Aspects of Categorical Physics: A category for modelling dependence relations and a generalised Entropy functor

Vaia Patta

St Cross College

DPhil thesis

Michaelmas Term 2017

Department of Computer Science

University of Oxford

June 19, 2018

Contents

In loving memory of my grandfather.

Acknowledgements

Firstly, I would like to thank everyone who offered emotional support; I couldn't possibly list them all here.

I would like to acknowledge the financial contribution of EPSRC, which funded my doctorate fees for 3.5 years (studentship 1239597), as well as my mother and grandfather, who paid most of my insane living expenses during my study in Oxford. In a cruel twist of fate, grandpa passed away not long before this doctorate started to go well, never getting to see the fruits of his trust and substantial material investment. He will always live in my heart.

I would like to thank my supervisors, Jamie Vicary and Chris Heunen, who guided me throughout this work. Special thanks goes to Jamie, who saved my doctorate by helping me get a badly needed retroactive suspension of one year. I would also like to thank David Kay, Director of Studies, who supported me in this. I'm also grateful to Adrian Taylor, Director of Studies in the Materials Department, for releasing the EPSRC funds to the Computer Science department so that I could keep my funding.

I would like to thank my officemates, Dominic Verdon and Antonin Delpeuch, for their occasional help with both scientific and technical issues. I acknowledge the contribution of Mario Alvarez-Picallo to my understanding of abstract algebra concepts used in Chapter 3.

I would like to thank my colleague Rui Soares Barbosa, who listened to an idea of mine, supported me and contributed feedback. This idea acted as motivation for parts of Chapter 3. On that note, I would also like to thank Samson Abramsky for listening.

I would like to thank Nathan Bowler, Jeffrey Giansiracusa and most of all an anonymous referee for their valuable input on the paper that eventually morphed into Chapter 2 of this thesis.

I would like to warmly thank Martin Hyland for the time he took to offer me some great insights, criticisms and observations on Chapter 3. On the same note, I would also like to thank Bob Coecke and Dan Marsden, who offered me some great advice at my confirmation exam. Finally, I would like to thank Dominic Horsman and – again – Bob Coecke for offering me similarly constructive feedback at my final viva.

Chapter 1

Introduction

1.1 General context

This thesis covers two topics pertaining to the potential use of Category Theory for Physics. Familiarity with the basic concepts of Category Theory is presumed, as would be covered in a standard Category Theory textbook [2]. Such topics include functors, equivalences, adjunctions, natural transformations, monads, (co)limits, etc.

Category Theory has seen various applications in Physics. For detailed guides to such applications, there are comprehensive review papers such as Coecke and Paquette's "Categories for the practising physicist" [23] or Baez and Lauda's "A prehistory of n-categorical physics" [15] or Baez and Stay's "Physics, topology, logic and computation: a Rosetta Stone" [9]. Here we present a summary thereof.

William Lawvere was perhaps the first person to realise the potential of Category Theory in Physics. His "Categorical Dynamics" formalised the relation between laws of force and laws of motion in terms of a cartesian closed category [43]. His ideas have since bloomed into a substantial body of knowledge called Synthetic Differential Geometry, which formalises various aspects of Continuum Physics [44]; notably, it introduces a way of axiomatising infinitesimals, long used in Physics and Engineering, through constructive logic [49].

These developments started a categorical tradition in Physics, where the objects of a category are used to describe kinematics and morphisms are used to describe dynamics. A functor is then a physical theory[1] in the sense that its domain category describes the "rules" of an abstract process and its codomain category describes the transformations of the physical system at hand. Categorical Physics confers certain advantages over other perspectives:

- Deep mathematical connections between different physical phenomena, or between a physical phenomenon and an algorithmic process, can become apparent; furthermore, the similarities are formalised as equivalences of categories. The most successful instance of this is in Quantum Algorithms, where categorical descriptions have led to improvements and generalisations [77].

- Calculations can be simplified. One example here is gauge field theories, where categorical tools have helped with complicated group operations [26].

- Theories can be reconstructed with fewer conceptual requirements. One such case is General Relativity, where the topology of spacetime can be recovered without the notion of smoothness [55].

- The elegance of some theories is revealed more clearly. This is not merely a matter of aesthetics, but a way of approaching foundational research: at least for the more mathematically-minded physicists, the idea that a mathematically elegant physical theory must have some element of truth is pervasive.

The most prominent representative of this tradition, and currently the most active area of research in Categorical Physics, is Categorical Quantum Mechanics. The aim of

[1]Lawvere used the term "model" for the functor and the term "theory" for its domain.

Categorical Quantum Mechanics is to model quantum systems and quantum processes. The codomain of the "theory" functor in this case is often **Hilb**, the category of finite-dimensional Hilbert spaces and linear operators. Depending on the problem to be modelled, the domain can be, for instance:

- a group seen as a one-object category, which is the group of symmetries of the system[2];
- the category of unions of parametrised circles and certain 2-dimensional cobordisms[3], which yields a conformal field theory as part of String Theory [24];
- the category of compact oriented manifolds and oriented cobordisms, to describe a Topological Quantum Field Theory [45].

Other such functors in Categorical Quantum Mechanics have as codomain a category of certain representations of some sort of group and as domain a category whose morphisms are a network-like structure. Such "theories" encompass structures like Feynman diagrams [12] and Penrose's spin networks [10]; for the latter, the reference given also describes a 2-categorical structure with morphisms between spin networks (called "spin foams") that describes processes in Loop Quantum Gravity.

There is a distinctive characteristic of approaches to Quantum Mechanics, as opposed to Classical Physics, which has necessitated some new notation. In Classical Physics, the structures that tend to arise are cartesian closed categories, such as the ones Lawvere worked with. By contrast, researchers in Categorical Quantum Mechanics tend to work with braided monoidal structures instead.[4] Monoidal structures are best

[2]This is the simplest example, mainly used as an introduction [15, 9] to more intricate and useful cases, such as the rest on this list.

[3]Due to space constraints, this description is a simplification of a simplification, itself taken from Baez's review paper [15]. The paper referenced actually involves infinity-categories.

[4]This is not a clear-cut dichotomy; the reasons for the tendency to prefer either structure depending on context have been addressed by Abramsky and Coecke [1].

visualised and handled using string diagrammatic language. We briefly use this notation in Chapter 3, even though we are not working in a quantum setting. We assume familiarity with the basic principles of string diagrams [71].

Beyond Categorical Quantum Mechanics, there are also similar applications of Category Theory to Relativity [55] and gauge theories [14]. In these cases, the domain of the "theory" is a groupoid whose objects are points in space and morphisms are paths between them, and the codomain has objects corresponding to sets of possible internal states of a particle and morphisms corresponding to maps between them.

This is the general framework within which this thesis aims to aid the potential application of Category Theory to Physics. The thesis consists of two subprojects dealing with different aspects of the discipline, each treated in a separate chapter. Chapters 2 and 3 are independent enough of each other that they can be read in either order. The connection between them will become apparent in the sequel and will be made explicit in Chapter 4.

1.2 Background to our work on matroids

Matroids are abstract structures which are widely used, their prime application being in Combinatorics; as such, in Physics they have found use in a range of areas, from Civil Engineering [67] to Electrical Circuits [65]. They are also useful as a means to compute dependent probabilities [53]; the latter use gives us a connection to Entropy: Shannon Entropy is the rank function of a generalisation of matroids called polymatroids [30], and is also connected to matroids [56].

Matroid theory lies within set theory. Informally, a matroid is a set with some added structure in the form of a distinguished family of subsets, which describes a dependence relation. Historically, matroids were introduced by Whitney in 1935. His motivation was to generalise linear independence and acyclicity in graphs [80]. It was Mac Lane who discovered the connection between matroids and projective geometry a year later [54]. Since then, matroids have found application in geometry, topology, combinatorial optimisation, network theory, and coding theory [79, 62].

While matroids themselves have been studied extensively, there is much less literature focussing on maps between them. Furthermore, there has been very little explicitly categorical work on matroids, which we now list.

The most closely related work to this thesis is Al Hawary's doctoral thesis from 1997 [5], which was an attempt to characterise one of the categories we discuss here: the category of loopless pointed matroids and pointed strong maps. That thesis purports to prove many related results, mainly concerning free objects, limits and colimits; some corrected statements of theorems therein can be found in Propositions 30 and 42 of this thesis, and others in Al Hawary's papers of 2000 [6] (concerning free objects) and 2003 [7] (revolving around epimorphisms, monomorphisms, limits and colimits).

Before 2000, there was some categorical work on matroid bundles [8], which are meant to imitate vector bundles. Matroid bundles are defined on the basis of weak maps, which we do not examine in this thesis. We note, however, that the results on matroid bundles find categorical implications for matroid representability, which features prominently in this thesis.

Moving on to more recent work, there was a paper in 2010 [52] that examined categorical properties of and adjunctions between a category of matroids and corresponding categories of matroid generalisations called fuzzy matroids and fuzzifying matroids. A notable result was that the category of matroids embeds in that of fuzzifying matroids as a simultaneously concretely reflective and coreflective subcategory. The mappings of this matroid category were again the weak maps, so the results of this paper, while interesting, have no implications for the category we examine in this thesis.

Lastly, there was a paper in 2013 [73] on topological representations of matroid maps. While the author examines both weak and strong maps, he focusses on the former. The most notable result of this paper is that weak maps between matroids induce topological mappings between their representations via a functor from the category of matroids and weak maps to the homotopy category of topological spaces. Since we do not examine these relations between matroids and topological spaces, we make no use of this paper; however, for the reader's reference we note that the paper also contains a partial result about the category of matroids and strong maps.

Chapter 2 (which has been previously published [33]; the cited paper constitutes joint work with Chris Heunen) aims to explore the categorical properties of matroids. In this thesis we do not go into actual use of this category as a domain for a "theory" functor for a specific type of system (in the sense described in the previous section), but have laid the foundations for such an application.

1.3 Background to our work on entropy in Thermodynamics

We begin with a brief overview of Thermodynamics and Statistical Mechanics. The latter, while not explicitly tackled in Chapter 3, partly motivates it and is linked to our future goals, as will be discussed in Chapter 4. Also note the connection to Shannon Entropy, which as we have seen is expressible in terms of matroids, hence links this aspect of the thesis to the work on matroids.

In examining the physical properties of a system composed of N particles in the *thermodynamic limit*, *i.e.* when $N \to \infty$, we may assume that quantities such as energy, temperature, mass, volume, pressure, etc. vary continuously with each other. Thermodynamics concerns itself solely with these interdependencies; from the point of view of Thermodynamics, a system is in a *macrostate*, that is, a set of values of such macroscopic variables. By contrast, Statistical Mechanics connects these quantities to the statistical properties of the underlying energy states of the individual particles of the system, hence is concerned with *microstates*, which specify the energy of each individual particle.

A basic postulate in Statistical Mechanics is that of "equal a priori probabilities", which states that all microstates that correspond to a given macrostate are equally likely. The number Ω of all possible microstates is a function of the macroscopic properties of the system. This function completely defines the thermodynamical behaviour of the system. Systems generally tend to thermal equilibrium, which is the macrostate corresponding to the largest number of microstates; it is an experimental fact that this macrostate is typically orders of magnitude more likely than the next likeliest macrostate. The fact that this is the macrostate which corresponds to uniform temperature reveals a deep connection between Statistics and Thermodynamics: the heat flow divided by temperature is a quantity proportional to $\ln \Omega$. This fundamental quantity is called Entropy.

In the classical regime, the specific statistics of the particles of the system depend on many parameters: whether they are distinguishable, how many energy levels are allowed, how many states are allowed on each level, how many particles are allowed to be in each state, etc. Things become significantly more complicated when the particles are considered quantum, in which case new concepts emerge. The quantum regime is not explicitly tackled in this thesis, but is within the scope of its applicability.

After this brief overview, we now move on to the central theme of Chapter 3, Entropy. The notion of Entropy is fundamental, both in Physics (Thermodynamical Entropy) and in Information Theory (Shannon Entropy). As regards Physics, Thermodynamics was revolutionised as soon as the significance of Entropy, defined as the heat flow divided by temperature, was discovered. In Information Theory, Entropy is an indispensable measure of information. It is of course no mere coincidence that these two quantities share the same name; employing the Statistical Thermodynamics formulation of Entropy we see that the two Entropies have the same defining formula $\left(-\sum_i p_i \log p_i\right)$ (where p_i are probabilities in both contexts), the only difference being in the basis of the logarithm and in an overall positive multiplicative factor. There is a deep conceptual link between them, explained by Statistical Mechanics.

In this work we examine Thermodynamical Entropy through the prism of the Second Law of Thermodynamics, which states that Entropy is a nondecreasing function in any process taking place on a thermally isolated system (*i.e.* disallowing heat exchange). However, there is an equivalent[5] formulation of the Second Law: Entropy is a nondecreasing function in any process which is realised by means of an interaction with some device consisting of some auxiliary system and a weight, in such a way that the auxiliary system returns to its initial state at the end of the process whereas the weight may have risen or fallen. Traditionally, the term "adiabatic" is used for thermally isolated processes, but in this work we use "adiabatic" to refer to the processes described in the latter formulation; explicitly:

Definition 1 ([51])**.** For a given system undergoing a process, we shall call the process *adiabatic* if it is realised by means of an interaction with some device consisting of some auxiliary system and a weight, in such a way that the auxiliary system returns to its initial state at the end of the process whereas the weight may have risen or fallen.

This type of process is one where only the *net* heat exchange is required to be zero, and the definition has the advantage that it avoids explicitly referring to heat exchange, thus has less conceptual requirements. For more clarification and further discussion on

[5]The equivalence holds when the systems in question are assumed to be finite (*i.e.* the auxiliary system cannot be a heat bath).

how different authors interpret the term, along with constructive criticism of different frameworks, see a relevant review paper [75].

From a mathematical viewpoint, Faddeev [28] has shown that the Entropy formula, as defined on some set of probability measures, arises from just three simple properties:

Theorem 2 (Faddeev)**.** *Suppose I is a map sending any probability measure on any finite set to a nonnegative real number. Suppose that:*

1. *I is invariant under bijections.*
2. *I is continuous.*
3. *For any probability measure p on a set of the form $\{1, \ldots, n\}$, and any number $0 \leq t \leq 1$,*

$$I((tp_1, (1-t)p_1, p_2, \ldots, p_n)) = I((p_1, \ldots, p_n)) + p_1 I((t, 1-t)).$$

Then I is a constant nonnegative multiple of Shannon entropy.

This result arises from a simple fact: the logarithmic function is the only function f for which $f(ab) = f(a) + f(b)$. Faddeev's theorem implies that Entropy reduces to fundamental properties such as additivity, monotonicity etc and should therefore lend itself to a simple axiomatisation; that is, it should arise from a small set of simple, intuitive axioms. This in turn may open the door to all sorts of mathematical tools at our disposal.

