The space of measurement outcomes as a non-commutative spectrum

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Abstract

Bohrification defines a locale of hidden variables internal in a topos. We find that externally this is the space of *partial* measurement outcomes. By considering the $\neg\neg$-sheafification, we obtain the space of measurement outcomes, a genuine generalization of the spectrum of a C*-algebra.

1 Introduction

By combining Bohr's philosophy of quantum mechanics, Connes' non-commutative geometry [Con94], constructive Gelfand duality [BM00b, BM00a, Coq05, CS09] and inspiration from Doering and Isham's spectral presheaf [DI08], we proposed Bohrification as a spatial quantum logic [HLS09a, HLS09b]. Given a C*-algebra A, modeling a quantum system, consider the poset of Bohr's classical concepts

$$\mathcal{C}(A) := \{C \mid C \text{ is a commutative C*-subalgebra of } A\}.$$

In the functor topos $\mathbf{Sets}^{\mathcal{C}(A)}$ we consider the *Bohrification* $\underline{A}$: the trivial functor $C \mapsto C$. This is an internal C*-algebra of which we can compute the spectrum, an internal locale Σ in the topos $\mathbf{Sets}^{\mathcal{C}(A)}$. This locale, or its externalization, is our proposal for an intuitionistic quantum logic. In section 3 we compute the externalization of this locale. It is the space of partial measurement outcomes: the points are pairs of a C*-subalgebra together with a point of its spectrum. This construction raises two natural questions:

- Can we restrict to the maximal commutative subalgebras, i.e. total measurement frames?
- Are we allowed to use classical logic internally?

In section 4 we will see that, in a sense, the answers to both of these questions are positive. The collection of maximal commutative subalgebras covers the space in the dense topology and this dense, or double negation, topology forces (sic) the logic to be classical. By considering the $\neg\neg$-sheafification, we obtain a genuine generalization of the spectrum. Moreover, our previous constructions [HLS09a] of the phase space (Σ) and the state space still apply essentially unchanged.

2 Preliminaries

An extensive introduction to the context of the present paper can be found in [HLS09b, HLS09a] and the references therein. Here we will just repeat the bare minimum of definitions.

A site on an poset defines a covering relation. To simplify the presentation we restrict to the case of a meet-semilattice.

Definition 1 *Let L be a meet-semilattice. A* covering relation *on L is a relation $\lhd \subset L \times P(L)$ satisfying:*

1. *if $x \in U$ then $x \lhd U$;*
2. *if $x \lhd U$ and $U \lhd V$ (i.e. $y \lhd V$ for all $y \in U$) then $x \lhd V$;*
3. *if $x \lhd U$ then $x \wedge y \lhd U$;*
4. *if $x \lhd U$ and $x \lhd V$, then $x \lhd U \wedge V$, where $U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$.*

Such a pair $(L, \lhd)$ is called a formal topology.

Every formal topology defines a locale, conversely every locale can be presented in such a way.

Definition 2 *Let $(L, \lhd)$ be a formal topology. A* point *is an inhabited $\alpha \subset L$ that is filtering with respect to $\leq$, and such that for each $a \in \alpha$ if $a \lhd U$, then $U \cap \alpha$ is inhabited. In short, it is a completely prime filter.*

The spectrum Σ of a C*-algebra A can be described directly as a lattice $L(A)$ together with covering a relation; see [CS09].

3 The space of partial measurement outcomes

Iterated topos constructions, similar to iterated forcing in set theory were studied by Moerdijk [Moe86][Joh02, C.2.5]. To wit, let $\mathcal{S}$ be the ambient topos. One may think of the topos **Sets**, but we envision applications where a different choice for $\mathcal{S}$ is appropriate [HLS09b].

