger can be derived from the context.

2.1.2 Networks

As the precise shape of wires in a diagram does not matter topologically and boxes can be pulled along wires, the topological structure can be captured by just memorizing which boxes appear and which output is connected to which input.

Definition 2.2 *Let* $\mathscr{S} = (S, F, \mathrm{dom}, \mathrm{cod})$ *be a signature. A* (traced symmetric monoidal) $\mathscr{S}$-network N *from* $\mathcal{A}$ *to* $\mathcal{B}$ *where* $\mathcal{A}, \mathcal{B} \in S^*$ *is a triple* $N = (B, \ell, \pi)$ *consisting of a finite set of* boxes B, *a function* $\ell : B \to F_\bullet$ *labeling every box with a sort and a bijection* π. *When denoting* $B \cup \{\bullet_N\}$ *with* $B_\bullet$ *and defining*

$$\mathrm{cod}(\ell(\bullet_N)) = \mathcal{A}, \qquad \mathrm{dom}(\ell(\bullet_N)) = \mathcal{B}$$ [2]

π *shall be a bijection of type*

$$\pi : \mathrm{Out}_N \to \mathrm{In}_N$$

where

$$\begin{aligned}\mathrm{Out}_N &= \{(b, i) \mid b \in B_\bullet,\ 1 \leq i \leq |\mathrm{cod}(\ell(b))|\},\\ \mathrm{In}_N &= \{(j, b) \mid b \in B_\bullet,\ 1 \leq j \leq |\mathrm{dom}(\ell(b))|\},\end{aligned}$$

satisfying the typing condition

$$\pi(b, i) = (j, b') \quad \Rightarrow \quad \mathrm{cod}(\ell(b))_i = \mathrm{dom}(\ell(b'))_j$$

[2]Note that - contrary to what this definition suggests - $\ell(\bullet)$ itself is not defined.

for all $(b,i) \in \mathrm{Out}_N$, $(j,b') \in \mathrm{In}_N$. *When misunderstandings are impossible we omit the index of* $\bullet_N$. *We will also write* $N : \mathcal{A} \to \mathcal{B}$. *If* $\mathscr{S}$ *is a dagger-signature, we will also speak of* (traced symmetric monoidal) dagger networks. N *is called* closed, *if* $\mathcal{A} = \mathcal{B} = \epsilon_0$, *it is called* simple *if* $\ell(B) \subset F$, *i.e. if no boxes of* N *are labelled with* $\bullet_A$ *for any* A. *If the network* N_0 *arises from* N *by dropping all its trivial cycles, then we call* N_0 *the* kernel *of* N. *If we just speak of (dagger) networks without specifying* $\mathscr{S}$ *we implicitly refer to* $\mathscr{S}_\infty$.

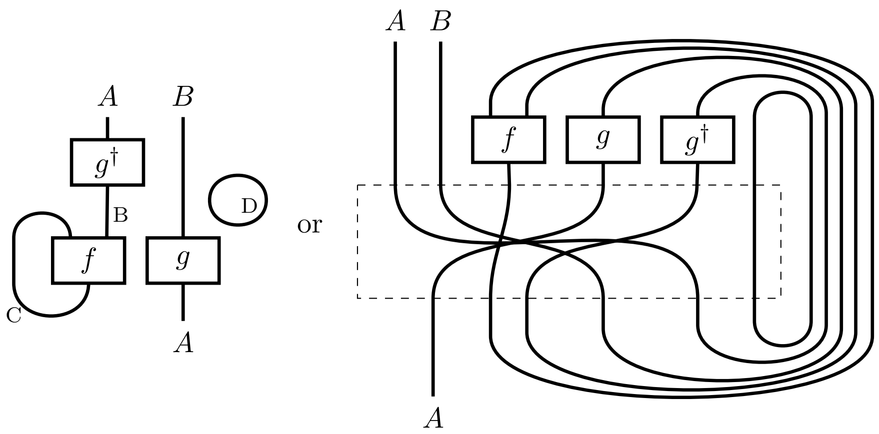
The following example illustrates how (dagger) networks match our intuition of diagrams and why they belong to the language of traced (dagger) categories.

Example 2.3 Consider the network $N = (B, \ell, \pi) : A \to AB$ with

$$B = \{b_0, b_1, b_2, b_3\}, \quad \ell(b_0) = \bullet_D,\ \ell(b_1) = f,\ \ell(b_2) = g,\ \ell(b_3) = g^\dagger$$

x	$(\bullet, 1)$	$(b_1, 1)$	$(b_1, 2)$	$(b_2, 1)$	$(b_3, 1)$
$\pi(x)$	$(1, b_2)$	$(1, b_1)$	$(1, b_3)$	$(2, \bullet)$	$(1, \bullet)$

over the dagger signature $\mathscr{S} = (\{A, B, C, D\}, \{f, g, f^\dagger, g^\dagger\}, \mathrm{dom}, \mathrm{cod})$ assigning to f, g the types $f : C \to CB$ and $g : A \to B$. Drawing this data leads to the diagram



The right version indicates already how a network can be interpreted as a diagram belonging to the graphical language of traced (dagger) categories. We will get back to the framed part of the right diagram later.

Apparently a diagram can be expressed as the trace of a morphism if and only if every output of a box is connected to an input and vice versa, which explains the definition of the bijection π of a network. Clearly we can define (dagger) compact closed or symmetric monoidal (dagger) networks in a similar manner. For compact closed categories e.g. we could work over a signature with an additional fixpoint-free involution $* : S \to S$ and define π as an involutive fixpoint-free permutation of $\mathrm{Out}_N \cup \mathrm{In}_N$ (memorizing the ends of the appearing wires). Similarly we identify symmetric monoidal (dagger) networks as those simple traced (dagger) networks that can be endowed with a function ord : $B \to \{1, 2, \ldots, |B|\}$ subject to the additional condition

$$\pi(b, i) = (j, b') \quad \Rightarrow \quad \mathrm{ord}(b) < \mathrm{ord}(b') \qquad \forall b, b' \in B$$

to ensure that no wire is going back to previous (w.r.t. ord) wires (which cannot be done without traces). However, we will focus on traced (dagger) networks in the following (and call them just (dagger) networks) as they will turn out to be sufficient for a completeness analysis of (dagger) compact closed categories (cf. Section 2.4.2).

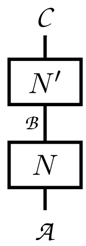
Before we continue with the question of how to formalize the transition from a formal network to a diagram in a certain category, we observe that the graphical intuition for networks - as demonstrated in Example 2.3 - makes it evident that (dagger) networks form a traced (dagger) category itself. Indeed when we represent a network abstractly as a box with an input and output wire like morphisms in the graphical calculus, we can read off the formal definitions of the composition, tensor products and traces of networks from the graphical language for diagrams (cf. [3] p.8).

2.1.3 The category of traced (dagger) networks

Definition 2.4 *Let $\mathscr{S}$ be a (dagger) signature.*

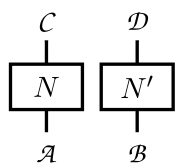
(i) *For any $\mathcal{A} = A_1 \ldots A_l \in S^*$ we define the* identity network $\mathrm{id}_{\mathcal{A}}$ *of $\mathcal{A}$ as $(\emptyset, \emptyset, \pi)$ where $\pi(\bullet, i) = \pi(i, \bullet)$ for all $1 \leq i \leq r$.*

(ii) *For networks $N = (B, \ell, \pi) : \mathcal{A} \to \mathcal{B}$, $N' = (B', \ell', \pi') : \mathcal{B} \to \mathcal{C}$ we define their* composition $N' \circ N$ *as the network $(B \mathbin{\dot\cup} B', \ell \mathbin{\dot\cup} \ell', \Pi) : \mathcal{A} \to \mathcal{C}$ where*



$$\Pi(b, i) = \begin{cases} \pi'(b, i) & \text{if } b \in B' \\ \pi'(\bullet_{N'}, j) & \text{if } \pi(b, i) = (j, \bullet_N) \text{ for some } j \leq |\mathcal{B}| \\ \pi(b, i) & \text{otherwise} \end{cases}$$

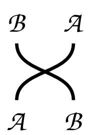
(iii) *For networks $N = (B, \ell, \pi) : \mathcal{A} \to \mathcal{C}$, $N' = (B', \ell', \pi') : \mathcal{B} \to \mathcal{D}$ we define their* tensor network *as*



$$N \otimes N' = (B \mathbin{\dot\cup} B', \ell \mathbin{\dot\cup} \ell', \pi \mathbin{\dot\cup} \pi') : \mathcal{AB} \to \mathcal{CD}$$

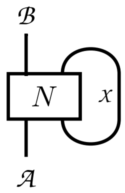
(where the input $(\bullet, 1), \ldots, (\bullet, |\mathcal{A}| + |\mathcal{B}|)$ is interpreted as $(\bullet_N, 1)$, $\ldots, (\bullet_N, |\mathcal{A}|), (\bullet_{N'}, 1), \ldots, (\bullet_{N'}, |\mathcal{B}|)$ and analogously for $(-, \bullet)$.)

(iv) *For strings $\mathcal{A}, \mathcal{B} \in S^*$ we define its* swap network *as $(\emptyset, \emptyset, \pi) : \mathcal{AB} \to \mathcal{BA}$ with*



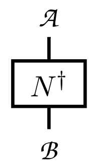
$$\pi(\bullet, i) = \begin{cases} (i + |\mathcal{B}|, \bullet) & \text{if } i \leq |\mathcal{A}| \\ (i - |\mathcal{A}|, \bullet) & \text{otherwise} \end{cases}$$

(v) *For a network* $N = (B, \ell, \pi) : \mathcal{A}X \to \mathcal{B}X$ *we define its trace as* $\mathrm{Tr}^X_{\mathcal{A},\mathcal{B}}(N) = (B, \ell, \Pi) : \mathcal{A} \to \mathcal{B}$ *where*



$$\Pi(b,i) = \begin{cases} \pi(\bullet, j') & \textit{if } \pi(b,i) = (j, \bullet) \textit{ for some } j > |\mathcal{A}| \\ & \textit{where } j' = j + |\mathcal{B}| - |\mathcal{A}| \\ \pi(b,i) & \textit{otherwise} \end{cases}$$

(vi) *If* $\mathscr{S}$ *is a dagger-network we define for a network* $N = (B, \ell, \pi) : \mathcal{A} \to \mathcal{B}$ *its* dagger network *as* $N^\dagger : (B, \ell^\dagger, \pi^\dagger) : \mathcal{B} \to \mathcal{A}$ *where*



$$\ell^\dagger(b) = \ell(b)^\dagger, \qquad \pi^\dagger(b,i) = (j, b') \;\text{ iff }\; \pi(b', j) = (i, b).$$

As we figured out in the last chapter, all axioms of traced (dagger) categories do not change the topological structure of their diagrams. Also the above constructions apparently comply with the diagrammatic intuition. Thus we see immediately that networks form a traced (dagger) category.

Theorem/Definition 2.5 *Let* $\mathscr{S}$ *be a (dagger) signature. Traced (dagger) networks over* $\mathscr{S}$ *form a* strict *traced symmetric monoidal (dagger) category with* S^* *as the set of objects, concatenation as the tensor product of objects, networks* $N : \mathcal{A} \to \mathcal{B}$ *as morphisms from* $\mathcal{A}$ *and* $\mathcal{B}$ *and the identity and swap networks as well as the composition, tensor and trace operation are given in Definition 2.4. In the following we denote the category of traced (dagger) networks with* $\mathbf{Net}\,\mathscr{S}$.

Clearly one could also verify the axioms formally without relying on their visualization.

2.1.4 Network homomorphisms

Finally we have to specify when two networks are isomorphic. This leads to the notion of a network homomorphism and therewith to a 2-category structure of $\mathbf{Net}\,\mathscr{S}$.

Proposition/Definition 2.6 *Let* $\mathscr{S}$ *be a (dagger) signature and* $N = (B, \ell, \pi)$, $N' = (B', \ell', \pi')$ *two* $\mathscr{S}$*-networks of equal types. A* network homomorphism ψ *from* N *to* N' *is a map* $\psi : B \to B'$ *satisfying* $\ell'(\psi(b)) = \ell(b)$ *for all* $b \in B$ *and*

$$\pi(b,i) = (j, b') \;\Rightarrow\; \pi'(\psi(b), i) = (j, \psi(b')) \qquad \forall (b,i) \in \mathrm{In}_N,\; (j, b') \in \mathrm{Out}_N \tag{2.1}$$

where we set $\psi(\bullet) = \bullet$. $\mathbf{Net}\,\mathscr{S}$ *together with network homomorphisms forms a* 2*-category.*

Proof. The vertical composition is given by the usual composition of functions, while the horizontal composition of two homomorphisms ψ, ψ' is given by $\psi \,\dot\cup\, \psi'$. It is straightforward to verify that this structure indeed satisfies the axioms of a 2-category. $\square$

In the following we will write $N \cong N'$ in order to indicate the existence of a network isomorphism between N and N'. Observe that a network homomorphism ψ is already a network isomorphism when ψ is bijective.

2.2 Network Interpretations

2.2.1 Definitions

The transition from abstract networks to concrete morphisms (resp. diagrams) in a specific category must be done in a structure preserving way. The traced structure of $\mathbf{Net}\,\mathscr{S}$ enables us to realize this by means of functors.

Definition 2.7 *Let $\mathscr{S}$ be a (dagger) signature and $\mathcal{C}$ a traced (dagger) category. A* $\mathcal{C}$-interpretation *for* $\mathbf{Net}\,\mathscr{S}$ *or just* $\mathcal{C}$-$\mathcal{C}$-interpretation *or $\mathscr{S}$- is a traced (dagger) functor*

$$[\![\ \ ]\!] : \mathbf{Net}\,\mathscr{S} \to \mathcal{C}.$$

We denote the category of interpretations $\mathbf{Net}\,\mathscr{S} \to \mathcal{C}$ and monoidal transformations between them with $\mathbf{Int}(\mathscr{S}, \mathcal{C})$. In the case $\mathscr{S} = \mathscr{S}_\infty$ we will usually just speak of $\mathcal{C}$-interpretations.

As we noted in section 1.4 $\mathbf{Int}(\mathscr{S}, \mathcal{C})$ is indeed a category. Before considering some examples, we aim to classify $\mathbf{Int}(\mathscr{S}, \mathcal{C})$, i.e. to understand in terms of what data a general interpretation can be described. As it turns out an interpretation is essentially determined by its actions on the object and arrow labels of $\mathscr{S}$. In order to make this precise we introduce the notion of a *model* of a signature (cf. [3] p.9).

Definition 2.8 *Let $\mathscr{S} = (S, F, \mathrm{dom}, \mathrm{cod})$ be a signature and $\mathcal{C}$ a monoidal category. A* $\mathcal{C}$-model *of $\mathscr{S}$ is a map*

$$[\![\ \ ]\!] : \begin{array}{c} S \to \mathrm{Ob}\,\mathcal{C} \\ F \to \mathrm{Mor}\,\mathcal{C} \end{array}$$

such that for all $f \in F$ of type $A_1 \dots A_m \to B_1 \dots B_n$ the map $[\![f]\!]$ is of type

$$[\![A_1]\!] \otimes \cdots \otimes [\![A_m]\!] \to [\![B_1]\!] \otimes \cdots \otimes [\![B_n]\!].^3$$

If $\mathscr{S}$ is a dagger signature and $\mathcal{C}$ a monoidal dagger category we additionally require $[\![f^\dagger]\!] = [\![f]\!]^\dagger$ for all $f \in F$. A model homomorphism $\theta : [\![\ \ ]\!]_0 \Rightarrow [\![\ \ ]\!]_1$ *from the $\mathcal{C}$-model $[\![\ \ ]\!]_0$ to the $\mathcal{C}$-model $[\![\ \ ]\!]_1$ is a bunch of maps*

$$\{\theta_A : [\![A]\!]_0 \to [\![A]\!]_1 \mid A \in S\}$$

[3] We generally associate two most left tensor factors when brackets are omitted.

such that all diagrams

$$\begin{array}{ccc} [\![A_1]\!]_0 \otimes \cdots \otimes [\![A_m]\!]_0 & \xrightarrow{[\![f]\!]_0} & [\![B_1]\!]_0 \otimes \cdots \otimes [\![B_n]\!]_0 \\ {\scriptstyle \theta_{A_1} \otimes \cdots \otimes \theta_{A_m}} \downarrow & & \downarrow {\scriptstyle \theta_{B_1} \otimes \cdots \otimes \theta_{B_n}} \\ [\![A_1]\!]_1 \otimes \cdots \otimes [\![A_m]\!]_1 & \xrightarrow{[\![f]\!]_1} & [\![B_1]\!]_1 \otimes \cdots \otimes [\![B_n]\!]_1 \end{array} \tag{2.2}$$

commute. Models together with model homomorphism form a category, which we will denote as $\mathbf{Mod}(\mathscr{S}, \mathcal{C})$*.*

In fact, for model homomorphisms $\theta : [\![\ ]\!]_0 \Rightarrow [\![\ ]\!]_1$, $\theta' : [\![\ ]\!]_1 \Rightarrow [\![\ ]\!]_2$ we can define $\theta' \circ \theta : [\![\ ]\!]_0 \Rightarrow [\![\ ]\!]_2$ by $(\theta' \circ \theta)_A : \theta'_A \circ \theta_A$. When writing $\theta_{\mathcal{A}}$ for $\theta_{A_1} \otimes \cdots \otimes \theta_{A_m}$ where $\mathcal{A} = A_1 \ldots A_m$ the interchange law yields $(\theta' \circ \theta)_{\mathcal{A}} = \theta'_{\mathcal{A}} \circ \theta_{\mathcal{A}}$ ensuring that (2.2) also commutes for $\theta' \circ \theta$.

2.2.2 Equivalence of models and interpretations

Theorem 2.9 *There is an equivalence of categories establishing*

$$\mathbf{Mod}(\mathscr{S}, \mathcal{C}) \simeq \mathbf{Int}(\mathscr{S}, \mathcal{C}).$$

Before proving this Theorem we should recognize its connection to Theorem 1.5. While Proposition 2.5 - implicitly stating that all axioms of traced (dagger) categories hold in the graphical calculus - expresses completeness of the graphical language, this Theorem is a consequence of soundness of the graphical language. Indeed as networks capture only the topological data of a diagram one might assume that a network can be interpreted in multiple ways since there are (formally) different diagrams with the same topological structure. But Theorem 2.9 shows that a given $\mathcal{C}$-model has an essentially unique extension to a traced (dagger) functor, so that all diagrams with the same topological structure must represent the same morphism (up to a unique isomorphism provided by the monoidal structure).

Proof. We construct a fully faithful functor

$$\begin{array}{cccc} \mathscr{F} : & \mathbf{Mod}(\mathscr{S}, \mathcal{C}) & \to & \mathbf{Int}(\mathscr{S}, \mathcal{C}) \\ & [\![\ ]\!] & \mapsto & [\![\ ]\!]^{\mathscr{F}} \\ & \theta : [\![\ ]\!]_0 \Rightarrow [\![\ ]\!]_1 & \mapsto & \mathscr{F}\theta : [\![\ ]\!]_0^{\mathscr{F}} \Rightarrow [\![\ ]\!]_1^{\mathscr{F}} \end{array}$$

that is essentially surjective, witnessing $\mathbf{Mod}(\mathscr{S}, \mathcal{C}) \simeq \mathbf{Int}(\mathscr{S}, \mathcal{C})$. Let $[\![\ ]\!] \in \operatorname{Ob} \mathbf{Mod}(\mathscr{S}, \mathcal{C})$ be a $\mathcal{C}$-model. For $\mathcal{A} = A_1 \ldots A_m \in \operatorname{Ob} \mathbf{Net}\, \mathscr{S}$ we set

$$[\![\mathcal{A}]\!]^{\mathscr{F}} = [\![A_1]\!] \otimes \cdots \otimes [\![A_m]\!].$$

Now for a $\mathscr{S}$-network $N = (B, \ell, \pi) : \mathcal{A} \to \mathcal{B}$ we define

$$[\![N]\!]^{\mathscr{F}} = \mathrm{Tr}^{\bigotimes_{b\in B}[\![\mathrm{cod}(\ell(b))]\!]}_{[\![\mathcal{A}]\!]^{\mathscr{F}},[\![\mathcal{B}]\!]^{\mathscr{F}}}\left(\left(\mathrm{id}_{[\![\mathcal{B}]\!]^{\mathscr{F}}} \otimes \bigotimes_{b\in B}[\![\ell(b)]\!]\right) \circ \hat{\pi}\right) \tag{2.3}$$

where

$$\begin{array}{rcccl} \hat{\pi} : & \displaystyle\bigotimes_{b\in B_\bullet}[\![\mathrm{cod}(\ell(b))]\!] & \longrightarrow & \displaystyle\bigotimes_{b\in B_\bullet}[\![\mathrm{dom}(\ell(b))]\!] & \quad \textit{resp.} \\ & [\![\mathcal{A}]\!]^{\mathscr{F}} \otimes \displaystyle\bigotimes_{b\in B}[\![\mathrm{cod}(\ell(b))]\!] & \longrightarrow & [\![\mathcal{B}]\!]^{\mathscr{F}} \otimes \displaystyle\bigotimes_{b\in B}[\![\mathrm{dom}(\ell(b))]\!] & \end{array}$$

denotes the isomorphism induced by π, i.e. the map whose diagrammatic expression in the graphical language consists only of wires connecting the objects belonging to $(b,i) \in \mathrm{Out}_N$ and $(j,b') \in \mathrm{In}_N$ iff $\pi(b,i) = (j,b')$ (cf. [3], p.7). Unpacking this definition shows that $[\![N]\!]^{\mathscr{F}}$ is a morphism in $\mathcal{C}$ which - represented as a diagram - indeed has the same topological structure, i.e. for every $b \in B$ it contains a box labelled with $\ell(b)$ and the wires proceed according to π. Example 2.3 can be consulted as an illustration for how the trace expression (2.3) looks like in the graphical language and how an arbitrary (dagger) network can be deformed into a diagram of shape (2.3). In 2.3 the framed part of the right diagram visualizes $\hat{\pi}$.

Soundness and completeness of the graphical calculus for traced (dagger) categories (cf. Theorem 1.5) imply that $[\![\ \ ]\!]^{\mathscr{F}}$ is a traced (dagger) functor when observing that $[\![\mathrm{id}_{\mathcal{A}}]\!]^{\mathscr{F}} = \mathrm{id}_{[\![\mathcal{A}]\!]^{\mathscr{F}}}$ and choosing

$$\phi_{\mathcal{A},\mathcal{B}} : [\![\mathcal{A}]\!]^{\mathscr{F}} \otimes [\![\mathcal{B}]\!]^{\mathscr{F}} \to [\![\mathcal{A}\mathcal{B}]\!]^{\mathscr{F}}\ [4]$$

as the unique isomorphism provided by the monoidal structure of $\mathcal{C}$.

Now let $[\![\ \ ]\!]_0, [\![\ \ ]\!]_1$ be two $\mathcal{C}$-models and $\theta : [\![\ \ ]\!]_0 \to [\![\ \ ]\!]_1$ a model homomorphism between them. Defining $\mathscr{F}\phi : [\![\ \ ]\!]_0^{\mathscr{F}} \Rightarrow [\![\ \ ]\!]_1^{\mathscr{F}}$ as the bunch of maps given by

$$(\mathscr{F}\theta)_{\mathcal{A}} = \theta_{\mathcal{A}} = \theta_{A_1} \otimes \cdots \otimes \theta_{A_m} : [\![\mathcal{A}]\!]_0^{\mathscr{F}} \to [\![\mathcal{B}]\!]_1^{\mathscr{F}}$$

for all $\mathcal{A} = A_1 \dots A_m$ yields a monoidal natural transformation. Indeed

$$\begin{tikzcd} {[\![\mathcal{A}]\!]_0} \arrow[r, "{[\![N]\!]_0}"] \arrow[d, "(\mathscr{F}\phi)_{\mathcal{A}}"'] & {[\![\mathcal{B}]\!]_0} \arrow[d, "(\mathscr{F}\phi)_{\mathcal{B}}"] \\ {[\![\mathcal{A}]\!]_1} \arrow[r, "{[\![N]\!]_1}"] & {[\![\mathcal{B}]\!]_1} \end{tikzcd} \tag{2.4}$$

commutes for all networks N representing a single morphism by definition of a model homomorphism. It also commutes if $[\![N]\!]_0, [\![N]\!]_1$ can be expressed in terms of the traced

[4]An inductive definition like e.g. $[\![\mathcal{A}\mathcal{B}]\!]^{\mathscr{F}} = [\![\mathcal{A}]\!]^{\mathscr{F}} \otimes [\![\mathcal{B}]\!]^{\mathscr{F}}$ is not possible as strings do not contain inner brackets. Therefore the isomorphisms $\phi_{\mathcal{A},\mathcal{B}}$ witnessing monoidality of $[\![\ \ ]\!]^{\mathscr{F}}$ will in general not be the identity.

(dagger) structure of $\mathcal{C}$ due to coherence for traced (dagger) categories (Thm. 1.5). Hence the diagrams (1.11) (in our context) commute and also (2.4) commutes for general N since $\llbracket\ \rrbracket_0$ and $\llbracket\ \rrbracket_1$ are compatible with the traced (dagger) structure of $\mathbf{Net}\,\mathscr{S}$ and $\mathcal{C}$. Thus $\mathscr{F}\phi$ is a monoidal natural transformation. So we have seen that $\mathscr{F}$ in fact maps into $\mathbf{Int}\,(\mathscr{S},\mathcal{C})$ and due to $\mathscr{F}\theta = \theta$ we see that $\mathscr{F}$ is a functor.

In order to show that $\mathscr{F}$ is essentially surjective we consider an arbitrary interpretation $\llbracket\ \rrbracket \in \operatorname{Ob}\mathbf{Int}(\mathscr{S},\mathcal{C})$ and define $\llbracket\ \rrbracket_0 \in \operatorname{Ob}\mathbf{Mod}(\mathscr{S},\mathcal{C})$ as the model induced by $\llbracket\ \rrbracket$. The property of $\llbracket\ \rrbracket$ to be a traced (dagger) functor provides unique isomorphisms

$$\varphi_{\mathscr{A}} : \llbracket\mathscr{A}\rrbracket_0^{\mathscr{F}} = \llbracket A_1\rrbracket \otimes \cdots \otimes \llbracket A_m\rrbracket \to \llbracket\mathscr{A}\rrbracket = \llbracket A_1 \ldots A_m\rrbracket$$

for all $\mathscr{A} = A_1 \ldots A_m$. Hence for all networks $N : \mathscr{A} \to \mathscr{B}$ the diagram

$$\begin{tikzcd} \llbracket\mathscr{A}\rrbracket_0^{\mathscr{F}} \arrow[r, "\llbracket N\rrbracket_0^{\mathscr{F}}"] \arrow[d, "\varphi_{\mathscr{A}}"'] & \llbracket\mathscr{B}\rrbracket_0^{\mathscr{F}} \arrow[d, "\varphi_{\mathscr{B}}"] \\ \llbracket\mathscr{A}\rrbracket \arrow[r, "\llbracket N\rrbracket"] & \llbracket\mathscr{B}\rrbracket \end{tikzcd} \tag{2.5}$$

commutes as it is another way of expressing compatibility of $\llbracket\ \rrbracket$ with the traced (dagger) structure of $\mathbf{Net}\,\mathscr{S}$ and $\mathcal{C}$. Here we took soundness of the graphical calculus into account ensuring that $\llbracket N\rrbracket_0^{\mathscr{F}}$ and $\llbracket N\rrbracket$ are equal up to the positions of appearing applications of $\llbracket\ \rrbracket$. Thus compatibility indeed applies. Likewise compatibility proves commutativity of the diagrams (1.11). Therefore $\varphi : \llbracket\ \rrbracket_0^{\mathscr{F}} \to \llbracket\ \rrbracket$ is a monoidal natural isomorphism.