As we shall see, one of the main concepts we introduce in this work is what we shall call an adiabatic category. The foundation of these categories is the paper "The mathematical structure of the second law of thermodynamics", written by Lieb and Yngvason [51]. They use a preorder to model existence of adiabatic processes between thermodynamic systems in specific states and derive a notion of thermodynamical entropy from a small number of first principles; concretely, they recover a *unique* (up to some factor) subadditive nondecreasing function, which must therefore be Entropy. Although their work is not categorical, it is conducive to categorical treatment and provides the motivation for adiabatic categories, which are defined in Section 3.3.1. Lieb and Yngvason's main theorem gives rise to an entropy functor from an LY-adiabatic category (which is a specific sort of adiabatic category) to the nonnegative real numbers (seen as a category). We shall see this in Theorems 225 and 230.

In the first part of Chapter 3, we use Category Theory to generalise Lieb and Yngvason's framework in order to model *classes* of adiabatic processes, whereas Lieb and Yngvason only examined *existence* of adiabatic processes with given initial and final state; we check that their proofs still hold in this setting (Subsection 3.3.4). This opens up the possibility of proving properties of different classes of adiabatic processes, something that Lieb and Yngvason's simple model did not cover (we give a concrete physical example at the end of Chapter 3).

Let us for a moment consider the possible uses of this new model. We shall call our model *thick*, and Lieb and Yngvason's model *thin*[6]. In a thick model, what could the different classes of adiabatic processes be? Naïvely we may consider using the displacement d of the weight as a classifier; but that would not work: the parameter d indirectly measures the amount of energy exchanged with the system, therefore processes p_1 and p_2 with the same initial state corresponding to displacements $d_1 \neq d_2$ would necessarily have final states of different energies. We may consider "cheating" by somehow omitting this energy from the state description (that is, by using this energy to change a parameter that is not included in the properties we consider as part of the state) but then we may pay the price of violating the Second Law! As described by Jaynes in 1996

[6]The name comes from the fact that Lieb and Yngvason's model gives rise to a thin category.

in an illuminating paper [36], this "hidden" information can be exploited to seemingly convert heat into work without any other apparent effects, which is explicitly forbidden by the Second Law. While Jaynes's example referred specifically to distinguishing between different substances, his argument applies to any property of the system, as long as one may find a way to use it to do work.

So in conclusion the classifier cannot be the displacement of the weight; it cannot be the initial or final *position* of the weight, either, because then composability (*i.e.* the principle that we may successively perform two processes as long as the final state of the former is the initial state of the latter) would be violated. Furthermore, in light of the above discussion, if we are to use a property of the system as a classifier, we not only have to exclude this property from the state description, but we also have to be careful to exclude from our setup all processes that can exploit this property to do work. The latter requirement is not placed on the model itself, but rather is a restriction on its applicability; in order for the model to give us the Entropy that corresponds to the actual experimental measurements of the underlying physical system, it has to contain all the data relevant to energy conversions. As pointed out by Jaynes in his aforementioned paper, this is in fact a way for a physicist to discover an unknown phenomenon: they may apply this or a similar model to processes on states differing by some property, which unbeknownst to them is somehow used to do work, and discover that their model does not validly describe the system.

To avoid such considerations, the classifier must either be a property of the auxiliary system (such as what tools are available) or a property of the noninteracting environment (such as a clock). The example we use in this work (Subsubsection 3.4.2) can be interpreted as either of these, that is, we may label the processes either by the time measured on a clock or by the average power of the stirrer we use.

So far we have spoken of the immediate connections between this thesis and previous work. Taking a step back, we can see that this is situated within a broader landscape; the use of Category Theory in Thermodynamics is of course nothing new. Our generalisation of Lieb and Yngvason's framework [51], which we discussed above, falls loosely under the umbrella of Resource Theories [29, 22], where the objects of a category are "resources" and its morphisms correspond to ways to "convert" one resource into another.

Perhaps the closest to our treatment is a paper very similar to Lieb and Yngvason's, namely "A characterization of entropy in terms of information loss" by Baez, Fritz and Leinster [13]. This paper defines the categories **FinProb** and **FinMeas**, which involve sets with measures and measure-preserving functions between them, and derives a notion of Shannon entropy from axioms similar to those of Lieb and Yngvason. They explicitly define entropy functors from **FinProb** and **FinMeas** to the nonnegative real numbers.

Although the two papers (Lieb-Yngvason and Baez-Fritz-Leinster) refer to different scientific fields, their motivation is similar. Lieb and Yngvason's goal was to derive Entropy as a nondecreasing function in a certain sort of physical process, namely adiabatic processes, where the thermodynamical states involved formed a preorder; Baez, Fritz and Leinster's goal was to derive Entropy as a nonincreasing function in a category of deterministic functions. Even though one of these settings lies in Physics and the other in Information Theory, both aim to tackle the more abstract question of what sort of systems display Entropy.

Furthermore, these two papers use overlapping methodologies. Specifically, they take the following steps to retrieve an appropriate functor (unique up to affine equivalence):

1. Establish a category with a symmetric monoidal product $\boxplus$ and a family of covariant strict monoidal endofunctors λ that correspond to the nonnegative reals

and whose composition corresponds to multiplication.

2. Ensure that the functor respects additivity (or convex linearity) and homogeneity (in Lieb-Yngvason this is done by inserting morphisms between X and $(1-\lambda)X \boxplus \lambda X$ both ways).[7]

3. Ensure that the functor is continuous[8] (in Lieb-Yngvason this is done by building continuity into the category).[9]

A notion of entropy can be defined if the objects of the category correspond to vectors and the λ functors act as multiplication by a constant; one can then apply these theorems in order to categorically characterise entropy-increasing and entropy-decreasing chains. Lieb-Yngvason focussed on a macroscopic treatment so they were satisfied to leave it at that, but if they had been working with probability distributions as in the Baez-Fritz-Leinster paper, they could have got the statistical formulation of Entropy by applying Faddeev's theorem (which Baez-Fritz-Leinster did explicitly); this is in part what motivates this work, but there is a caveat that we shall return to shortly. To see how this works in general, we refer back to Theorem 2; the three numbered properties are reflected in, respectively, the following properties that both of these constructions have in common:

1. Functoriality of Entropy, along with the fact that the target category is a poset.

2. Continuity of the Entropy functor.

3. Monoidality of the Entropy functor and strict commutativity between the Entropy functor and the λ endofunctors.

[7]The precise definition of additivity, convex linearity and homogeneity need not worry us here; a precise mathematical definition of an appropriate category is given in Section 3.3.1.

[8]Here we mean continuous in the topological sense.

[9]Again, the precise definition of continuity need not worry us here; see definition of adiabatic categories in Section 3.3.1. We note, however, that there is some nuance here. Continuity is defined and used slightly differently in each of these papers to achieve the same end result.

A more complete explanation of how to get the Entropy formula from an appropriate functor is found in Baez-Fritz-Leinster; a proof of existence of such a functor on appropriate preorders is found in Lieb-Yngvason and repeated in our treatment in a more general context (the main difference being that we no longer require a preorder).

The similarity of these two results is not to detract from their significance. Lieb-Yngvason may have restricted their treatment to preorders (in a way a "decategorified" version of Baez-Fritz-Leinster's setting), but it is remarkable that they reached their conclusion by making use of only the data of a category (*i.e.* instead of positing an appropriate functor, they showed that it exists). This highlights the simplest possible setting in which Entropy can arise. Baez-Fritz-Leinster, on the other hand, while they included existence of such a functor in their assumptions, were the first to bring the full force of categories into the picture and explore what sort of category is needed to recover the explicit formula of Entropy.

It certainly looks like these two papers should complement each other, and this is the contribution of this work. We aim to unite these two results into a seamless proof of the necessity of a unique Entropy, given by the statistical formulation. However, as we mentioned earlier, there is a caveat. This arises from the continuous-discrete dichotomy.

In our work, the Entropy functor we recover satisfies the properties of Faddeev's theorem on an abstract level; that is, even though the function's domain is not probability measures, it is invariant under bijections, continuous, and satisfies the same convexity-type property. At this point it looks like we have done most of the work: then surely all that remains is to set the domain to be a set of probability measures (or, more accurately, set up a functor from our category to the category of probability measures and stochastic maps) and we can apply Faddeev's theorem to recover the statistical formulation of Entropy! Unfortunately, however, things are not so simple. Such a categorical application has complications due to the fact that Lieb and Yngvason assume that their systems are continuous, whereas Faddeev's theorem only applies to probability measures on finite sets. This construction therefore demands a setup that recovers the continuous case as a limit; in fact, it turns out that this requires setting up intricate bicategories (meant as "approximations" of an adiabatic category) and taking the categorical colimit of a chain of these bicategories in an appropriate category of bicategories. While we have attempted this problem, it will not be further discussed in this thesis; it is a large undertaking, reserved for a future project.

Chapter 2

The category of matroids

2.1 Introduction

Note on previously published material

The material in this chapter has also been published in a relevant paper [33], which was coauthored with Chris Heunen. A small amount of material has been contributed by the anonymous reviewer, as we note in the appropriate places in this thesis.

Note on notation

Throughout this chapter, the term *source* will be used interchangeably with *domain* and the term *target* will be used interchangeably with *codomain*. We caution the reader that this differs from the use of the term *source* elsewhere in the literature [3].

2.1.1 Overview

As mentioned in Chapter 1, matroids abstract dependence and have found many uses in Physics and Engineering. This is primarily through their connection to Linear Algebra (where they encapsulate linear dependence) and to Graph Theory (where they encapsulate cycles). We treat these areas in Subsections 2.7.2 and 2.7.3 respectively. For the moment, we note that the connection of matroids to Linear Algebra concerns the so-called *representable* matroids, on which there are currently many open problems.

Of particular interest to us is the connection of matroids to Shannon Entropy [56]; this connection arises because matroids encapsulate probabilistic conditional dependence. However, this involves a more advanced notion of representability, the so-called *probabilistic representability*. We therefore opted not to tackle it in the context of this thesis, but in future work on categories of representable matroids. The thesis contains an expression of matroid representation as a functor, which can serve as the foundation for future work on probabilistic representability.

In the most general terms, the purpose of this work is to survey the properties of the category of matroids. Of course, speaking of "the" category of matroids entails a choice of morphisms. There are mainly three candidates [79]: weak maps, strong maps and comaps[1]. While there is some structure associated with the category of matroids and weak maps [79, Chapter 9], here we have chosen strong maps for various compelling reasons:

- They are a natural choice of structure-preserving functions.
- The resulting category is much "nicer" than the two other choices of morphisms; we shall see that it has interesting factorisation properties (see Section 2.6).
- A few commonly used matroid constructions become functorial in this setting (see Sections 2.5 and 2.8).
- There is a functor from vector spaces and linear maps to matroids and strong maps, arising from matroid representation.

[1]Here, the prefix co- bears no relation to its common use in Category Theory.

- As we show in Section 2.9, this category serves to characterise matroids in terms of optimality of the greedy algorithm via limits.

Having motivated the category, we now give an outline of the chapter:

- In Section 2.2 we define the matroid categories we examine and look at some basic properties.
- In Section 2.3 we look at limits and colimits of these categories.
- In Section 2.4 we examine chains of adjunctions between various notable subcategories of matroids.
- In Section 2.5 we look at categorical properties of two basic matroid operations, deletion and contraction.
- In Section 2.6 we look at factorisation systems in the category of matroids.
- In Section 2.7 we examine functors between matroids and other notable categories (vector spaces, geometric lattices and graphs).
- In Section 2.8 we examine functoriality of common matroid operations.
- Lastly, in Section 2.9 we have a categorical characterisation of the greedy algorithm.

2.2 The category

The category we shall be examining has matroids as its objects. Matroids, like topological spaces, have many equivalent definitions; as in the case of topology, this is a testament to their usefulness.

It is an unfortunate tradition in the matroids literature that these equivalent definitions are called "cryptomorphisms", to denote that the equivalence of these definitions is not obvious. We find that this term places the emphasis on a trivial aspect of the theory (*i.e.* whether a mathematical fact is intuitive), as well as alienates new researchers in the area. In this work, we shall not use the term "cryptomorphism", hoping that it will eventually be abandoned.

We go on to give some of these equivalent definitions of a matroid. These have not been chosen according to their ubiquitousness or general usefulness; they are merely the ones used in our categorical treatment. For a comprehensive list and to gain a better understanding of the intricate relations between the various notions involved, the reader is encouraged to consult relevant textbooks [62, 79].

Throughout this chapter, we shall use the notation $\#X$ to denote the cardinality of the set X; furthermore, this cardinality is always understood to be finite for the purposes of our treatment. We do note, however, that there exists a notion of matroids of infinite cardinality [20] and even matroids of infinite rank (see below for the definition of rank); many important matroid theorems still hold for these cases.

Definition 3. A *matroid* M consists of a finite *ground set* $|M|$ with, equivalently:

- a family of $\mathcal{I}$ of subsets of $|M|$, called the *independent sets*, satisfying:
  - nontrivial: the empty set is independent[2];
  - downward closed: if $I \in \mathcal{I}$ and $J \subseteq I$ then also $J \in \mathcal{I}$;
  - *independence augmentation*: if $I, J \in \mathcal{I}$ and $\#I < \#J$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$.
- a family $\mathcal{F}$ of subsets of $|M|$, called the *closed sets* or *flats*, satisfying:

[2] In view of the second axiom in this bullet point, this axiom is equivalent to "$\mathcal{I}$ is nonempty"; hence the name "nontrivial".

  - greatest element: $|M|$ itself is closed;
  - closed under intersection: if $F, G \in \mathcal{F}$ then also $F \cap G \in \mathcal{F}$;
  - *partitioning*: if $\{F_1, F_2, \ldots\}$ are the minimal flats properly containing a flat F, then $\{F_1 \setminus F, F_2 \setminus F, \ldots\}$ partitions $|M| \setminus F$.

- a *rank function* $\mathrm{rk}\colon 2^{|M|} \to \mathbb{N}$, satisfying:
  - bounded: $0 \leq \mathrm{rk}(X) \leq \#|M|$ for all $X \subseteq |M|$;
  - monotonic: if $X \subseteq Y \subseteq |M|$, then $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$;
  - semimodular: if $X, Y \subseteq |M|$, then $\mathrm{rk}(X \cup Y) + r(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

The following three notions are also essential to matroid theory:

Definition 4. A maximal independent set is called a *basis*; the collection of bases is denoted $\mathcal{B}$.