Theorem 3 (Moerdijk) *Let $\mathbb{C}$ be a site in $\mathcal{S}$ and $\mathbb{D}$ be a site in $\mathcal{S}[\mathbb{C}]$, the topos of sheaves over $\mathbb{C}$. Then there is a site*[1] $\mathbb{C} \ltimes \mathbb{D}$ *such that*

$$\mathcal{S}[\mathbb{C}][\mathbb{D}] = \mathcal{S}[\mathbb{C} \ltimes \mathbb{D}].$$

We will specialize to sites on a poset and without further ado focus on our main example. As before, let

$$\mathcal{C}(A) := \{C \mid C \text{ is a commutative C*-subalgebra of } A\}.$$

Let $\mathbb{C} := \mathcal{C}(A)^{\mathrm{op}}$ and $\mathbb{D} = \Sigma$ the spectrum of the Bohrification, we compute $\mathbb{C} \ltimes \mathbb{D}$. The objects are pairs (C, u), where $C \in \mathcal{C}(A)$ and u in $L(C)$. Define

[1] The notation $\ltimes$ is motivated by the special case where $\mathbb{C}$ is a group G considered as a category with one object and $\mathbb{D}$ is a group H in $\mathbf{Sets}^G$. Then $\mathbb{C} \ltimes \mathbb{D}$ is indeed the semi-direct product $H \ltimes G$

the order $(D,v) \leq (C,u)$ as $D \supset C$ and $v \subset u$. In terms of forcing, this is the information order and the objects are forcing conditions. We add a covering relation $(C,u) \lhd (D_i,v_i)$ as for all i, $C \subset D_i$ and $C \Vdash u \lhd V$, where V is the presheaf[2] generated by the conditions $D_i \Vdash v_i \in V$. It follows from the general theory that this is a Grothendieck topology.

We simplify: the presheaf V is generated by the conditions $D_i \Vdash v_i \in V$ means V is defined by $v_i \in V(D)$ iff $D \supset D_i$. Hence,

$$C \Vdash u \lhd V \text{ iff } u \lhd \{v_i \mid D_i = C\}$$

by the following lemma.

Lemma 4 [HLS09a] *Let V be an internal sublattice of L. Then $C \Vdash u \lhd V$ iff $u \lhd V(C)$.*

We claim that the points of $\mathbb{C} \ltimes \mathbb{D}$ are partial measurement outcomes.

Definition 5 *A* measurement outcome *is a point in the spectrum of a maximal commutative subalgebra. A* partial measurement outcome *is a point in the spectrum of a commutative subalgebra.*

Theorem 6 *The locale generated by $\mathbb{C} \ltimes \mathbb{D}$ is the classifying space of partial measurement outcomes.*

Proof Let τ be a point, that is a completely prime filter. Suppose that $(D,u) \in \tau$, then by the covering relation for $\mathbb{D}$, τ defines a point of the spectrum $\Sigma(D)$. This point is defined consistently: If $u \in L(C) \subset L(D)$, then $(D,u) \leq (C,u)$. Hence, if $(D,u) \in \tau$, so is (C,u) and the point in $\Sigma(D)$ defines a point in $\Sigma(C)$ as a restriction of functionals. When both (C,u) and (C',u') are in τ, then, by directedness, there exists (D,v) in τ such that $C, C' \subset D$ and $v \subset u, u'$. Moreover, $(C,u) \in \tau$ implies $(C,\top) \in \tau$. Hence $\bigcup\{C \mid (C,\top) \in \tau\}$ is the required subalgebra.

Conversely, let τ be a partial measurement outcome, then

$$\{(C,u) \mid C \subset \mathrm{dom}\tau \text{ and } \tau \in u\}$$

defines a completely prime filter. □

Let us call this locale pMO for partial measurement outcomes. For commutative C*-algebras pMO is similar, but not equal, to the spectrum:

Corollary 7 *For a compact regular X, the points of $pMO(C(X))$ are points of the spectrum of a C^*-*sub*algebra of $C(X)$.*

An explicit external description of the locale may be found in [HLS09b]. The present computation gives an alternative description which makes it easy to compute the points.