Finally $\mathcal{F}$ is faithful since it acts as the identity on the bunch of morphisms describing a model homomorphism and $\mathcal{F}$ is full because every monoidal transformation $\theta : \llbracket\ \rrbracket_0^{\mathscr{F}} \Rightarrow \llbracket\ \rrbracket_1^{\mathscr{F}}$ between the images of $\mathcal{C}$-models $\llbracket\ \rrbracket_0, \llbracket\ \rrbracket_1$ under $\mathscr{F}$ induces a model homomorphism $\theta_0 : \llbracket\ \rrbracket_0 \to \llbracket\ \rrbracket_1$ that gets mapped to θ under $\mathcal{F}$. Indeed we have $(\mathcal{F}\theta_0)_{\mathscr{A}} = (\theta_0)_{\mathscr{A}} = \theta_{\mathscr{A}}$ where $(\theta_0)_{\mathscr{A}} = \theta_{\mathscr{A}}$ follows from commutativity of the diagrams (1.11). $\square$

The next statement expresses the intuitive fact that morphisms are equal (up to a unique isomorphism provided by the monoidal structure) if the wires of their diagrammatic expressions are pairwise differently labelled. We explicitly state it as it will simplify later considerations about the **Int**-construction.

Corollary 2.10 *Let $f_1, \ldots, f_n$ be arrows of a traced (dagger) category $\mathcal{C}$ and p, q two equally typed arrows of $\mathcal{C}$ that can be algebraically expressed in terms of the traced symmetric monoidal (dagger) structure (i.e. $\circ, \otimes, \mathrm{id}, \alpha, \lambda, \rho, c, \mathrm{tr}, \dagger$) by using $f_1, \ldots, f_n$ at most once. Moreover assume every object of $\mathcal{C}$ appears at most once (as a tensor factor) among* $\mathrm{dom}(f_1), \ldots, \mathrm{dom}(f_n), \mathrm{cod}(p) = \mathrm{cod}(q)$ *and at most once among* $\mathrm{cod}(f_1), \ldots, \mathrm{cod}(f_n), \mathrm{dom}(p) = \mathrm{dom}(q)$. *Then $p = q$.*

Although it is formally a consequence of the previous Theorem (and therewith called a Corollary) we only state the key observation that the typing constraints require p, q having the same diagrammatic expression up to topological equivalence as the given data $f_1, \ldots, f_n$ considered as typed morphism labels of a signature only allows the expression of a unique network. Thus $p = q$ follows from soundness of the graphical calculus.

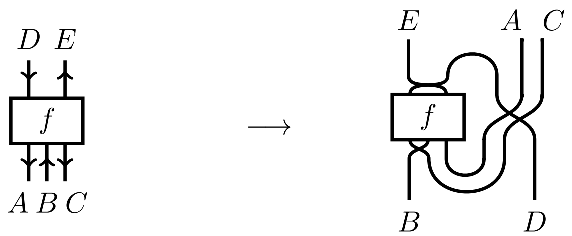
2.3 The Int-Construction

2.3.1 Overall Idea

We have seen how a compact closed structure on $\mathcal{C}$ induces trace operations making $\mathcal{C}$ to a traced category. Joyal, Street and Verity elaborated in [4] that the opposite direction can be gone by embedding $\mathcal{C}$ into a category $\mathbf{Int}\,\mathcal{C}$ whose objects are pairs of objects of $\mathcal{C}$. Although this construction was orignially done for balanced monoidal categories we will only discuss it for symmetric monoidal categories and observe that it equally works in the presence of a dagger.

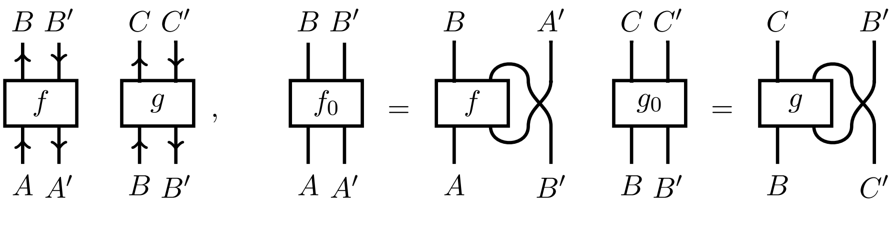
The **Int**-construction[5] relies on the general thought that arrows whose domain or codomain contain duals can reversibly transformed into arrows without duals in their types by swapping duals to the right and applying cups and caps[6] afterwards. Consider as an example

$$f : A^* \otimes B \otimes C^* \to D^* \otimes E \quad \longrightarrow \quad f_0 : B \otimes D \to E \otimes A \otimes C$$



(2.6)

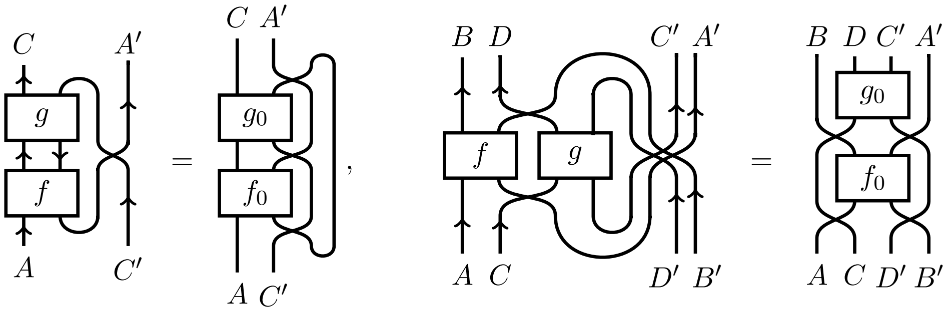
In the following we will say f_0 is *induced* by f. As we will see now, expressions in compact closed categories containing those f can be imitated by means of the f_0 using a traced structure only. Indeed when having two maps f, g in compact closed categories as well as their induced maps f_0, g_0



[5]Despite similar terminology the category $\mathbf{Int}\,\mathcal{C}$ and the category of interpretations $\mathbf{Int}(\mathscr{S}, \mathcal{C})$ are of course entirely different objects.

[6]In order to keep diagrams as clear as possible we will also make use of counterclockwise directed caps. If the corresponding category does not provide a dagger this shall be understood as an abbreviation for $\epsilon_A \circ c_{A^*,A}$.

we can express the map induced by $g \circ f$ in terms of f_0, g_0 as the next diagram illustrates. Likewise in the presence of another map $f' : C \otimes C'^* \to D \otimes D'^*$ the induced map of $f \otimes f'$ can be written in terms of f_0, f_0'.



(2.7)

Hence when starting with a traced category, we can consider its maps as induced maps f_0 without duals in their types and try to recover the compact closed structure initiating the transition (2.6). Then (2.7) describes how the composition and tensor operation have to be defined for f_0, g_0 resp. f_0, f_0' in order to match the corresponding notions for f, g resp. f, f'. In order to memorize which tensor factors of a morphisms domain and codomain are duals the objects of the hidden compact closed structure must be of kind (A, A') where the right coordinate stores the dual part. Hence $f_0 : A \otimes B' \to B \otimes A'$ - representing f - is of type $(A, A') \to (B, B')$ in the compact closed context. Likewise the right hand side of the second equation in (2.7) is of type

$$(A \otimes C) \otimes (C' \otimes A') \;\to\; (B \otimes D) \otimes (D' \otimes B')$$

enforcing the definition $(A, A') \otimes (C, C') = (A \otimes C, C' \otimes A')$. We are now in a position to understand the next definitions.

2.3.2 Formal Construction

For a traced category $\mathcal{C}$ we define $\mathbf{Int}\,\mathcal{C}$ in several steps:

- An object (A, A') of $\mathbf{Int}\,\mathcal{C}$ shall be a pair of objects in $\mathcal{C}$: $\mathrm{Ob}\,\mathbf{Int}\,\mathcal{C} := \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}$.

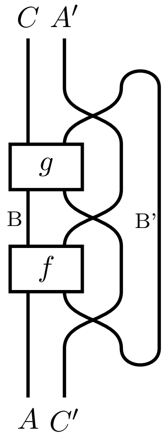
- An arrow $\mathfrak{f} : (A, A') \to (B, B')$ shall be a map f in $\mathcal{C}$ of type $A \otimes B' \to B \otimes A'$:

$$\mathrm{Hom}_{\mathbf{Int}\,\mathcal{C}}((A, A'), (B, B')) = \mathrm{Hom}_{\mathcal{C}}(A \otimes B', B \otimes A')$$

We will often notationally distinguish between $\mathfrak{f}$ and f in order to indicate whether we work in $\mathbf{Int}\,\mathcal{C}$ or $\mathcal{C}$. This will be helpful since the composition and tensor product will differ in $\mathbf{Int}\,\mathcal{C}$ and $\mathcal{C}$. Interestingly - due to Corollary 2.10 - the above definitions already determine that the composition of

$$(A, A') \xrightarrow{\mathfrak{f}} (B, B') \xrightarrow{\mathfrak{g}} (C, C')$$

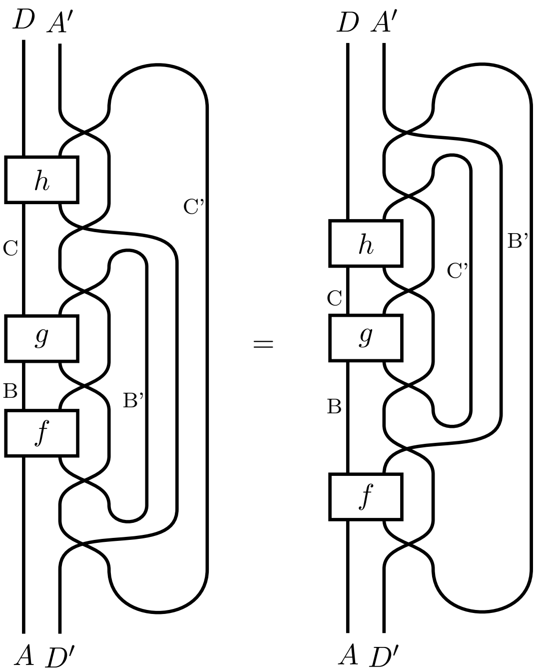
must be defined like suggested by (2.7) to be of type $(A, A') \to (C, C')$, namely

$\mathfrak{g} \circ \mathfrak{f} :=$  (2.8)

Also $\mathrm{id}_{(A,A')} := \mathrm{id}_{A \otimes A'}$ is the only possible definition of a map from (A, A') to (A, A') that can be defined by means of the traced structure of $\mathcal{C}$. The category axioms

$$(\mathfrak{h} \circ \mathfrak{g}) \circ \mathfrak{f} = \mathfrak{h} \circ (\mathfrak{g} \circ \mathfrak{f}), \qquad \mathfrak{f} \circ \mathrm{id}_{(A,A')} = \mathrm{id}_{(B,B')} \circ \mathfrak{f}$$

could be verified by using the graphical calculus. We illustrate this for associativity of the composition:

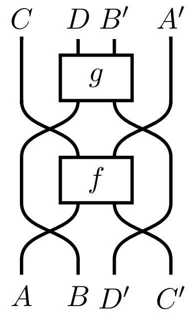
$\mathfrak{h} \circ (\mathfrak{g} \circ \mathfrak{f}) =$  $= (\mathfrak{h} \circ \mathfrak{g}) \circ \mathfrak{f}$ (2.9)

But this equality is obvious due to Corollary 2.10 as no object $X \in \mathrm{Ob}\,\mathcal{C}$ appears multiple times among $\mathrm{dom}(\mathfrak{f}), \mathrm{dom}(\mathfrak{g}), \mathrm{dom}(\mathfrak{h}), \mathrm{cod}((\mathfrak{h} \circ \mathfrak{g}) \circ \mathfrak{f})$ and among $\mathrm{cod}(\mathfrak{f}), \mathrm{cod}(\mathfrak{g}), \mathrm{cod}(\mathfrak{h})$, $\mathrm{dom}((\mathfrak{h} \circ \mathfrak{g}) \circ \mathfrak{f})$. Hence instead of drawing diagrams and observing that the labelling of wires makes equality obvious, we will rely on Corollary 2.10 immediately for $\mathfrak{f} \circ \mathrm{id}_{(A,A')} = \mathrm{id}_{(B,B')} \circ \mathfrak{f}$ and later equations.

For objects $(A, A'), (B, B')$ we define its tensor product

$$(A, A') \otimes (B, B') = (A \otimes B, B' \otimes A')$$

as explained in the first subsection, and (I, I) as the tensor unit. Again Corollary 2.10 ensures that, having morphisms $f : A \otimes C' \to C \otimes A'$ and $g : B \otimes D' \to D \otimes B'$ there is only one way to define a map of type $(A \otimes B) \otimes (D' \otimes C') \to (C \otimes D) \otimes (B' \otimes A')$, namely in the way (2.7) indicates:

$\mathfrak{f} \otimes \mathfrak{g} \quad =$ 

(2.10)

Also, associators, unitors, and swaps are determined already. An evaluation of their type requirements reveals their types:

$$\begin{array}{llll}
\alpha_{(A,A'),(B,B'),(C,C')} : & & & \\
((A, A') \otimes (B, B')) \otimes (C, C') & \to & (A, A') \otimes ((B, B') \otimes (C, C')) & \text{in } \mathbf{Int}\,\mathcal{C} \\
((A \otimes B) \otimes C) \otimes ((C' \otimes B') \otimes A') & \to & (A \otimes (B \otimes C)) \otimes (C' \otimes (B' \otimes A')) & \text{in } \mathcal{C}
\end{array}$$

$$\begin{array}{lcccl}
\lambda_{(A,A')} : & (I, I) \otimes (A, A') & \to & (A, A') & \text{in } \mathbf{Int}\,\mathcal{C} \\
 & (I \otimes A) \otimes A' & \to & A \otimes (A' \otimes I) & \text{in } \mathcal{C} \\
\rho_{(A,A')} : & (A, A') \otimes (I, I) & \to & (A, A') & \text{in } \mathbf{Int}\,\mathcal{C} \\
 & (A \otimes I) \otimes A' & \to & A \otimes (I \otimes A') & \text{in } \mathcal{C} \\
c_{(A,A'),(B,B')} : & (A, A') \otimes (B, B') & \to & (B, B') \otimes (A, A') & \text{in } \mathbf{Int}\,\mathcal{C} \\
 & (A \otimes B) \otimes (A' \otimes B') & \to & (B \otimes A) \otimes (B' \otimes A') & \text{in } \mathcal{C}
\end{array}$$

Due to Mac Lane's Coherence Theorem 1.2 there are unique isomorphisms provided by the monoidal strcuture of $\mathcal{C}$ with the required types which serve as the definition for α, λ, ρ, c. Corollary 2.10 shows that they are natural. We demonstrate this for naturality of c.

$$\begin{tikzcd}
(A, A') \otimes (B, B') \arrow[r, "c_{(A,A'),(B,B')}"] \arrow[d, "\mathfrak{f} \otimes \mathfrak{g}"'] & (B, B') \otimes (A, A') \arrow[d, "\mathfrak{g} \otimes \mathfrak{f}"] \\
(C, C') \otimes (D, D') \arrow[r, "c_{(C,C'),(D,D')}"] & (D, D') \otimes (C, C')
\end{tikzcd}$$

Apparently both composed morphisms are of the same type and the only boxes their diagrammatic expressions contain are labelled with f, g. But $A, A', B, B', C, C', D, D'$ appear exactly once among $\mathrm{dom}(f), \mathrm{dom}(g), \mathrm{cod}(c_{(C,C'),(D,D')} \circ (\mathfrak{f} \otimes \mathfrak{g}))$ and among $\mathrm{cod}(f), \mathrm{cod}(g)$, $\mathrm{dom}(c_{(C,C'),(D,D')} \circ (\mathfrak{f} \otimes \mathfrak{g}))$. Hence the above diagram commutes.

Also due to Mac Lane's Coherence Theorem the triangle, pentagon and hexagon diagrams commute as well as $c^{-1}_{-1,-2} = c_{-2,-1}$ turning $\mathbf{Int}\,\mathcal{C}$ into a symmetric monoidal category. Unit and counit are also determined by the monoidal structure of $\mathcal{C}$ when defining

$$(A, A')^* = (A', A).$$

Then η and ϵ have to be of the following types:

$$\eta_{(A,A')}: \begin{array}{ccccc} (I, I) & \to & (A', A) \otimes (A, A') & \text{in } \mathbf{Int}\,\mathcal{C} \\ I \otimes (A' \otimes A) & \to & (A' \otimes A) \otimes I & \text{in } \mathcal{C} \end{array}$$

$$\epsilon_{(A,A')}: \begin{array}{ccccc} (A, A') \otimes (A', A) & \to & (I, I) & \text{in } \mathbf{Int}\,\mathcal{C} \\ (A \otimes A') \otimes I & \to & I \otimes (A \otimes A') & \text{in } \mathcal{C} \end{array}$$

Thus Mac Lane's Coherence Theorem provides the definitions for η and ϵ and shows why the snake equations are valid. Hence $\mathbf{Int}\,\mathcal{C}$ holds a compact closed structure.

If $\mathcal{C}$ is a traced dagger category, we can convey the dagger structure to $\mathbf{Int}\,\mathcal{C}$. Indeed for an arrow $\mathfrak{f} : (A, A') \to (B, B')$ given by $f : A \otimes B' \to B \otimes A'$ we gain a dagger functor by defining $\mathfrak{f}^\dagger : (B, B') \to (A, A')$ as $f^\dagger : B \otimes A' \to A \otimes B'$. Compatibilty of $\dagger$ and $\otimes$ follows from Corollary 2.10 while all other axioms in (ii),(iii),(v) of Definition 1.9 hold due to coherence for dagger compact closed categories (cf. Theorem 1.10). We summarize our results.

Proposition 2.11 *Let $\mathcal{C}$ be a traced (dagger) category. Then the category $\mathbf{Int}\,\mathcal{C}$ as defined above is a (dagger) compact closed category.*

The following fact explains how $\mathcal{C}$ can be regarded as a subcategory of $\mathbf{Int}\,\mathcal{C}$.

Proposition 2.12 *Let $\mathcal{C}$ be a traced (dagger) category. Then the map*

$$A \in \mathrm{Ob}\,\mathcal{C} \quad \mapsto \quad \mathfrak{I}(A) = (A, I) \in \mathrm{Ob}\,\mathbf{Int}\,\mathcal{C}$$

induces a traced (dagger) functor $\mathfrak{I} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ acting on functions as

$$f : A \to B \quad \mapsto \quad \mathfrak{I}(f) : (A, I) \to (B, I) \text{ given by } \rho_B^{-1} \circ f \circ \rho_A : A \otimes I \to B \otimes I$$

Proof. Obviously $\rho_B^{-1} \circ f \circ \rho_A$ is the only way of defining a map of type $(A, I) \to (B, I)$ for a given $f : A \to B$. Clearly $\mathfrak{I}(\mathrm{id}_A) = \mathrm{id}_{(A,I)}$ and compatibility with the composition follows from (2.8) showing that the compositions in $\mathcal{C}$ and $\mathbf{Int}\,\mathcal{C}$ coincide if the right wires represent the tensor unit I. The unique way to define ϕ and ϕ_0 by means of the monoidal structure of $\mathcal{C}$ is $\phi_0 : \mathfrak{I}I \to (I,I)$ as the identity $\mathrm{id}_{(I,I)}$ and

$$\phi_{A,B} : \mathfrak{I}A \otimes \mathfrak{I}B \to \mathfrak{I}(A \otimes B)$$

as the unique isomorphism of type

$$(A \otimes B) \otimes I \to (A \otimes B) \otimes (I \otimes I)$$

The diagrams (1.7), (1.8) and (1.9) commute due to Mac Lane's Coherence Theorem while $\mathfrak{I}(f^\dagger) = \mathfrak{I}(f)^\dagger$ is straightforward and compatibility with the trace operations (1.10) as well as naturality of ϕ follow from Corollary 2.10. □

2.4 Correspondence between traced and compact closed categories

2.4.1 $\mathscr{U}$ and Int are adjoints

The **Int**-construction can be considered as the map

$$\mathbf{Int} : \mathrm{Ob}\,\mathbf{TrSM(D)Cat} \to \mathrm{Ob}\,\mathbf{(D)CompCCat}, \qquad \mathcal{C} \mapsto \mathbf{Int}\,\mathcal{C}$$

raising the question whether **Int** can be generalized to a functor. An analysis of appearing typing requirements reveals that only one definition is possible.

Proposition/Definition 2.13 *Let $(F, \phi, \phi_0) : \mathcal{C} \to \mathcal{D}$ be a traced (dagger) functor. We define $\overline{F}$ as the functor*

$$\begin{array}{llcl} \overline{F}: & \mathbf{Int}\,\mathcal{C} & \to & \mathbf{Int}\,\mathcal{D} \\ & (A, A') & \mapsto & (FA, FA') \\ & \mathfrak{f} : (A, A') \to (B, B') & \mapsto & \overline{F}\mathfrak{f} : (FA, FA') \to (FB, FB') \\ & & & \text{given by } \phi^{-1}_{B,A'} \circ Ff \circ \phi_{A,B'} : FA \otimes FB' \to FB \otimes FA' \end{array}$$

Then **Int** *defined as*

$$\begin{array}{lccc} \mathbf{Int}: & \mathbf{TrSM(D)Cat} & \to & \mathbf{(D)CompCCat} \\ & \mathcal{C} & \mapsto & \mathbf{Int}\,\mathcal{C} \\ & F : \mathcal{C} \to \mathcal{D} & \mapsto & \mathbf{Int}\,F := \overline{F} : \mathbf{Int}\,\mathcal{C} \to \mathbf{Int}\,\mathcal{D} \end{array}$$

is a functor.

Proof. Type requirements force us to define

$$\begin{array}{lccl} \overline{\phi}_{(A,A'),(B,B')} = \phi_{A,B} \otimes \phi^{-1}_{B',A'} : & & & \\ \overline{F}(A, A') \otimes \overline{F}(B, B') & \to & \overline{F}((A, A') \otimes (B, B')) & \text{in } \mathbf{Int}\,\mathcal{D} \\ (FA \otimes FB) \otimes F(B' \otimes A') & \to & F(A \otimes B) \otimes (FB' \otimes FA') & \text{in } \mathcal{D} \end{array}$$

$$\overline{\phi}_0 = \phi_0 \otimes \phi_0^{-1} : \quad \begin{array}{cccl} (I_\mathcal{D}, I_\mathcal{D}) & \to & \overline{F}(I_\mathcal{C}, I_\mathcal{C}) & \text{in } \mathbf{Int}\,\mathcal{D} \\ I_\mathcal{D} \otimes FI_\mathcal{C} & \to & FI_\mathcal{C} \otimes I_\mathcal{D} & \text{in } \mathcal{D} \end{array}$$

Now $(\overline{F}, \overline{\phi}, \overline{\phi}_0)$ is in fact a symmetric monoidal (dagger) functor, as all axioms (including functoriality of $\overline{F}$) are special instances of compatibility of F with the traced (dagger) structures of $\mathcal{C}$ and $\mathcal{D}$. For traced (dagger) functors $(F, \phi, \phi_0) : \mathcal{C} \to \mathcal{D}$, $(G, \phi', \phi'_0) : \mathcal{D} \to \mathcal{E}$ functoriality of **Int** follows from taking the definition of the composition for monoidal functors into account. In fact the functors $\mathbf{Int}(G \circ F)$ and $\mathbf{Int}\,G \circ \mathbf{Int}\,F$ obviously coincide on objects, map identities to identities and for every morphism $\mathfrak{f} : (A, A') \to (B, B')$ of $\mathbf{Int}\,\mathcal{C}$ we have

$$\begin{aligned} \mathbf{Int}(G \circ F)(f) &= (G\phi_{B,A'} \circ \phi'_{B,A'})^{-1} \circ GFf \circ (G\phi_{A,B'} \circ \phi'_{A,B'}) \\ &= \phi'^{-1}_{B,A'} \circ G\left(\phi^{-1}_{B,A'} \circ Ff \circ \phi_{A,B'}\right) \circ \phi'_{A,B'} = (\mathbf{Int}\,G \circ \mathbf{Int}\,F)(\mathfrak{f}). \end{aligned}$$ □

Theorem 2.14 *The functors*

$$\mathbf{Int} : \mathbf{TrSM(D)Cat} \to \mathbf{(D)CompCCat}, \quad \mathscr{U} : \mathbf{(D)CompCCat} \to \mathbf{TrSM(D)Cat}$$

form an adjunction up to monoidal isomorphisms *from* **TrSM(D)Cat** *to* **(D)CompCCat** *with universal arrows*

$$\mathfrak{I} : \mathcal{C} \to \mathscr{U}\,\mathbf{Int}\,\mathcal{C}$$

for all traced (dagger) categories $\mathcal{C}$.

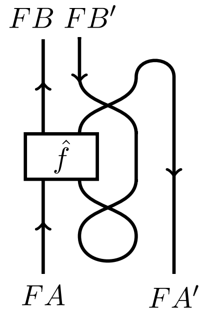
Before proving this Theorem we remind that an *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, called *left* and *right adjoints* together with natural bijections $\theta_{A,B} : \mathrm{Hom}(FA, B) \xrightarrow{\cong} \mathrm{Hom}(A, GB)$ for $A \in \mathrm{Ob}\,\mathcal{C}$, $B \in \mathrm{Ob}\,\mathcal{D}$. This is equivalent to having a bunch of *universal arrows* $\eta_C : C \to GD_C$ that is natural in $C \in \mathrm{Ob}\,\mathcal{C}$ where universality means that for every $D \in \mathrm{Ob}\,\mathcal{D}$ and arrow $f : C \to GD$ in $\mathcal{C}$ there is a unique arrow $\hat{f} : D_C \to D$ in $\mathcal{D}$ such that $f = G\hat{f} \circ \eta_C$. Indeed in this case one can define F as $FC = D_C$ and Ff as the unique arrow satisfying $\eta_{C'} \circ f = GFf \circ \eta_C$ for all maps $f : C \to C'$ in $\mathcal{C}$ (cf. [2] pp.79-86).