Definition 5. The *closure* operation $\mathrm{cl}\colon 2^{|M|} \to 2^{|M|}$ is defined by

$$\mathrm{cl}(X) = \big\{x \in |M| \;\big|\; \mathrm{rk}(X \cup \{x\}) = \mathrm{rk}(X)\big\}.$$

Definition 6. The maximal flats properly contained in the ground set are called *hyperplanes*; the collection of hyperplanes is denoted by $\mathcal{H}$.

Remark 7. The connection between the different equivalent definitions is as follows: A flat is precisely a subset of $|M|$ which equals its closure (hence the alternative term "closed set") and $\mathrm{rk}(X)$ is precisely the size of the largest independent set contained in $X \subseteq |M|$.

We note for the sake of completeness that a matroid may equally be defined by axiomatising its family of bases, its closure function or its family of hyperplanes instead of its family of independent sets, its family of flats or its rank function.

Remark 8. At this point the astute reader may observe, based on the above definitions, that a matroid may be defined as a functor from some appropriate poset to the category **Sub** of sets and inclusions. This idea will play a role in Section 2.9, when we examine the connections to the greedy algorithm.

Matroids are fundamentally set-theoretic constructs, hence the elements themselves are immaterial; we only deal with the distinguished subsets of the matroid that arise from its defining family (eg independent sets, flats) or function (eg rank). However, there are certain elements that exhibit notable properties. These are the loops, isthmuses and parallel elements.

Definition 9. A *loop* is an element of a matroid that is not contained in any independent set, or equivalently, an element that is contained in all flats. An *isthmus* is an element that is included in every basis. Nonloop elements of the same rank-1 flat are called *parallel*.

The following special types of matroids are of special interest in matroid theory.

Definition 10. A matroid is *pointed* when it has a distinguished loop, denoted $\bullet$ and called *the point*. A (pointed) matroid is:

- *loopless* when it has no loops (other than the point);
- *simple* when it has no loops (other than the point) or parallel elements;
- *free* when every subset (not containing the point) is independent; equivalently, when every subset (containing the point) is closed.

In this work, we shall also use the following term, mirroring the notion of free matroids.

Definition 11. A matroid is *cofree* when the empty set is the only independent set; equivalently, when only the ground set is closed.

The following examples of matroids constitute the central themes of this work.

Example 12. Any finite[3] vector space V gives rise to a matroid $M(V)$, whose ground set is V, and whose independent sets are the linearly independent subsets of V; flats correspond to vector subspaces of V, the closure operation takes linear spans, and the rank function computes the dimension of the linear span. The matroid $M(V)$ becomes pointed by making the only possible choice for the distinguished loop $\bullet$; that is, setting the origin 0 as $\bullet$. The pointed matroid $M(V)$ is free only when V is zero-dimensional or when $V = \mathbb{Z}_2$.

The matroid in the next example is called the *cycle matroid* of a (multi)graph.

Example 13. Any undirected multigraph G gives rise to a matroid $M(G)$, whose ground set consists of the edges, and where the independent sets are the forests. Loops of $M(G)$ are precisely loops of G (*i.e.* edges between a vertex and itself), isthmuses of $M(G)$ are precisely isthmuses of G (*i.e.* edges that are not contained in any cycle), and parallel elements of $M(G)$ are precisely parallel edges of G (*i.e.* edges with both adjacent vertices in common); this is, in fact, the origin of the terminology. We can point $M(G)$ by choosing a loop. The matroid $M(G)$ is simple when it has no loops and no parallel edges; that is, when G is a graph. $M(G)$ is free when G is a forest.

For the next example, recall the following definitions:

Definition 14.

- An element z in a partially ordered set (poset) *covers* x when $x \leq z$, and if $x \leq y \leq z$ then $x = y$ or $y = z$. In a poset P, a *chain* from x_0 to x_n is a subset $\{x_0, x_1, \ldots, x_n\}$ of P such that $x_0 < x_1 < \ldots < x_n$. The *length* of this chain is n. The chain is *maximal* if each of its elements x_i covers x_{i-1} ($i > 0$). The poset P

[3] According to the usual definition of a matroid, infinite vector spaces also qualify, but we do not consider the infinite setting here.

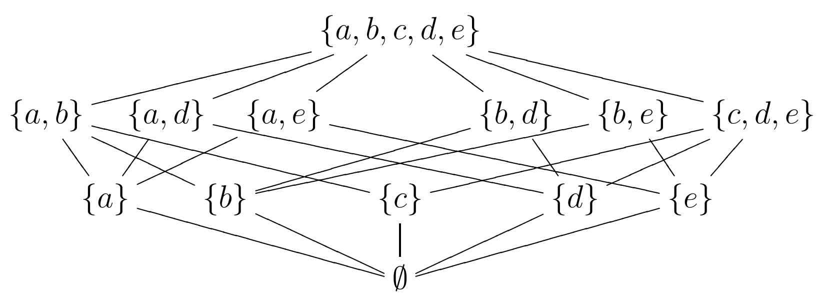
satisfies the *Jordan-Dedekind chain condition* if for every pair $a < b$ all maximal chains from a to b have the same length.

- A *lattice* is a poset with all pairwise least upper bounds (*joins*) and all pairwise greatest lower bounds (*meets*). The least element of the lattice is called the *bottom* and the greatest element of the lattice is called the *top*.

- The *height* $h(y)$ of an element y of a poset is the maximum length of a chain from 0 to y (elements of height 1 are also called *atoms*). A lattice is *semimodular* if it satisfies the Jordan-Dedekind chain condition and $h(x)+h(y) \geq h(x \vee y)+h(x \wedge y)$ for any pair of elements x, y.

- A lattice is *geometric* if it is semimodular and every element is a join of a (possibly empty) set of atoms.

Example 15. A matroid M is specified by (the Hasse diagram of) its partially ordered set $L(M)$ of flats, ordered by inclusion. For example:



is a matroid with ground set $\{a, b, c, d, e\}$. As we will see later, any geometric lattice L gives rise to a matroid $M(L)$, which furthermore is simple; if L is the lattice of flats of some matroid M', then $M(L)$ may or may not be equal to M'. The matroid $M(L)$ is free only when L is isomorphic to a full powerset lattice.

We now specify the morphisms of the category. These will be the so-called strong maps.

Definition 16. A *strong map* from M to N is a function $f\colon |M| \to |N|$ such that the inverse image of any flat in N is a flat in M. Write **Matr** for the category of matroids and strong maps, and **LMatr**, **SMatr**, **FMatr** for the full subcategories of loopless, simple, and free matroids.

A strong map between pointed matroids is *pointed* when it sends the point to the point. Write $\mathbf{Matr}_\bullet$ for the category of pointed matroids and pointed strong maps, and $\mathbf{LMatr}_\bullet$, $\mathbf{SMatr}_\bullet$, $\mathbf{FMatr}_\bullet$ for the full subcategories of loopless, simple, and free matroids.

The flats of a matroid M, when ordered by inclusion, form a geometric lattice $L(M)$, where the height of an element is the rank of the corresponding flat; and conversely, every geometric lattice is isomorphic to the lattice of some matroid [62, Theorem 1.7.5]. This fact yields a characterisation of strong maps between matroids.

Lemma 17. *For $M, N \in \mathbf{Matr}$ and $f\colon |M| \to |N|$ the following are equivalent:*

(a) f is a strong map;

(b) $\mathrm{rk}(f(Y)) - \mathrm{rk}(f(X)) \leq \mathrm{rk}(Y) - \mathrm{rk}(X)$ *for all $X \subseteq Y \subseteq |M|$;*

(c) the function $L(f)\colon L(M) \to L(N)$ given by $X \mapsto \mathrm{cl}(f(X))$ preserves joins and sends elements of height 1 to elements of height 0 or 1.

Proof. This is a well-known result in matroid theory [79, Propositions 8.1.3 and 8.1.6]. □

Of particular interest are submatroids:

Definition 18. Given a matroid M, a *submatroid* N of M is a matroid such that $|N| \subseteq |M|$ and $\mathcal{F}_N = \{F \cap |N| : F \in \mathcal{F}_M\}$.

It is immediate that if N is a submatroid of M then the map $i : N \to M$ acting as the identity on elements is a strong map, and furthermore any strong map with target N extends to a strong map with target M via composition with i.

The assignment $M \mapsto |M|$ forms a forgetful functor $|-|\colon \mathbf{Matr} \to \mathbf{FinSet}$ from the category of matroids to the category **FinSet** of finite sets and functions. The following observation justifies the use of the terms "free" and "cofree".

Theorem 19. *There is a series of adjunctions $F \dashv |-| \dashv C \dashv (-)_0$ given by*

$$\begin{aligned} \mathcal{F}_{F(X)} &= 2^X, & F(f) &= f, \\ \mathcal{F}_{C(X)} &= \{X\}, & C(f) &= f, \\ (M)_0 &= F_0, & (f)_0 &= f|_{F_0}. \end{aligned}$$

where F_0 denotes the matroid with only one flat, which is the rank-0 flat of M. There are no further adjoints.

Proof. Functoriality is immediate in all cases. We show that the adjunctions hold.

For $F \dashv |-|$: Observe that matroids of the form $F(X)$ for some finite set X are precisely free matroids, and that any matroid mapping whose source is free is a strong map. This yields a universal arrow; that is, for every set X, every matroid M, and every function $f\colon X \to |M|$, there must exist a function $\eta_X\colon X \to X$ and a unique strong map $\hat{f}\colon F(X) \to M$ satisfying $f = \hat{f} \circ \eta_X$, namely $\eta_X = \mathrm{id}_X$ and $\hat{f} = f$.

For $|-| \dashv C$: Observe that matroids of the form $C(X)$ are precisely the cofree matroids, and that any matroid mapping whose target is cofree is a strong map. This again yields a universal arrow: for every strong map $f\colon M \to C(X)$ there must exist a strong map $\eta_M\colon M \to C(|M|)$ and a unique function $\hat{f}\colon |M| \to X$ satisfying $f = \hat{f} \circ \eta_M$, namely $\eta_M(x) = x$ and $\hat{f} = f$.

For $C \dashv (-)_0$: observe that any strong map whose source is cofree must have a cofree matroid as its target. Therefore, we again recover a universal arrow (set $\eta_X = \mathrm{id}_X$ and $\hat{f} = f$).

We now show that there are no further adjoints.

Suppose K were left adjoint to F. Let M be the matroid with flats

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\},$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}\}.$$

We will count the morphisms $K(M) \to X$ and $M \to F(X)$ (for the following suppose that $\#X > 1$). It is straightforward that there are $\#X^{\#K(M)}$ morphisms $K(M) \to X$. As for the morphisms $M \to F(X)$, in order for the preimage of each subset of X to be a flat of M, either all elements of M must map to the same element of X, which can be done in $\#X$ ways, or precisely 2 of the elements of M must map to each of 2 elements of X, which can be done in $3\#X(\#X - 1)$ ways; in the end we get $3\#X^2 - 2\#X$ maps in total. So $K(M)$ must satisfy $\#X^{\#K(M)} = 3\#X^2 - 2\#X$; but this clearly cannot hold for every choice of X. Thus F has no left adjoint.

Finally, suppose B were right adjoint to $(-)_0$. Let M be a free matroid. Then there is a unique function $(M)_0 \to X$, so there can only exist one strong map $f\colon M \to B(X)$. Hence $\#B(X) = 1$ for all X. Now let M' be a matroid with at least one loop and let $\#X > 1$; there are multiple functions $(M')_0 \to X$, so $\#B(X) > 1$. We have reached a contradiction, therefore $(-)_0$ cannot have a right adjoint. □

The anonymous reviewer of our relevant paper [33] was the first to point out the following link to topology. The preimage of any finite set X under $|-|$ is partially ordered: if M and N are matroids with $|M| = |N| = X$, then $M \leq N$ if and only if $\mathcal{F}_M \subseteq \mathcal{F}_N$. This resembles the situation in general topology, with $\leq$ indicating a "finer" matroid structure, $F(X)$ being the finest (most closed sets) one, and $C(X)$ being the coarsest (fewest closed sets) one.

2.3 Limits and colimits

We now examine limits and colimits in the matroid categories defined in Definition 16. We give proofs and counterexamples to accommodate different variations and to repair mistakes in the literature.

Firstly, observe that **FMatr** and $\mathbf{FMatr}_\bullet$ are finitely complete and cocomplete.

Remark 20. The functor $|-|\colon \mathbf{Matr} \to \mathbf{FinSet}$ restricts to an isomorphism of categories $\mathbf{FMatr} \to \mathbf{FinSet}$. It follows that **FMatr** has all finite limits and colimits.

Similarly, the category $\mathbf{FMatr}_\bullet$ is isomorphic to the category $\mathbf{FinSet}_\bullet$ of pointed finite sets and pointed functions, and so has all finite limits and colimits [3, 28.9.5].

Before continuing to the other matroid categories of interest, we offer a contextualising remark, which describes a way to construct limits and colimits from **FinSet**. The following remark was contributed by the anonymous reviewer of our relevant paper [33], who observed that the situation was similar to topology. Note that the latter part of our independent observation in Remark 20 also arises as a corollary of Remark 21.

Remark 21. In some ways, including the computation of limits and colimits, the category of matroids is analogous to the category of topological spaces and continuous functions.

Let $\mathbf{D}$ be a diagram in **Matr**. To construct its limit (if it exists) first take the limit L of $|\mathbf{D}|$ in **FinSet** and denote the limit cone by $\lambda_X\colon L \to |X|$ for $X \in \mathbf{D}$. Then the limit of $\mathbf{D}$ exists if and only if there is a coarsest matroid structure on L making the λ_X strong, that is, if and only if there is a coarsest matroid structure M on L such that $\{\lambda_X^{-1}(S) \mid X \in \mathbf{D}, S \in \mathcal{F}_X\} \subseteq \mathcal{F}_M$.

Similarly, for the colimit of $\mathbf{D}$, take the colimit K in **FinSet** and form a colimit cocone $\kappa_X : |X| \to K$ in **FinSet**. Then the colimit exists if and only if there is a finest matroid structure N on K such that $\mathcal{F}_N \subseteq \{S \subseteq K \mid \forall X \in \mathbf{D}, \kappa_X^{-1}(S) \in \mathcal{F}_X\}$.

In contrast to topological spaces, there is an obstacle to the existence of all finite limits and colimits of matroids: matroid flat structures on a set X are not closed under finite intersections in 2^{2^X} because of the partition property. Thus the category of "generalised matroids", with objects defined via closed subsets by removing the partition axiom and strong maps as morphisms, gives a finitely complete and cocomplete category containing **Matr**. The inclusion preserves coproducts and equalisers. As we shall see, products and coequalisers are not reflected.