[2] In the general setting we use a sheaf.

4 Maximal commutative subalgebras, classical logic and the spectrum

As stated in the introduction, we address the following questions:

- Can we restrict to the *maximal* commutative subalgebras?
- Are we allowed to use classical logic internally?

In a sense, the answers to both of these questions are positive. The collection of maximal commutative subalgebras covers the space $\mathcal{C}(A)$ in the dense topology and this dense, or double negation, topology forces the logic to be classical.

Sheaves for the dense topology may be used to present classical set theoretic forcing or Boolean valued models. In set theoretic forcing one considers the topos $\mathrm{Sh}(P, \neg\neg)$ [MM92, p.277]. The dense topology on a poset P is defined as $p \lhd D$ if D is dense below p: for all $q \leq p$, there exists a $d \in D$ such that $d \leq q$.[3] The locale presented by this site is a Boolean algebra, the topos is a Boolean valued model. This topos satisfies the axiom of choice [MM92, VI.2.9] when our base topos does. The associated sheaf functor sends the presheaf topos $\hat{P}$ to the sheaves $\mathrm{Sh}(P, \neg\neg)$. The sheafification can be described explicitly [MM92, p.273] for $V \rightarrowtail W$:

$$\neg\neg V(p) = \{x \in W(p) \mid \text{for all } q \leq p \text{ there exists } r \leq q \text{ such that } x \in V(r)\}.$$

We apply this to the poset $\mathcal{C}(A)$. We write A for the constant functor $C \mapsto A$. Then $\underline{A} \subset A$ in $\mathbf{Sets}^{\mathcal{C}(A)}$.

For commutative A, $\mathcal{C}(A)$ has A as bottom element. For all C, $\underline{A}_{\neg\neg}(C) = A$.

For the general case, we observe that each C is covered by the collection of all its supersets. By Zorn[4], each commutative subalgebra is contained in a maximal commutative one. Hence the collection of maximal commutative subalgebras is dense. So, $\underline{A}_{\neg\neg}(C)$ is the intersection of all maximal commutative subalgebras containing C.

The covering relation for $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ is $(C, u) \lhd (D_i, v_i)$ iff $C \subset D_i$ and $C \Vdash u \lhd V_{\neg\neg}$, where $V_{\neg\neg}$ is the sheafification of the presheaf V generated by the conditions $D_i \Vdash v_i \in V$. Now, $V \rightarrowtail L$, where L is the spectral lattice of the *presheaf* $\underline{A}$.

$$V_{\neg\neg}(C) \quad = \quad \{u \in L(C) \mid \forall D \leq C \exists E \leq D . u \in V(E)\}.$$

So, $(C, u) \lhd (D_i, v_i)$ iff

$$\forall D \leq C \exists D_i \leq D . u \lhd V(D_i).$$

Theorem 8 *The locale MO generated by $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ classifies measurement outcomes. It is a (dense) sublocale of pMO.*

Proof In the context of Theorem 6 we suppose that $(C, \top) \in \tau$. The subalgebra C is covered by all the maximal commutative subalgebras containing it, so by directedness we conclude that $(M, \top) \in \tau$ for some maximal M.

[3] This description uses classical meta-logic. In general it is only a necessary condition.

[4] Here we use classical meta-logic.

The MO construction is a non-commutative generalization of the spectrum. In this sense it behaves better then pMO; compare Corollary 7.

Corollary 9 *For a compact regular X, $X \cong MO(C(X))$.*

Proof $C(X)$ is the only maximal commutative subalgebra of $C(X)$. □

By considering the double negation we may use classical logic internally in our Boolean valued model. A *global* point in the internal spectrum still defines a unique value for each a in A and the relation with Kochen-Specker is as before [HLS09b]: global points do not exist. Internally the axiom of choice holds, so Σ is a compact Hausdorff space. Still, the spectrum does not have a *global* point and the algebra does not have a global element. This underlines the convenience of the double negation: the value is uniquely defined. By contrast, this is not the case when we trivialize the poset $\mathcal{C}(A)$ by restricting it to its maximal elements.