The phrase *up to monoidal isomorphisms* in Theorem 2.14 means that we identify two arrows if there is a monoidal isomorphism between them. Hence formally **Int** and $\mathscr{U}$ consititute an adjunction between the categories $\mathbf{TrSM(D)Cat}^-$ and $\mathbf{(D)CompCCat}^-$ whose arrows are equivalence classes of monoidally isomorphic maps (cf. Remark 1.22). Here we make implicit use of the observation that the images of two monoidally isomorphic arrows under **Int** resp. $\mathscr{U}$ remain monoidally isomorphic. While this is trivial for $\mathscr{U}$ in the case $F \cong G$ witnessed by θ we obtain $\overline{F} \cong \overline{G}$ by means of θ where $\theta_{(A,A')} = \theta_A \otimes \theta^{-1}_{A'}$.

Proof. Let $\mathcal{C}$ be a traced (dagger) category, $\mathcal{D}$ a (dagger) compact closed category and $(F, \phi, \phi_0) : \mathcal{C} \to \mathscr{U}\mathcal{D} = \mathcal{D}$ a traced (dagger) functor between them. In the following we refer to a preliminarily chosen duality structure $(^*, \eta, \epsilon)$ of $\mathcal{D}$. We have to show that up to a monoidal isomorphism there is a unique symmetric monoidal (dagger) functor

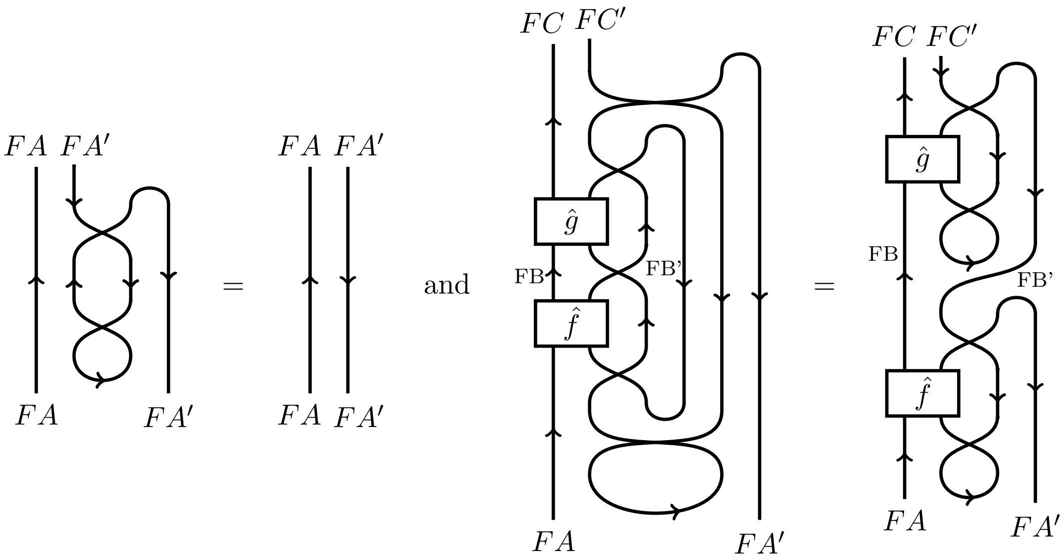
$\hat{F} : \mathbf{Int}\,\mathcal{C} \to \mathcal{D}$ satisfying $\mathfrak{I} \circ \mathscr{U}\hat{F} = \mathfrak{I} \circ \hat{F} \cong F$. We set $\hat{F}(A, A') := FA \otimes (FA')^*$ for all objects (A, A') of $\mathbf{Int}\,\mathcal{C}$. For arrows $\mathfrak{f} : (A, A') \to (B, B')$ we define $\hat{f} = \phi^{-1}_{B,A'} \circ Ff \circ \phi_{A,B'}$ and

$\hat{F}\mathfrak{f} :=$



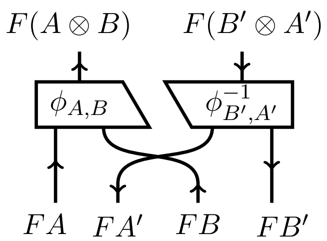
. (2.11)

Functoriality of $\hat{F}$ follows from



for all $\mathfrak{f} : (A, A') \to (B, B')$, $\mathfrak{g} : (B, B') \to (C, C')$. Again instead of drawing diagrams explicitly we can also argue that the given data permits only one topological structure since all wires are differently labelled. Hence in the following we will make use of this more concise argument[7]. We extend $\hat{F}$ to a monoidal functor by defining

$\Phi_{(A,A'),(B,B')} :=$



$: \hat{F}(A, A') \otimes \hat{F}(B, B') \to \hat{F}(A \otimes B, B' \otimes A')$

[7]We do not formalize this way of reasoning unlike for the case of traced (dagger) networks (cf. Corollary 2.10) as a formal proof would require a statement analogous to Theorem 2.9 for (dagger) compact closed categories which we do not formally elaborate in this work.

for all objects $(A, A'), (B, B')$ of $\mathbf{Int}\,\mathcal{C}$ and

$$\Phi_0 = (\phi_0 \otimes (\phi_0^{-1})^*) \circ c_{I_\mathcal{D}^*, I_\mathcal{D}} \circ \eta_{I_\mathcal{D}} : I_\mathcal{D} \to \hat{F}(I_\mathcal{C}, I_\mathcal{C}).$$

The diagrams (1.7), (1.8) and (1.9) commute since after plugging in all definitions these diagrams only contain morphisms that can be expressed in terms of (F, ϕ, ϕ_0) and the (dagger) compact closed structure of $\mathcal{D}$. Their commutativity follows then from coherence for (dagger) compact closed categories as well as compatibility of F with the traced (dagger) structures of $\mathcal{C}$ and $\mathcal{D}$. Naturality of Φ is obvious since analyzing the types of the appearing morphisms reveals that the wires in the corresponding diagrams are pairwise differently labelled so that they must be topologically equivalent. Finally in the presence of a dagger the same argument works for $\hat{F}(f^\dagger) = (\hat{F}f)^\dagger$ when observing $(\phi_{B,A'}^{-1} \circ Ff \circ \phi_{A,B'})^\dagger = \phi_{A,B'}^{-1} \circ Ff^\dagger \circ \phi_{B,A'}$ for all $\hat{f} \in \mathrm{Mor}\,\mathbf{Int}\,\mathcal{C}$ and it is straightforward to verify $\Phi^\dagger = \Phi^{-1}$ as well as $\Phi_0^\dagger = \Phi_0^{-1}$ when using $\phi^\dagger = \phi^{-1}, \phi_0^\dagger = \phi_0^{-1}$. Hence $\hat{F}$ is a morphism in $(\mathbf{D})\mathbf{CompCCat}$.

We clearly have

$$\hat{F}\mathfrak{I}C = \hat{F}(C, I_\mathcal{C}) = FC \otimes (FI_\mathcal{C})^*$$

for all $C \in \mathcal{C}$, thus $F \cong \hat{F}\mathfrak{I}$ is witnessed by

$$\{\theta_C : FC \to FC \otimes (FI_\mathcal{C})^* \mid C \in \mathrm{Ob}\,\mathcal{C}\}$$

where the θ_C denote the unique isomorphisms provided by the (dagger) compact closed structure of $\mathcal{D}$ of the required type. We verified in Example 1.19 that θ and similar expressions are in fact monoidal isomorphisms.

Now suppose there is another symmetric monoidal (dagger) functor $G : \mathbf{Int}\,\mathcal{C} \to \mathcal{D}$ with $F \cong G \circ \mathfrak{I}$. We consider the duality structure of $\mathcal{D}$ induced by G (cf. Example 1.16). As $F \cong \widehat{F} \circ \mathfrak{I}$ we gain $\widehat{F} \circ \mathfrak{I} \cong G \circ \mathfrak{I}$ which shall be witnessed by θ. Then

$$\overline{\theta}_{(C,C')} := \theta_C \otimes (\theta_{C'}^{-1})^* : \; \widehat{F}(C, I_\mathcal{C}) \otimes (\widehat{F}(C', I_\mathcal{C}))^* \;\longrightarrow\; G(C, I_\mathcal{C}) \otimes (G(C', I_\mathcal{C}))^*$$

is a monoidal isomorphism as analogous reasoning to 1.19 together with natural monoidality of θ show. We have

$$(\widehat{F}(-, I_\mathcal{C}))^* = (F - \otimes (FI_\mathcal{C})^*)^* \cong FI_\mathcal{C} \otimes (F-)^* = \widehat{F}(I_\mathcal{C}, -)$$

and subsequently

$$\widehat{F}(-, I_\mathcal{C}) \otimes (\widehat{F}(-, I_\mathcal{C}))^* \cong \widehat{F}\left((-, I_\mathcal{C}) \otimes (I_\mathcal{C}, -)\right) \cong \widehat{F}(-, -).$$

Moreover - since the duality structure of $\mathcal{D}$ is induced by G - we obtain

$$G(-, I_\mathcal{C}) \otimes (G(-, I_\mathcal{C}))^* = G(-, I_\mathcal{C}) \otimes G(I_\mathcal{C}, -) \cong G((-, I_\mathcal{C}) \otimes (I_\mathcal{C}, -)) \cong G(-, -).$$

Thus $\overline{\theta}$ witnesses $\widehat{F} \cong G$. $\square$

The definition (2.11) of $\hat{F}$ is motivated by the observation that it reverses the construction (2.6) for any map $\hat{f} : FA \otimes (FB')^* \to FB \otimes (FA')^*$.

2.4.2 Compact Closed Networks

If $\mathcal{C}$ is (dagger) compact closed already, this Theorem provides a monoidal equivalence verifying $\mathcal{C} \simeq \mathbf{Int}\,\mathcal{C}$. Indeed when considering $F = \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ universality of $\mathfrak{I}$ reveals $\widehat{\mathrm{id}_{\mathcal{C}}} \circ \mathfrak{I} \cong \mathrm{id}_{\mathcal{C}}$. On the other hand $\mathfrak{I} \circ \widehat{\mathrm{id}_{\mathcal{C}}} \cong \mathrm{id}_{\mathbf{Int}\,\mathcal{C}}$ is witnessed by the monoidal isomorphism

$$\left\{\theta_{(A,A')} = \alpha^{-1}_{A,A'^*,A'} \circ \mathrm{id}_A \otimes \eta_{A'} : (A, A') \to (A \otimes A'^*, I) \;\middle|\; (A, A') \in \mathrm{Ob}\,\mathbf{Int}\,\mathcal{C}\right\}.$$

Corollary 2.15 *Let $\mathcal{C}$ be a (dagger) compact closed category. Then $\mathfrak{I} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ and $\widehat{\mathrm{id}_{\mathcal{C}}} : \mathbf{Int}\,\mathcal{C} \to \mathcal{C}$ establish the monoidal equivalence $\mathcal{C} \simeq \mathbf{Int}\,\mathcal{C}$.*

An adjunction can be understood as the categorical manifestation of inverting a process. Hence if a forgetful functor is involved an adjoint functor describes the optimal way of recovering the concerned structure. Thus instead of defining (dagger) compact closed networks explicitly, we might proceed as in the next definition, which is stated as the notion of (dagger) compact closed networks will be necessary in the next chapter to introduce completeness.

Definition 2.16 *Let $\mathscr{S}$ be a (dagger) signature. We define the category of (dagger) compact closed networks by*

$$\mathbf{CompCNet}\,\mathscr{S} := \mathbf{Int}\,\mathbf{Net}\,\mathscr{S}.$$

Moreover for a given (dagger) compact closed category $\mathcal{C}$ we denote the category of compact closed $\mathcal{C}$-interpretations of $\mathbf{CompCNet}\,\mathscr{S}$ *with* $\mathbf{CompCInt}(\mathscr{S}, \mathcal{C})$*, i.e. the category of symmetric monoidal (dagger) functors $\mathbf{CompCNet}\,\mathscr{S} \to \mathcal{C}$ and monoidal transformations between them.*

Chapter 3

Completeness results for interpretations with unbounded dimensions

Using the formalization of diagrams and interpretations as networks and functors we are now able to capture the conecept of completeness formally. It means the existence of interpretations yielding unequal outcomes when applied on non-isomorphic diagrams. We will distinguish between several degrees of completeness accommodating to what extent interpretations depend on the diagrams they separate. We will treat both cases of compact closed and dagger compact closed categories separately and deliberate on essential completeness, i.e. completeness just for simple diagrams.

Theorems 2.9 and 2.14 will then simplify completeness analyses as they allow us to focus on closed traced (dagger) networks and to characterize an interpretation by its action on object and morphism labels. Bearing this in mind we discuss Selinger's results in [1], i.e. we derive completeness of **FHilb**-interpretations in the dagger case and of **FVect**-interpretations over fields with transcendentals for compact closed categories. We generalize this to free finite-dimensional semimodules over semi-rings containing $\mathbb{N}$ by taking density of $\mathbb{Q}$ in $\mathbb{R}$ into account.

In the fifth section we will deal explicitly with dagger compact closed categories and demonstrate that when working over a semi-ring with transcendentals the presence of a dagger makes no difference. Hence it will suffice for all further considerations concerning e.g. $\mathbb{C}$ to work just with compact closed categories. Moreover we proof completeness of $\mathbf{FProd}_R$-interpretations also for several discrete (semi-)rings like $\mathbb{Z}[i]$ and $\mathbb{Z}[X]$.

The strongest form of completeness - the existence of a unique interpretation separating all non-isomorphic diagrams from one another - will finally turn out to be equivalent to weaker forms of completeness when demanding an upper limit for the dimensions of the interpretations of object labels.

3.1 The concept of Completeness

3.1.1 Different notions of completeness

Definition 3.1 *Let $\mathscr{S}$ be a (dagger) signature and $(\mathcal{C}, \mathrm{Obj}_0)$ a pair consisting of a (dagger) compact closed category $\mathcal{C}$ and a subset $\mathrm{Obj}_0 \subset \mathrm{Ob}\,\mathcal{C}$ of the set of objects of $\mathcal{C}$. We say*

(i) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* relatively complete *for* $\mathbf{CompCNet}\,\mathscr{S}$ *if for all non-isomorphic ((dagger) compact closed* $\mathscr{S}$*-)networks* $M, N \in \mathrm{Ob}\,\mathbf{CompCNet}\,\mathscr{S}$ *there is an interpretation* $[\![\ \ ]\!]_{M,N} \in \mathbf{CompCInt}(\mathscr{S}, \mathcal{C})$ *mapping every object label of* $\mathscr{S}$ *into* Obj_0*, with* $[\![M]\!]_{M,N} \neq [\![N]\!]_{M,N}$.

(ii) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* semi-relatively complete *for* $\mathbf{CompCNet}\,\mathscr{S}$ *if for every network* $M \in \mathrm{Ob}\,\mathbf{CompCNet}\,\mathscr{S}$ *there is an interpretation* $[\![\ \ ]\!]_M \in \mathbf{CompCInt}(\mathscr{S}, \mathcal{C})$ *mapping every object label of* $\mathscr{S}$ *into* Obj_0*, such that for all networks* $N \in \mathrm{Ob}\,\mathbf{CompCNet}\,\mathscr{S}$ *we have* $[\![M]\!]_M = [\![N]\!]_M$ *if and only if* M *and* N *are isomorphic.*

(iii) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* fully complete *for* $\mathbf{CompCNet}\,\mathscr{S}$ *if there is an interpretation* $[\![\ \ ]\!] \in \mathbf{CompCInt}(\mathscr{S}, \mathcal{C})$ *mapping every object label of* $\mathscr{S}$ *into* Obj_0*, such that for all networks* $M, N \in \mathrm{Ob}\,\mathbf{CompCNet}\,\mathscr{S}$ *we have* $[\![M]\!] = [\![N]\!]$ *if and only if* M *and* N *are isomorphic.*

If $\mathscr{S}$ *is not stated explicitly, then we presume* $\mathscr{S} = \mathscr{S}_\infty$*. Also instead of mentioning* $\mathbf{CompCNet}\,\mathscr{S}_\infty$ *we will speak more generally of* completeness for (dagger) compact closed categories*. If the above conditions only hold for simple networks* $M, N \in \mathrm{Ob}\,\mathbf{CompCNet}\,\mathscr{S}$ *we speak of* essential completeness*. Finally if no* Obj_0 *is specified, then* Obj_0 *is supposed to be the set/class of all objects of* $\mathcal{C}$*.*

The introduced notions obey a couple of trivial implications:

(i) full completeness $\Rightarrow$ semi-relative completeness $\Rightarrow$ relative completeness

(ii) completeness $\Rightarrow$ essential completeness

(iii) If $\mathrm{Obj}_0 \subset \mathrm{Obj}_1 \subset \mathrm{Ob}\,\mathcal{C}$ then:

$(\mathcal{C}, \mathrm{Obj}_0)$-interpretations are complete $\Rightarrow$ $(\mathcal{C}, \mathrm{Obj}_1)$-interpretations are complete

(iv) If $\mathcal{C}_1$ is a (dagger) compact closed category with $\mathcal{C}_0 \subset \mathcal{C}_1$ - i.e. there is a faithful symmetric monoidal (dagger) functor $\mathrm{Inc} : \mathcal{C}_0 \to \mathcal{C}_1$ acting injectively on objects - then:

$(\mathcal{C}_0, \mathrm{Obj}_0)$-interpretations are complete $\Rightarrow$ $(\mathcal{C}_1, \mathrm{Obj}_0)$-interpretations are complete

(v) If $\mathscr{S} \subset \mathscr{S}'$ - i.e. $\mathscr{S}'$ contains all sorts and morphism labels of $\mathscr{S}$, the latter have the same type and if $\mathscr{S}$ is a dagger signature then so is $\mathscr{S}'$ - then:

$$\text{completeness for } \mathbf{CompCNet}\,\mathscr{S}' \quad \Rightarrow \quad \text{completeness for } \mathbf{CompCNet}\,\mathscr{S}$$

The distinction between full, semi-relative and relative completeness is motivated by Selinger's results (cf. [1] p.4) which require to distinguish between whether an interpretation seperating two networks M, N does in general depend on both networks or only one of them or does not depend on M, N at all. In order to restrict interpretations of sorts to e.g. spaces of dimensions smaller than a certain upper bound, we specify a set of objects $\mathrm{Obj}_0 \subset \mathrm{Ob}\,\mathcal{C}$, sorts are allowed to be mapped into. In this case of bounded dimensions some kinds of completeness will not hold although their essential counterparts do, giving rise for considering this weakened kind of completeness.

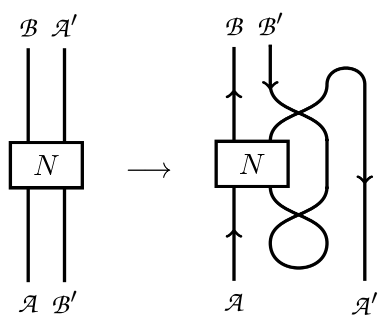
3.1.2 Reductions

Before considering examples we simplify the definition of completeness. We first observe that a network $\mathscr{N} : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}')$ in $\mathbf{CompCNet}\,\mathscr{S}$ corresponds to a network $N : \mathcal{A}\mathcal{B}' \to \mathcal{B}\mathcal{A}'$ of $\mathbf{Net}\,\mathscr{S}$. Therefore it is sufficient to consider only traced (dagger) networks M, N in Definition 3.1.

The work behind this definition was already done in Theorem 2.14 and Definition 2.16. Assume we had defined compact closed networks in a more self-evident way like suggested in the comments to Definition 2.2. Then the resulting category - let us call it $\mathbf{CompCDiag}\,\mathscr{S}$ - would contain networks with duals as well as cups and caps which are not part of traces. However, supposing that a good definition of $\mathbf{CompCDiag}\,\mathscr{S}$ makes it to a smallest (dagger) compact closed category containing $\mathbf{Net}\,\mathscr{S}$ (which we do not know to be unique yet) we obtain a monoidal equivalence $\mathbf{CompCDiag}\,\mathscr{S} \simeq \mathbf{CompCNet}\,\mathscr{S}$ as Corollary 2.15 reveals

$$\mathbf{CompCDiag}\,\mathscr{S} \simeq \mathbf{Int}\,\mathbf{CompCDiag}\,\mathscr{S} \supset \mathbf{Int}\,\mathbf{Net}\,\mathscr{S} = \mathbf{CompCNet}\,\mathscr{S}.$$

Therefore Definition 2.16 adequately captures what we intuitively expect from the category of compact closed networks (over a given signature). This monoidal equivalence acts on $\mathscr{N}$ like



(3.1)

(cf. (2.11)) where we implicitly assume that the network on the right hand side is formalized in **CompCDiag** $\mathscr{S}$. Hence the transition (3.1) and its counter-construction (2.6) lie at the heart of the trick allowing us to focus on **Net** $\mathscr{S}$.

Furthermore, if the underlying signature $\mathscr{S}$ provides infinitely many arrow labels of type $\epsilon \to \mathcal{A}$ and $\mathcal{A} \to \epsilon$ for every string of sorts $\mathcal{A}$ then it suffices to consider only closed networks M, N in Definition 3.1. In fact two traced (dagger) $\mathscr{S}$-networks $N, N' : \mathcal{A} \to \mathcal{B}$ are isomorphic if and only if so are $\boxed{f} \circ N \circ \boxed{g}$, $\boxed{f} \circ N' \circ \boxed{g}$, where $\boxed{f}$ resp. $\boxed{g}$ denote the networks consisting of a single box labelled with function labels $f : \mathcal{B} \to \epsilon$ resp. $g : \epsilon \to \mathcal{A}$ not appearing in N, N' (cf. [1], p.2).

Finally we only have to consider interpretations induced by a model, i.e. on interpretations acting like (2.3) since Theorem 2.9 shows that every $\mathcal{C}$-interpretation in $\mathbf{Int}(\mathscr{S}, \mathcal{C})$ is induced by a $\mathcal{C}$-model up to a natural isomorphism. But for a natural isomorphism $\theta : [\![\ ]\!] \to [\![\ ]\!]'$ between two interpretations $[\![\ ]\!], [\![\ ]\!]' \in \operatorname{Ob} \mathbf{Int}(\mathscr{S}, \mathcal{C})$ and $\mathscr{S}$-networks $N, N' : \mathcal{A} \to \mathcal{B}$ the diagrams

$$\begin{tikzcd} {[\![\mathcal{A}]\!]} \arrow[r, shift left, "\theta_{\mathcal{A}}"] \arrow[d, "{[\![N]\!]}"'] & {[\![\mathcal{A}]\!]'} \arrow[l, shift left, "\theta_{\mathcal{A}}^{-1}"] \arrow[d, "{[\![N]\!]'}"] \\ {[\![\mathcal{B}]\!]} \arrow[r, shift left, "\theta_{\mathcal{B}}"] & {[\![\mathcal{B}]\!]'} \arrow[l, shift left, "\theta_{\mathcal{B}}^{-1}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[\![\mathcal{A}]\!]} \arrow[r, shift left, "\theta_{\mathcal{A}}"] \arrow[d, "{[\![N']\!]}"'] & {[\![\mathcal{A}]\!]'} \arrow[l, shift left, "\theta_{\mathcal{A}}^{-1}"] \arrow[d, "{[\![N']\!]'}"] \\ {[\![\mathcal{B}]\!]} \arrow[r, shift left, "\theta_{\mathcal{B}}"] & {[\![\mathcal{B}]\!]'} \arrow[l, shift left, "\theta_{\mathcal{B}}^{-1}"] \end{tikzcd}$$

commute so that $[\![N]\!] = [\![N']\!]$ if and only if $[\![N]\!]' = [\![N']\!]'$. We summarize these simplifications for the case $\mathscr{S} = \mathscr{S}_\infty$.

Proposition 3.2 *Let* $(\mathcal{C}, \mathrm{Obj}_0)$ *be a pair consisting of a (dagger) compact closed category* $\mathcal{C}$ *and a subset* $\mathrm{Obj}_0 \subset \operatorname{Ob} \mathcal{C}$ *of the set of objects of* $\mathcal{C}$.

(i) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* relatively complete *for (dagger) compact closed categories if for all non-isomorphic closed networks* $M, N (\in \operatorname{Ob} \mathbf{Net}\, \mathscr{S}_\infty)$ *there is a* $\mathcal{C}$*-interpretation* $[\![\ ]\!]_{M,N} (\in \mathbf{Int}(\mathscr{S}_\infty, \mathcal{C}))$ *induced by a* $\mathcal{C}$*-model that maps every object label into* Obj_0*, with* $[\![M]\!]_{M,N} \neq [\![N]\!]_{M,N}$.

(ii) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* semi-relatively complete *for (dagger) compact closed categories if for every closed network* M *there is a* $\mathcal{C}$*-interpretation* $[\![\ ]\!]_M$ *induced by a* $\mathcal{C}$*-model that maps every object label into* Obj_0*, such that for all closed networks* N *we have* $[\![M]\!]_M = [\![N]\!]_M$ *if and only if* $M \cong N$.

(iii) $(\mathcal{C}, \mathrm{Obj}_0)$*-interpretations are* fully complete *for (dagger) compact closed categories if there is a* $\mathcal{C}$*-interpretation* $[\![\ ]\!]$ *induced by a* $\mathcal{C}$*-model that maps every object label into* Obj_0*, such that for all closed networks* M, N *we have* $[\![M]\!] = [\![N]\!]$ *if and only if* $M \cong N$.

3.2 The Denotation

In order to obtain a clearer impression of how model induced interpretatons $[\![\;]\!]$ act we evaluate $[\![N]\!]$ for general closed networks N in the case $\mathcal{C} = \mathbf{FProd}_R$.

Example 3.3 Consider $\mathcal{C} = \mathbf{FProd}_R$, a $\mathcal{C}$-model $[\![\;]\!]_0$ with induced interpretation $[\![\;]\!]$ as well as we a closed network $N = (B, \ell, \pi)$. The following calculations require a couple of abbreviating notations.