Clearly the empty matroid is an initial object in all of the unpointed matroid categories we consider. The one-element matroid where the element is a loop is a terminal object in all categories of matroids we consider.

Proposition 22. *The categories* **SMatr**, **LMatr**, *and* **Matr** *have coproducts.*

Proof. Coproducts have been found by Brylawski [21] and Crapo [25, Proposition 4]; they are the same construction in all the unpointed matroid categories we examine. The coproduct $M + N$ has ground set $|M + N| = |M| \sqcup |N|$ and flats $\{F \cup G \mid F \in \mathcal{F}_M, G \in \mathcal{F}_N\}$. It is easy to see that if M and N are simple or loopless, then so is $M + N$. The coprojections are the inclusions $M \to M + N$ and $N \to M + N$, and it is easy to see that for strong maps $f : M \to P$ and $g : N \to P$ there is a unique strong map $[f, g] : M + N \to P$. □

Corollary 23. *The categories* $\mathbf{SMatr}_\bullet$, $\mathbf{LMatr}_\bullet$, *and* $\mathbf{Matr}_\bullet$ *have coproducts, defined in the same way as in the unpointed categories but additionally identifying the points of the constituent matroids.*

Proposition 24. *The categories* **SMatr**, **LMatr**, *and* **Matr** *have equalisers.*

Proof. Equalisers have been shown to exist in the pointed loopless category [5, Theorem 53] and are the same construction in all the matroid categories we examine. The equaliser of $f, g\colon M \to N$ is the inclusion of the matroid E with $|E| = \{x \in |M| \mid f(x) = g(x)\}$ and $\mathcal{F}_E = \{F \cap |E| \mid F \in \mathcal{F}_M\}$. This is a well-defined matroid and clearly satisfies the universal property. Finally, if M is simple or loopless, then so is E. $\square$

Corollary 25. *The categories* $\mathbf{SMatr}_\bullet$*,* $\mathbf{LMatr}_\bullet$*, and* $\mathbf{Matr}_\bullet$ *have equalisers, defined in the same way as in the unpointed categories.*

Proposition 26. *The categories* **SMatr***,* **LMatr***, and* **Matr** *do not have all products.*

Proof. This was shown by Crapo [25, Proposition 5]; we repeat his argument here. In any of **SMatr**, **LMatr** or **Matr**, consider the simple matroid M of rank 2 with $|M| = \{a, b, c, d\}$ and suppose $M \times M$ existed. For any pair $(x, y) \in |M| \times |M|$ we have two strong maps $x, y\colon 1 \to M$, and therefore a unique map $\langle x, y\rangle\colon 1 \to M \times M$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Hence $M \times M$ has at least 16 elements. Take any permutation $\sigma\colon |M| \to |M|$; this yields a strong map $\langle \sigma, \mathrm{id}_M\rangle : M \to M \times M$, whose image $\{(a, \sigma(a)), (b, \sigma(b)), (c, \sigma(c)), (d, \sigma(d))\}$ must be a flat. But $\{(a, a), (b, b), (c, c), (d, d)\}$, $\{(a, a), (b, b), (c, d), (d, c)\}$ cannot both be flats. $\square$

Corollary 27. *The categories* $\mathbf{SMatr}_\bullet$*,* $\mathbf{LMatr}_\bullet$*, and* $\mathbf{Matr}_\bullet$ *do not have all products.*

Proof. Consider the pointed matroid arising from the matroid M in the above proof by adding a point. The same argument holds. $\square$

It follows that these categories do not have pullbacks or exponentials either.

It was the anonymous reviewer of our relevant paper [33] who first pointed out the existence of pushouts under cofree matroids, though they did not provide a proof.

Proposition 28. *The categories* **SMatr***,* **LMatr***, and* **Matr** *do not have all pushouts, but pushouts under cofree matroids exist.*

Proof. For the first part, we transcribe the proof in Crapo's lecture notes [25, Proposition 7], who proves the lack of all pushouts in **SMatr**. In any of **SMatr**, **LMatr** or **Matr**, consider the following simple matroids M, M_1 and M_2, defined by their hyperplanes:

$$
\begin{aligned}
M: \quad & |M| = \{a,b,c,d,e,f\}, \\
& \mathcal{H}_M = \{\{i,j\} | i \neq j, i \in |M|, j \in |M|\} \\
M_1: \quad & |M_1| = \{a,b,c,d,e,f,g\}, \\
& \mathcal{H}_{M_1} = \{\{a,c\},\{a,d\},\{a,f\},\{b,c\},\{b,d\},\{b,f\},\{c,e\},\{c,f\},\{d,e\},\{d,f\}, \\
& \qquad \{e,f\},\{e,g\},\{f,g\},\{a,b,g\},\{c,d,g\}\} \\
M_2: \quad & |M_2| = \{a,b,c,d,e,f,h\}, \\
& \mathcal{H}_{M_2} = \{\{a,c\},\{a,d\},\{a,f\},\{b,c\},\{b,d\},\{b,f\},\{c,e\},\{c,f\},\{d,e\},\{d,f\}, \\
& \qquad \{a,b,h\},\{c,d,h\},\{e,f,h\}\}
\end{aligned}
$$

Suppose the pushout P of the embeddings $f_1\colon M \to M_1$ and $f_2\colon M \to M_2$ existed. Then the image of M in the colimit cone P' cannot be of rank 3, because then M_1 and M_2 would be embedded in P with overlap M. The closures of the sets $\{a,b\}$ and $\{d,e\}$ in P contain an element which is not in the closure of $\{e,f\}$ of M_1 but is in the closure of $\{e,f\}$ in M_2. Therefore the image of M in P has rank at most 1. But there exist different choices P', such that no rank-1 flat has a strong map to both choices P'; specifically, consider:

$$
\begin{aligned}
|P'_1| &= \{abgh, c, d, e, f\}, & \mathcal{H}_{P'_1} &= \{\{abgh\},\{c\},\{d\},\{e\},\{f\}\} \\
|P'_2| &= \{a,b,c,d,e,f,gh\}, & \mathcal{H}_{P'_2} &= \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{gh\}\}
\end{aligned}
$$

For the second part: Morphisms $f\colon C(X) \to M$ and $g\colon C(X) \to N$ in **Matr** correspond to $\hat{f}\colon X \to (M)_0$ and $\hat{g}\colon X \to (N)_0$ in **FinSet** by Theorem 19. These have a pushout Y in **FinSet**, and it follows that $C(Y)$ is the pushout of f and g in **Matr**. $\square$

Corollary 29. *The categories* $\mathbf{SMatr}_\bullet$*,* $\mathbf{LMatr}_\bullet$ *and* $\mathbf{Matr}_\bullet$ *do not have all pushouts.*

Proof. Consider the pointed simple matroid with the same flat structure as M in the above proof; the same argument holds. $\square$

Coequalisers are a bit messier, as we need more counterexamples. We start with the pointed categories.

Proposition 30. *The categories* $\mathbf{SMatr}_\bullet$*,* $\mathbf{LMatr}_\bullet$*, and* $\mathbf{Matr}_\bullet$ *do not have all coequalisers.*

Proof. Consider the following objects and morphisms:

$$\begin{aligned}
|M| &= |N| = \{\bullet, 1, 2, 3, 4\}, \\
\mathcal{H}_M &= \mathcal{H}_N = \big\{\{\bullet, i, j\} | i, j \in |M|, \bullet \neq i \neq j \neq \bullet\big\}, \\
f &= \mathrm{id}, \\
g &= \big(\bullet \mapsto \bullet, 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4\big).
\end{aligned}$$

If the coequaliser $c\colon N \to C$ of $f, g\colon M \to N$ existed, then [5, Theorem 54][4] its ground set must be (in bijection with) the quotient $|N|/\sim$ by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in |N|$, and c must map x to its equivalence class $[x]$. Explicitly, $|C| = \{\bullet, [12], [3], [4]\}$.

$$\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "c"] \arrow[dr, "c'"'] & C \arrow[d, dotted, "k"] \\
& & C'
\end{tikzcd}$$

[4]That proof is invalid, as the map $y \mapsto [y]$ need not be strong. However, it does hold for the choices in our proof below; that is, there are strong maps from N to options (2.1) and (2.3).

Each flat G of C must be of the form $[F] = \{[x] \mid x \in F\}$ for some flat F of N, because $G = c(c^{-1}(G)) = [c^{-1}(G)]$. In fact, each flat F of C must satisfy $F = [F]$ in N. Thus we do not allow the flats $\{\bullet, [12], [3]\}$ and $\{\bullet, [12], [4]\}$ in C. We are further constrained by the three axioms that the closed sets of a matroid must satisfy, so we can only choose $\mathcal{F}_C$ to be one of the following (for $\mathbf{SMatr}_\bullet$, consider only the first and third options):

$$\big\{\{\bullet\}, \{\bullet, [12], [3], [4]\}\big\}, \tag{2.1}$$

$$\big\{\{\bullet\}, \{\bullet, [12]\}, \{\bullet, [3], [4]\}, \{\bullet, [12], [3], [4]\}\big\}, \text{ or} \tag{2.2}$$

$$\big\{\{\bullet\}, \{\bullet, [12]\}, \{\bullet, [3]\}, \{\bullet, [4]\}, \{\bullet, [12], [3], [4]\}\big\}. \tag{2.3}$$

Option (2.1) fails when we set C' to have the same ground set as C with flats (2.3), because then the inverse image under k of any rank-1 flat is not a flat. Option (2.2) fails when we set $|C'| = \{[\bullet 4], [12], [3]\}$ and $\mathcal{F}_{C'} = \big\{\{[\bullet 4]\}, \{[\bullet 4], [12], [3]\}\big\}$, because then the inverse image under k of the rank-0 flat is not a flat. Option (2.3) fails when we set $|C'| = \{[\bullet 34], [12]\}$ and $\mathcal{F}_{C'} = \big\{\{[\bullet 34]\}, \{[\bullet 34], [12]\}\big\}$, because then the inverse image under k of the rank-0 flat is not a flat. □

Corollary 31. *The categories* **LMatr** *and* **Matr** *do not have all coequalisers.*

Proof. In the above proof, remove the point from every matroid and set involved; amend the last paragraph as follows:

Set C' to have the same ground set as C. Option (2.1) fails when we give C' flats (2.3), because then the inverse image under k of any rank-1 flat is not a flat; option (2.2) fails when we give C' flats (2.3), because then the inverse image under k of two of the rank-1 flats is not a flat; option (2.3) fails when we give C' flats (2.2), because then the inverse image under k of one of the rank-1 flats is not a flat. □

Lastly, the case of unpointed simple matroids requires a larger counterexample.

Proposition 32. *The category* **SMatr** *does not have all coequalisers.*

Proof. Consider the following objects and morphisms:

$$
\begin{aligned}
|M| &= |N| = \{1,2,3,4,5\},\\
\mathcal{H}_M &= \mathcal{H}_N = \big\{\{i,j,k\} | i,j,k \in |M|, k \neq i \neq j \neq k\big\},\\
f &= \mathrm{id},\\
g &= \big(1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 5\big).
\end{aligned}
$$

If the coequaliser $c\colon N \to C$ of $f, g\colon M \to N$ existed, then (by the same reasoning as above) its ground set must be (in bijection with) the quotient $|N|/\sim$ by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in |N|$, and c must map x to its equivalence class $[x]$. Explicitly, $|C| = \{[12],[3],[4],[5]\}$. Because C is simple, its rank-1 flats are $\{[12]\}, \{[3]\}, \{[4]\}, \{[5]\}$.

$$
\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "c"] \arrow[dr, "c'"'] & C \arrow[d, dotted, "k"] \\
 & & C'
\end{tikzcd}
$$

Each flat F of C must satisfy $F = [F]$ in N, therefore we do not allow flats $\{[12],[3],[4]\}$, $\{[12],[3],[5]\}$ and $\{[12],[4],[5]\}$ in C. We are further constrained by the three axioms that the closed sets of a matroid must satisfy, so we can only choose $\mathcal{F}_C$ to be one of the following:

$$\big\{\emptyset, \{[12]\}, \{[3]\}, \{[4]\}, \{[5]\}, \{[12],[3],[4],[5]\}\big\}, \tag{2.4}$$

$$\big\{\emptyset, \{[12]\}, \{[3]\}, \{[4]\}, \{[5]\}, \{[12],[3]\}, \{[12],[4]\}, \{[12],[5]\}, \{[3],[4]\}, \{[3],[5]\}, \{[4],[5]\}, \{[12],[3],[4],[5]\}\big\}, \text{ or} \tag{2.5}$$

$$\big\{\emptyset, \{[12]\}, \{[3]\}, \{[4]\}, \{[5]\}, \{[3],[4],[5]\}, \{[12],[3]\}, \{[12],[4]\}, \{[12],[5]\}, \{[12],[3],[4],[5]\}\big\}. \tag{2.6}$$

Set $|C'| = |C|$. Option (2.4) fails when we give C' flats (2.5), because then the inverse image under k of any rank-1 flat is not a flat. Option (2.5) fails when we give C' flats (2.6), because then there is one rank-1 flat whose inverse image under k is not a flat. Option (2.6) fails when we give C' flats (2.5), because then there are three rank-1 flats whose inverse image under k is not a flat. $\square$

So far, we have reasoned directly about pointed categories by applying the same (perhaps slightly amended) proof as their unpointed counterparts. This is a simple method, but there exists in fact a more elegant line of reasoning, again contributed by the anonymous reviewer of our relevant paper [33]:

Remark 33. A loop in a matroid M is (the image of) a strong morphism $C(1) \to M$ (where 1 denotes the singleton set). Hence $\mathbf{Matr}_\bullet$ is (isomorphic to) the coslice category $C(1)/\mathbf{Matr}$. Since $\mathbf{Matr}$ has coproducts by Proposition 22, the forgetful functor $\mathbf{Matr}_\bullet \to \mathbf{Matr}$ is monadic and its left adjoint $(-)_\bullet$ sends each object $M \in \mathbf{Matr}$ to the coproduct inclusion $C(1) \to M + C(1)$.[5] Moreover, this monad preserves connected colimits. Thus the forgetful functor $\mathbf{Matr}_\bullet \to \mathbf{Matr}$ creates limits and connected colimits. In particular it preserves and reflects monomorphisms and epimorphisms.

The above observations seem to refer to well-known facts, but some of the corresponding proofs could not be located in the literature. We therefore append a proof for the sake of completeness.

We first need an auxiliary definition and an auxiliary lemma.

Definition 34. A *fibred coproduct* is a colimit of a nonempty, possibly infinite, family of morphisms with the same domain. A *fibred product* is a limit of a nonempty, possibly infinite, family of morphisms with the same codomain.

[5]We had independently found this left adjoint, but not expressed in terms of the coslice category. That proof is straightforward and omitted.