As an example, consider the matrix algebra M_n. Let D_n be the n-dimensional diagonal matrix. The maximal subalgebras of M_n are $\{\varphi D_n \mid \varphi \in SU_n\}$; see [CHLS09]. Moreover, $\Sigma \cong \{1, \ldots, n\}$ in $\mathrm{Sh}(\mathcal{C}(A), \neg\neg)$. This is a complete Boolean algebra. We have arrived at the setting of iterated forcing as in set theory. Iterated forcing in set theory may be presented as follows; see Moerdijk [Moe86, Ex 1.3a]. If P is a poset in **Sets**, and Q is a poset in $\hat{P}$, then $P \ltimes Q$ is the poset in **Sets** of pairs (p, q) with $p \in P$, $p \Vdash q \in Q$, and $(p, q) \leq (p', q')$ iff $p \leq p'$ and $p \Vdash q \leq q'$. If $\mathcal{E} = \mathrm{Sh}(P, \neg\neg)$, and Q is a poset in $\mathcal{E}$, $\mathcal{F} := \mathrm{Sh}_{\mathcal{E}}(Q, \neg\neg)$, then $\mathcal{F} \cong \mathrm{Sh}(P \ltimes Q, \neg\neg)$. In other words, $(P, \neg\neg) \ltimes (Q, \neg\neg) \cong (P \ltimes Q, \neg\neg)$. If, as in the case of M_n, Q is a cBA in $\mathcal{E}$, then $(Q, \neg\neg) \cong Q$. So $(P \ltimes Q, \neg\neg) \cong (P, \neg\neg) \ltimes Q$. We expect similar simplifications when starting from a Rickart C*-algebra [HLS09c].

A similar $\neg\neg$-transformation can be applied to our Bohrification of OMLs. In the example studied in [HLS09b], we compute a 17 element *Heyting* algebra from an OML. Adding the double negation we obtain a 16 element *Boolean* algebra. The function $f(0) = 0$ and $f(i) = 1$ is 'eventually' equal to the constant function 1. As a result, we obtain the product of 4 Boolean algebras, the spectrum is the copruduct of the corresponding locales.

5 Conclusions and further research

We have presented a non-commutative generalization of the spectrum motivated by physical considerations.

We suggest another way to restrict to maximal subalgebras, while preserving the possibility to compute a unique functional from a global section. Consider a matrix algebra. Let $p \in C$ be a projection and suppose that $M \mapsto \sigma_M \in \Sigma(M)$ is continuous with respect to the unitary group action. Then $\sigma(p) \in \{0, 1\}$, say it is 0. Since the unitary group is connected and acts transitively on the maximal subalgebras, $\sigma(u^* p) = 0$ for all u. Suppose that $p \in M_1, M_2$. Let u transform M_1 into M_2, but leave p fixed. We see that $\sigma(p) = 0$ independent of the choice of maximal subalgebra. By linearity and density, this extends from projections to general elements: σ may be uniquely defined on all elements. This suggests that, at least for matrix algebras, the independence guaranteed by the poset, may also be guaranteed by the group action. We leave this issue to future research.

Bohrification, i.e. the *pMO* construction, is not functorial when we equip C*-algebras with their usual morphisms. The construction *is* functorial when we change the notion of morphism [vdBH10]. More work seems to be need for the MO construction: We have $(I_2, \top) \lhd (C(2), \{(0,1),(1,0)\})$. However, this no longer holds when we map $C(2)$ into M_2. In short, covers need not be preserved under natural notions of morphism.

Bohrification may be described as a (co)limit [vdBH10]. While technically different the intuitive meaning is similar: we are only interested in what happens eventually.

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