We write $d_A = \dim[\![A]\!]_0$ for all sorts A. We also presuppose an enumeration of the wires $w_1, \dots, w_n$ of N and identify them with the outputs in Out_N of the boxes they start from, i.e. we define $\mathrm{Out}_N = \{w_1, \dots, w_n\}$. We will write $\ell(w_i)$ for the object label belonging to w_i. We denote basis elements of A with $|e_i^{(A)}\rangle$ for $1 \leq i \leq d_A$. Furthermore for every $b \in B$ we write $\mathscr{d}(b) \in \mathbb{N}^*$ for the string consisting of the indices of these wires ending in the inputs of $\ell(b)$. Analogously $\mathscr{c}(b) \in \mathbb{N}^*$ shall denote the indices of those wires starting from the outputs of b. Hence

$$
\begin{aligned}
\mathscr{d}(b) = i_1 \dots i_{|\mathrm{dom}\,\ell(b)|} \quad &\Leftrightarrow \quad w_{i_k} = \pi^{-1}(k, b) \quad \forall 1 \leq k \leq |\mathrm{dom}\,\ell(b)|, \\
\mathscr{c}(b) = j_1 \dots j_{|\mathrm{cod}\,\ell(b)|} \quad &\Leftrightarrow \quad w_{j_l} = (b, l) \qquad \forall 1 \leq l \leq |\mathrm{cod}\,\ell(b)|.
\end{aligned}
$$

We set

$$
\mathrm{Idx} = \{\phi : \{1, \dots, n\} \to \mathbb{N}^{>0} \mid \phi(i) \leq d_{\ell(w_i)}\}
$$

and regard a $\phi \in \mathrm{Idx}$ as a simultaneous picking of the basis vectors $|e_{\phi(i)}^{(\ell(w_i))}\rangle$ of $\ell(w_i)$ for all wires w_i of N. For a string of indices $i = i_1 \dots i_l \in \{1, \dots, n\}^*$ and a $\phi \in \mathrm{Idx}$ we define

$$
|e_{\phi(i)}\rangle = \left|e_{\phi(i_1)}^{(\ell(w_{i_1}))}\right\rangle \otimes \dots \otimes \left|e_{\phi(i_l)}^{(\ell(w_{i_l}))}\right\rangle
$$

as well as

$$
[\![\ell(b)]\!]_{N_0}(\phi) = \left\langle e_{\phi(\mathscr{c}(b))}\right| [\![\ell(b)]\!]_{N_0} \left|e_{\phi(\mathscr{d}(b))}\right\rangle.
$$

In other words $[\![\ell(b)]\!]_{N_0}(\phi)$ denotes the entry of $[\![\ell(b)]\!]_{N_0}$ that is addressed by the choice of basis vectors encoded by ϕ. The definition of Idx allows us to write down bases

$$
\left|\bigotimes_{b \in B} e_{\phi(\mathscr{c}(b))}\right\rangle, \; \phi \in \mathrm{Idx} \quad \text{of} \quad \bigotimes_{b \in B} [\![\mathrm{cod}(\ell(b))]\!] \qquad \text{and}
$$

$$
\left|\bigotimes_{b \in B} e_{\phi(\mathscr{d}(b))}\right\rangle, \; \phi \in \mathrm{Idx} \quad \text{of} \quad \bigotimes_{b \in B} [\![\mathrm{dom}(\ell(b))]\!].
$$

The isomorphism $\hat{\pi}$ of (2.3) can now be characterized by

$$
\hat{\pi}\left|\bigotimes_{b \in B} e_{\phi(\mathscr{c}(b))}\right\rangle = \left|\bigotimes_{b \in B} e_{\phi(\mathscr{d}(b))}\right\rangle
$$

for all $\phi \in \mathrm{Idx}$.

Subsequently (2.3) reveals

$$
\begin{aligned}
[\![N]\!] &= \mathrm{Tr}^{\bigotimes_{b\in B}[\![\mathrm{cod}(\ell(b))]\!]}\left(\bigotimes_{b\in B}[\![\ell(b)]\!]_0 \,\circ\, \hat{\pi}\right) \\
&= \sum_{\phi\in\mathrm{Idx}} \left\langle \bigotimes_{b\in B} e_{\phi(c(b))} \middle| \left(\bigotimes_{b\in B}[\![\ell(b)]\!]_0 \,\circ\, \hat{\pi}\right) \middle| \bigotimes_{b\in B} e_{\phi(c(b))} \right\rangle \\
&= \sum_{\phi\in\mathrm{Idx}} \left\langle \bigotimes_{b\in B} e_{\phi(c(b))} \middle| \bigotimes_{b\in B}[\![\ell(b)]\!]_0 \middle| \bigotimes_{b\in B} e_{\phi(d(b))} \right\rangle \\
&= \sum_{\phi\in\mathrm{Idx}} \prod_{b\in B} \left\langle e_{\phi(c(b))} \middle| \, [\![\ell(b)]\!]_0 \middle| e_{\phi(d(b))} \right\rangle \\
&= \sum_{\phi\in\mathrm{Idx}} \prod_{b\in B} [\![\ell(b)]\!]_0(\phi)
\end{aligned}
\tag{3.2}
$$

We call this formula (3.2) the *denotation* and note that it is a homogeneous polynomial in the entries $[\![\ell(b)]\!]_0(\phi)$ of the interpretations of the box labels $\ell(b)$ appearing in N. Its degree equals the number of boxes in N. Moreover it will be crucial for later considerations that all of its coefficients are natural numbers. We also note that the denotation does not depend on the choice of basis vectors $|e_i^{(A)}\rangle$ as traces remain invariant under basis changes. Ultimately (3.2) makes it obvious that isomorphic networks coincide after applying any interpretation as a network isomorphism is just a permutation of B compatible with wire labels.

3.3 The Selinger Interpretation

3.3.1 Construction

Using our different notions of completeness Selinger's result states that **FHilb**-interpretations are (essentially) semi-relatively complete for dagger compact closed categories. As the construction he has done will be helpful in different contexts we will introduce it here.

Definition 3.4 *Let $N = (B, \ell, \pi)$ be a* closed *(dagger) $\mathscr{S}_\infty$-network. We adopt the notations $d(b), c(b)$ as well as the explicit reference $\mathbf{In}_N = \{w_1, \ldots, w_n\}$ to the wires of N from 3.3. Moreover let R be a semiring with a conjugation $^- : R \to R$ and set $R[B] = R[(x_b)_{b\in B}]$ resp. $R[B, \overline{B}] = R[(x_b, \overline{x}_b)_{b\in B}]$ We define the* Selinger Interpretation

$$[\![\ ]\!]_N : \mathbf{Net}\,\mathscr{S}_\infty \to \mathbf{FMod}_{R[B]} \qquad \text{resp.} \qquad [\![\ ]\!]_N : \mathbf{Net}\,\mathscr{S}_\infty \to \mathbf{FProd}_{R[B,\overline{B}]}$$

in the dagger case, by defining its action on object and morphism labels. For all objects $A \in \mathrm{Ob}\,\mathcal{C}$ we set $[\![A]\!]_N = R[B]^{d_A}$ resp. $[\![A]\!]_N = R[B,\overline{B}]^{d_A}$ where

$$d_A \;=\; \#\text{ of wires in } N \text{ labelled with } A \;=\; \#\,\{1 \le i \le n \mid \ell(w_i) = A\}.$$

Now let $|e_{i_j}\rangle$ be the basis elements of A for $1 \leq j \leq d_A$ where the i_j are the indices belonging to these wires that are labelled with A. For all $b \in B$ we define the linear map[1]

$$m_b : [\![\mathrm{dom}(\ell(b))]\!] \to [\![\mathrm{cod}(\ell(b))]\!]$$

by

$$\left\langle \bigotimes_{1\leq l\leq|c(b)|} e_{j_l} \middle| m_b \middle| \bigotimes_{1\leq k\leq|d(b)|} e_{i_k} \right\rangle = \begin{cases} x_b & \text{if } w_{i_k} = \pi^{-1}(k,b)\ \forall 1 \leq k \leq |d(b)| \text{ and} \\ & w_{j_l} = (b,l)\ \forall 1 \leq l \leq |c(b)| \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

for all basis vectors $|e_{i_k}\rangle \in [\![(\mathrm{dom}\,\ell(b))_k]\!]_N$ and $|e_{j_l}\rangle \in [\![(\mathrm{cod}\,\ell(b))_l]\!]_N$. For every morphism label $f \in F_\infty$ we set

$$[\![f]\!]_N = \sum_{\substack{b\in B,\\ \ell(b)=f}} m_b \qquad \textit{resp.} \qquad [\![f]\!]_N = \sum_{\substack{b\in B,\\ \ell(b)=f}} m_b + \sum_{\substack{b\in B,\\ \ell(b)=f^\dagger}} m_b^\dagger \tag{3.4}$$

in the dagger case where $m_b^\dagger = \overline{m_b}^t$. In particular, objects and morphism labels that are not part of N get interpreted as the zero-object resp. the zero-morphism.

We note that due to (3.2) the Selinger Interpretation $[\![M]\!]_N$ of any network M can be regarded as a polynomial in $\mathbb{N}[X_1, \ldots, X_{|B|}] \subset \mathbb{N}[(X_i)_{i\in\mathbb{N}}]$ respectively in $\mathbb{N}[X_1, \ldots, X_{|B|}, \overline{X}_1, \ldots, \overline{X}_{|B|}] \subset \mathbb{N}[(X_i)_{i\in\mathbb{N}}]$.

3.3.2 Semi-relative Completeness in the presence of transcendentals

Selinger's completeness theorem relies on the following fact he has proven (cf. [1] p.4, 8ff).

Proposition 3.5 *Let $N_0 = (B_0, \ell_0, \pi_0)$ be a simple closed (dagger) network. Then for all simple closed (dagger) networks $N = (B, \ell, \pi)$ we have $[\![N]\!]_{N_0} = [\![N_0]\!]_{N_0}$ if and only if N and N_0 are isomorphic. Moreover the coefficient of the monomial $\sum_{b'\in B_0} x_{b'}$ of $[\![N]\!]_{N_0}$ equals the number of network isomorphisms between N and N_0.*

An application of the following argument to an example network can be found in [1] p.4f. Essentially Proposition 3.5 relies on the fact that due to (3.3) the Selinger Interpretation $[\![\ \ ]\!]_{N_0}$ memorizes what labels are the inputs and outputs of $\ell(b')$ for every $b' \in B_0$. More concretely, Selinger's Interpretation chooses sufficiently large dimensions in order to make ϕ reconstructable when having $[\![\ell(b)]\!](\phi)$ in (3.2). Subsequently the appearance of $\prod_{b'\in B_0} x_{b'}$ in $[\![N]\!]_{N_0}$ allows us to extract enough information for verifying $N \cong N_0$.

Proof. Write $\mathrm{Out}_{N_0} = \{w_1^0, \ldots, w_{n_0}^0\}$ and $\mathrm{Out}_N = \{w_1, \ldots, w_n\}$ for the wires of N_0 and N.

[1] The definition of m_b as well as (3.3) and (3.4) are taken from [1] p.8. We modified the notations.

We adopt the notations of Definition 3.4 and particularly denote the basis vectors of $[\![A]\!]_{N_0}$ with $|e_{i_1}\rangle, \ldots, |e_{i_{d_A}}\rangle$ if $w_{i_1}, \ldots, w_{i_{d_A}}$ are precisely these wires labelled with A. As the indexing of the basis vectors differs from how it was done in 3.3 we need to adjust the remaining notations in order to keep (3.2) valid. For a string of indices $i = i_1 \ldots i_l \in \{1, \ldots, n_0\}^*$ we define $|e_i\rangle = |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_l}\rangle$. We set

$$\mathrm{Idx} = \Big\{ \phi : \{1, \ldots, n\} \to \{1, \ldots, n_0\} \ \Big| \ \ell(w_i) = \ell(w^0_{\phi(i)}) \text{ for all } 1 \leq i \leq n \Big\}$$

and $\phi(i)$ shall denote the string $\phi(i_1) \ldots \phi(i_l)$ where $\phi \in \mathrm{Idx}$. For a $b \in B$ we again define

$$[\![\ell(b)]\!]_{N_0}(\phi) = \Big\langle e_{\phi(c(b))} \Big| \, [\![\ell(b)]\!]_{N_0} \Big| e_{\phi(d(b))} \Big\rangle.$$

Using the Kronecker-δ notation $\delta(i, j) = \delta_{ij}$ we can rewrite (3.3) as

$$\Big\langle e_{\phi(c(b))} \Big| \, m_{b'} \Big| e_{\phi(d(b))} \Big\rangle = x_{b'} \cdot \delta\Big(\phi(d(b)), d(b')\Big) \cdot \delta\Big(\phi(c(b)), c(b')\Big) \tag{3.5}$$

for all $b \in B,\ b' \in B_0$. The condition $\ell(w_i) = \ell(w^0_{\phi(i)})$ in the definition of Idx allows us to regard an element of Idx as choosing for every wire in N a basis vector of the (interpretation of the) object it is labelled with. Hence the formula (3.2) for the denotation remains valid and shows[2]

$$\begin{aligned}
[\![N]\!]_{N_0} &= \sum_{\phi \in \mathrm{Idx}} \prod_{b \in B} [\![\ell(b)]\!]_{N_0}(\phi) \\
&= \sum_{\phi \in \mathrm{Idx}} \prod_{b \in B} \sum_{\substack{b' \in B_0, \\ \ell_0(b') = \ell(b)}} \Big\langle e_{\phi(c(b))} \Big| \, m_{b'} \Big| e_{\phi(d(b))} \Big\rangle \\
&= \sum_{\phi \in \mathrm{Idx}} \prod_{b \in B} \sum_{b' \in B_0} x_{b'} \cdot \delta\Big(\phi(d(b)), d(b')\Big) \cdot \delta\Big(\phi(c(b)), c(b')\Big) \cdot \delta\Big(\ell_0(b'), \ell(b)\Big) \\
&= \sum_{\phi \in \mathrm{Idx}} \sum_{\psi : B \to B_0} \prod_{b \in B} x_{\psi(b)} \cdot \delta\Big(\phi(d(b)), d(\psi(b))\Big) \cdot \delta\Big(\phi(c(b)), c(\psi(b))\Big) \cdot \delta\Big(\ell_0(\psi(b)), \ell(b)\Big)
\end{aligned}$$

The second and third equation follow from (3.4) and (3.5) while the last one is a consequence of distributivity. In the dagger case we similarly obtain

$$\begin{aligned}
[\![N]\!]_{N_0} = &\sum_{\phi \in \mathrm{Idx}} \prod_{b \in B} \left(\sum_{\substack{b' \in B_0, \\ \ell_0(b') = \ell(b)}} \Big\langle e_{\phi(c(b))} \Big| \, m_{b'} \Big| e_{\phi(d(b))} \Big\rangle + \sum_{\substack{b' \in B_0, \\ \ell_0(b') = \ell(b)^\dagger}} \Big\langle e_{\phi(c(b))} \Big| \, m^\dagger_{b'} \Big| e_{\phi(d(b))} \Big\rangle \right) \\
= &\sum_{\phi \in \mathrm{Idx}} \sum_{\psi : B \to B_0} \Bigg(\prod_{b \in B} x_{\psi(b)} \cdot \delta\Big(\phi(d(b)), d(\psi(b))\Big) \cdot \delta\Big(\phi(c(b)), c(\psi(b))\Big) \cdot \delta\Big(\ell_0(\psi(b)), \ell(b)\Big) \\
&\qquad + \prod_{b \in B} \overline{x}_{\psi(b)} \cdot \delta\Big(\phi(c(b)), d(\psi(b))\Big) \cdot \delta\Big(\phi(d(b)), c(\psi(b))\Big) \cdot \delta\Big(\ell_0(\psi(b)), \ell(b)\Big) \Bigg)
\end{aligned}$$

[2]This calculation can be found in [1] p.8f with different notations.

Every $\phi \in \mathrm{Idx}$ can be considered as a map from the wires of N to the wires of N_0 when defining $\phi(w_i) = w^0_{\phi(i)}$. In both cases the (first) monomial belonging to (ϕ, ψ) equals $\prod_{b' \in B_0} x_{b'}$ if and only if (ϕ, ψ) is a network isomorphism establishing $N \cong N_0$. Here (ϕ, ψ) is called a network isomorphism resp. homomorphism if so is ψ and ϕ associates wires of N and N_0 in this way that is enforced by ψ. Indeed

$$\prod_{b \in B} x_{\psi(b)} \cdot \delta\Big(\phi(\mathscr{d}(b)), \mathscr{d}(\psi(b))\Big) \cdot \delta\Big(\phi(\mathscr{c}(b)), \mathscr{c}(\psi(b))\Big) \cdot \delta\Big(\ell_0(\psi(b)), \ell(b)\Big) = \prod_{b' \in B_0} x_{b'} \tag{3.6}$$

requires ψ to be bijective as otherwise some of the $x_{b'}$ would appear several times or would not appear at all on the right hand side. Also - as the left hand side is unequal to 0 - we must have

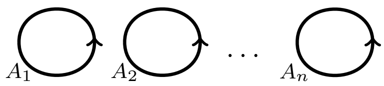
$$\phi(\mathscr{d}(b)) = \mathscr{d}(\psi(b)), \qquad \phi(\mathscr{c}(b)) = \mathscr{c}(\psi(b)), \qquad \ell_0(\psi(b)) = \ell(b)$$

for all $b \in B$, ensuring that (ϕ, ψ) is in fact a network homomorphism. Thus if $[\![N]\!]_{N_0}$ and $[\![N_0]\!]_{N_0}$ contain the same monomials there is already an isomorphism between N and N_0 (as $[\![N_0]\!]_{N_0}$ trivially contains the right hand side of (3.6)) and the number of different isomorphisms $\psi : N \xrightarrow{\cong} N_0$ matches the number of appearances of $\prod_{b' \in B_0} x_{b'}$. $\square$

In the following we will make use of the notation $\widehat{R}$ for the semi-ring of polynomials $R[(X_i)_{i \in \mathbb{N}}]$ with coefficients from a given semi-ring R. Note that the $X_i \in \widehat{R}$ form a countable algebraically independent set of transcendentals[3] over R. As so do $\mathbb{C}$ and $\mathbb{R}$ these cases are always included when speaking of a general $\widehat{R}$.

Corollary 3.6 *Let R be a semi-ring containing $\mathbb{N}$, i.e. there is an injective semi-ring homomorphism $\phi : \mathbb{N} \to R$. Then $\mathbf{FMod}_{\widehat{R}}$-interpretations are semi-relatively complete for compact closed categories. If endowing $\widehat{R}$ with the dagger induced by a conjugation $\overline{} : \widehat{R} \to \widehat{R}$ that acts as the identity on R and as a fixpoint-free involution on $(X_i)_{i \in \mathbb{N}}$, e.g. $\overline{X_k} = X_{k+(-1)^k}$, then $\mathbf{FProd}_{\widehat{R}}$-interpretations are semi-relatively complete for dagger compact closed categories. In particular, $\mathbf{FVect}_{\mathbb{R}}$-interpretations resp. $\mathbf{FHilb}$-interpretations are semi-relatively complete for compact closed resp. dagger compact closed categories.*

Proof. Let M be a simple closed network. The interpretation that arises from $[\![\ ]\!]_M$ by substituting the $x_b, b \in B$ with an algebraically independent subset of $\widehat{R}$ - e.g. $X_0, \ldots, X_{|B|-1} \in \widehat{R}$ resp. $X_0, X_2, X_4, \ldots, X_{2|B|-2} \in \widehat{R}$ in the dagger case - witnesses essential relative completeness due to the last Proposition. By modifying Selinger's Interpretation this approach also works for general closed networks M. Let M_0 be the kernel of M, i.e. M can be written as

$M \ = \ M_0$ 

[3] We implicitly generalize the notion of algebraic independence to semi-rings in the obvious way.

where M_0 is simple. Now consider the interpretation $[\![\ ]\!]_{(M)}$ arising from Selinger's Interpretation $[\![\ ]\!]_{M_0}$ by defining $\dim [\![A]\!]_{(M)}$ as pairwise different prime numbers p_A with $p_A \geq \dim [\![A]\!]_{M_0}$ for all object labels A appearing in M. For all other object labels A we set $[\![A]\!]_{(M)} = [\![A]\!]_{M_0} = \{0\}$ and for any arrow label f the matrix $[\![f]\!]_{(M)}$ is supposed to arise from $[\![f]\!]_M$ by filling up the additional entries with zeros, i.e. $[\![f]\!]_{(M)}$ is also defined by (3.3) when extending it to all suitable $|e_{i_*}\rangle, |e_{j_*}\rangle$. Now we have

$$[\![M]\!]_{(M)} = [\![M_0]\!]_{M_0} \cdot \prod_{1 \leq i \leq n} p_{A_i}.$$

Likewise for any other traced network N we have $[\![N]\!]_{(M)} = \text{const.} \cdot [\![N_0]\!]_{M_0}$, so that the proof of Proposition 3.5 shows

$$[\![N]\!]_{(M)} = [\![M]\!]_{(M)} \qquad \Rightarrow \qquad N_0 \cong M_0$$

since then $[\![N_0]\!]_{M_0}$ and $[\![M_0]\!]_{M_0}$ contain the same monomials. Uniqueness of prime factorization then shows that in the case $[\![N]\!]_{(M)} = [\![M]\!]_{(M)}$ must also contain the same trivial cycles. Now the same replacement of the $x_b, b \in B$ appearing in $[\![\ ]\!]_{(M)}$ finishes the proof when observing that in the dagger case $(X_{2i}, \overline{X}_{2i})_{0 \leq i < |B|} = (X_i)_{0 \leq i < 2|B|-1}$ is still algebraically independent. □

The interpretation $[\![\ ]\!]_{(M)}$ we have constructed here will play an important role in later proofs. Hence in the following we will refer to it as the *modified version* of the Selinger Interpretation $[\![\ ]\!]_{M_0}$.

Corollary 3.7 *Let R be a semi-ring containing $\mathbb{N}$. Then $\mathbf{FMod}_R$-interpretations are relatively complete for compact closed categories.*

Proof. For two closed networks $M = (B, \ell, \pi)$ and N consider the polynomial $p_{M,N} = [\![M]\!]_{(M)} - [\![N]\!]_{(M)} \in R[B]$ using the modified version $[\![\ ]\!]_{(M)}$ of the Selinger Interpretation. If $M \not\cong N$ then Proposition 3.5 shows that $p_{M,N}$ is not the zero-polynomial. As $\mathbb{N} \subset R$ there is a non-root $(n_i)_i \subset R$ of $p_{M,N}$. □

In so far as essential completeness is concerned, these consequences were already mentioned by Selinger (cf. [1] p.10f, although he stated the last corollary only for the case of infinite fields). But we will see in the next section that Proposition 3 in fact allows stronger conclusions by utilizing density of $\mathbb{Q}$ in $\mathbb{R}$.

3.4 Semi-relative completeness for semi-rings containing $\mathbb{N}$

The generalization of semi-relative completeness to general semi-rings R containing $\mathbb{N}$ as well as to dagger compact closed categories will require a technical preparation.

Lemma 3.8 *Let $t \in \mathbb{N}$ be fixed and $p_0 \in \mathbb{N}[X_1, \ldots, X_t]$ a homogeneous polynomial. Moreover let $\mathcal{P} \subset \mathbb{N}[X_1, \ldots, X_t]$ be a subset of homogeneous polynomials with $p_0 \notin \mathcal{P}$.*

(i) *There are rational numbers $q_1, \ldots, q_t \in \mathbb{Q}$ with*

$$p(q_1, \ldots, q_n) \neq p_0(q_1, \ldots, q_t)$$

for all $p \in \mathcal{P}$.

(ii) *Suppose there are only finitely many $p \in \mathcal{P}$ with $\deg p < \deg p_0$. Then there are natural numbers $n_1, \ldots, n_t \in \mathbb{N}$ with*

$$p(n_1, \ldots, n_t) \neq p_0(n_1, \ldots, n_t)$$

for all $p \in \mathcal{P}$.

Proof. (i): For any homogeneous polynomial $p \in \mathbb{N}[X_1, \ldots, X_t]$ we denote the sum of the coefficients of all monomials appearing in p with s_p and we set $d_p = \deg p$. Assume w.l.o.g. that p_0 contains the variables $X_1, \ldots, X_m$. Then replacing the X_i by some $r_i \in \mathbb{R}$ with $2 \leq r_i$ for all $1 \leq i \leq t$ and $2 \leq r_i \leq 3$ for $1 \leq i \leq m$ ensures that $p_0(r_1, \ldots, r_t) \leq s_{d_0} \cdot 3^{d_{p_0}} \leq C$ for some constant $C \in \mathbb{R}^{>0}$. We also obtain the estimation

$$p(r_1, \ldots, r_t) \geq s_p \cdot 2^{d_p} \qquad \Rightarrow \qquad p(r_1, \ldots, r_t) - p_0(r_1, \ldots, r_t) \geq s_p \cdot 2^{d_p} - C. \tag{3.7}$$

for all homogeneous $p \in \mathbb{N}[X_1, \ldots, X_t]$. Hence we see $p(r_1, \ldots, r_t) > 0$ for all homogeneous p with s_p or d_p sufficiently large. As p must have natural coefficients, there are only finitely many different polynomials p left with $s_p \cdot 2^{d_p} \leq R$ and we may define

$$\{p_1, \ldots, p_k\} = \{p \in \mathcal{P} \mid s_p \cdot 2^{d_p} \leq R\}.$$

Due to this finiteness there is a sufficiently small $\varepsilon > 0$ such that $p_j(q_1, \ldots q_t)$ remains unequal to $p_0(q_1, \ldots, q_t)$ for all $1 \leq j \leq k$ and all q_i who differ from r_i by no more than ε and such that for all those q_i we have $2 \leq q_i$ for all $1 \leq i \leq t$ as well as $2 \leq q_i \leq 3$ for all $1 \leq i \leq m$. Then (3.7) remains true for these q_i so that all $p \in \mathcal{P}$ differ from p_0 at $(q_1, \ldots, q_t)$. Density of $\mathbb{Q} \subset \mathbb{R}$ allows us to choose the q_i as rationals.
(ii): When choosing $L \in \mathbb{N}$ large enough, we may assume that the chosen rationals q_i have common denominator L (e.g. $L = \lceil \frac{1}{\varepsilon} \rceil$). Now consider

$$\begin{aligned} \Delta_p(\lambda) &= p(\lambda \cdot Lq_1, \ldots, \lambda \cdot Lq_t) - p_0(\lambda \cdot Lq_1, \ldots, \lambda \cdot Lq_t) \\ &= (\lambda L)^{d_p} \cdot p(q_1, \ldots, q_t) - (\lambda L)^{d_{p_0}} \cdot p_0(q_1, \ldots, q_t) \end{aligned}$$

for $\lambda \in \mathbb{N}$ and $p \in \mathcal{P}$. If $d_p = d_{p_0}$ then

$$\Delta_p(\lambda) = (\lambda L)^{d_{p_0}} \cdot (p(q_1, \ldots, q_t) - p_0(q_1, \ldots, q_t)) \neq 0$$

for all $\lambda \in \mathbb{N}$ by choice of the q_i. If $d_p > d_{p_0}$ then

$$\Delta_p(\lambda) > (\lambda L)^{d_{p_0}+1} \cdot 2^{d_{p_0}+1} - (\lambda L)^{d_{p_0}} \cdot s_{p_0} 3^{d_0} > 0 \tag{3.8}$$

when choosing λ sufficiently large. This can be done independently of p so that (3.8) holds for some $\lambda_0 \in \mathbb{N}$ and all $p \in \mathbb{P}$ with $d_p > d_{p_0}$. Finally in the case $d_p < d_{p_0}$ we have $\Delta_p(\lambda) \neq 0$ for all $\lambda \in \mathbb{N}^{>0}$ apart from at most one exception since $p(q_1, \ldots, q_t) \neq p_0(q_1, \ldots, q_t)$. Hence due to the presupposition that $\mathcal{P}$ contains only finitely many p with $d_p < d_{p_0}$ we can choose a $\lambda_1 \geq \lambda_0$ with $\Delta_p(\lambda_1) \neq 0$ for all those p. Thus

$$p(\lambda_1 \cdot Lq_1, \ldots, \lambda_1 \cdot Lq_t) \neq p_0(\lambda_1 \cdot Lq_1, \ldots, \lambda_1 \cdot Lq_t)$$

for all $p \in \mathcal{P}$ and $\lambda_1 \cdot Lq_i \in \mathbb{N}$ for all $1 \leq i \leq t$. □

Proposition 3.9 **$\mathbf{FVect}_{\mathbb{Q}}$**-*interpretations are semi-relatively complete for compact closed categories.*

Proof. Let $N_0 = (B_0, \ell_0, \pi_0)$ be a closed network. Then define $p_0 := [\![N_0]\!]_{(N_0)} \in \mathbb{N}[B_0]$ and set

$$\mathcal{P} = \{p_N = [\![N]\!]_{(N_0)} \in \mathbb{N}[B_0] \mid N \text{ closed network with } N \ncong N_0\}$$

where $[\![\ ]\!]_{(N_0)}$ refers to the modified version of the Selinger Interpretation. The term (3.2) for the denotation shows that p_0 as well as all p_N are homogeneous and due to Corollary 3.7 we must have $p_0 \notin \mathcal{P}$. Hence Lemma 3.8 (i) provides an assignment $q_1, \ldots, q_{|B_0|}$ for the $x_b, b \in B_0$ that turns $[\![\ ]\!]_{(N_0)}$ after replacement into a $\mathbf{FVect}_{\mathbb{Q}}$-interpretation $[\![\ ]\!]^{q \to x}_{(N_0)}$ witnessing semi-relative completeness as

$$N \ncong N_0 \quad \Rightarrow \quad [\![N]\!]^{q \to x}_{(N_0)} = p_N(q_1, \ldots, q_t) \neq p_0(q_1, \ldots, q_t) = [\![N_0]\!]^{q \to x}_{(N_0)}$$ □

Part (ii) of Lemma 3.8 will also enable us to prove semi-relative completeness, but it has to be refined to make work it for diagrams containing trivial cycles.