Lemma 35. *A category* $\mathbf{C}$ *has every connected colimit if* $\mathbf{C}$ *has the two following colimits:*

1. *coequalisers,*
2. *fibred coproducts.*

Proof. A result by Paré [63] states that a category $\mathbf{C}$ has all connected limits if it has fibred products and is "simply connected", meaning that the category $\mathbf{C}'$ resulting from adding inverses to every morphism in $\mathbf{C}$ is a preorder. Dualising this result we conclude that a category has all connected colimits if it has fibred coproducts and is simply connected. We need only show that if $\mathbf{C}$ has coequalisers then $\mathbf{C}'$ is a preorder; for this, we need the observation that there exists a functor $\pi_1 : \mathbf{C} \to \mathbf{C}'$ sending every morphism to its invertible version, as composition and identity are clearly preserved[6]. The result then follows: taking $f, g : A \to B$ in $\mathbf{C}$ with coequaliser c, we get $\pi_1(f) = \pi_1(c)^{-1} \circ \pi_1(c) \circ \pi_1(f) = \pi_1(c)^{-1} \circ \pi_1(c \circ f) = \pi_1(c)^{-1} \circ \pi_1(c \circ g) = \pi_1(c)^{-1} \circ \pi_1(c) \circ \pi_1(g) = \pi_1(g)$; taking $f : A \to B$ and $g : B \to A$ in $\mathbf{C}$, let c_1 be the coequaliser of $f \circ g$ and id_B and let c_2 be the coequaliser of $g \circ f$ and id_A, getting $\pi_1(f) \circ \pi_1(g) = \pi_1(f \circ g) = \pi_1(c_1)^{-1} \circ \pi_1(c_1) \circ \pi_1(f \circ g) = \pi_1(c_1)^{-1} \circ \pi_1(c_1 \circ f \circ g) = \pi_1(c_1)^{-1} \circ \pi_1(c_1) = \mathrm{id}_{\pi_1(B)}$ and $\pi_1(g) \circ \pi_1(f) = \pi_1(g \circ f) = \pi_1(c_2)^{-1} \circ \pi_1(c_2) \circ \pi_1(g \circ f) = \pi_1(c_2)^{-1} \circ \pi_1(c_2 \circ g \circ f) = \pi_1(c_2)^{-1} \circ \pi_1(c_2) = \mathrm{id}_{\pi_1(A)}$, therefore $\pi_1(f) = \pi_1(g)^{-1}$. So in the end $\mathbf{C}'$ is a preorder. $\square$

Now we give the proof.

Lemma 36. *Let* $\mathbf{C}$ *be a category with coproducts and* $C \in \mathrm{Ob}(\mathbf{C})$*. Then the forgetful functor* $U : C/\mathbf{C} \to \mathbf{C}$ *has a left adjoint* $F : \mathbf{C} \to C/\mathbf{C}$ *that sends each object* D *to the coproduct inclusion* $C \to C + D$*. Furthermore, the adjunction is monadic and the*

[6] This is a well-known functor, as is evident from Paré's paper [63], but details need not concern as here.

monad preserves connected colimits. Then U creates limits and connected colimits, and in particular it preserves and reflects monomorphisms and epimorphisms.

Proof. **For $F \dashv U$:** Explicitly, the forgetful functor U sends each object $f : C \to D$ of $C/\mathbf{C}$ to target(f) and each morphism of $C/\mathbf{C}$ to itself, and the functor F sends each object D of $\mathbf{C}$ to $p_1(C+D)$ and each morphism $f : A \to B$ of $\mathbf{C}$ to $\mathrm{id}_C + f$. We note that these are both well-defined functors. Then $U(F(D)) = C + D$ and $U(F(f)) = \mathrm{id}_C + f$. We verify the existence of a universal morphism η_A, namely the coproduct inclusion $A \to C + A$. For every $A \in \mathrm{Ob}(\mathbf{C})$ and any maps $g : C \to B$ and $f : A \to B$ in $\mathbf{C}$, consider the following diagrams in $\mathbf{C}$:

$$\begin{tikzcd}
A \arrow[r, "\eta_A = p_2(C+A)"] \arrow[dr, "f"'] & U(F(A)) \arrow[r, leftrightarrow, "\mathrm{id}"] & C + A \arrow[d, "\hat{f}"] \\
 & U(g) \arrow[r, leftrightarrow, "\mathrm{id}"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "F(A) = p_1(C+A)"] \arrow[dr, "g"'] & C + A \arrow[d, "\hat{f}"] \\
 & B
\end{tikzcd}$$

The right-hand commutative diagram is a morphism $\hat{f} : FA \to g$ in $C/\mathbf{C}$. By the universal property of the coproduct, there is a one-to-one correspondence between morphisms $\hat{f}$, morphisms f, commutative diagrams of the left-hand form and commutative diagrams of the right-hand form (in fact, $\hat{f} = [g, f]$ by definition of the coproduct). This establishes the result.

Adjunction is monadic: Observe that the counit of the triple $(U \circ F, \eta, \mu)$ is defined by $\mu_A = [\mathrm{id}_C, \mathrm{id}_C] + \mathrm{id}_A$. Then an algebra (A, a) of the monad corresponds to a morphism $f : C \to A$, setting $a = [f, \mathrm{id}_A]$, and $\mathrm{Hom}((A, a), (B, b)) = \mathrm{Hom}(A, B)$. So the Eilenberg-Moore category of $U \circ F$ is isomorphic to $C/\mathbf{C}$, making the adjunction monadic.

$U \circ F$ preserves connected colimits: A proof of the dual proposition can be found online[7]. Here we essentially dualise that proof.[8]

Observe that the monad $U \circ F$ is precisely the functor $C + \_$. We show that $C + \_$ preserves connected colimits. By Lemma 35, we need only show that $C + \_$ preserves coequalisers and fibred coproducts. As every colimit functor, the coproduct functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is left adjoint to a constant diagram functor, hence preserves arbitrary colimits. So we have

$$C + \mathrm{colim}(c \overset{g_i}{\to} c_i) = \mathrm{colim}(C \overset{f_i = \mathrm{id}_C}{\to} C_i = C) + \mathrm{colim}(c \overset{g_i}{\to} c_i) =$$
$$\mathrm{colim}(C + c \overset{\mathrm{id}_C + g_i}{\to} C + c_i),$$

so $C + \_$ preserves fibred coproducts. Similarly

$$C + \mathrm{colim}(c \overset{f||g}{\to} c') = \mathrm{colim}(C \overset{\mathrm{id}_C || \mathrm{id}_C}{\to} C' = C) + \mathrm{colim}(c \overset{f||g}{\to} c') =$$
$$\mathrm{colim}(C + c \overset{(\mathrm{id}_C + f)||(\mathrm{id}_C + g)}{\to} C + c'),$$

so $C + \_$ preserves coequalisers. Hence the proposition is proven.

U **creates limits and connected colimits:** Since U is monadic, it creates limits [3, Proposition 20.12]. Monadic functors also create any colimits preserved by the monad [3, Proof of Proposition 20.16][9], thus U creates connected colimits.

U **preserves and reflects monomorphisms and epimorphisms:** A morphism

[7] See Theorem 3.2 on `https://ncatlab.org/nlab/show/connected+limit` (as it was in 2017).

[8] The theorem we cite requires the category to be complete (which in our case would mean cocomplete, as we prove the dual statement), but that is only because they prove something stronger. Here we only use coproducts, which exist by assumption.

[9] The proposition cited makes a weaker claim, but its proof also proves the claim we are making.

$f : X \to Y$ is an epimorphism precisely when the following diagram is a pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd}$$

A morphism $f : X \to Y$ is a monomorphism precisely when the following diagram is a pullback:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Since limits and connected colimits are created by U, it follows that monomorphisms and epimorphisms are preserved and reflected. $\square$

Corollaries 27 and 29 follow from Remark 33, as right adjoints preserve limits. Corollary 23 follows from Remark 33 and Proposition 28, noting that the coproduct of $M_\bullet$ and $N_\bullet$ in $\mathbf{Matr}_\bullet$ is the pushout in $C(1)/\mathbf{Matr}$ of $M_\bullet : C(1) \to M + C(1)$ and $N_\bullet : C(1) \to N + C(1)$.

Lemma 37. *A morphism of* **Matr** *is monic if and only if it is injective, and epic if and only if it is surjective. The same holds for* $\mathbf{Matr}_\bullet$.

Proof. A direct proof was given by Crapo [25] $\square$

Proof. (Contributed by the anonymous reviewer of our relevant paper [33])

The functor $|-| : \mathbf{Matr} \to \mathbf{FinSet}$ reflects pullback and pushout diagrams of the form:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "\mathrm{id}_Y"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

It follows that $|-|$ preserves and reflects monomorphisms and epimorphisms. $\square$

To add a bit of detail to the reviewer's proof: The preservation property follows from the fact that $|-|$ has both a left and right adjoint, hence preserves limits and colimits. The reflection property follows from faithfulness.

We end this section with a related lemma.

Lemma 38. *A morphism of* $\mathbf{Matr}$ *is an isomorphism if and only if it is bijective and the direct image of any flat is a flat.*

Proof. If $f: M \to N$ is an isomorphism, it is bijective by Lemma 37. The direct image under f of a flat F equals the inverse image under f^{-1} of F, and therefore is a flat in N because f^{-1} is strong. The latter argument also establishes the converse. $\square$

2.4 Adjunctions between subcategories of matroids

We have seen various classes of matroids: all matroids, simple matroids, free matroids, and loopless matroids. We now study free and cofree constructions translating between these classes. Theorem 19 already showed that free and cofree matroids over sets exist, and are precisely what we have been calling free and cofree matroids. We go on to consider whether the inclusions

$$\mathbf{FMatr} \hookrightarrow \mathbf{SMatr} \hookrightarrow \mathbf{LMatr} \hookrightarrow \mathbf{Matr}$$

and

$$\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{SMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$$

have adjoints. Before embarking on this task, we note the following.

Lemma 39. *The forgetful functor* $\mathbf{Matr}_\bullet \to \mathbf{Matr}$ *has no right adjoint.*

Proof. Let $R : \mathbf{Matr} \to \mathbf{Matr}_\bullet$ be any functor. Consider the empty matroid O with $|O| = \emptyset$. For any pointed matroid N, there are no functions $N \to O$. However, there does exist a pointed strong map $N \to R(O)$, namely the map that sends every element of N to $\bullet$. Therefore, R cannot be a right adjoint to the forgetful functor. ☐

Remark 40. It can be seen that the categories $\mathbf{Matr}_\bullet$ and $\mathbf{Matr}$ are not equivalent by noting that only $\mathbf{Matr}_\bullet$ has a zero object: the one-element matroid is both initial and terminal. This matroid is a terminal object in $\mathbf{Matr}$, but not initial.

We shall first focus on pointed matroids. As we have shown in Section 2.3, $\mathbf{Matr}_\bullet$ does not have many colimits apart from coproducts. Because of this, we cannot invoke the adjoint functor theorem. We will reason concretely below, to avoid and repair mistakes in the literature.

2.4.1 Free pointed matroids

We start with right adjoints of functors out of the category of free pointed matroids.

Theorem 41. *The category* $\mathbf{FMatr}_\bullet$ *is a coreflective subcategory of* $\mathbf{LMatr}_\bullet$*; that is to say, the inclusion* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet$ *has a right adjoint* F *defined by*

$$|F(M)| = |M|, \qquad \mathcal{F}_{F(M)} = 2^{|M|} \setminus 2^{|M| \setminus \{\bullet\}}, \qquad F(f) = f.$$

Proof. This has been proven in Al Hawary's thesis [5, Theorem 86]; the proof can also be found in two of his papers [7, 6]. ☐

It is straightforward to check that the above functor F extends the right adjoint to the inclusion $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{SMatr}_\bullet$ and extends to the right adjoint of the inclusion $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$. We now examine whether the functor F itself has a right adjoint for each of those three cases.

Proposition 42. *The functors* $F\colon \mathbf{SMatr_\bullet} \to \mathbf{FMatr_\bullet}$ *and* $F\colon \mathbf{LMatr_\bullet} \to \mathbf{FMatr_\bullet}$ *have no right adjoints.*[10]

Proof. Suppose V were a right adjoint; then $\#\mathrm{Hom}(F(N), N) = \#\mathrm{Hom}(M, V(N))$ for every loopless/simple matroid M and every free matroid N. We find a simple counterexample by counting morphisms in homsets.

Let N be the matroid with flats $\big\{\{\bullet\}, \{\bullet, e\}\big\}$, and let M be the matroid with flats $\big\{\{\bullet\}, \{\bullet, a\}, \{\bullet, b\}, \{\bullet, c\}, \{\bullet, a, b, c\}\big\}$. There are $\#|N|^{\#|M|-1} = 8$ morphisms $F(M) \to N$. Now let us count the morphisms $M \to V(N)$. For $\mathbf{SMatr_\bullet}$:

- If $\#V(N) = 2$, $V(N)$ necessarily has flats $\big\{\{\bullet\}, \{\bullet, e\}\big\}$. Noting that the strong property excludes maps that send precisely 2 nonloop elements to the loop, we arrive at 5 morphisms.

- If $\#V(N) = 3$, $V(N)$ necessarily has flats $\big\{\{\bullet\}, \{\bullet, e_1\}, \{\bullet, e_2\}, \{\bullet, e_1, e_2\}\big\}$. Of the 27 possible maps between the underlying pointed sets, it is easy to see that we must exclude the 6 that send precisely 2 nonloop elements to the loop and the 10 whose restriction to nonloop elements is surjective, arriving at 11 morphisms.

- If $\#V(N) > 3$, there are at least 11 morphisms.