Proposition 3.10 *Let R be a semi-ring containing $\mathbb{N}$. Then* **$\mathbf{FMod}_R$**-*interpretations are semi-relatively complete for compact closed categories.*

Proof. Let $N_0 = (B_0, \ell_0, \pi)$ be a *simple* closed network and define $p_0 = [\![N_0]\!]_{N_0}$ as well as

$$\mathcal{P} = \{p_N = [\![N]\!]_{N_0} \in \mathbb{N}[B_0] \mid N \textit{ simple} \text{ closed network with } N \ncong N_0\}.$$

The formula for the denotation (3.2) shows that there are only finitely many $p_N \in \mathcal{P}$ with $\deg p_N < \deg p_0$ resp. $\deg p_N \leq \deg p_0$. Indeed only those finitely many boxes labelled with some $f \in F_\infty$ that appear in N_0 are mapped under the Selinger Interpretation to non-zero matrices. The number of appearing boxes in N equals $\deg p_N$ and is therefore limited when requiring $\deg p_N \leq \deg p_0$. Finally when not allowing trivial cycles then there are only finitely many closed networks N containing a preliminary given collection of boxes. Hence the requirements of Lemma 3.8 (ii) are satisfied and we obtain natural numbers $n_i = \lambda_1 \cdot Lq_i \in \mathbb{N}$ yielding an interpretation witnessing essential semi-relative completeness when replacing the $x_b, b \in B$ appearing in the definition of $\llbracket\ \rrbracket_{N_0}$ with the n_i. Here we made use of the notations λ_1, L and q_i of the proof of Lemma 3.8 (ii). Now we extend this result to all closed networks and start with an arbitrary closed network $\overline{N_0}$ whose kernel is N_0 without loss of generality. Consider this modified version $\llbracket\ \rrbracket_{(\overline{N_0})}$ of the Selinger Interpretation $\llbracket\ \rrbracket_{N_0}$ that interprets all object labels appearing in $\overline{N_0}$ with pairwise different prime numbers $s_1, \ldots, s_m$ satisfying

$$s_i > C := \max_{\substack{p \in \mathcal{P} \cup \{p_0\} \\ \text{with } \deg p \leq \deg p_0}} (\lambda_1 L)^{\deg p} \cdot p(q_1, \ldots, q_t) \tag{3.9}$$

for all $1 \leq i \leq m$. C exists as due to the previous comments the maximum is taken over a finite set of polynomials. Furthermore we define $S \subset \mathbb{N}^{>0}$ as the set of natural numbers whose prime factors all appear among $s_1, \ldots, s_m$. We now define $\overline{p_0} = \llbracket \overline{N_0} \rrbracket_{(\overline{N_0})}$ which obviously equals $c \cdot p_0$ for some $c \in S$, as well as

$$\overline{\mathcal{P}} = \{\overline{p_N} = \llbracket N \rrbracket_{(\overline{N_0})} \in \mathbb{N}[B_0] \mid N \text{ closed network with } N \ncong \overline{N_0}\}. \tag{3.10}$$

As for every closed network N with kernel N' we have $\overline{p_N} = c' \cdot p_{N'}$ for some $c' \in S$ the estimation

$$\begin{aligned} \Delta_{\overline{p_N}}(\overline{\lambda}) &:= \overline{p_N}(\overline{\lambda} \cdot Lq_1, \ldots, \overline{\lambda} \cdot Lq_t) - \overline{p_0}(\overline{\lambda} \cdot Lq_1, \ldots, \overline{\lambda} \cdot Lq_t) \\ &\geq (\overline{\lambda} \cdot L)^{d_{p_0}+1} \cdot c' \cdot p_{N'}(q_1, \ldots, q_t) - (\overline{\lambda} \cdot L)^{d_{p_0}} \cdot c \cdot p_0(q_1, \ldots, q_t) > 0 \end{aligned} \tag{3.11}$$

still holds in the case $\deg \overline{p_N} > \deg \overline{p_0}$ when choosing $\overline{\lambda} = \overline{\lambda}_0 \in \mathbb{N}$ sufficiently large (which might be larger than λ_1). The choice can again be taken independently of all these $\overline{p_N}$. Here we wrote t for $|B_0|$. If $\deg \overline{p_N} = \deg \overline{p_0}$ then we have

$$\begin{aligned} \Delta_{\overline{p_N}}(\overline{\lambda}) &= \overline{p_N}(\overline{\lambda} \cdot Lq_1, \ldots, \overline{\lambda} \cdot Lq_t) - \overline{p_0}(\overline{\lambda} \cdot Lq_1, \ldots, \overline{\lambda} \cdot Lq_t) \\ &= \left(\frac{\overline{\lambda}}{\lambda_1}\right)^{d_{p_0}} \cdot \left(c' \cdot p_{N'}(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t) - c \cdot p_0(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t)\right) \neq 0. \end{aligned} \tag{3.12}$$

In fact

$$c' \cdot p_{N'}(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t) = c \cdot p_0(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t) \tag{3.13}$$

is impossible for $c = c'$ due to the construction of the $\lambda_1 Lq_i$ and for $c \neq c'$ (3.13) would imply that some of the $p_{N'}(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t)$, $p_0(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t)$ are divisible by some of the s_i contradicting (3.9). Finally in the case $\deg \overline{p_N} < \deg \overline{p_0}$ we can argue like in the proof of Lemma 3.8 and choose a sufficiently large $\overline{\lambda_1} \geq \overline{\lambda}_0$ which can be chosen such that $\overline{\lambda}_1/\lambda_1$ is a power of 2 with

$$p(\overline{\lambda_1} Lq_1, \ldots, \overline{\lambda_1} Lq_t) \neq p_0(\overline{\lambda_1} Lq_1, \ldots, \overline{\lambda_1} Lq_t)$$

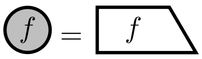
for all $p \in \mathcal{P}$ with $\deg p < \deg p_0$. But then

$$\begin{aligned} & c' \cdot p(\overline{\lambda_1} \cdot Lq_1, \ldots, \overline{\lambda_1} \cdot Lq_t) \neq c \cdot p_0(\overline{\lambda_1} \cdot Lq_1, \ldots, \overline{\lambda_1} \cdot Lq_t) \\ \Leftrightarrow \quad & c' \cdot \left(\frac{\overline{\lambda_1}}{\lambda_1}\right)^{d_p} \cdot p(\lambda_1 \cdot Lq_1, \ldots, \lambda_1 \cdot Lq_t) \neq c \cdot \left(\frac{\overline{\lambda_1}}{\lambda_1}\right)^{d_{p_0}} \cdot p_0(\lambda_1 \cdot Lq_1, \ldots, \lambda_1 \cdot Lq_t) \end{aligned} \tag{3.14}$$

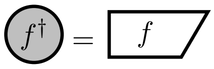
is obvious for $c = c'$ and otherwise we deduce the same contradiction that some of the $p_{N'}(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t), p_0(\lambda_1 Lq_1, \ldots, \lambda_1 Lq_t)$ are divisible by some of the s_i since 2 is the only prime factor appearing in $\overline{\lambda_1}/\lambda_1$. Therefore substituting the $x_b, b \in B$ of the modified Selinger-Interpretation $[\![\ ]\!]_{(N_0)}$ with $\overline{\lambda}_1 Lq_i$ yields an $\mathbf{FMod}_{\mathbb{N}}$-interpretation witnessing semi-relative completeness. □

3.5 Including the Dagger

Corollary 3.7 cannot be generalized to dagger compact closed categories, as the diagrams



and



cannot be seperated by means of the trivial conjugation. Hence no kind of completeness for dagger compact closed categories can hold for a sub-semiring $R \subset \mathbb{Q}$. However, due to the complex conjugation we do not have to distuinguish between completeness for compact closed and dagger compact closed categories in the case $R = \mathbb{C}$.

3.5.1 Equivalences between the dagger and non-dagger case

Proposition 3.11a *Let $\mathfrak{H}$ be a class of finite-dimensional Hilbert spaces. Then $(\mathfrak{H}, \mathbf{FHilb})$-interpretations are (essentially) fully/semi-relatively/relatively complete for dagger compact closed categories if and only if so are $(\mathfrak{H}, \mathbf{FVect}_{\mathbb{C}})$-interpretations for compact closed categories.*

The proof will suggest a new terminology we introduce here. Let $[\![\ ]\!]$ be a $\mathbf{FMod}_R$-interpretation. The *abstract interpretation* induced by $[\![\ ]\!]$ is this $\mathbf{FMod}_{\widehat{R}}$-interpretation that equals $[\![\ ]\!]$ on objects but maps arrow labels to matrices with parwise different indeterminate entries X_i, i.e. any X_i appears at most once among *all* abstract interpretations of arrow labels. If $[\![\ ]\!]$ is an $\mathbf{FProd}_R$-interpretation then its induced abstract interpretation shall only map non-dagger arrow labels to those matrices with indeterminate entries, while the image of dagger labels is determined by dagger functoriality of the abstract interpretation. We denote the abstract interpretation of a diagram M with p^M (in order to indicate that it is effectively a polynomial in the X_i) and we write $p^M((c_i)_i) \in R$ for this element of R that arises from p^M when replacing the X_i with $c_i \in R$.

Proof. Let M, N be two (simple) closed dagger networks. Define M', N' as the networks arising from M, N by bijectively replacing all dagger labels $\{f_1^\dagger, \dots, f_m^\dagger\}$ appearing in M, N with new labels $\{g_1, \dots, g_m\}$ of the same type. Clearly M and N are isomorphic if and only if so are M', N' and in the other case we suppose there is an $\mathbf{FVect}_\mathbb{C}$-interpretation $[\![\ ]\!]$ with $[\![M']\!] \neq [\![N']\!]$. Then $p^{M'}, p^{N'} \in \widehat{\mathbb{C}}$ are different polynomials as they disagree on the assignment for the X_i that $[\![\ ]\!]$ provides. Especially $p^{M'}((z_i)_i) \neq p^{N'}((z_i)_i)$ for all algebraically independent $(z_i)_i \subset \mathbb{C}$. Now when choosing $(z_i)_i \subset \mathbb{C}$ such that e.g. real and imaginary parts together form an algebraically independent subset of $\mathbb{R}$, then it is a property of the complex conjugation that also $(z_i, \overline{z_i})_i \subset \mathbb{C}$ is algebraically independent. Hence the $(\mathfrak{H}, \mathbf{FHilb})$-interpretation $[\![\ ]\!]^\star$ arising from the abstract interpretation when substituting all X_i appearing in the $[\![f]\!]$ where $f \in F_\infty$ by z_i and (necessarily) interpreting $[\![f^\dagger]\!] = \overline{[\![f]\!]}^t$ separates M and N. As $[\![\ ]\!]^\star$ does not directly depend on M, N but only on $[\![\ ]\!]$ we have proven the non-trivial implication for all types of completeness. □

The proof crucially depends on the possibility to choose $(z_i)_i$ in a way ensuring that $(z_i, \overline{z_i})_i$ is an algebraically independent set. Hence the same proof is valid when e.g. considering conjugations that interchange algebraically independent transcendentals.

Proposition 3.11b *Let R be a semi-ring. Endow $\widehat{R}$ with a conjugation $^- : \widehat{R} \to \widehat{R}$ that acts as the identity on R and as a fixpoint-free involution on $(X_i)_{i\in\mathbb{N}}$, e.g. $\overline{X_k} = X_{k+(-1)^k}$. Then for every class $\mathfrak{M}$ of free finite-dimensional $\widehat{R}$-semimodules $(\mathfrak{M}, \mathbf{FProd}_{\widehat{R}})$-interpretations are (essentially) fully/semi-relatively/relatively complete for dagger compact closed categories if and only if so are $(\mathfrak{M}, \mathbf{FMod}_{\widehat{R}})$-interpretations for compact closed categories.*

By means of this Proposition we obtain semi-relative completeness for dagger compact closed categories as it is stated in Corollary 3.6 a second time, when just using the non-dagger case of this corollary. However in the third section we have treated both cases simultaneously since the dagger version of Selinger's Interpretation is simpler than the interpretation $[\![\ ]\!]^\star$ which is induced by the non-dagger version of Selinger's Interpretation. The next observation illustrates another application of abstract interpretations.

Proposition 3.12 *Let R be a semi-ring containing $\mathbb{N}$ and $n \in \mathbb{N}_\infty$. Then $(\mathcal{B}_n R, \mathbf{FProd}_R)$-interpretations are relatively complete for compact closed categories if and only if so are $(\mathcal{B}_n\mathbb{N}, \mathbf{FMod}_\mathbb{N})$-interpretations. Likewise if $\mathbb{Z}[i] \subset R$ then $(\mathcal{B}_n R, \mathbf{FProd}_R)$-interpretations are relatively complete for compact closed categories if and only if so are $(\mathcal{B}_n\mathbb{Z}[i], \mathbf{FProd}_R)$-interpretations (where the underlying conjugation is the complex conjugation).*

Proof. The if-part is trivial as $\mathbb{N} \subset R$ resp. $\mathbb{Z}[i] \subset R$. For the only-if-part note that for two given diagrams M, N and a seperating $(\mathcal{B}_n R, \mathbf{FMod}_R)$-interpretation $[\![\ ]\!]_{M,N}$ we may consider their abstract interpretations $p^M, p^N \in \mathbb{N}[(X_i)_{i\in\mathbb{N}}]$ induced by $[\![\ ]\!]_{M,N}$. As $[\![\ ]\!]_{M,N}$ provides an assignment for the X_i distinguishing p^M from p^N these polynomials are different and therefore also have a common non-root in $\mathbb{N}$. In the dagger case we similarly consider $p^M, p^N \in \mathbb{N}[(X_i, \overline{X}_i)_{i\in\mathbb{N}}]$ and choose a common non-root in $\mathbb{Z}[i]$. □

3.5.2 Semi-relative completeness for several discrete semi-rings with non-trivial conjugation

Proposition 3.13 $\mathbf{FProd}_{\mathbb{Z}[i]}$*-interpretations are semi-relatively complete for dagger compact closed categories.*

Before proving this proposition we observe that the proof of Lemma 3.8 equally works in the case of finitely many homogeneous polynomials $p_0^{(1)}, \dots, p_0^{(l)} \notin \mathcal{P}$ that shall be *simultaneously* separated by a rational resp. a natural assignment from all polynomials in $\mathcal{P}$. As we will now use the same arguments as the proof of Proposition 3.10 we adopt its notations as far as possible.

Proof. Let $\overline{N_0}$ be an arbitrary closed dagger network with kernel $N_0 = (B_0, \ell_0, \pi_0)$. We denote the corresponding dagger version of the Selinger Interpretation with $[\![\ ]\!]_{N_0}$. Define for every simple closed dagger network N with $N \not\cong N_0$ the polynomial $p_N = [\![N]\!]_{N_0} \in \mathbb{N}[(X_b, \overline{X_b})_{b \in B}]$. When defining $X_b = Y_b + iZ_b$ and subsequently $\overline{X_b} = Y_b - iZ_b$ we may write $p_N = p_N^0 + ip_N^1$ with polynomials $p_N^0, p_N^1 \in \mathbb{N}[(Y_b, Z_b)_{b \in B}]$. Likewise define $p_0 = [\![N_0]\!]_{N_0} = p_0^0 + ip_0^1$ with $p_0^0, p_0^1 \in \mathbb{N}[(Y_b, Z_b)_{b \in B}]$. We know that not both $p_N^0 - p_0^0$ and $p_N^1 - p_0^1$ vanish as otherwise $p_N - p_0$ would vanish even for every input in $\mathbb{C}$. But this is impossible as the Selinger Interpretation witnesses semi-relative completeness for dagger compact closed categories when underlying $\mathbb{C}$ (cf. 3.6) and therewith provides an assignment for the X_b separating p_N from p_0. Let p'_N be one of the polynomials p_N^i $(i = 1, 2)$ with $p_N^i - p_0^i \neq 0$. The idea is now to repeat the proof of Proposition 3.10 with regard to the p'_N. We define again

$$\mathcal{P} = \{p'_N \in \mathbb{N}[(Y_b, Z_b)_{b \in B}] \mid N \text{ simple closed network with } N \not\cong N_0\}$$

and apply the second part of Lemma 3.8 on $\mathcal{P}$, $\{p_0^0, p_0^1\}$ to obtain $n_i = \lambda_1 \cdot Lq_i \in \mathbb{N}$, $1 \leq i \leq t = 2|B|$ separating all $p'_N \in \mathcal{P}$ from *each* p_0^0 and p_0^1 when replacing the Y_b, Z_b by the n_i. Hence this replacement of the Y_b, Z_b appearing in $[\![\ ]\!]_{N_0}$ yields an $\mathbf{FProd}_{\mathbb{Z}[i]}$-interpretation witnessing essential semi-relative completeness.

The extension to all closed dagger networks will be done analogously to the proof of Theorem 3.10. We consider the modified version $[\![\ ]\!]_{(N_0)}$ of the Selinger Interpretation sending object labels appearing in N_0 to pairwise different prime numbers $s_1, \dots, s_m$ satisfying the equivalent of (3.9). We generalize our previous definitions by setting

$$\begin{aligned}
&\overline{p}_N = [\![N]\!]_{(N_0)} \in \mathbb{N}[(X_b, \overline{X_b})_{b \in B}] \quad \text{for all closed dagger networks } N \qquad \text{and} \\
&\overline{p}_N = \overline{p}_N^0 + i\overline{p}_N^1 \quad \text{with} \quad \overline{p}_N^0, \overline{p}_N^1 \in \mathbb{N}[(Y_b, Z_b)_{b \in B}].
\end{aligned}$$

and likewise $\overline{p}_0 = [\![\overline{N}_0]\!]_{(N_0)} = \overline{p}_0 + i\overline{p}_1$. Define $\overline{p}'_N$ again as one of the polynomials $\overline{p}_N^i$ with $\overline{p}_N^i - \overline{p}_0^i \neq 0$. Note that this can be done in a way ensuring that for all closed dagger networks N with kernel N' we have $\overline{p}'_N = \overline{p}_N^i$ implies $\overline{p}'_{N'} = \overline{p}_{N'}^i$ for the same i. Then we obtain again $\overline{p}_N = c' \cdot p_{N'}$ for some $c \in \mathbb{N}$ whose prime factors all appear among $s_1, \dots, s_m$.

We set

$$\overline{\mathcal{P}} = \{\overline{p}'_N \in \mathbb{N}[(Y_b, Z_b)_{b\in B}] \mid N \text{ closed dagger network with } N \not\cong \overline{N_0}\}$$

and define analogously to (3.11)

$$\Delta_{\overline{p}'_N}(\overline{\lambda}) := \overline{p}_N(\overline{\lambda}\cdot Lq_1,\ldots,\overline{\lambda}\cdot Lq_t) - \overline{p}^i_0(\overline{\lambda}\cdot Lq_1,\ldots,\overline{\lambda}\cdot Lq_t)$$

where in this case $i \in \{0,1\}$ is this bit depending on N with $\overline{p}'_N = \overline{p}^i_N$. Due to the fact that the $\lambda_1 \cdot Lq_t$ separate all p'_N from both p_0 and p_1 together with $\overline{p}_N = c' \cdot p_{N'}$ we can argue exactly like in the proof of Proposition 3.10 and derive (3.11), (3.12) and (3.14) in the same way for some sufficiently large $\overline{\lambda}_1 \in \mathbb{N}$. Hence replacing the Y_b and Z_b by the $\overline{\lambda}_1 \cdot Lq_i$ and substituting the resulting values for the X_b into $[\![\ ]\!]_{(N_0)}$ yields an $\mathbf{FProd}_{\mathbb{Z}[i]}$-interpretation witnessing semi-relative completeness for dagger compact closed categories. □

Completeness of a lot of other semi-rings with a non-trivial conjugation containing $\mathbb{N}$ is indeed weaker than completeness for $\mathbb{Z}[i]$ as the following corollary shows.

Corollary 3.14 *Let R be a semi-ring containing $\mathbb{N}$.*

(i) *If $R \subset \mathbb{C}$ and $i \in R$ then $\mathbf{FProd}_R$-interpretations are semi-relatively complete for dagger compact closed categories.*

(ii) *If $\mathbb{Z} \subset R$ then $\mathbf{FProd}_{R[X]}$-interpretations are semi-relatively complete for dagger compact closed categories. This especially includes the cases $\mathbb{Z}[\zeta]$ for any transcendental $\zeta \in \mathbb{R}$*

(iii) *$\mathbf{FProd}_{R[X,\overline{X}]}$-interpretations are semi-relatively complete for dagger compact closed categories.*

Proof. (i): The conditions on R enforce $\mathbb{Z}[i] \subset R$.
(ii): Due to the last Proposition we have semi-relative completeness in the case $\mathbb{Z}[i] = \mathbb{Z}[X]/(X^2+1) \subset R[X]/(X^2+1)$. The existence of the modulo-homomorphism

$$\varphi : R[X] \to R[X]/(X^2+1), \qquad p \ \mapsto\ p \bmod X^2+1$$

now ensures that a collection of $\mathbf{FProd}_{\mathbb{Z}[i]}$-interpretations witnessing semi-relative completeness can be converted to a collection of $\mathbf{FProd}_{R[X]}$-interpretations witnessing semi-relative completeness when replacing every appearance of i with X.
(iii): Argue analogously with

$$\varphi' : R[X,\overline{X}] \to R[X] = R[X,\overline{X}]/(X+\overline{X}), \qquad p \ \mapsto\ p \bmod X+\overline{X}$$ □

We note that this transition to $\mathbb{Z}[X]$, $\mathbb{N}[X,\overline{X}]$ can analogously be done for Proposition 3.12.

3.6 Interconnections between full completeness and completeness for bounded dimensions

After clarifying when we have (essential) relative and semi-relative completeness we might ask whether full completeness holds for $\mathbf{FMod}_R$-interpretations. This turns out to be a harder problem. As we will see, (essential) full completeness of **FHilb**-interpretations is equivalent to essential full and (essential) relative completeness of $(\mathcal{B}_n\mathbb{C}, \mathbf{FHilb})$-interpretations for sufficiently large $n \in \mathbb{N}$. Here for any semi-ring R we denote the class of all free finite-dimensional R-semimodules with a dimension less than or equal to n with $\mathcal{B}_n R$ where $n \in \mathbb{N}$. If $n = \infty$ then $\mathcal{B}_\infty R$ shall denote the class of all free finite-dimensional R-semimodules.

Before stating the next result we remind of the notation $\mathscr{S}_\infty = (S_\infty, F_\infty, \mathrm{dom}, \mathrm{cod})$ and write $\mathscr{S}_\infty(\mathcal{A}, \mathcal{B})$ for the set of all $f \in F_\infty$ of type $\mathcal{A} \to \mathcal{B}$ in the non-dagger case. In the dagger case we presuppose a splitting into non-dagger labels $\mathscr{S}_\infty(\mathcal{A}, \mathcal{B})$ and corresponding dagger labels $\mathscr{S}^\dagger_\infty(\mathcal{A}, \mathcal{B})$ of type $\mathcal{A} \to \mathcal{B}$.

Proposition 3.15 *Let R be a semi-ring with a fixed conjugation $^-$ and $n \in \mathbb{N}_\infty$. If $\mathbf{FMod}_R$- resp. $\mathbf{FProd}_R$-interpretations are essentially fully complete for (dagger) compact closed categories then $(\mathcal{B}_n R, \mathbf{FMod}_R)$- resp. $(\mathcal{B}_n R, \mathbf{FProd}_R)$-interpretations are essentially relatively complete for (dagger) compact closed categories, where*

$$n \geq \min_{\substack{[\![\ ]\!] \text{ interpretation witnessing} \\ \text{essential full completeness}}} \min_{A \in S_\infty} [\![A]\!].$$

Proof. Let essential full completeness be witnessed by $[\![\ ]\!]$ and let M, N be simple closed networks with $M \not\cong N$. Also let $A \in S_\infty$ be one of the sorts, minimizing $\dim[\![X]\!]$ for $X \in S_\infty$. Let $\Phi : \mathbf{Net}\,\mathscr{S}_\infty \to \mathbf{Net}\,\mathscr{S}_\infty$ be a traced (dagger) functor that it characterized by the following information. It maps all object labels to A and every non-dagger arrow label f to a non-dagger arrow label Φf of type $A^{|\mathrm{dom}(f)|} \to A^{|\mathrm{cod}(f)|}$ in a way ensuring that

$$\begin{array}{ccc} \{f \in F_\infty \mid |\mathrm{dom}(f)| = i,\ |\mathrm{cod}(f)| = j\} & \to & \{f \in F_\infty \mid \mathrm{dom}(f) = A^i,\ \mathrm{cod}(f) = A^j\} \\ f & \mapsto & \Phi f \end{array}$$

is injective for all $i, j \in \mathbb{N}$. This can be realized as all S_∞ and all $\mathscr{S}_\infty$-homsets are countable, the previous function is of type

$$\bigcup_{\substack{\mathcal{X} = X_1 \dots X_i \in S^i_\infty \\ \mathcal{Y} = Y_1 \dots Y_j \in S^j_\infty}} \mathscr{S}_\infty(\mathcal{X}, \mathcal{Y}) \quad \to \quad \mathscr{S}_\infty(A^i, A^j)$$

and countable unions as well as finite cartesian products of countable sets remain countable. In the dagger case we necessarily set $\Phi(f^\dagger) = (\Phi f)^\dagger$. Due to this injectivity (and the absence of trivial cycles) we obtain $\Phi M \not\cong \Phi N$ and therefore $[\![\Phi M]\!] \neq [\![\Phi N]\!]$. Defining

$[\![\ \ ]\!]_{M,N} = [\![\ \ ]\!] \circ \Phi$ yields a seperating interpretation as

$$[\![M]\!]_{M,N} = [\![\Phi M]\!] \neq [\![\Phi N]\!] = [\![N]\!]_{M,N}. \qquad \square$$

Proposition 3.16 *Let R be a semi-ring and $n \in \mathbb{N}$. $(\mathcal{B}_n\widehat{R}, \mathbf{FMod}_{\widehat{R}})$-interpretations are essentially fully complete if and only if they are essentially relatively complete for compact closed categories.*

Proof. Suppose $(\mathcal{B}_n\widehat{R}, \mathbf{FMod}_{\widehat{R}})$-interpretations are essentially relatively complete for compact closed categories. Then every pair M, N of non-isomorphic simple closed networks can be separated by an interpretation $[\![\ \ ]\!]_{M,N}$ that maps all object labels to $\widehat{R}^n$ (extend matrices with zero elements if necessary). Hence the abstract interpretations $p^M \in \widehat{R}[(Y_i)_i]$ of diagrams M induced by $[\![\ \ ]\!]_{M,N}$ do not depend on M, N as we assume $\dim[\![A]\!] = n$ for all object labels A. The interpretations $[\![\ \ ]\!]_{M,N}$ provide assignments for the Y_i demonstrating that the p^M are pairwise different when M ranges over all simple diagrams. Replacing the Y_i by a countable algebraically independent set of elements in $\widehat{R}$ whose elements w.l.o.g. do not appear in the p^M, yields an interpretation witnessing essential full completeness. $\square$

We stated Proposition 3.16 only for the non-dagger case as the analogous dagger statement automatically follows from Proposition 3.11. Now we will see that Proposition 3.15 and 3.16 do not have to be restricted to essential completeness.

3.7 Classification of Essential Completeness

Proposition 3.17 *Let R be a semi-ring containing $\mathbb{N}$ with a fixed conjugation $^-$ and $n \in \mathbb{N}_\infty$.*

(i) *If $(\mathcal{B}_nR, \mathbf{FMod}_R)$- resp. $(\mathcal{B}_nR, \mathbf{FProd}_R)$-interpretations are essentially relatively complete for (dagger) compact closed categories, then they are relatively complete for (dagger) compact closed categories.*

(ii) *If $(\mathcal{B}_n\widehat{R}, \mathbf{FMod}_{\widehat{R}})$-interpretations are essentially fully complete for compact closed categories, then $\mathbf{FMod}_{\widehat{R}}$-interpretations are fully complete for compact closed categories.*

Proof. (i): Let M, N be two closed networks with kernels M_0, N_0. If $M_0 \cong N_0$, then there is an object label X appearing unequally often in M and N. Setting $[\![X]\!]_{M,N} = R^2$, $[\![A]\!]_{M,N} = R$ for all other objects A and defining $[\![f]\!]_{M,N}$ for all arrows f as matrices with natural entries such that $[\![M_0]\!]_{M,N} = [\![N_0]\!]_{M,N} > 0$, yields

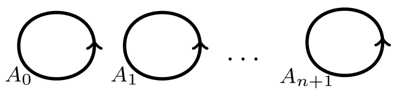
$$\begin{aligned}[\![M]\!]_{M,N} &= 2^{\#\text{ of trivial cycles in } M \text{ labelled with } X} \cdot [\![M_0]\!]_{M,N} \\ &\neq 2^{\#\text{ of trivial cycles of } N \text{ labelled with } X} \cdot [\![N_0]\!]_{M,N} = [\![N]\!]_{M,N}.\end{aligned}$$

If $M_0 \ncong N_0$ then their abstract interpretations p^{M_0}, p^{N_0} induced by an interpretation $[\![\ \ ]\!]_{M_0,N_0}$ which separates M_0 from N_0, are not multiples of each other. Indeed, considering the denotation (3.2) shows that the X_i appearing in the monomials of p^{M_0}, p^{N_0} determine how often which box appears in M_0, N_0. This data in turn determines how often wires with a certain label appear. Thus if p^{M_0}, p^{N_0} consist of the same monomials, then they must have an equal number of summands, making $c \cdot p^{M_0} = c' \cdot p^{N_0}$ impossible for $c \neq c'$. Therefore $p^M = \text{const.} \cdot p^{M_0} \neq \text{const.} \cdot p^{N_0} = p^N$ and we can choose a non-root of $p^M - p^N$.
(ii): Let $[\![\ \ ]\!]$ be a $(\mathcal{B}_n\widehat{R}, \mathbf{FMod}_{\widehat{R}})$-interpretation witnessing essential full completeness. Then we may assume w.l.o.g. that the $\dim[\![A]\!]$ for $A \in S_\infty$ are pairwise different prime numbers, otherwise increase the $\dim[\![A]\!]$ appropriately and fill up the additional entries of the matrices $[\![f]\!]$, $f \in F_\infty$ with zeros. Due to uniqueness of prime factorization we can recover what trivial cycles M contains when having $p^M \in \widehat{R}[(Y_i)_i]$. Therefore all p^M are (pairwise) different and replacing the Y_i by algebraically independent transcendentals from $\widehat{R}$ yields an interpretation proving full completeness. □

Again 3.11 makes it redundant to treat the dagger case of (ii) as well. The second implication only speaks of full completeness for interpretations with unbounded dimensions. This is in fact the best we can achieve as the following observation illustrates.

Proposition 3.18 *Let R be a semi-ring and $n \in \mathbb{N}$ a natural number. Then $(\mathcal{B}_nR, \mathbf{FMod}_R)$-interpretations are not semi-relatively complete for compact closed categories.*

Proof. Suppose there is an $(\mathcal{B}_nR, \mathbf{FMod}_R)$-interpretation $[\![\ \ ]\!]_M$ for every closed network M witnessing semi-relative completeness. Consider

$M =$ 

.

As $\dim[\![A_i]\!]_M \leq n$ for all $0 \leq i \leq n+1$ the interpretations of some of the A_i - w.l.o.g. A_0 and A_1 - have the same dimension. But then we have $[\![M]\!]_M = [\![M']\!]_M$ for the diagram M' consisting of $n+2$ trivial cycles labelled with $A_0, A_0, A_2, A_3, \dots, A_{n+1}$ although $M \ncong M'$. ↯ □

The various implications we have proven are visualized in the following diagram.

$$\begin{tikzcd}
\text{full completeness} \arrow[r, shift right] & \text{essential full completeness} \arrow[l, dashed, shift right] \arrow[ddd, bend left] \\
\cancel{\text{full completeness with bounded dimensions}} & \text{essential full completeness with bounded dimensions} \arrow[u] \arrow[ul, dashed] \arrow[d] \\
\cancel{\text{semi-relative completeness with bounded dimensions}} & \text{essential semi-relative completeness with bounded dimensions} \arrow[d] \\
\text{relative completeness with bounded dimensions} \arrow[r, shift right] & \text{essential relative completeness with bounded dimensions} \arrow[l, shift right] \arrow[uu, dashed, bend left]
\end{tikzcd} \tag{3.15}$$

Here completeness means both completeness of $\mathbf{FMod}_R$- and $\mathbf{FProd}_R$-interpretations with $\mathbb{N} \subset R$ for compact closed resp. dagger compact closed categories. The dashed arrows only hold for $\mathbf{FMod}_{\widehat{R}}$-interpretations. We particularly observe that in this case all stated kinds of completeness are equivalent. Also Proposition 3.18 explains why it is necessary to differentiate between completeness and essential completeness. Furthermore we know for $\mathbb{N} \subset R$ that $\mathbf{FMod}_R$-interpretations are always semi-relatively complete. Hence in this case also semi-relative and essential semi-relative completeness are equivalent. Hence completeness for interpretations with bounded dimensions is the remaining open question we have to deal with.

Chapter 4

Non-trivial trace equations

As we have seen that for $\mathbf{FMod}_{\hat{R}}$-interpretations full completeness is equivalent to relative completeness for interpretations with bounded dimensions we will now focus on the question when two non-isomorphic diagrams can be distinguished in the latter case. For this it will be useful to focus first on networks representing the trace of a composition of functions. Note that we do not need to consider dagger networks separately as Proposition 3.11 shows how any non-trivial equation between non-isomorphic diagrams including daggers can be transformed into such an equation without daggers when underlying a semi-ring R providing countably many transcendentals.

Selinger resp. Pare pointed out already that Hilbert spaces restricted to dimensions of at most 2 are not essentially relatively complete for compact closed and therewith dagger compact closed categories because the trace equation

$$\mathrm{tr}(A^2B^2AB) = \mathrm{tr}(B^2A^2BA) \tag{4.1}$$

holds for all $A, B \in \mathbb{C}^{2\times 2}$ but not in the graphical calculus. We will now aim to find out whether similar trace equations (over $\mathbb{C}$) can be found for higher dimensional matrices. This will require a lot of notational preparations we will deal with in the first section. Then we will deduce that validity of non-trivial trace equations like (4.1) is equivalent to a combinatorial problem allowing us to exclude existence of those equations in a lot of special instances. In the third section we will discuss related questions, namely what trace equations exist for two dimensional matrices, under what operations the collection of trace equations is closed and how far non-existence of non-trivial trace equations would bring us in terms of completeness for all closed networks.

Finally we finish our completeness analyses by dealing with $\mathbf{FMod}_R$-interpretations for semi-rings R with $\mathbb{N} \nsubseteq R$. In this case we will derive several non-trivial trace equations proving that then in many cases not even essential relative completeness can be achieved. Our considerations will also turn out to be directly applicable to $\mathbf{Rel}$-interpretations.

4.1 Semantical Preparations

For notational convenience we will work in the context of *words* over *alphabets* and regard a product $P = M_{i_1}^{e_1} \cdot \ldots \cdot M_{i_n}^{e_n}$ consisting only of the matrices $M_1, \ldots, M_l$, as the string $\mathcal{P} = M_{i_1}^{e_1} \ldots M_{i_n}^{e_n}$ over the alphabet $\{M_1, \ldots, M_l\}$. The following preliminary remark evaluates $\mathrm{tr}(\mathcal{P})$.

Lemma 4.1 *Let $M_1, \ldots, M_l$ be $n \times n$-matrices (over a general semi-ring R) and let $m_k(i,j)$ denote the ij-th entry of M_k. Then the ij-th entry of $\prod_{1 \le k \le l} M_k$ equals*

$$\sum_{d_1,\ldots,d_{l-1}=1}^{n} m_1(i,d_1)m_2(d_1,d_2)\cdot\ldots\cdot m_{l-1}(d_{l-2},d_{l-1})m_l(d_{l-1},j) = \sum_{d_1,\ldots,d_{l-1}=1}^{n} \prod_{k=1}^{l} m_k(d_{k-1},d_k)$$

where $d_0 = i$ and $d_l = j$.

Proof. The claim is easily verified by induction on l. The case $l = 1$ is trivial and if the Lemma holds for $M_1, \ldots, M_l$ then the definition of matrix multiplication shows for an additional M_{l+1}:

$$\begin{aligned}\left(\prod_{k=1}^{l+1} M_k\right)_{ij} &= \sum_{d_l=1}^{n} \left(\sum_{d_1,\ldots,d_{l-1}=1}^{n} m_1(i,d_1)m_2(d_1,d_2)\cdot\ldots\cdot m_l(d_{l-1},d_l) \right) \cdot m_{l+1}(d_l,j) \\ &= \sum_{d_1,\ldots,d_l=1}^{n} m_1(i,d_1)m_2(d_1,d_2)\cdot\ldots\cdot m_l(d_{l-1},d_l)m_{l+1}(d_l,j)\end{aligned}$$

□

Corollary 4.2 *Let $A = (a_{ij})_{ij}$ be an $n \times n$-matrix and $l \ge 1$. Then*

$$\mathrm{tr}(A^l) = \sum_{d_1,\ldots,d_l=1}^{n} a_{d_l d_1} a_{d_1 d_2} \cdot \ldots \cdot a_{d_{l-1} d_l}. \tag{4.2}$$

We observe that (4.2) could also be derived from the formula (3.2) for the denotation. The previous Corollary already suggests to introduce shorter notations in order to abbreviate the appearing summands. The next definition provides all terminology we need to state the results of this chapter in a concise manner.

Definition 4.3 *Let Σ be an alphabet and $\mathcal{d} = d_1 \ldots d_l \in \Sigma^*$ a string over Σ.*

(i) *$\mathcal{d}' \in \Sigma^*$ is called a* cyclic permutation *of $\mathcal{d}$ iff $\mathcal{d}' = d_i \ldots d_l d_1 \ldots d_{i-1}$ for some $1 \le i \le l$. In this case we write $\mathcal{d} \sim_c \mathcal{d}'$. If Σ is a set of matrices and $\mathcal{P}, \mathcal{P}' \in \Sigma^*$ can be regarded as matrix products (i.e. the matrices have appropriate dimensions) then we call the trace equation* $\mathrm{tr}(\mathcal{P}) = \mathrm{tr}(\mathcal{P}')$ trivial *iff $\mathcal{P} \sim_c \mathcal{P}'$, otherwise it is called* non-trivial.

(ii) *We define*

$$\langle d \rangle := \langle (d_i, d_{i\oplus 1}) \rangle_{i=1,\dots,n} = \langle (d_1, d_2), \dots, (d_{n-1}, d_n), (d_n, d_1) \rangle$$

and for another $d' \in \Sigma^*$ *we write* $d \sim_2 d'$ *iff* $\langle d \rangle = \langle d' \rangle$*. If* Σ *is a set of indices we abbreviate the product* $a_{d_l d_1} a_{d_1 d_2} \cdot \ldots \cdot a_{d_{l-1} d_l}$ *with* a_d*.*

(iii) d *is called* primitive, *if there is no shorter string* $s \in \Sigma^*$ *with* $d = s^k$ *for some* $k \in \mathbb{N}^{>1}$.

Clearly $\sim_c$ and $\sim_2$ are equivalence relations and $d \sim_c d'$ implies $d \sim_2 d'$. Furthermore we write $[d]_c$ resp. $[d]_2$ for the equivalence classes of d with respect to $\sim_c$ resp. $\sim_2$. It is also obvious that any trace equation

$$\operatorname{tr}(\mathcal{P}) = \operatorname{tr}(\mathcal{P}') \tag{4.3}$$

for $\mathcal{P}, \mathcal{P}' \in \Sigma^*$ holds in the graphical language if and only if $\mathcal{P}$ and $\mathcal{P}'$ are cyclic permutations of each other (and hence trivial trace equations are in fact valid). The definition of $\langle d \rangle$ is motivated by the observation that the product a_d does not completely depend on d but only on $\langle d \rangle$, while the notion of primitivity will be useful later to prove a non-trivial trace equation for $R = \mathbb{Z}/p^l\mathbb{Z}$.

Now suppose some equation of kind (4.3) holds for all square matrices $M_0, \dots, M_l$ of the same dimension. Let A, B be new matrix variables. Then we can substitute M_i by AB^i for all $0 \leq i \leq l$ and get a new equation only including the matrix variables A and B. Moreover the map

$$\{M_0, \dots, M_k\}^* \xrightarrow{\text{subs}} \{A, B\}^*, \qquad M_i \mapsto AB^i \tag{4.4}$$

(whose values for longer words are determined by its compatibility with concatenation) is obviously bijective and compatible with $\sim_c$, i.e. $\mathcal{P} \sim_c \mathcal{P}'$ if and only if $\operatorname{subs}(\mathcal{P}) \sim_c \operatorname{subs}(\mathcal{P}')$. Hence (4.3) is non-trivial if and only if so is $\operatorname{tr}(\operatorname{subs}(\mathcal{P})) = \operatorname{tr}(\operatorname{subs}(\mathcal{P}'))$. Therefore if there is a non-trivial trace equation for matrices with restricted dimensions at all, then there is also such an equation with words over the alphabet $\{A, B\}$. Because of this we focus on the search for trace equations containing only two different types of maps. It will also be convenient to use the abbreviation

$$\mathcal{P}(k) = AB^{k_1} AB^{k_2} \cdot \ldots \cdot AB^{k_l}$$

where $k \in \mathbb{N}^*$.

4.2 Equivalence to a combinatorial problem

4.2.1 Reconstructability of combinatorial information

We are now able to deduce the main result stating that finding non-trivial trace equations over $\mathbb{C}$ is equal to solving a combinatorial problem. Before stating the main result we observe that the lengths $|\mathscr{d}|, |\mathscr{d}'|$ of two strings coincide if $\mathscr{d} \sim_2 \mathscr{d}'$.

Theorem 4.4 *Let $n \in \mathbb{N}$ and A, B be two complex $n \times n$-matrices. Then for all strings of natural numbers $\mathscr{k} = k_1, \dots k_l, \mathscr{k}' = k'_1 \dots k'_{l'} \in \mathbb{N}^*$ we have: The trace equation*

$$\operatorname{tr}\left(\mathscr{P}(\mathscr{k})\right) = \operatorname{tr}\left(\mathscr{P}(\mathscr{k}')\right) \tag{4.5}$$

holds if and only if $l = l'$ and for every equivalence class $[\mathscr{d}_0]_2 \in \{1, \dots, n\}^* / \sim_2$ with $|\mathscr{d}_0| = l$ we have

$$\left\langle \left(\sum_{j : d_j = i} k_j \right)_{i=1,\dots,n} \right\rangle_{\mathscr{d}=d_1 \dots d_l \in [\mathscr{d}_0]_2} = \left\langle \left(\sum_{j : d_j = i} k'_j \right)_{i=1,\dots,n} \right\rangle_{\mathscr{d}=d_1 \dots d_l \in [\mathscr{d}_0]_2} \tag{4.6}$$

Example 4.5 We illustrate the meaning of this Theorem by applying it on (4.1) which is equivalent to $\operatorname{tr}(\mathscr{P}(012)) = \operatorname{tr}(\mathscr{P}(210))$. The multisets (4.6) are given in the following table for different n and $\mathscr{d}_0$.

n	$\mathscr{d}_0$	$\mathscr{k} = 012$	$\mathscr{k}' = 210$
2	111	$\langle(3,0)\rangle$	$\langle(3,0)\rangle$
2	112	$\langle(1,2),(3,0),(2,1)\rangle$	$\langle(3,0),(1,2),(2,1)\rangle$
2	122	$\langle(0,3),(1,2),(2,1)\rangle$	$\langle(2,1),(1,2),(0,3)\rangle$
2	222	$\langle(0,3)\rangle$	$\langle(0,3)\rangle$
3	123	$\langle(0,1,2),(1,2,0),(2,0,1)\rangle$	$\langle(2,1,0),(1,0,2),(0,2,1)\rangle$

As the multisets for $n = 2$ coincide (4.1) is valid for 2-dimensional matrices, but it does not hold for 3×3-matrices as (4.6) is not satisfied for $\mathscr{d}_0 = 123$.

Proof of Theorem 4.4. Suppose (4.5) holds for given words $\mathscr{k}, \mathscr{k}' \in \mathbb{N}^*$. Then we may choose B as a diagonal matrix

$$B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and the entries $a_{ij} \in \mathbb{C}$, $1 \leq i, j \leq n$ of A such that the set $\{a_{ij}, \lambda_i \,|\, 1 \leq i, j \leq n\} \subset \mathbb{C}$ is algebraically independent over $\mathbb{Q}$. Define $M_\iota = AB^{k_\iota}$ and $M_\kappa = AB^{k'_\kappa}$ for all $1 \leq \iota \leq l, 1 \leq \kappa \leq l'$. Then clearly $M_\iota = (\lambda_j^{k_\iota} a_{ij})_{ij}$, $M'_\kappa = (\lambda_j^{k'_\kappa} a_{ij})_{ij}$. Therefore, using Lemma 1, we obtain

$$\begin{aligned}
\operatorname{tr}(\mathcal{P}(\mathcal{K})) = \operatorname{tr}(M_1 \dots M_l) &= \sum_{d_1,\dots,d_l=1}^{n} \lambda_{d_1}^{k_1} a_{d_l d_1} \lambda_{d_2}^{k_2} a_{d_1 d_2} \cdot \ldots \cdot \lambda_{d_l}^{k_l} a_{d_{l-1} d_l} \\
&= \sum_{\substack{d \in \{1,\dots,n\}^*,\\ |d|=l}} \lambda_{d_1}^{k_1} \cdot \ldots \cdot \lambda_{d_l}^{k_l} a_d = \sum_{[d_0]_2 \in \{1,\dots,n\}/\sim_2} \left(\sum_{\substack{d\in[d_0]_2,\\ d=d_1\dots d_l}} \lambda_{d_1}^{k_1} \cdot \ldots \cdot \lambda_{d_l}^{k_l} \right) a_{d_0}.
\end{aligned} \tag{4.7}$$

When arguing analogously for the right hand side of (4.5) we reformulate this equation as

$$\sum_{\substack{[d_0]_2 \in \{1,\dots,n\}/\sim_2\\ |d_0|=l}} \left(\sum_{\substack{d\in[d_0]_2,\\ d=d_1\dots d_l}} \lambda_{d_1}^{k_1} \cdot \ldots \cdot \lambda_{d_l}^{k_l} \right) a_{d_0} = \sum_{\substack{[d_0]_2 \in \{1,\dots,n\}/\sim_2\\ |d_0|=l'}} \left(\sum_{\substack{d\in[d_0]_2,\\ d=d_1\dots d_{l'}}} \lambda_{d_1}^{k'_1} \cdot \ldots \cdot \lambda_{d_{l'}}^{k'_{l'}} \right) a_{d_0} \tag{4.8}$$

As all factors a_{ij}, λ_i are algebraically independent, the coefficients of both sides of (4.8) interpreted as polynomials in the a_{ij}, must be equal. Hence we must have $l = l'$ (otherwise both sides contain different monomials) and it holds

$$\sum_{d=d_1\dots d_l \in [d_0]_2} \lambda_{d_1}^{k_1} \cdot \ldots \cdot \lambda_{d_l}^{k_l} = \sum_{d=d_1\dots d_{l'} \in [d_0]_2} \lambda_{d_1}^{k'_1} \cdot \ldots \cdot \lambda_{d_{l'}}^{k'_{l'}}$$

for all $[d_0]_2 \in \{1,\dots,n\}/\sim_2$. Due to the algbraic independence of the λ_i again, the n-tuple of exponents of the $\lambda_1,\dots,\lambda_n$, which are

$$\sum_{j:d_j=1} k_j, \dots, \sum_{j:d_j=n} k_j \qquad \text{resp.} \qquad \sum_{j:d_j=1} k'_j, \dots, \sum_{j:d_j=n} k'_j$$

must be the same. This proves (4.6).

Now conversely assume that (4.6) holds. The expression

$$\operatorname{tr}(AB^{k_1}AB^{k_2} \cdot \ldots \cdot AB^{k_l}) - \operatorname{tr}(AB^{k'_1}AB^{k'_2} \cdot \ldots \cdot AB^{k'_{l'}})$$

can be considered as a polynomial in the $2n^2$ variables $a_{ij}, b_{ij}, 1 \le i,j \le n$. Let $Z_0 \subset \mathbb{C}^{2n^2}$ be the set of its zeros. As the trace is invariant under change of bases, we have

$$\operatorname{tr}(AB^{k_1}AB^{k_2} \cdot \ldots \cdot AB^{k_l}) = \operatorname{tr}(A'B'^{k_1}A'B'^{k_2} \cdot \ldots \cdot A'B'^{k_l})$$

where $A' = T^{-1}AT$ and $B' = T^{-1}BT$. Hence - together with the previous consderation - we are able to deduce, that also (4.5) will hold if the eigenvalues of B together with the (multi)set of entries of $T^{-1}AT$ (where T depends on B) are algebraically independent. Thus it remains to show that in the case $Z_0 \neq \mathbb{C}^{2n^2}$ the set $Z \subset \mathbb{C}^{2n^2}$ of matrix pairs

(A, B) getting mapped to an algebraically independent (multi)set

$$(A, B) \mapsto \langle \text{entries of } T^{-1}AT, \text{ eigenvalues of } B \rangle \tag{4.9}$$

is strictly larger then Z_0. Due to $Z_0 \neq \mathbb{C}^{2n^2}$ it is the set of zeros of a non-zero polynomial. Hence it has Lebesgue measure 0. But the image of (4.9) will be algebraically independent in the generic case so that Z must have infinite Lebesgue measure. Therefore $Z \neq Z_0$. $\square$

Informally this proposition tells us that the multisets (4.6) are exactly this information about $\mathpzc{k}$ one can deduce from having $\mathrm{tr}(AB^{k_1} \cdot \ldots \cdot AB^{k_l})$. The question whether non-trivial trace formulas exist therefore equals the problem whether the information $[\mathpzc{k}]_c$ can be deduced from the "datasets" (4.6). In order to capture this mathematically, we introduce the concept of n-reconstructability:

Definition 4.6 *Let $\mathpzc{k} = k_1 \ldots k_l \in \mathbb{N}^*$ be a word and $\mathpzc{P}_{\mathpzc{k}} = AB^{k_1} \cdot \ldots \cdot AB^{k_l}$ be the corresponding matrix product. Let also* req *be a map*[1] *defined on $\mathbb{N}^*$ or - more generally - on a subset of $\mathbb{N}^*$. We call* req n-reconstructable *if for all $\mathpzc{k}, \mathpzc{k}' \in \mathbb{N}^*$ satisfying the trace equation $\mathrm{tr}(\mathpzc{P}_{\mathpzc{k}}) = \mathrm{tr}(\mathpzc{P}_{\mathpzc{k}'})$ for all $n \times n$-matrices A, B, we have*

$$\mathrm{req}(\mathpzc{k}) = \mathrm{req}(\mathpzc{k}').$$

Using this terminology, Proposition 4.4 claims the n-reconstructability of

$$\mathpzc{k}\langle \mathpzc{d}_0, n\rangle := \left\langle \left(\sum_{j:d_j=i} k_j \right)_{i=1,\ldots,n} \right\rangle_{\mathpzc{d}=d_1\ldots d_l \in [\mathpzc{d}_0]_2} \tag{4.10}$$

and the non-existence of non-trivial trace equations for $n \times n$-matrices is equivalent to the n-reconstructability of $[\mathpzc{k}]_c$. Obviously n-reconstructability implies N-reconstructability if $n \leq N$.

4.2.2 Consequences

Before stating the next result, note that whenever indices might be out of range they have to be considered up to a multiple of the length l of the corresponding string $\mathpzc{k}$. Moreover we regard the first and last digit of $\mathpzc{k}$ as neighboured (i.e. we think modulo cyclic permutations). Therefore $\mathpzc{k} = 22122$ contains e.g. the substring 2222. Furthermore we say that $\mathpzc{k}$ contains a k^e-*block*, if either $\mathpzc{k} = k^e$ or $\mathpzc{k}$ contains the substring k^e that is bounded by letters different from k. Therefore 22122 contains a 2^4-block but not a 2^2-block. Finally we will focus on the issue of 3-reconstructability as 3 is the least dimension for which the question about existence of non-trivial trace equations is open. However, most statements have an immediate n-reconstructability analogue.