For $\mathbf{LMatr_\bullet}$ we have to consider one extra case: $\mathcal{F}_{V(N)} = \big\{\{\bullet\}, \{\bullet, e_1, e_2\}\big\}$. In this case we need only exclude the 6 maps that send precisely 2 nonloop elements to the loop, resulting in 21 morphisms $M \to V(N)$. $\square$

Theorem 43. *The functor* $F\colon \mathbf{Matr_\bullet} \to \mathbf{FMatr_\bullet}$ *has adjoints* $F \dashv V \dashv H$ *given by*

$$\mathcal{F}_{V(M)} = \{|M|\}, \qquad V(f) = f,$$

$$H(M) = F(M_0), \qquad H(f) = f|_{|M_0|}.$$

[10]The purported right adjoint in Al Hawary's thesis [5, Theorem 126] fails for every function mapping $1 < n < \#M$ elements to $\bullet$.

where M_0 is the matroid with only one flat, which is the rank-0 flat of M. The functor H has no right adjoint. The inclusions $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{SMatr}_\bullet$*,* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet$*, and* $\mathbf{FMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *have no left adjoint.*

Proof. The proof is almost identical to that of Theorem 19; the nonexistence of the left adjoint to the inclusions requires the following adjustment:

Suppose K were left adjoint to F. Let M be the matroid with flats

$$\big\{\{\bullet\}, \{\bullet, a\}, \{\bullet, b\}, \{\bullet, c\}, \{\bullet, d\},$$
$$\{\bullet, a, b\}, \{\bullet, a, c\}, \{\bullet, a, d\}, \{\bullet, b, c\}, \{\bullet, b, d\}, \{\bullet, c, d\}, \{\bullet, a, b, c, d\}\big\}.$$

We will count the morphisms $K(M) \to X$ and $M \to F(X)$. There are $\#X^{\#K(M)-1}$ morphisms $K(M) \to X$. The morphisms $M \to F(X)$ are those that map the elements as in the unpointed case (see Theorem 19) plus those maps that map exactly one nonloop element to the loop and everything else to the same nonloop element; there are $4(\#X - 1)$ maps of the latter sort. In total there are $3\#X^2 - 6\#X + 1$ maps $M \to F(X)$. So $K(M)$ must satisfy $\#X^{\#K(M)-1} = 3\#X^2 - 6\#X + 1$, which cannot be true for every choice of X. Thus F has no left adjoint.

□

2.4.2 Simple pointed matroids

Next, we turn to the inclusion of simple matroids into larger categories.

Proposition 44. *The category* $\mathbf{SMatr}_\bullet$ *is a reflective subcategory of* $\mathbf{Matr}_\bullet$*: the inclusion* $\mathbf{SMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has a left adjoint* $\mathrm{si}_\bullet$*. The functor* $\mathrm{si}_\bullet$ *has no left adjoint.*

Proof. The first statement follows from Theorem 80, which states that $\mathbf{SMatr}_\bullet$ is (isomorphic to) the Eilenberg-Moore category of a monad $\mathrm{si}_\bullet$ on $\mathbf{Matr}_\bullet$; here, the inclusion is the canonical forgetful functor sending an algebra (A, a) to A, hence has a left adjoint sending A to $(\mathrm{si}_\bullet(A), \mu A)$, where μ is the algebra multiplication [16]. We shall examine this monad in more detail in subsection 2.7.1; for now, it suffices to note that the nonloop elements of $\mathrm{si}_\bullet(M)$ are the rank-1 flats of M.

For the second statement, suppose $K \dashv \mathrm{si}_\bullet$. Take M to be the pointed matroid with two parallel elements a, b and the loop $\bullet$; then $\mathcal{F}_{\mathrm{si}_\bullet(M)} = \big\{\{\bullet\}, \{\bullet, F_1\}\big\}$, where we write F_1 for the rank-1 flat of M. Let ϵ_M denote the couniversal morphism of M. Take S to be the pointed simple matroid with elements e and $\bullet$ and consider the map $f : S \to \mathrm{si}_\bullet(M)$ that maps e to F_1. Its transpose $\hat{f}$ must map some element e' of $K(S)$ to either a or b. But if $e' \mapsto a$ satisfies $f = \epsilon_M \circ \mathrm{si}_\bullet(\hat{f})$ then so does $e' \mapsto b$, and vice versa. Therefore, $\hat{f}$ cannot be unique. We conclude that the couniversal morphism cannot exist, hence neither does the adjoint. $\square$

Proposition 45. *The inclusion* $\mathbf{SMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has no right adjoint.*

Proof. If $R\colon \mathbf{Matr}_\bullet \to \mathbf{SMatr}_\bullet$ were a right adjoint, there would be a natural isomorphism $F \cong F \circ R\colon \mathbf{Matr}_\bullet \to \mathbf{FMatr}_\bullet$, whence R must (1) preserve cardinality and (2) reflect surjectivity. Now let S be the simple pointed matroid with $\mathcal{F}_S = \big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2, 3\}\big\}$ and let M be the pointed matroid with $\mathcal{F}_M = \big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2, 3\}, \{\bullet, 1, 2, 3\}\big\}$. By property (1), $R(M)$ has 4 elements. Without loss of generality, we fix $|R(M)| = |M|$, which leaves only two possible simple pointed matroids:

$$\mathcal{F}_{R(M)} = \big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2\}, \{\bullet, 2, 3\}, \{\bullet, 1, 3\}, \{\bullet, 1, 2, 3\}\big\} := \mathcal{F}_1, \text{ or}$$
$$\mathcal{F}_{R(M)} = \big\{\{\bullet\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 3\}, \{\bullet, 1, 2, 3\}\big\} = \mathcal{F}_S.$$

We can immediately reject the case $\mathcal{F}_{R(M)} = \mathcal{F}_1$, as then clearly $\#\mathrm{Hom}(S, R(M)) < \#\mathrm{Hom}(S, M)$ (because of the added flats to $R(M)$ which some maps will no longer reflect). Suppose then that $R(M) = S$. The transpose $\hat{f}$ of any surjective map f must be surjective by property (2); yet there are no surjective strong maps $S \to M$, whereas $\mathrm{id}_S : S \to R(M)$ is surjective. Thus this case is also rejected. We conclude that the adjoint does not exist. $\square$

Propositions 44 and 45 straightforwardly also apply to the inclusion $\mathbf{SMatr}_\bullet \hookrightarrow \mathbf{LMatr}_\bullet$.

2.4.3 Loopless pointed matroids

All that remains to consider is the inclusion $\mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$.

Theorem 46. *The category* $\mathbf{LMatr}_\bullet$ *is a reflective subcategory of* $\mathbf{Matr}_\bullet$*: the inclusion* $\mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has a left adjoint J that deletes every loop except $\bullet$ from objects and acts on morphisms $f\colon M \to N$ as*

$$J(f)(e) = \begin{cases} f(e) & \textit{if } f(e) \in J(N) \\ \bullet & \textit{if } f(e) \notin J(N) \end{cases}$$

The functor $J\colon \mathbf{Matr}_\bullet \to \mathbf{LMatr}_\bullet$ *has no left adjoint.*

Proof. For the universal morphism $\eta_M\colon M \to J(M)$ we may take the strong map that sends every loop to $\bullet$ and every nonloop element to itself. Then morphisms $f\colon M \to N$ correspond bijectively to $\hat{f} = J(f)$ satisfying $\hat{f} \circ \eta_M = f$.

Suppose $G \dashv J$. It is easy to see that G cannot be the constant functor $\bullet$, so we may pick $K \in \mathrm{Ob}(\mathbf{LMatr}_\bullet)$ with $\bullet \neq e \in |G(K)|$. Let M be the matroid with loops $*$ and $\bullet$. Then any function $G(K) \to M$ (of which there exist at least two, interchanging the mappings $e \mapsto *$ and $e \mapsto \bullet$) is strong. However, there exists only one strong map $f\colon K \to J(M)$, namely the one that maps every element of K to $\bullet$. So there is no bijection between the two homsets, which means that G could not have been the left adjoint to J. $\square$

Theorem 47. *The inclusion* $\mathbf{LMatr}_\bullet \hookrightarrow \mathbf{Matr}_\bullet$ *has no right adjoint.*

Proof. If N were a right adjoint, there would be a natural isomorphism $F \cong F \circ N \colon \mathbf{Matr}_\bullet \to \mathbf{FMatr}_\bullet$, whence N must (1) preserve cardinality and (2) reflect surjectivity. Let D be the matroid with flats $\{\{\bullet\}, \{\bullet, c\}, \{\bullet, a, b\}, \{\bullet, a, b, c\}\}$, and M the matroid with flats $\{\{\bullet, *\}, \{\bullet, *, e\}\}$. Now $\#|N(M)| = 3$ by property (1); without loss of generality we may set $|N(M)| = |M|$, leaving two possible choices for $\mathcal{F}_{N(M)}$. The first choice is $\big\{\{\bullet\}, \{\bullet, *\}, \{\bullet, e\}, \{\bullet, *, e\}\big\}$; this is rejected because there are 9 strong maps $D \to N(M)$ but 15 strong maps $D \to M$, so the two homsets are not isomorphic. The second choice is $\big\{\{\bullet\}, \{\bullet, *, e\}\big\}$; by property (2), epimorphisms $D \to N(M)$ correspond to epimorphisms $D \to M$, but there are 8 of the former and 4 of the latter. In the end, either choice results in a contradiction, so the adjoint does not exist. $\square$

The following theorem summarises all adjunctions in the pointed case.

Theorem 48. *The inclusions have the following adjunctions:*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{Matr}_\bullet \arrow[r, shift left] \arrow[d, shift right] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] \arrow[dr, shift left] \arrow[dr, shift right=2] & \mathbf{LMatr}_\bullet \arrow[l, hook', shift left] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \arrow[dl, shift left] \\
\mathbf{SMatr}_\bullet \arrow[u, hook, shift right] \arrow[r, shift right] \arrow[r, phantom, "\perp"] \arrow[ur, hook, shift left] & \mathbf{FMatr}_\bullet \arrow[l, hook', shift right] \arrow[u, hook, shift left] \arrow[ul, hook', shift left] \arrow[ul, shift right=2]
\end{tikzcd}
$$

The functors in the above diagram have no adjoints other than those indicated.

Proof. Collate the previous results in this section. $\square$

2.4.4 Unpointed categories

As we have seen in Remark 40, the categories $\mathbf{Matr}_\bullet$ and $\mathbf{Matr}$ are not equivalent, so our results for the pointed categories do not translate into the unpointed versions. We now reason directly for the (non)existence of adjoints between unpointed matroid categories.

Proposition 49. *The category* $\mathbf{FMatr}$ *is a coreflective subcategory of* $\mathbf{SMatr}$*; that is to say, the inclusion* $\mathbf{FMatr} \hookrightarrow \mathbf{SMatr}$ *has a right adjoint* F *defined by:*

$$|F(M)| = |M|, \qquad \mathcal{F}_{F(M)} = 2^{|M|}, \qquad F(f) = f.$$

It extends to right adjoints of the inclusions $\mathbf{FMatr} \hookrightarrow \mathbf{LMatr}$ *and* $\mathbf{FMatr} \hookrightarrow \mathbf{Matr}$*.*

Proof. Setting $\hat{f} = f$, one can easily check that the map $\eta_M : M \to F(M)$ defined as the identity on elements is a universal arrow. $\square$

The functor $F : \mathbf{Matr} \to \mathbf{FMatr}$ has a right adjoint $V : \mathbf{FMatr} \to \mathbf{Matr}$, which in turn has a right adjoint $H : \mathbf{Matr} \to \mathbf{FMatr}$, both defined as in Theorem 43, and H has no right adjoint; the proof is identical to Theorem 19.

Proposition 50. *The functor* $F : \mathbf{SMatr} \to \mathbf{FMatr}$ *has no right adjoint.*

Proof. Suppose G were a right adjoint. Let M be the free matroid on 2 elements, and let D be the matroid with flats $\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\big\}$. There are $2^3 = 8$ strong maps $F(D) \to M$.

If $G(M)$ contains D as a submatroid, then there are at least 9 maps $D \to G(M)$, because there are 9 maps $D \to D$; since adjunction requires the homsets $\mathrm{Hom}(F(D), M)$

and $\text{Hom}(D, G(M))$ to be isomorphic, this case is rejected. Assume then that $G(M)$ does not contain D as a submatroid, so that we cannot map each of the elements of D to a separate flat of $G(M)$. Since pairs are not flats in D, for the map f to be strong, all elements of D must map to the same flat of $G(M)$; since there are no loops or parallel elements, it follows that every element of D must map to the same element of $G(M)$. Therefore there are $\#G(M)$ maps $D \to G(M)$, hence for the two homsets to be isomorphic we must then have $\#G(M) = 8$.

Now take D to be the simple matroid with $|D| = \{e\}$. There are 2 maps $F(D) \to M$, but 8 maps $D \to G(M)$, which again contradicts the hypothesis that these two homsets are isomorphic. Thus G could not have been right adjoint to F. □

Proposition 51. *The functor $F\colon$ **LMatr** $\to$ **FMatr** has a right adjoint U given by:*

$$|U(M)| = |M|, \qquad \mathcal{F}_{U(M)} = \{\emptyset, |M|\}, \qquad U(f) = f.$$

The functor U has no right adjoint.

Proof. For the former statement, take the universal morphism to be the morphism acting as the identity on elements and $\hat{f} = f$ to establish $F \dashv U$; this is similar to the adjunction $|-| \dashv C$ established in Theorem 19, with parallel elements here replacing loops as "interchangeable target elements".

For the latter statement, suppose $U \dashv G$. Take D to be the free matroid on 2 elements, so $U(D)$ is the matroid with 2 parallel elements. If we take $M = D$ we have exactly 2 morphisms $U(D) \to M$, whereas there are $\#|G(M)|^2$ morphisms $D \to G(M)$. Therefore the homsets $\text{Hom}(U(D), M)$ and $\text{Hom}(D, G(M))$ are not isomorphic, hence G could not have been right adjoint to U. □

None of the inclusions **FMatr** $\hookrightarrow$ **SMatr**, **FMatr** $\hookrightarrow$ **LMatr**, and **FMatr** $\hookrightarrow$ **Matr** have a left adjoint; the proof is identical to Theorem 19. Furthermore, the inclusions **SMatr** $\hookrightarrow$ **Matr** and **LMatr** $\hookrightarrow$ **Matr** have no left adjoint K: if M has at least one loop then there are no morphisms $M \to K(M)$ at all. The category **SMatr** is a reflective subcategory of **LMatr**, as the inclusion has a left adjoint si, and furthermore the functor si has no left adjoint; existence of si and nonexistence of a further left adjoint can be proven as in Proposition 44, noting that si can be defined on loopless matroids identically to $\mathrm{si}_\bullet$. The inclusions **SMatr** $\hookrightarrow$ **Matr** and **SMatr** $\hookrightarrow$ **LMatr** have no right adjoint; the proof is identical to Proposition 45.