[1] We envisage req as a method of *requesting* information about $\mathpzc{k}$. Instead of mentioning req explicitly we will usually refer to it by its value $\mathrm{req}(\mathpzc{k})$ for an indeterminate $\mathpzc{k} = k_1 \ldots k_l$.

Corollary 4.7 *Let $\underset{\sim}{k} = k_1 \dots k_l \in \mathbb{N}^*$ be a string of natural numbers.*

(i) *For every $1 \leq i < l$ the multiset*

$$\underset{\sim}{k}\langle i \rangle := \left\langle \sum_{\iota=\kappa}^{\kappa+i-1} k_\iota \right\rangle_{\kappa=1,\dots,l}$$

is 2-reconstructable.

(ii) *Set $k = \max(k_1, \dots, k_l)$. Then for every $1 \leq i < l$ the number of subtrings of the form k^i in $\underset{\sim}{k}$ is 2-reconstructable. Hence also the number of k^i-blocks in k is 2-reconstructable.*

(iii) *The multiset of pairs of neighboured digits $\langle (k_1, k_2), (k_2, k_3), \dots, (k_l, k_1) \rangle$ is 3-reconstructable.*

(iv) *For every $1 \leq i < l$ the multiset $\langle k_1 + k_{i+1}, k_2 + k_{i+2}, \dots, k_l + k_i \rangle$ is 3-reconstructable.*

(v) *For all $1 \leq i, j < l$ with $i + j < l$ the multiset of pairs*

$$\left\langle \left(\sum_{\iota=\kappa}^{\kappa+i-1} k_\iota, \sum_{\iota=\kappa+i}^{\kappa+i+j-1} k_\iota \right) \right\rangle_{\kappa=1,\dots,l}$$

is 3-reconstructable.

Proof. First of all we observe that $\sim_2$ identifies two strings $\underset{\sim}{d}, \underset{\sim}{d}' \in \mathbb{N}^*$ with each other only if every $d \in \mathbb{N}$ appears in $\underset{\sim}{d}$ and $\underset{\sim}{d}'$ equally often and if the number of blocks containing d is equal for $\underset{\sim}{d}$ and $\underset{\sim}{d}'$. Now we can verify (i) to (v) by using the reconstructability of (4.10) for a specific $\underset{\sim}{d}_0$:

(i) Choose $\underset{\sim}{d}_0 = 1^i 2^{l-i}$. Then the multiset of the first coordinates of the pairs in $\underset{\sim}{k}\langle \underset{\sim}{d}_0, 2 \rangle$ equals the multiset $\underset{\sim}{k}\langle i \rangle$ stated in (i) because $[1^i 2^{l-i}]_2 = [1^i 2^{l-i}]_c$ due to the above comments.

(ii) Due to (i) the maximum k is 2-reconstructable. Then choosing $\underset{\sim}{d}_0 = 1^i 2^{l-i}$ returns a multiset of pairs of exponents. If the first coordinate is $k \cdot i$, then we know that k's were added up i times, as k is maximal. Hence the number of pairs with first coordinate $k \cdot i$ equals the number of runs k^i appearing in $\underset{\sim}{k}$. As the number c_i of k^i-blocks and the number d_i of runs of the form k^i are related by

$$d_i = \sum_{j=i}^{\infty} (j - i + 1) c_j$$

the c_i can be inductively reconstructed as $d_i = 0$ for i large enough.

(iii) Choose $\mathscr{d}_0 = 123^{l-2}$. Then the multiset of the first two coordinates of the elements of $\mathscr{k}\langle \mathscr{d}_0, 3\rangle$ equals the requested multiset and again $[123^{l-2}]_2 = [123^{l-2}]_c$ due to the previous explanations.

(iv) Choose $\mathscr{d}_0 = 12^{i-1}13^{l-i-1}$ and argue analogously.

(v) Choose $\mathscr{d}_0 = 1^i 2^j 3^{l-i-j}$ and argue analogously. □

At least the 3-reconstructable properties (i) - (iv) do in general not suffice to reconstruct $[\mathscr{k}]_c$ since the strings

$$\mathscr{k} = 110100 \qquad \text{and} \qquad \mathscr{k}' = 110010$$

which are not equal up to cyclic permutations, agree on all multisets of (i) - (iv). In fact, $\mathscr{k}$ is a cyclic permutation of the reverse of $\mathscr{k}'$ and the multisets of (i), (ii) and (iv) obviously coincide for a string and its reverse. For (iii) it follows from $\mathscr{k} \sim_2 \mathscr{k}'$. However although they not constitute a non-trivial trace equation as substituting AB by B in $\operatorname{tr}(\mathcal{P}(\mathscr{k})) = \operatorname{tr}(\mathcal{P}(\mathscr{k}'))$ yields (4.1) again. Now we will deal with two more sophisticated applications of Proposition 3.

Corollary 4.8 *Suppose for $\mathscr{k} = k_1 \dots k_l \in \mathbb{N}^*$ there is a $1 \le i < l$ such that the multiset $\mathscr{k}\langle i\rangle$ contains a unique element. Then $[\mathscr{k}]_c$ is 3-reconstructable*[2]*. If $\mathscr{k} = \mathscr{k}_0^e$ for some $\mathscr{k}_0 \in \mathbb{N}^*$ and some of the $\mathscr{k}_0\langle i\rangle$ contain a unique element then $[\mathscr{k}]_c$ is 3-reconstructable.*

Proof. Assume $K = k_j + \dots + k_{j+i-1}$ is (one of) the unique element(s). Then consider $\mathscr{k}\langle \mathscr{d}_0, 3\rangle$ for $\mathscr{d}_0 = 1^i 2 3^{l-i-1}$. The uniqueness of K implies that only one 3-tuple in $\mathscr{k}\langle \mathscr{d}_0, 3\rangle$ has K as the first coordinate. Then the second coordinate must be k_{j+i}. Now we argue analogously for $\mathscr{d}_0 = 1^i 2^2 3^{l-i-2}$ and obtain the sum $k_{j+i} + k_{j+i+1}$. Therefore we know both k_{j+i}, k_{j+i+1} and considering $\mathscr{k}\langle \mathscr{d}_0, 3\rangle$ for all $\mathscr{d}_0 = 1^i 2^{i'} 3^{l-i-i'}$ yields the ordered tuple $(k_{j+i}, k_{j+i+1}, \dots, k_{j-1})$. Uniqueness of K implies that also $\sum_m k_m - K = k_{j+i} + \dots + k_{j-1}$ is unique in $\mathscr{k}\langle l-i\rangle$. Hence we can apply the same procedure to identify $(k_j, \dots, k_{j+i-1})$. Putting both together shows 3-reconstructability of $[\mathscr{k}]_c$ (and not $\mathscr{k}$ as j is unknown). If $\mathscr{k} = \mathscr{k}_0^e$ and K is a unique element of some $\mathscr{k}_0\langle i\rangle$ then the proof is still valid when observing that the sum K appears exactly e-times in $\mathscr{k}_0^e\langle i\rangle$. But due to the periodicity of $\mathscr{k}_0^e$ the second entries of the 3-tuples in $\mathscr{k}\langle \mathscr{d}_0, 3\rangle$ belonging to $\mathscr{d}_0 = 1^i 2^{i'} 3^{l-i-i'}$ are always the same if the first entry equals K. □

By means of this Corollary we would have disproven the existence of non-trivial trace equations for 3×3 matrices if every primitive $\mathscr{k}$ obeyed the stated uniqueness condition. But a computational analysis shows that there are counter-examples.

Example 4.9 The shortest primitive string $\mathscr{k} \in \{0,1\}^*$ containing only two characters

[2]Formally the function req : $\mathscr{k} \to [\mathscr{k}]_c$, which is defined on the subset of $\mathbb{N}^*$ of strings such that one of the multisets $\mathscr{k}\langle i\rangle$ contains a unique element, is 3-reconstructable.

such that $\mathscr{k}\langle i\rangle$ contains no unique element for all $1 \leq i \leq |\mathscr{k}|$, is

$$\mathscr{k} = k_1 \dots k_{14} = 10010101100110.$$

Indeed this holds for all of the strings $100(10)^m1100110$ with $m \in \mathbb{N}^{>1}$. Moreover it is easy to see that the condition of Corollary 4.8 remains unsatisfied if we proceed to the string of distances between appearing ones, i.e. $\mathscr{k} = 21^m0201$ for $m \in \mathbb{N}^{>1}$.

Corollary 4.10 *For a given $\mathscr{k} = k_1 \dots k_l \in \mathbb{N}^*$ define $k = \max(k_1, \dots, k_l)$ and let $e \in \mathbb{N}$ be the largest number such that some k^e-blocks appear in $\mathscr{k}$. Let $K_1, \dots, K_r$ be the numbers of digits between neighboured k^e-blocks in $\mathscr{k}$ where K_i has to be the distance between the i-th and the $(i+1)$-th k^e-block of $\mathscr{k}$ (and the first k^e-block of $\mathscr{k}$ is arbitrarily chosen). Define $\mathscr{K} = K_1 \dots K_r$. Then for any fixed $\mathscr{s}_0 = s_1 \dots s_r \in \{1,2\}^*$ the map $\mathscr{k} \to \mathscr{K}\langle \mathscr{s}_0, 2\rangle$ is 3-reconstructable.*

Proof. For any $\mathscr{s} = s_1 \dots s_r \in \{1,2\}^*$ define

$$\mathscr{D}(\mathscr{s}, \mathbf{e}) = 3^e s_1^{e_1} 3^e s_2^{e_2} \dots 3^e s_r^{e_r}$$

where $\mathbf{e} = (e_1, \dots, e_r)$ is an r-tuple of exponents with $e_1, \dots, e_r \geq 1$ and $\sum_i e_i = l - re$, as well as

$$D_1 = \left\{ \mathscr{d} = d_1 \dots d_l \in \{1,2,3\}^* \;\middle|\; \sum_{j:d_j=3} k_j = r \cdot e \cdot k, \begin{array}{c} \text{neither 12 nor 21 are substrings of } \mathscr{d} \\ \text{and } \mathscr{d} \text{ contains } r \text{ many } 3^e\text{-blocks} \end{array} \right\},$$

$$D(\mathscr{s}) = \left\{ \mathscr{d} = d_1 \dots d_l \in [\mathscr{D}(\mathscr{s}, \mathscr{K})]_2 \;\middle|\; \sum_{j:d_j=3} k_j = r \cdot e \cdot k \right\}$$

where $\mathscr{s} \in \{1,2\}^*$ is an arbitrary string of length r. Since $\sim_2$-equivalence implies invariance in the number of appearances of the letter 3 and the number of blocks containing 3 we deduce from the maximality of k and e that the strings in $D(\mathscr{s})$ contain exactly r-many 3^e-blocks that match the k^e-blocks of $\mathscr{k}$. This proves

$$D_1 = \bigcup_{\substack{\mathscr{s} \in \{1,2\}^*, \\ |\mathscr{s}| = r}} D(\mathscr{s}).$$

If $\mathscr{d} \in D(\mathscr{s})$ then obviously for every $(i,j) \in \langle \mathscr{s}\rangle$ the set $\langle \mathscr{d}\rangle$ contains the pairs $(i,3),(3,j)$ (in compliance with multiplicities) while every other pair in $\langle \mathscr{d}\rangle$ has equal coordinates. Therefore knowledge of $\langle \mathscr{d}\rangle$ resp. $[\mathscr{d}]_2$ allows us to reconstruct $\langle \mathscr{s}\rangle$ and hence $[\mathscr{s}]_2$. But we are able to 3-reconstruct $[D_1]_2 := \{[\mathscr{d}]_2 \mid \mathscr{d} \in D_1\}$ by searching through all coefficients of the polynomial (4.7), i.e.

$$\operatorname{tr}(\mathscr{P}(\mathscr{k})) = \sum_{\mathscr{d} \in \{1,\dots,n\}^*, |\mathscr{d}| = l} \lambda_{d_1}^{k_1} \cdot \ldots \cdot \lambda_{d_l}^{k_l} a_{\mathscr{d}}$$

and storing all $[\mathscr{d}]_2$ for which the exponent of λ_3 is maximal, i.e. it equals $r \cdot e \cdot k$, the factors a_{12}, a_{21} do not appear in $a_{\mathscr{d}}$ and for which $\mathscr{d}$ contains r many 3^e-blocks and no other appearances of 3 (which can be deduced from $[\mathscr{d}]_2$). Here we implicitly use that according to Corollary 4.7 (i) and (ii) the numbers r, e and k are 2-reconstructable. Therefore for a specific $\mathscr{s}_0 \in \{1,2\}^*$ we are now able to 3-reconstruct

$$\mathscr{D} = \bigcup_{\mathscr{s} \sim_2 \mathscr{s}_0} D(\mathscr{s}) \tag{4.11}$$

by filtering all $[\mathscr{d}]_2 \in [D_1]_2$ with $\mathscr{d} \in D(\mathscr{s})$ for some $\mathscr{s}$ with $\mathscr{s} \sim_2 \mathscr{s}_0$. Lastly we can 3-reconstruct a final set $\mathscr{D}_0 \subset \mathscr{D}$ which shall be a system of representatives of the $\sim_2$-equivalence classes of the elements in $\mathscr{D}$ by partitioning its elements into their $\sim_2$-equivalence classes. This means for every $\mathscr{s} \in [\mathscr{s}_0]$ there is exactly one $\mathscr{d} \in \mathscr{D}_0$ with $\mathscr{d} \in D(\mathscr{s})$ (although in general we cannot construct $\mathscr{s}$ out of a given $\mathscr{d}$).Then the observation

$$\mathscr{K}\langle \mathscr{s}_0, 2\rangle = \Big\langle (n_1, n_2) \;\Big|\; \mathscr{d} \in \mathscr{D}_0,\ n_i = \#\text{ of } i\text{'s in } \mathscr{d} \text{ (for } i = 1,2)\Big\rangle$$

proves 3-reconstructability of $\mathscr{K}\langle \mathscr{s}_0, 2\rangle$. ◻

A consequence of this corollary is that - with the above notations - if $\mathscr{K} \mapsto [\mathscr{K}]_c$ is 2-reconstructible then $\mathscr{k} \mapsto [\mathscr{K}]_c$ will be 3-reconstructible. By means of the next corollary we will see that the previous examples $\mathscr{k} = 100(10)^m 1100110$ for $m \in \mathbb{N}^{>1}$ are 3-reconstructable.

Corollary 4.11 *Taking the notations from above, assume that $r = 2$ and $K_2 \geq 2K_1$ where w.l.o.g. the K_i are chosen such that $K_2 \geq K_1$. Then $[\mathscr{k}]_c$ is 3-reconstructable.*

Proof. Assume

$$\mathscr{k} \sim_c k^e \mathscr{k}_{e+1} \dots \mathscr{k}_{e+K_2} k^e \mathscr{k}_{2e+K_2+1} \dots \mathscr{k}_l$$

for notational convenience. The sums

$$S_1 = \mathscr{k}_{e+1} + \cdots + \mathscr{k}_{e+K_2} \qquad \text{and} \qquad S_2 = \mathscr{k}_{2e+K_2+1} + \cdots + \mathscr{k}_l$$

can easily be 3-reconstructed by considering $\mathscr{k}\langle \mathscr{d}, 3\rangle$ for $\mathscr{d} = 3^e 1^{K_1} 3^e 2^{K_2}$ and picking out this 3-tuple with maximal third coordinate. Here we took advantage of 3-reconstructability of $\langle K_1, K_2\rangle$ which follows from the previous corollary. Now consider $\mathscr{k}\langle \mathscr{d}, 3\rangle$ for all

$$\mathscr{d} = 3^e 1^{K_1} 3^e 1^j 2^{l-2e-K_1-j} \quad \text{and} \quad \mathscr{d} = 2^{l-2e-K_1-j} 1^j 3^e 1^{K_1} 3^e$$

with $1 \leq j \leq K_2 - K_1$. The condition on j guarantees $[\mathscr{d}]_2^{3\to\max} = [\mathscr{d}]_2^{3\to\max}$ for both cases, where we set

$$[\mathscr{d}]_2^{3\to\max} = \left\{ d' = d'_1 \dots d'_l \in [\mathscr{d}]_2 \;\middle|\; \sum_{j : d'_j = 3} \mathscr{k}_j = 2 \cdot e \cdot k \right\}$$

and analogously for $[\mathscr{d}]_c^{3\to\max}$. Hence when picking the 3-tuples of $\mathscr{k}\langle \mathscr{d}, 3\rangle$ with maximal third coordinate the corresponding first coordinates are

$$S_1 + k_{e+1} + \cdots + k_{e+j} \qquad \text{and} \qquad S_1 + k_{e+K_2} + \cdots + k_{e+K_2-j+1}.$$

After evaluating suitable subtractions we obtain 3-reconstructability of

$$k_{e+1}, k_{e+2}, \ldots, k_{e+K_2-K_1} \qquad \text{and} \qquad k_{e+K_2}, k_{e+K_2-1}, \ldots, k_{e+K_1+1}$$

as S_1 is 3-reconstructable as well. Due to $2K_2 \geq K_1$ we have shown that $k_{e+1}\ldots k_{e+K_2}$ is 3-reconstructable. Now consider $\mathscr{k}\langle \mathscr{d}, 3\rangle$ for all

$$\mathscr{d} = 3^e 1^{K_2} 3^e 1^j 2^{l-2e-K_2-j}$$

with $1 \leq j \leq l - 2e - K_2$. Because of

$$[\mathscr{d}]_2^{3\to\max} = [\mathscr{d}]_c^{3\to\max} \cup [3^e 1^{K_1} 3^e 1^{j+K_2-K_1} 2^{l-2e-K_2-j}]_c^{3\to\max}$$

$\mathscr{k}\langle \mathscr{d}, 3\rangle$ contains two 3-tupels with maximal third coordinate. The respective first coordinates are

$$S_2 + k_{2e+K_2+1} + \cdots + k_{2e+K_2+j} \qquad \text{and} \qquad S_1 + k_{e+1} + \cdots + k_{e+K_2-K_1+j}.$$

But we can differentiate between these two cases as we can identify the sums on the right side due to 3-reconstructability of S_1 and $k_{e+1}\ldots k_{e+K_2}$. Hence the collection of sums

$$S_2 + k_{2e+K_2+1} + \cdots + k_{2e+K_2+j}$$

and therewith $k_{2e+K_2+1}\ldots k_l$ is 3-reconstructable since so is S_2. $\square$

In the case $\mathscr{k} = 100(10)^m 1100110$ we clearly have $r = 2$ and $K_1 = 2, K_2 = 4 + 2m$. As $4 + 2m \geq 2 \cdot 2$ we have proven 3-reconstructability of $[\mathscr{k}]_c$ for all $m \in \mathbb{N}$. We can argue similarly for $\mathscr{k} = 121^m 020$ if $m \geq 3$. For $\mathscr{k} = 1211020$ a computational analysis shows that this string does not satisfy any non-trivial trace equation.

4.3 Further results

4.3.1 Properties of the class of non-trivial trace equations

Although the previous corollaries seem to be sufficient to exclude almost every given string from being part of a 3-dimensional non-trivial trace equation it is not clear whether there is a general scheme for reconstructing $[\mathscr{k}]_c$ from the multisets $\mathscr{k}\langle \mathscr{d}, 3\rangle$ in the general case. Therefore we try to derive properties of the set of non-reconstructable strings.

Proposition 4.12 *Suppose $\mathscr{k} = k_1 \ldots k_l, \mathscr{k}' = k'_1 \ldots k'_l \in \mathbb{N}^*$ satisfy the trace equation $\operatorname{tr}(\mathscr{P}(\mathscr{k})) = \operatorname{tr}(\mathscr{P}(\mathscr{k}'))$ for n-dimensional square matrices. Then for all $s_0, s_1 \in \mathbb{N}^*$ the* *strings*

$$\mathcal{K} = s_0 s_1^{k_1} s_0 s_1^{k_2} \dots s_0 s_1^{k_l}, \qquad \mathcal{K}' = s_0 s_1^{k_1'} s_0 s_1^{k_2'} \dots s_0 s_1^{k_l'}$$

satisfy $\operatorname{tr}(\mathcal{P}(\mathcal{K})) = \operatorname{tr}(\mathcal{P}(\mathcal{K}'))$ *for dimension* n. *Additionally, for every* $a, b \in \mathbb{N}$ *the linear transformations*

$$a\mathcal{k} + b = (ak_1 + b)\dots(ak_l + b), \qquad a\mathcal{k}' + b = (ak_1' + b)\dots(ak_l' + b)$$

satisfy $\operatorname{tr}(\mathcal{P}(a\mathcal{k} + b)) = \operatorname{tr}(\mathcal{P}(a\mathcal{k}' + b))$.

Proof. When replacing all A's by $\mathcal{P}(s_0)$ and all B's by $\mathcal{P}(s_1)$ then $\mathcal{P}(\mathcal{k})$ resp. $\mathcal{P}(\mathcal{k}')$ become $\mathcal{P}(\mathcal{K})$ resp. $\mathcal{P}(\mathcal{K}')$. Similarly the case of a linear transformation follows from substituting A and B in $\operatorname{tr}(\mathcal{P}(\mathcal{k}))$ by AB^b and B^a. $\square$

Also, Pare's counterexample to completeness for 2×2-matrices raises the question, how the class of non-trivial trace equations looks like in the 2×2 case. His equation (4.1) resp. $\operatorname{tr}(\mathcal{P}(210)) = \operatorname{tr}(\mathcal{P}(012))$ has the following immediate generalization.

Corollary 4.13 *For any* $\mathcal{k} = k_1 \dots k_l \in \mathbb{N}^*$ *let* $\operatorname{rev}(\mathcal{k}) = k_l \dots k_1 \in \mathbb{N}^*$ *be its reverse string. Then*

$$\operatorname{tr}(\mathcal{P}(\mathcal{k})) = \operatorname{tr}\left(\mathcal{P}(\mathcal{k}')\right)$$

for 2*-dimensional matrices.*

Proof. For every $\mathcal{d} \in \{1,2\}^*$ the multiset $\langle \mathcal{d} \rangle$ must contain $(2,1)$ as often as $(1,2)$ since for every 1^*-block beginning there is exactly one 1^*-block end. Thus $\mathcal{d} \sim_2 \operatorname{rev}(\mathcal{d})$ and subsequently we obtain for all $\mathcal{k} \in \mathbb{N}^*$ and $\mathcal{d} \in \{1,2\}^*$ with $|\mathcal{d}| = |\mathcal{k}|$

$$\operatorname{rev}(\mathcal{k})\langle \mathcal{d}, 2\rangle = \operatorname{rev}(\mathcal{k})\langle \operatorname{rev}(\mathcal{d}), 2\rangle = \mathcal{k}\langle \mathcal{d}, 2\rangle.$$

Hence Theorem 4.4 finishes the proof. $\square$

According to Proposition 4.12 the class of $\mathcal{k} \in \mathbb{N}^*$ with $\operatorname{tr}(\mathcal{P}(\mathcal{k})) = \operatorname{tr}(\mathcal{P}(\mathcal{k}_0))$ in 2 dimensions for some fixed $\mathcal{k}_0 \in \mathbb{N}^*$ is in general strictly larger than the one generated by cyclic permutations and reversions. Indeed it yields e.g.

$$\operatorname{tr}(\mathcal{P}(s_0 s_1^{k_1} \dots s_0 s_1^{k_l})) = \operatorname{tr}(\mathcal{P}(s_0 s_1^{k_l} \dots s_0 s_1^{k_1}))$$

for all $s_0, s_1, k_1 \dots k_l \in \mathbb{N}^*$ that cannot be obtained by cyclic permutations and reversions alone as long as $\operatorname{rev}(s_1) \neq s_1$ and $\operatorname{rev}(s_0) \neq s_0$. However it is still an open question whether

$$\operatorname{TrEq}_2 = \{(\mathcal{k}, \mathcal{k}') \in \mathbb{N}^* \times \mathbb{N}^* \mid \operatorname{tr}(\mathcal{P}(\mathcal{k})) = \operatorname{tr}(\mathcal{P}(\mathcal{k}')) \text{ in 2 dimensions}\}$$

is the smallest set generated by cyclic permutation, reversion and substitution, i.e. whether it is the smallest set X satisfying

- $\forall \mathcal{k}, \mathcal{k}' \in \mathbb{N}^* : \quad \mathcal{k} \sim \mathcal{k}' \;\Rightarrow\; (\mathcal{k}, \mathcal{k}'), (\mathcal{k}, \mathrm{rev}(\mathcal{k}')) \in X$
- $\forall s_0, s_1 \in \mathbb{N}^* : \quad (k_1 \dots k_l, k'_1 \dots k'_l) \in X \;\Rightarrow\; (s_0 s_1^{k_1} \dots s_0 s_1^{k_1}, s_0 s_1^{k'_1} \dots s_0 s_1^{k'_l}) \in X$

4.3.2 Products of traces

We finally observe that non-existence of non-trivial trace equations would even exclude products of traces from being counterexamples to essential relative completeness. Hence the considerations of this chapter go beyond the question whether just non-trivial trace equations exist.

Proposition 4.14 *Let $n \in \mathbb{N}^{>1}$ and A, B be the underlying matrices of the $\mathcal{P}$-notation where $A = (a_{ij})_{1\leq i,j\leq n}$ is an $n \times n$-matrix with indeterminate entries and B a diagonal $n \times n$-matrix with indeterminate diagonal entries $\lambda_1, \dots, \lambda_n$. For any $\mathcal{k} = k_1 \dots k_l \in \mathbb{N}^*$ the trace $\mathrm{tr}(\mathcal{P}(\mathcal{k}))$ is prime considered as a polynomial in $\mathbb{Z}[(a_{ij})_{1\leq i,j\leq n}][(\lambda_i)_{1\leq i\leq n}]$. Hence if there are no non-trivial trace equations for dimensions $\geq n$, all non-isomorphic diagrams consisting only of traces can be separated by interpretatons in dimensions $\geq n$.*

Proof. We have seen in (4.7) that

$$\mathrm{tr}\,(\mathcal{P}(\mathcal{k})) = \sum_{\substack{d \in \{1,\dots,n\}^*,\\ |d| = l}} \lambda_{d_1}^{k_1} \cdot \dots \cdot \lambda_{d_l}^{k_l} a_{d} \;\in\; \mathbb{N}[(a_{ij})_{1\leq i,j\leq n}][(\lambda_i)_{1\leq i\leq n}].$$

Suppose $\mathrm{tr}\,(\mathcal{P}(\mathcal{k})) = g \cdot h$ for some polynomials $g, h \in \mathbb{Z}[(a_{ij})_{ij}][(\lambda_i)_i]$. As $\mathrm{tr}\,(\mathcal{P}(\mathcal{k}))$ is both homogeneous as a polynomial in the a_{ij} with coefficients in $\mathbb{Z}[(\lambda_i)_i]$ and homogeneous as a polynomial in the λ_i with coefficients in $\mathbb{Z}[(a_{ij})_{ij}]$, so are g and h. We set $K = \sum_i k_i$. Choosing $d_0 = 1^l$ and $d_0 = 2^l$ shows that $\mathrm{tr}\,(\mathcal{P}(\mathcal{k}))$ contains the monomials $\lambda_1^K a_{11}^l$ and $\lambda_2^K a_{22}^l$. We laxly write

$$\mathrm{tr}\,(\mathcal{P}(\mathcal{k})) = \lambda_1^K a_{11}^l + \lambda_2^K a_{22}^l + \dots$$

Hence g, h must be of the following form

$$g = \lambda_1^{e_1} a_{11}^{e_2} + \lambda_2^{e_1} a_{22}^{e_2} + \dots, \qquad h = \lambda_1^{K-e_1} a_{11}^{l-e_2} + \lambda_2^{K-e_1} a_{22}^{l-e_2} + \dots$$

Thus the product $g \cdot h$ also contains $\lambda_1^{e_1} \lambda_2^{K-e_1} a_{11}^{e_2} a_{22}^{l-e_2}$, but $a_{11}^{e_2} a_{22}^{l-e_2}$ is unequal to all a_{d_0} unless $l = 0$ or $l = e_2$. Assuming w.l.o.g. the latter, we deduce

$$g = \lambda_1^{e_1} a_{11}^l + \lambda_2^{e_1} a_{22}^l + \dots, \qquad h = \lambda_1^{K-e_1} + \lambda_2^{K-e_1} + \dots$$

so that $g \cdot h$ contains $\lambda_1^{e_1} \lambda_2^{K-e_1} a_{11}^l$ which can only appear in (4.7) if $e_1 = K$. Therefore h must be constant. We even have $h \equiv 1$ since $\mathrm{tr}\,(\mathcal{P}(\mathcal{k}))$ contains the monomial $\lambda_1^K a_{11}^l$ exactly once.

Now suppose M, N are simple closed networks consisting only of traces, i.e. every appearing box has exactly one input and one output wire, with $M \ncong N$. By considering a seperating interpretation $[\![\;]\!]_{M,N}$ mapping all appearing object labels to the n-dimensional space and interpreting all function labels according to the substitution

$$\{f_0, \dots, f_k\}^* \xrightarrow{\text{subs}} \{f, g\}^*, \qquad f_i \mapsto f g^i$$

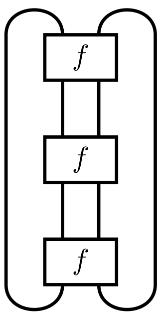
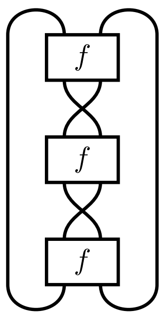
(cf. (4.4)) we gain

$$[\![M]\!]_{M,N} = \prod_{i=1}^{m} \operatorname{tr}\left(\mathcal{P}(\mathcal{k}_i)\right), \qquad [\![N]\!]_{M,N} = \prod_{i=1}^{m'} \operatorname{tr}\left(\mathcal{P}(\mathcal{k}'_i)\right)$$

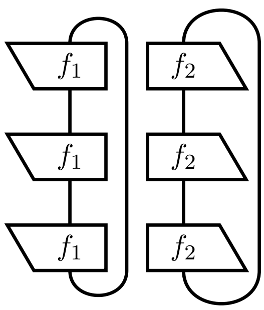
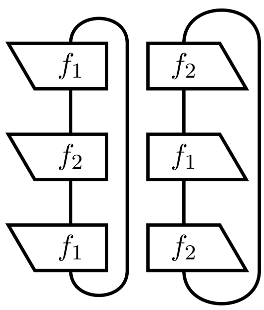
for some $m, m' \in \mathbb{N}$ and $\mathcal{k}_1, \dots, \mathcal{k}_m, \mathcal{k}'_1, \dots, \mathcal{k}'_{m'} \in \mathbb{N}^*$. But those products cannot be equal unless they consist of the same factors, as all factors are prime and $\mathbb{Z}[(a_{ij})_{ij}][(\lambda_i)_i]$ is a factorial ring(cf. Gauss' Theorem [30] p.198f). But in the absence of non-trivial trace equations for dimensions $\geq n$ equality of the factors implies the contradiction $M \cong N$. The case for not necessarily simple diagrams follows from Proposition 3.16(i). □

Although this Proposition only deals with products of trace diagrams we can use it to find seperating interpretations for a much larger class of networks. The next example illustrates two of those applications.

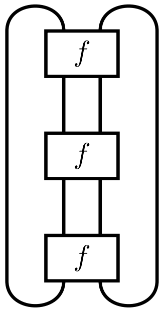
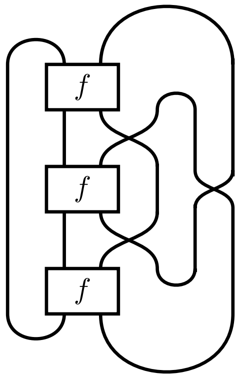
Example 4.15 As we will only work with an arrow f of type $A \otimes A \to A \otimes A$, we omit labels of appearing wires. Suppose there are no non-trivial trace equations for $n \times n$-matrices where $n \in \mathbb{N}^{>2}$. Then there is a separating interpretation with n-dimensional matrices separating the diagrams

$M =$  , $N =$  .

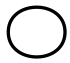
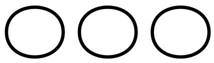
Indeed choosing $f = f_1 \otimes f_2$ leads to the diagrams

$M =$  , $N =$  .

which are distinguishable by means of Proposition 4.14. This approach is obviously not working for

$M =$  , $N =$  .

Here the choice $f = c_{A,A}$ yields two distinguishable diagrams consisting only of trivial cycles:

$M =$  , $N =$  .

4.4 Non-completeness for semi-rings which do not contain $\mathbb{N}$

4.4.1 Non-trivial equations for $\mathbb{Z}/p^l\mathbb{Z}$ and rings with prime characteristic

We have seen in the last chapter that also semi-relative completeness holds for all semi-rings containing $\mathbb{N}$ (in the non-dagger case). Similar completeness results do not hold for other semi-rings. In many cases we will even derive counterexamples to essential relative completeness. As the proofs will consist of verifying non-trivial trace equations and rely on the terminology we introduced in the first section, the following results are stated here instead of in the last chapter. We first analyse the case $R = \mathbb{Z}/p^l\mathbb{Z}$ where m is a prime power:

Proposition 4.16 *Define $R = \mathbb{Z}/p^l\mathbb{Z}$ with $l \in \mathbb{N}^{>0}$ and let p be a prime number. Then for every square matrix $A \in R^{n\times n}$ we have*

$$\operatorname{tr}\left(A^{p^l}\right) = \operatorname{tr}\left(A^{p^{l-1}}\right).$$

Before proving this identity, we remind that a word $\mathscr{d} \in \Sigma^*$ is called *primitive*, if there is no shorter $\mathscr{s} \in \Sigma^*$ with $\mathscr{d} = \mathscr{s}^k$ for some $k \in \mathbb{N}^{>1}$. If e.g. $\mathscr{d}$ has prime length p, then $\mathscr{d}$ is obviously primitive unless $\mathscr{d} = a^p$ for some $a \in \Sigma$. The notion of primitivity is useful because it is equivalent to $\#[\mathscr{d}]_c = |\mathscr{d}|$, i.e. the cyclic permutations $d_{0\oplus k} \dots d_{(n-1)\oplus k}$ of $\mathscr{d} = d_0 \dots d_{n-1}$ are mutually different for all $0 \le k < n$. Also, if $\mathscr{d} = \mathscr{s}^k$ for some primitive $\mathscr{s}$, then $\#[\mathscr{d}]_c = |\mathscr{s}|$ and $[d]_c$ consists of all strings of the form $\mathscr{s}'^k$, where $\mathscr{s}' \in [\mathscr{s}]_c$. If we additionally have $\Sigma = \{1, \dots, n\}$ and $A = (a_{ij})_{ij} \in R^{n\times n}$, then $a_{\mathscr{d}} = a_{\mathscr{s}}^k$. Now we are prepared for proving the proposition.

Proof. When denoting the ij-th entry of A with a_{ij} Corollary 4.2 together with the above comments reveal

$$\operatorname{tr}\left(A^{p^l}\right) = \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p^l}} a_{\mathscr{d}} = \sum_{k=0}^{l} \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p^k,\\ \mathscr{d} \text{ primitive}}} a_{\mathscr{d}^{p^{l-k}}} = \sum_{k=0}^{l} \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p^k,\\ \mathscr{d} \text{ primitive}}} a_{\mathscr{d}}^{p^{l-k}}.$$

Analogous reasoning for $\operatorname{tr}\left(A^{p^{l-1}}\right)$ yields

$$\operatorname{tr}\left(A^{p^l}\right) - \operatorname{tr}\left(A^{p^{l-1}}\right) = \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p^l,\\ \mathscr{d} \text{ primitive}}} a_{\mathscr{d}} + \sum_{k=0}^{l-1} \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p^k,\\ \mathscr{d} \text{ primitive}}} \left(a_{\mathscr{d}}^{p^{l-k}} - a_{\mathscr{d}}^{p^{l-1-k}}\right).$$

Since $a_{\mathscr{d}}$ does not change under cyclic permutation of $\mathscr{d}$ we may sum over $\{1,\dots,n\}/\sim_c$:

$$\operatorname{tr}\left(A^{p^l}\right) - \operatorname{tr}\left(A^{p^{l-1}}\right) = \sum_{\substack{[\mathscr{d}]_c\in\{1,\dots,n\}/\sim_c,\\ |\mathscr{d}|=p^l,\\ \mathscr{d} \text{ primitive}}} p^l a_{\mathscr{d}} + \sum_{\substack{[\mathscr{d}]_c\in\{1,\dots,n\}/\sim_c,\\ |\mathscr{d}|=p^k,\\ \mathscr{d} \text{ primitive}}} p^k \left(a_{\mathscr{d}}^{p^{l-k}} - a_{\mathscr{d}}^{p^{l-1-k}}\right) = 0$$

The last equation holds in $\mathbb{Z}/p^l\mathbb{Z}$ because p^{l-k} divides $a_{\mathscr{d}}^{p^{l-k}} - a_{\mathscr{d}}^{p^{l-1-k}}$. This is obvious in the case $p|a_{\mathscr{d}}$ (as $l-k \le 2^{l-k-1}$), otherwise we may apply Euler's Theorem (cf. [26], p.28), which states

$$a_{\mathscr{d}}^{(p-1)p^{l-k-1}} \equiv a_{\mathscr{d}}^{\varphi(p^{l-k})} \equiv 1 \qquad \mod p^{l-k}.$$

Now a multiplication with $a_{\mathscr{d}}^{p^{l-1-k}}$ completes the proof. $\square$

Clearly $\operatorname{tr}\left(A^{p^l}\right) = \operatorname{tr}\left(A^{p^{l-1}}\right)$ does not hold in the language of compact closed categories, therefore free finite-dimensional $\mathbb{Z}/p^l\mathbb{Z}$-modules are not essentially relatively complete for compact closed categories.

Proposition 4.17 *Let R be a ring with prime characteristic* $\operatorname{char} R = p > 0$. *Then*

$$\operatorname{tr}(A^p) = \operatorname{tr}(A)^p \tag{4.12}$$

for all square matrices A with entries in R.

Proof. For an $n \times n$-matrix $A = (a_{ij})_{ij}$ we obtain

$$\operatorname{tr}(A^p) = \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=1}} a_{\mathscr{d}}^p + \sum_{\substack{\mathscr{d}\in\{1,\dots,n\},\\ |\mathscr{d}|=p,\\ \mathscr{d} \text{ primitive}}} a_{\mathscr{d}} = \sum_{i=1}^{n} a_{ii}^p + \sum_{\substack{[\mathscr{d}]_c\in\{1,\dots,n\}/\sim_c,\\ |\mathscr{d}|=p,\\ \mathscr{d} \text{ primitive}}} p a_{\mathscr{d}} = \sum_{i=1}^{n} a_{ii}^p$$

as well as

$$\operatorname{tr}(A)^p = \sum_{\substack{0 \le k_1, \ldots, k_n \le p, \\ k_1 + \cdots + k_n = p}} \frac{p!}{k_1! \cdot \ldots \cdot k_n!} a_{11}^{k_1} \cdot \ldots \cdot a_{nn}^{k_n} = \sum_{i=1}^{n} a_{ii}^{p}.$$

Indeed, all other summands are multiples of p as p is prime. □

The proof of equation (4.12) utilizes $p = 0$ only for evaluating the coefficients of the involved expressions, making it valid for all rings with prime characteristic. In particular, it holds for all fields with non-vanishing characteristic. However, an easy calculation reveals that the binomial coefficient $\binom{p^{l+1}}{p^l}$ for all prime numbers p and $l \in \mathbb{N}$ is divisible by p but not by p^2. Therefore it seems unlikely that (4.12) has an immediate generalization to rings R with $\operatorname{char} R = p^l$.

4.4.2 General semi-rings

In order to find equations for arbitrary semi-rings R we generalize the notion of the characteristic to all semi-rings. The semi-ring homomorphism

$$\phi : \mathbb{N} \to R, \qquad n \mapsto n_R = n \cdot 1_R = \underbrace{1_R + \cdots + 1_R}_{n} \tag{4.13}$$

is obviously either injective - in this case $\mathbb{N} \subset R$ - or there are minimal natural numbers $m, c \in \mathbb{N}$ with $c > 0$ such that

$$m_R + c_R = m_R.$$

In the former case we say R has characteristic $\operatorname{char} R = 0$, otherwise we define $\operatorname{char} R = c_R$. Note that m_R does not have to be 0 as R is not required to provide additive inverses.

Proposition 4.18 *Let R be a semi-ring with $\operatorname{char} R = p^l$ for a prime number p and some $l \in \mathbb{N}$. Then $\mathbf{FMod}_R$-interpretations are not relatively complete for compact closed categories. Moreover if the semi-ring homomorphism (4.13) is surjective, i.e. if R is generated by 1_R, then $\mathbf{FMod}_R$-interpretations are not even essentially relatively complete for compact closed categories.*

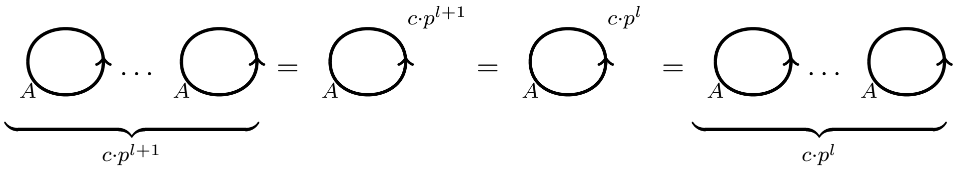
Proof. Let $m \in \mathbb{N}$ be minimal with $m_R + (p^l)_R = m_R$. For $k, n \in \mathbb{N}$ we will write $k\%n$ for the residue of k modulo n. When choosing $A = a$ as a one-dimensional matrix in Proposition 4.16 we obtain

$$a^{p^{l+1}} \equiv a^{p^l} \qquad \mod p^{l+1}$$

for all $a \in \mathbb{Z}/p^{l+1}\mathbb{Z}$. As for all $k \ge m$ we have $k_R = m_R + (k\%p^l)_R$ we can choose a $c \in \mathbb{N}$ with $2^c \ge m$ and get

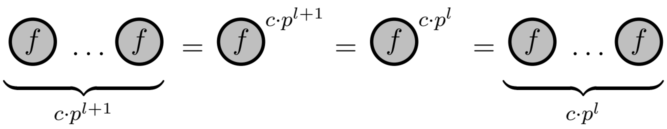
$$a_R^{c \cdot p^{l+1}} = a_R^{c \cdot p^l} \tag{4.14}$$

for all $a \in \mathbb{N}$. Hence the equation



(4.15)

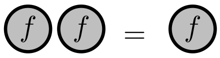
which is clearly not true in the graphical calculus, holds after applying an arbitrary interpretation $[\![\ \ ]\!]$ since $[\![\circlearrowleft]\!] = \operatorname{tr} \mathrm{id}_A = (\dim A)_R$. Here we worked modulo p^{l+1} in order to include the case $l = 0$. If ϕ_R is surjective then $R = \{0_R, 1_R, \ldots, m_R + (p^l - 1)_R\}$ and subsequently (4.15) is equivalent to



(4.16)

for all $f : I \to I$. $\square$

As $\mathbf{FRel} = \mathbf{FProd}_{B_0}$ for the boolean algebra $B_0 = (\{0,1\}, \vee, \wedge)$ we especially obtain that $\mathbf{FRel}$-interpretations are not essentially relatively complete since $\operatorname{char} B_0 = 1 = 2^0$. But a special case of (4.16) namely



(4.17)

holds for more interpretations. In fact it holds in $\mathbf{Rel}$ as $x^2 = x$ holds in its set of scalars $\mathrm{Hom}_{\mathbf{Rel}}(I, I) \cong (\mathbb{F}_2, \cdot) = (\{0,1\}, \wedge)$. (4.17) also holds for distributive lattices R with 0 and 1 as in them we equally have $x^2 = x$ for all x. We remind that a lattice L is a partially ordered set such that every pair of elements $a, b \in L$ has an *infimum/meet* $a \wedge b$ and a *supremum/join* $a \vee b$. 0 and 1 denote a global minum resp. global maximum while distributivity means

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

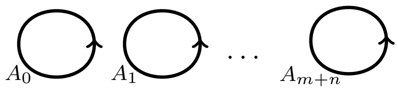
It is obvious that in this case L forms a (commutative) semi-ring.

In the case of (semi-)rings whose non-vanishing characteristic is not equal to a prime power we can at least exclude semi-relative completeness.

Proposition 4.19 *Let R be a semi-ring with non-vanishing characteristic. Then $\mathbf{FMod}_R$-interpretations are not semi-relatively complete for compact closed categories. If R is finite then they are not even essentially semi-relatively complete for compact closed categories.*

Proof. We argue similarly to Proposition 3.18. Suppose $\operatorname{char} R = n$ and let $m \in \mathbb{N}$

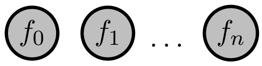
be minimal with $m_R + n_R = m_R$. Consider

$M =$ 

When ϕ denotes the homomorphism given by (4.13) then its image contains $m+n$ elements. Thus for any fixed interpretation $[\![\ \ ]\!]$ at least two of the A_i - w.l.o.g. A_0 and A_1 - have the same dimension under $[\![\ \ ]\!]$, i.e. w.l.o.g.

$$(\dim [\![A_0]\!])_R = (\dim [\![A_1]\!])_R\,.$$

Hence $[\![\ \ ]\!]$ does not separate M from the diagram consisting of trivial cycles labelled with $A_0, A_0, A_2, A_3, \ldots, A_{m+n}$. If $\#R = n < \infty$ then argue analogously with

$M =$  $\square$

4.4.3 Further observations and summary

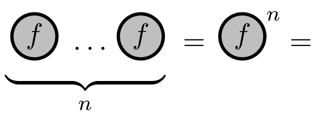
We close with a trace equation holding in some categories without a superposition rule and a summary of our results in this section.

Proposition 4.20 *Let $\mathcal{C}$ be a compact closed category with invertible and only finitely many (but more than one) scalars. Then $\mathcal{C}$-interpretations are not even essentially relatively complete for compact closed categories.*

Proof. The set of scalars $\mathcal{C}(I, I)$ forms a finite abelian group and is - according to the fundamental theorem of finitely generated abelian groups (cf. [26] p.39) - isomorphic to

$$\prod_{i=1}^{r}\prod_{j=1}^{t_i} (\mathbb{Z}/p_i^{e_{ij}}\mathbb{Z})^{f_{ij}}$$

for some $e_{ij}, f_{ij}, t_i, r \in \mathbb{N}$ and primes p_i. Here composition corresponds to addition. When writing $e_i = \max(e_{i1}, \ldots, e_{it_i})$ and $n = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$, the diagrammatic equation



where the empty diagram on the right hand side symbolizes the identity of the tensor unit id_I, does not hold in the graphical language but it holds after applying an arbitrary interpretation $[\![\ \ ]\!]$ since $a \cdot n = 0$ for all $a \in \mathbb{Z}/p_i^{e_i}\mathbb{Z},\ 1 \le i \le r$. $\square$

Proposition 4.21 $\mathbf{Rel}$*-interpretations and therewith* $\mathbf{FRel}$*-interpretations are not essentially relatively complete for compact closed categories. Now let R be a semi-ring not containing the natural numbers. Then* $\mathbf{FMod}_R$*-interpretations are not semi-relatively complete for compact closed categories. Also:*

- *If R is finite then* $\mathbf{FMod}_R$*-interpretations are not essentially semi-relatively complete for compact closed categories.*
- *If* $\mathrm{char}\, R$ *is a prime power then* $\mathbf{FMod}_R$*-interpretations are not relatively complete for compact closed categories.*
- *If (a) R is a ring with prime characteristic or if (b) $R = \mathbb{Z}/p^l\mathbb{Z}$ for a prime number p or if (c)* $\mathrm{char}\, R$ *is a prime power and R is generated by 1_R or if (d) R is a distributive lattice, then* $\mathbf{FMod}_R$*-interpretations are not essentially relatively complete for compact closed categories.*

Chapter 5

Summary and further work

5.1 Achieved Completeness Results

In this dissertation we have contributed to completeness studies for the graphical language of (dagger) compact closed categories in several ways. First we introduced various kinds of completeness regarding to (a) the dependence of interpretations on diagrams they separate, (b) the presence of a dagger, (c) whether spaces with bounded dimensions are sufficient and (d) whether only simple or general traced networks are considered. Originating from the formalization of diagrams and interpretations developed by Hasegawa, Hofmann and Plotkin in [3] we gave a formally rigorous definition and therewith provided a common fundament for speaking about the completeness results in [1] and [3]. Then we considered the categories of free finite-dimensional R-semimodules for arbitrary semi-rings R generalizing the considerations in [1, 3] for fields with transcendentals. Our results for $\mathbb{N} \subset R$ are illustrated in the following tables.

for compact closed categories ‖ for dagger compact closed categories	$\widehat{R}$ with $\mathbb{N} \subset R$	$\mathbb{N} \subset R$	$\widehat{R}$ with $\mathbb{N} \subset R$	$\mathbb{Z}[i] \subset R$ $R = \mathbb{Z}[X]$ $R = \mathbb{N}[X, \overline{X}]$
relative completeness	YES	YES	YES	YES
semi-relative completeness	YES	YES	YES	YES
full completeness	$\star$		$\star$	

While semi-relative completeness for $\widehat{R} = R[(X_i)_{i \in \mathbb{N}}]$ and relative completeness for R with $\mathbb{N} \subset R$ was implicitly shown by Selinger, semi-relative completeness for discrete (semi-)rings was not known before. Furthermore we were able to prove equivalence of all fields - also including the next table - marked with a $\star$. Especially we have proven that when working with interpretations with bounded dimensions the completeness questions essentially collapse to one open problem which is moreover equivalent to full completeness for $\widehat{R}$ when allowing arbitrary interpretations.

for compact closed categories ‖ for dagger compact closed categories with bounded dimensions	$\widehat{R}$ with $\mathbb{N} \subset R$	$\mathbb{N} \subset R$	$\widehat{R}$ with $\mathbb{N} \subset R$	$\mathbb{Z}[i] \subset R$ / $R = \mathbb{Z}[X]$ / $R = \mathbb{N}[X, \overline{X}]$
(essential) relative completeness	$\star$	$\star$	$\star$	$\star$
essential semi-relative completeness	$\star$		$\star$	
semi-relative completeness	NO	NO	NO	NO
essential full completeness	$\star$		$\star$	
full completeness	NO	NO	NO	NO

Besides we explored when completeness results require to restrict ourselves on diagrams without trivial cycles and gained the characterization (3.15) of essential completeness, explaining the distinction of cases in the left column. In particular completeness and essential completeness are equivalent for $\mathbb{N} \subset R$, with the potential exception of full completeness when R does not provide transcendentals. Also the derivation of non-trivial trace equations, exemplifying the following non-completeness relations, are our achievements.

for (dagger) compact closed categories (with bounded dimensions)	char$R \neq 0$	$\#R < \infty$	char$R$ $= p^l$	R ring & char$R = p$, / $R = \mathbb{Z}/p^l\mathbb{Z}$, / char$R = p^l$ and R generated by 1_R, / R distributive lattice, / **Rel**
relative completeness	might depend on R	might depend on R	NO	NO, not even essentially
semi-relative completeness	NO	NO, not even essentially	NO	NO, not even essentially
full completeness	NO	NO, not even essentially	NO	NO, not even essentially

The second part of our analyses dealt with the possibility of non-trivial trace equations when the maximal dimension of involved matrices is limited. We obtained that those equations do not exist if and only if the class of cyclic permutations $[k]_c$ is n-reconstructable, i.e. if the multisets (4.10) contain enough information to reconstruct $[k]_c$. This equivalence to a combinatorial problem allowed us to draw a variety of conclusion (cf. 4.7 - 4.11) sufficing to falsify equality of a large class - in fact any example we have considered - of trace expressions. Ultimately we deduced properties of the collection of valid trace equations and saw that non-existence would indeed imply separability of a much broader class of diagrams with bounded dimensions (cf. 4.12 -4.15).

5.2 Open questions

However, a couple of unsolved problems as well as further research opportunities remain. We discuss four of them.

- Is there any $n \geq 3$ such that $[k]_c$ is n-reconstructable for all $k \in \mathbb{N}^*$? In particular, is $n = 3$ already sufficient? Our Corollaries might indicate this, but none of them seems to be strong enough to apply for all $\Bbbk$. Moreover, would the absence of non-trivial trace equations already enable us to separate *all* diagrams by interpretations with bounded dimensions? This is equally suggested by 4.14 and 4.15 as they likewise demonstrate separability for a vast collection of diagrams. If both questions were true, we would gain an (almost) complete picture of when what kind of completeness holds for $\mathbb{N} \subset R$ as all cases marked with a $\star$ would have a positive answer.

- Is it possible to strengthen the results for semi-rings R not containing $\mathbb{N}$? Especially, can we establish more non-trivial trace-equations, e.g. for (semi-)rings R whose characteristic is the product of different primes? Up to now we only know for several specific examples of (semi-)rings that even (essential) relative completeness does not hold.

- Besides one could extend these completeness considerations to categories different from $\mathbf{FMod}_R/\mathbf{FProd}_R$ and $\mathbf{Rel}$. We hinted at this already with Proposition 4.20. Due to [24] categories without products and coproducts should be considered, e.g. categories of *cobordisms* (cf. [28], p.50-52).

- Ultimately further analyses could examine whether similar completeness results can be achieved for other types of categories like e.g. braided/symmetric/balanced monoidal (dagger) categories. A comprehensive survey of different categorical structures and their diagrammatic representations can be found in [14].

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