Theorem 52. *The inclusion* **LMatr** $\hookrightarrow$ **Matr** *has no right adjoint.*

Proof. The proof is similar to Theorem 47, except for the fact that we have to consider more cases for $N(M)$. Concretely:

If N were a right adjoint, there would be a natural isomorphism $F \cong F \circ N \colon \mathbf{Matr} \to \mathbf{FMatr}$, whence N must preserve cardinality; call this property (1) as in Theorem 47. Let D be the matroid with flats $\{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\}$, and M the matroid with flats $\{\{a,b\}, \{a,b,c\}\}$. Now $\#|N(D)| = \#|N(M)| = 3$ by property (1); without loss of generality we may set $|N(M)| = |M|$, leaving four possible choices for $\mathcal{F}_{N(M)}$:

	$\mathcal{F}_{N(M)}$	$\#\mathrm{Hom}(D, N(M))$
$\mathcal{F}_1 :=$	$\big\{\emptyset, \{a,b,c\}\big\}$	27
$\mathcal{F}_2 :=$	$\big\{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\big\}$	15
$\mathcal{F}_3 :=$	$\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b,c\}\big\}$	9
$\mathcal{F}_4 :=$	$\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\big\}$	9

By hypothesis the homsets $\mathrm{Hom}(D, N(M))$ and $\mathrm{Hom}(D, M)$ are isomorphic; since there are 15 strong maps $D \to M$, it follows that $\mathcal{F}_{N(M)} = \mathcal{F}_2$.

Now pick instead D with flats $\big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b,c\}\big\}$. Then there are 9 strong maps $D \to N(M)$ but 15 strong maps $D \to M$, making the homsets nonisomorphic. Therefore, N could not have been a right adjoint. □

The following theorem summarises all adjunctions in the unpointed case.

Theorem 53. *The inclusions have the following adjunctions:*

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbf{Matr} \arrow[dr, shift left=1] \arrow[dr, shift right=3] & \mathbf{LMatr} \arrow[l, hook'] \arrow[dl, shift right, "\perp"'] \arrow[d, "\dashv"] \\
\mathbf{SMatr} \arrow[u, hook] \arrow[ur, hook, shift right=2] \arrow[r, shift right, "\perp"'] & \mathbf{FMatr} \arrow[ul, hook, shift left=3, "\perp"'] \arrow[ul, shift right=1, "\perp"'] \arrow[l, hook', shift right] \arrow[u, hook, shift left=3] \arrow[u, shift right=3, "\dashv"']
\end{tikzcd}
$$

The functors in the above diagram have no adjoints other than those indicated.

Proof. Collate the previous results in this subsection. □

2.5 Deletion and contraction

Let us recall some standard terminology from matroid theory.

Definition 54. Let M be a matroid. The *deletion* of $Y \subseteq |M|$ from M is the matroid $M \setminus Y$ with ground set $|M| \setminus Y$ and rank function $X \mapsto \mathrm{rk}_M(X)$. The resulting matroid is said to be *embedded* in M, and the strong map $M \setminus Y \to M$ is called an *embedding*. The *contraction* of M by $Z \subseteq |M|$ is the matroid M/Z with ground set $|M| \setminus Z$ and rank function $X \mapsto \mathrm{rk}_M(X \cup Z) - \mathrm{rk}_M(Z)$. A *minor* of M is a the matroid resulting from a sequence of deletions and contractions of M.

Recall the definitions of categorical subobject and categorical quotient:

Definition 55. Consider a category $\mathbf{A}$ and an object $A \in \mathbf{A}$. Two monomorphisms $f : R \to A$ and $g : S \to A$ are equivalent when there exists an isomorphism $\tau : R \to S$

such that $g \circ \tau = f$. An equivalence class of monomorphisms with codomain A is called a *subobject* of A.

Dually, two epimorphisms $f : A \to R$ and $g : A \to S$ are equivalent when there exists an isomorphism $\tau : R \to S$ such that $\tau \circ g = f$. An equivalence class of epimorphisms with domain A is called a *quotient* of A.

In terms of their domains, subobjects of N correspond to the matroids from which there exists an injective strong map into N.[11] The case of quotients can be confusing, as the term "quotient" is already used in matroid theory.

Definition 56. A matroid Q is a *(matroid) quotient* of M if there exist a matroid N and some $X \subseteq |N|$ so that $M = N \setminus X$ and $Q = N/X$.

Hence by definition, (matroid) quotients are strong maps that are composed of a contraction after an embedding. The rest of this section proves that quotients are precisely the bijective strong maps, from which it follows by Lemma 37 that matroid quotients are not the categorical quotients in the category of matroids. This also leads us to a characterisation of subobjects; these are embeddings followed by matroid quotient maps.

Theorem 76 in a later section shows that matroid quotients do correspond to categorical quotients in a related category.

We can derive that contractions, like embeddings, are strong maps.

Corollary 57. *If M is a pointed matroid and $Z \subseteq |M|$, there is a strong map $M \to M/Z$.*

[11]Some publications [39, 46, 79] state that subobjects in this category coincide with matroid minors. This is incorrect; for example, the canonical map $F(M) \to M$ is injective, but $F(M)$ is not generally a minor of M.

Proof. For $A = X \cup Z$, it follows from Definition 3 that

$$\mathrm{rk}_M(A) + \mathrm{rk}_M(Y) \geq \mathrm{rk}_M(A \cup Y) + \mathrm{rk}_M(A \cap Y),$$

so $\mathrm{rk}_M(Y) - \mathrm{rk}_M(A \cap Y) \geq \mathrm{rk}_M(A \cup Y) - \mathrm{rk}_M(A)$, whence

$$\begin{aligned}
\mathrm{rk}_M(Y) - \mathrm{rk}_M(X) &\geq \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(X \cup Z) \\
&= \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(Z) - (\mathrm{rk}_M(X \cup Z) - \mathrm{rk}_M(Z)) \\
&= \mathrm{rk}_{M/Z}(Y) - \mathrm{rk}_{M/Z}(X).
\end{aligned}$$

Lemma 17 now establishes the result. ☐

By the standard definition of the contraction operation, the strong map corresponding to contraction acts as the identity on noncontracted elements and maps the rest to the distinguished loop. Alternatively, one may redefine the contracted matroid on the original ground set, keeping the original elements as loops. In the latter case, the contraction map acts as the identity on all elements.

Finally we establish that matroid quotients are precisely bijective strong maps.

Lemma 58. *A function $f\colon M \to N$ is a bijective strong map if and only if it factors as an embedding $M \to M \backslash X = Q$ followed by a contraction $Q = N/X \to N$.*

$$\begin{tikzcd}
& Q \arrow[dr, "\textit{contraction}"] & \\
M \arrow[ur, "\textit{embedding}"] \arrow[rr, "f"'] & & N
\end{tikzcd}$$

Proof. Sufficiency follows from Corollary 57, necessity is proven by Higgs [34]. ☐

The minimal matroid Q through which a quotient factors like this has been worked out by Kennedy [40]. Every surjective strong map factors as a bijective strong map followed by a strong map τ with $L(\tau) = \mathrm{id}$ [79, page 228], hence a surjection $f\colon M \to N$ is a strong map if and only if it factors as a matroid quotient followed by a map τ with $L(\tau) = \mathrm{id}$.

Lemma 59. *Every contraction is a coequaliser in* $\mathbf{Matr}_{\bullet}$*, but not conversely.*

Proof. Suppose $c\colon N \to N/Z$ is a contraction with $c(z) = \bullet$ for $z \in Z \subseteq |N|$. Let M be the free matroid on $|M| = |N|$. Define $f, g\colon M \to N$ by $f(x) = x$, and

$$g(x) = \begin{cases} x, & \text{if } x \notin Z, \\ \bullet, & \text{if } x \in Z. \end{cases}$$

Then c is a coequaliser of f and g.

Conversely, keeping f the same but letting g send all nonloop elements to the same nonloop element results in a coequaliser that is not a contraction. □

2.6 Factorisation

In this section we study how morphisms between matroids can be factored into easier classes of strong maps. Let us first recall the basic definition [3].

Definition 60. A *weak factorisation system* in a category consists of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ such that:

- every morphism f factors as $f = r \circ l$ for some $l \in \mathcal{L}$ and $r \in \mathcal{R}$;
- both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms;
- if $l, l' \in \mathcal{L}$, $r, r' \in \mathcal{R}$, and arbitrary morphisms f, g make the following diagram

commute, then there is a *fill-in* h making both squares commute:

$$\begin{tikzcd} M \arrow[r, "l"] \arrow[d, "f"'] & I \arrow[r, "r"] \arrow[d, dashed, "h"] & N \arrow[d, "g"] \\ M' \arrow[r, "l'"'] & I' \arrow[r, "r'"'] & N' \end{tikzcd} \tag{2.7}$$

In an *orthogonal factorisation system* the fill-in h is additionally unique.

The standard example of an orthogonal factorisation system is that every function between sets factors as an epimorphism followed by a monomorphism, where the fill-in is the restriction of the function to its image. The category of matroids has an analogous orthogonal factorisation system.

Lemma 61. *The category* $\mathbf{Matr}_\bullet$ *has an orthogonal factorisation system where* $\mathcal{L}$ *consists of epimorphisms and* $\mathcal{R}$ *consists of embeddings.*

Proof. The fill-in is the restriction of g to the image of l, which is a strong map. □

Call a morphism *lattice-preserving* if the lattice of flats of the domain is isomorphic to the lattice of flats of the image. Epimorphisms in $\mathbf{Matr}_\bullet$ can be further decomposed into a quotient followed by a lattice-preserving map [79, page 228]. This does not yield a factorisation system, but lattice-preserving maps are a left class of one particular factorisation.

Theorem 62. *The category* $\mathbf{Matr}_\bullet$ *has an orthogonal factorisation system where* $\mathcal{L}$ *consists of lattice-preserving maps and* $\mathcal{R}$ *consists of maps that are injective on elements of each rank-1 flat.*

Proof. For this proof we shall ignore lattice labellings; that is, we shall consider two lattices equal if and only if they are isomorphic. We note that we do not lose generality in doing this.

Observe that a matroid is uniquely determined by its lattice of flats and its rank-1 flats; equivalently, by its lattice of flats, its loops and its sets of parallel elements. Given a morphism $f\colon M \to N$, define the matroid I as follows:

- $L(I) \cong L(M)$.
- The loops of I are the loops of N.
- The nonloop elements of each rank-1 flat F_i of I are copies of the elements of $f(F_i)$, indexed by i (this includes extra copies of the loops).

Now f decomposes as $M \xrightarrow{l} I \xrightarrow{r} N$, where l is the lattice-preserving map that acts as f on elements when ignoring indices i, and r is the map with $L(r) \cong L(f)$ that sends each element of I to the unindexed version of the element in N. Then l is strong because $L(l) \cong \mathrm{id}_{L(M)}$, and r is strong because $L(r) \cong L(f)$. By construction l is lattice-preserving, and r injective on elements of each rank-1 flat. Both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms by Lemma 38.

For the fill-in h in (2.7), take the morphism with $L(h) \cong L(f)$ that acts as g on elements when ignoring indices. This is by construction the unique strong map that makes both squares commute, as we conclude by separately considering the effect of morphisms on lattices and on elements. □

Moreover, the category $\mathbf{Matr}_\bullet$ has a double factorisation system. Recall [64, Definition 2.1]:

Definition 63. A *double factorisation system* in a category $\mathbf{C}$ is given by a triple $(\mathcal{E}, \mathcal{J}, \mathcal{M})$ of classes of morphisms satisfying:

1. $\mathrm{Iso} \cdot \mathcal{E} \subseteq \mathcal{E}$, $\mathrm{Iso} \cdot \mathcal{J} \cdot \mathrm{Iso} \subseteq \mathcal{J}$ and $\mathcal{M} \cdot \mathrm{Iso} \subseteq \mathcal{M}$,
2. $\mathrm{Mor}(\mathbf{C}) = \mathcal{M} \cdot \mathcal{J} \cdot \mathcal{E}$,

3. for any commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "u"] & \bullet \arrow[r, "j"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "j'"] & \bullet \arrow[r, "m"] & \bullet
\end{tikzcd}$$

in $\mathbf{C}$ with $e \in \mathcal{E}$, $j, j' \in \mathcal{J}$, $m \in \mathcal{M}$ there are uniquely determined "diagonals" s and t with $s \circ e = u$, $j' \circ s = t \circ j$ and $m \circ t = v$. (Here, $\mathcal{A} \cdot \mathcal{B}$ denotes the set of all morphisms $a \circ b$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$; Iso denotes the set of all isomorphisms.)

Remark 64. The category $\mathbf{Matr}_\bullet$ has a double factorisation system [64]: every morphism decomposes as a lattice-preserving map followed by an epimorphism injective on elements of each rank-1 flat followed by an embedding.

Proof. As embeddings are by construction injective, the orthogonal factorisation systems of Lemma 61 and Theorem 62 are "comparable" [64, Proposition 2.6], hence form a double factorisation system [64, Theorem 2.7]. The middle class of morphisms are those that are both epic and injective on rank-1 flats, that is, strong maps that are bijective when restricted to rank-1 flats. □

The aforementioned paper goes on to define the orthogonal Quillen factorisation system [64, Definition 3.9]:

Definition 65. An *orthogonal Quillen factorisation system* of a category $\mathbf{C}$ is given by morphism classes $\mathcal{D}$ (*cofibrations*), $\mathcal{W}$ (*weak equivalences*), $\mathcal{N}$ (*fibrations*) such that:

1. $(\mathcal{W} \cap \mathcal{D}, \mathcal{N})$, $(\mathcal{D}, \mathcal{W} \cap \mathcal{N})$ are factorisation systems,

2. $\mathcal{W}$ has the 2-out-of-3 property (*i.e.* if any two of $f, g, g \circ f$ is in $\mathcal{W}$ then so is the third).

Unfortunately, the double factorisation system of Remark 64 is not an orthogonal Quillen factorisation system, as the second property is not satisfied. Specifically, the fibrations $\mathcal{N} = \mathcal{M} \cdot \mathcal{J}$ have to be the maps that are injective on elements of each rank-1 flat, the cofibrations $\mathcal{D} = \mathcal{J} \cdot \mathcal{E}$ have to be the epimorphisms, and the weak equivalences $\mathcal{W} = \mathcal{M} \cdot \mathcal{E}$ have to be the monomorphisms f such that $L(f)$ preserves rank (equivalently, the strong maps that factor as a lattice-preserving map followed by an embedding). Now take $\mathcal{F}_{M_1} = \{\{\bullet\}\}, \mathcal{F}_{M_2} = \{\{\bullet\}, \{\bullet, *\}\}, \mathcal{F}_{M_3} = \{\{\bullet, *\}\}$ and $f : M_1 \to M_2$, $g : M_2 \to M_3$ mapping each element to itself; f and $g \circ f$ are clearly embeddings whereas g cannot be expressed as a lattice-preserving map followed by an embedding.

Lastly, as we shall show later (Corollary 86), $\mathbf{Matr}_\bullet$ inherits orthogonal factorisation systems from a category of geometric lattices (see subsection 2.7.1 for the complete discussion).

2.7 Functors

This section considers functors between the category of matroids on the one hand and the categories of geometric lattices, vector spaces, and graphs on the other. We shall need the following auxiliary notions.

Definition 66. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is *nearly full* when any morphism g in $\mathbf{D}$ between objects in the image of F is of the form $g = F(f)$ for some morphism f in $\mathbf{C}$.

Recall that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is *full* if the functions $\mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$ are surjective for all objects A and B in $\mathbf{C}$. Hence every full functor is nearly full, but the converse is not true in general. Here is a counterexample:

Example 67. Define the categories $\mathbf{C}$ and $\mathbf{D}$ as follows:

$$\mathrm{Ob}(\mathbf{C}) = \{C_1, C_1', C_2, C_2'\}$$

$$\begin{aligned}
\mathrm{Mor}(\mathbf{C}) &= \{f_a : C_1 \to C_2, f_b : C_1' \to C_2'\} \cup \{\mathrm{id}_X | X \in \mathrm{Ob}(\mathbf{C})\} \\
\mathrm{Ob}(\mathbf{D}) &= \{D_1, D_2\} \\
\mathrm{Mor}(\mathbf{D}) &= \{f_a, f_b : D_1 \to D_2\} \cup \{\mathrm{id}_X | X \in \mathrm{Ob}(\mathbf{D})\}
\end{aligned}$$

and define the functor $F : \mathbf{C} \to \mathbf{D}$ as follows:

$$\begin{aligned}
C_i, C_i' &\mapsto D_i, & i &= 1, 2 \\
f_i &\mapsto f_i, & i &= a, b
\end{aligned}$$

In other words, "nearly full" is a strictly weaker notion than "full".

For the next notion, first recall [78, Chapter IV, Definition 4.7]:

Definition 68. A monoidal category $(\mathbf{S}, \cdot)$ is said to *act upon* a category $\mathbf{X}$ by a functor[12] $\cdot : \mathbf{S} \times \mathbf{X} \to \mathbf{X}$ if there are natural isomorphisms $s \cdot (t \cdot x) \cong (s \cdot t) \cdot x$ and $e \cdot x \cong x$ for $s, t \in \mathbf{S}$ and $x \in \mathbf{X}$, satisfying coherence conditions for the products $s \cdot t \cdot u \cdot x$ and $s \cdot e \cdot x$ analogous to the coherence conditions defining $\mathbf{S}$.

When a category $\mathbf{C}$ is such that a monoidal category $\mathbf{M}$ acts on $\mathbf{C}$, we shall call $\mathbf{C}$ *enriched* in left $\mathbf{M}$-actions.

Definition 69. Let $\mathbf{M}$ be a monoid M seen as a one-object category, and $\mathbf{C}$ a category enriched in left M-actions. A functor $F \colon \mathbf{C} \to \mathbf{D}$ is *nearly faithful* when $F(f) = F(g)$ implies $m_1 \cdot f = m_2 \cdot g$ for some $m_1, m_2 \in M$.

Any faithful functor is nearly faithful, but the converse it not true in general. Intuitively, a nearly faithful functor is 'faithful up to a scalar in M'.

[12]This notation is consistent, as $\mathbf{S}$ acts on itself by $\cdot$.

2.7.1 Geometric lattices

We start with a functor from the category of matroids to the following category of geometric lattices, extending Example 15.

Definition 70. Write **GLat** for the category whose objects are geometric lattices (where isomorphic lattices are identified) and whose morphisms are functions that preserve joins and map every atom to either an atom or the least element.

Remark 71. Here we have made the choice to identify isomorphic lattices in order to simplify the notation, as this rids us of composition with isomorphisms in many expressions and diagrams. There is no difference in any of our results if we do not make this identification; as we shall see, our principal results involving **GLat** concern factorisation (where the identification leaves the classes unchanged, as each contains all isomorphisms), (near-)fullness and (near-)faithfulness of functors between **GLat** and $\mathbf{Matr}_{(\bullet)}$ (that is, properties that remain unchanged) and a composite functor involving vector spaces, whose properties likewise remain unchanged.

As all elements of a geometric lattice are joins of atoms, morphisms in **GLat** are completely determined by their action on atoms.

Proposition 72. *There is a functor* $L\colon \mathbf{Matr} \to \mathbf{GLat}$ *(and a functor* $L_\bullet\colon \mathbf{Matr}_\bullet \to \mathbf{GLat}$*) that sends a matroid* M *to its lattice* $L(M)$ *of flats, and sends a strong map* $f\colon M \to N$ *to the function* $L(f)\colon L(M) \to L(N)$ *given by* $X \mapsto \mathrm{cl}(f(X))$.

Proof. Lemma 17 guarantees that $L(f)$ is a well-defined morphism in **GLat**. If $X \in L(M)$ then $X = \mathrm{cl}(X)$, so L preserves identities. It remains to show that L preserves composition.

Let $f\colon M \to N$ and $g\colon N \to P$ be strong maps. If $X \in L(M)$ then $\mathrm{cl}(f(M)) \in L(N)$, so because g is strong we get $\mathrm{cl}(g(f(X))) = \mathrm{cl}(g(\mathrm{cl}(f(X))))$. So in the end $L(g \circ f) = L(g) \circ L(f)$. $\square$

The functor L (functor $L_\bullet$) is surjective on objects, and injective on the objects of **SMatr** (of $\mathbf{SMatr}_\bullet$), but not injective on objects in general.

Lemma 73. *The functor L (functor $L_\bullet$) is not faithful.*

Proof. Write M for the matroid on the ground set $\{0, 1, 2\}$ with flats $\{\{0\}, \{0, 1, 2\}\}$. The function $f\colon M \to M$ given by $f(0) = 0$, $f(1) = 2$, and $f(2) = 1$ is a strong map. Now $L(f) = L(\mathrm{id}_M)$ both equal the identity function on $\mathcal{F}_M$. □

The functor L is not full. For a counterexample, consider the unique matroid on ground set $\emptyset$. Any matroid M allows a unique morphism $L(M) \to L(\emptyset)$, but clearly there is no strong map $M \to \emptyset$ for nonempty M. The absence of loops in some matroids is in fact the only reason that L is not full, as the following proposition shows.

Proposition 74. *The functor L is nearly full; the functor $L_\bullet$ is full.*

Proof. Let M and N be pointed matroids, and let $g\colon L_\bullet(M) \to L_\bullet(N)$ be a morphism in **GLat**. Construct a function $f\colon M \to N$ as follows. For every rank-1 flat X in M for which $g(X)$ is the least element of $L_\bullet(N)$, define $f(x) = \bullet$ for all $x \in X$. For every rank-1 flat X in M for which the flat $g(X)$ in N has rank 1, let f map all $x \in X$ that have not yet been accounted for to an element of $g(X)$ which is not a loop. Lemma 17 shows that f is a strong map, and by construction $L_\bullet(f) = g$. Hence $L_\bullet\colon \mathbf{Matr}_\bullet \to \mathbf{GLat}$ is full.

Finally, observe that in general $L(M) = L_\bullet(M_\bullet)$. It follows from the argument above that $L\colon \mathbf{Matr} \to \mathbf{GLat}$ is nearly full. □

A morphism f is lattice-preserving precisely when $L(f) = \mathrm{id}$. As promised in Section 2.2, we can now prove that matroid quotients are precisely categorical quotients in **GLat** via the functor $L_\bullet$.

Lemma 75. *Epimorphisms in* **GLat** *are precisely the surjective morphisms.*

Proof. The proof is similar to the first proof of Lemma 37. The proof that surjective maps are epimorphisms is identical; for the converse, the proof by contradiction is amended as follows. Assume $e : M \to N$ is an epimorphism. If all atoms of N are in the image of e then by join preservation so is every other element of N, so either e is surjective or there exists an atom $y \in N$ outside the image of e. Assume the latter case. Write P for the unique geometric lattice with two elements. Define $f : N \to P$ by sending y and the bottom to the bottom, and the rest to the top. Define $g : N \to P$ by sending the bottom to the bottom and everything else to the top. Both are well-defined morphisms in **GLat**, and satisfy $f \circ e = g \circ e$. But then e being epic implies $f = g$, which is not true. So y must have been in the image of e, making e surjective. ☐

Theorem 76. *The functor* $L_\bullet\colon \mathbf{Matr}_\bullet \to \mathbf{GLat}$ *maps matroid quotients to categorical quotients. The restricted functor from the subcategory of pointed matroid quotients to the subcategory of categorical quotients is not full but nearly full.*

Proof. For the first statement: By Lemma 58, matroid quotients f are surjective functions. Hence $L_\bullet(f)$ must also be surjective, which makes it a categorical quotient by Lemma 75.

For nonfullness: Not every surjective morphism $g\colon L_\bullet(M) \to L_\bullet(N)$ in **GLat** has a matroid quotient $f : M \to N$ with $L_\bullet(f) = g$; as a counterexample, if N is the matroid with $1 + \#|M|$ loops and no other elements, then all morphisms into $L_\bullet(N)$ are surjective, but there are no matroid quotients $M \to N$. Hence the restriction of $L_\bullet$ is not full.

For near-fullness: Let $g\colon L_\bullet(M) \to L_\bullet(N)$ be a surjective map in **GLat**. Take M' to have $L_\bullet(M') = L_\bullet(M)$ and ground set comprising $\bullet$ and the atoms of $L_\bullet(M)$. Take N' to have $L_\bullet(N') = L_\bullet(N)$ and $|N'| = |M'|$; for any rank-0 or rank-1 flat F, populate F with the elements of M' that map via g to F or to the bottom. Then there is clearly a matroid quotient $f\colon M' \to N'$ with $L_\bullet(f) = g$, making the restriction of $L_\bullet$ nearly full. ☐

There is also a functor in the opposite direction.

Proposition 77. *There is an embedding $S\colon \mathbf{GLat} \to \mathbf{Matr}_\bullet$ defined as follows:*

- *Action on an object G: $|S(G)|$ is the set of atoms of G together with a loop $\bullet$; $\mathcal{F}_{S(G)}$ is such that $L_\bullet(S(G)) = G$.*
- *Action on a morphism $f\colon G \to H$: if f sends an atom of G to the bottom of H, then $S(f)$ sends the corresponding element of $S(G)$ to $\bullet$; if f sends an atom A of G to an atom B of H, then $S(f)$ also sends A to B.*

Proof. Proposition 74 guarantees that $S(f)$ is a legitimate morphism in $\mathbf{Matr}_\bullet$, and clearly S preserves identities. To show that S preserves composition, note that for (pointed) simple matroids, a strong map is completely defined by its action on the lattice of flats of its domain.

Faithfulness is a direct consequence of the fact that there is a one-to-one correspondence between atoms of the lattice and nonloop elements of the matroid's ground set; different mappings between atoms therefore give rise to differents functions between ground sets. Fullness follows from Lemma 17. Finally, S is injective on objects because each matroid has a unique lattice of flats. ☐

Theorem 78. *There is a reflection $L_\bullet \dashv S$.*

Proof. When the codomain of a strong map f is a simple pointed matroid and the action of f on rank-1 flats is known, then f is known; there is only one element in each rank-1 flat and the rank-0 flat that each element of the domain can map to. Hence $\mathbf{Matr}_\bullet(M, N) \simeq \mathbf{GLat}(L_\bullet(M), L_\bullet(N))$ for pointed matroids M and N if N is simple. Moreover, if N is simple then it is in the image of S (up to isomorphism), and by the definition of the action of S on objects we have $G = L_\bullet(S(G))$ for all geometric lattices

G. Therefore $\mathbf{Matr}_\bullet(M, S(G)) \simeq \mathbf{GLat}(L_\bullet(M), G)$, and this bijection is natural. Finally, note that the counit $L_\bullet(S(G)) \to G$ is an identity, making the adjunction into a reflection. $\square$

Any adjunction gives rise to a monad, and in this case we recover the following standard matroid operation [62, Section 1.7].

Definition 79. A *simplification* $\mathrm{si}(M)$ of a matroid M is a matroid isomorphic to the one obtained by deleting all the loops and all but one element in each rank-1 flat.

For the pointed case, we add a loop $\bullet$ to the simplification; then the following result applies.

Theorem 80. *Simplification is a monad* $\mathrm{si}_\bullet = S \circ L_\bullet \colon \mathbf{Matr}_\bullet \to \mathbf{Matr}_\bullet$. *Its category of Eilenberg-Moore algebras is (isomorphic to)* $\mathbf{SMatr}_\bullet$.

Proof. By definition $S(L_\bullet(M))$ is a simplification of M. Because the monad unit $M \to S(L_\bullet(M))$ sends each nonloop element to its closure, the unit law for Eilenberg-Moore algebras $\mathrm{si}_\bullet(M) \to M$ implies that $\mathrm{cl}(x) = \{x, \bullet\}$ for every nonloop element in M; that is, M is simple pointed, and the structure map has to be the map sending $\{x, \bullet\} \mapsto x$ for each nonloop element $\{x, \bullet\}$. The morphisms of Eilenberg-Moore algebras are the strong maps between pointed simple matroids. $\square$

Remark 81. The image of $\mathrm{si}_\bullet$ is equivalent but not isomorphic to $\mathbf{SMatr}_\bullet$. This can be fixed. Let $(\mathrm{si}_\bullet, \eta, \mu)$ be the triple of $\mathrm{si}_\bullet$. Then η acts on each matroid M by sending each element to its closure; μ acts on each matroid M by sending $\{\{x, \bullet\}, \bullet\} \mapsto \{x, \bullet\}$ for each nonloop element $\{\{x, \bullet\}, \bullet\}$; both η and μ do the obvious thing on morphisms. We can define a functor $\mu' : \mathrm{si}_\bullet \mathbf{Matr}_\bullet \to \mathbf{Matr}_\bullet$ that acts on each matroid M by sending $\{x, \bullet\} \mapsto x$ for each nonloop element $\{x, \bullet\}$ (its action on morphisms is obvious). Then the functor $\mu' \circ \mathrm{si}_\bullet : \mathbf{Matr}_\bullet \to \mathbf{Matr}_\bullet$ is also a simplification monad, and moreover it is idempotent.

We now briefly turn our attention to factorisation systems, which we discussed in the previous section. Clearly factorisation systems in $\mathbf{Matr}_\bullet$ are strongly linked to decomposing morphisms in $\mathbf{GLat}$. Call a morphism $f\colon A \to B$ in $\mathbf{GLat}$ a *contraction* when its restriction to the interval $[\bigvee f^{-1}(0_B), 1_A]$ is an identity; call $g\colon A \to B$ an *embedding* when its corestriction to the interval $[0_B, g(1_A)]$ is an identity. This is in line with the definitions of contraction and embedding for matroids: $L_\bullet(f)$ is a contraction/embedding if f is a contraction/embedding, and the converse holds for simple matroids; $L_\bullet(M)$ is a subobject of $L_\bullet(N)$ if the matroid M is a minor of a matroid N, and the converse holds for simple matroids.

Before moving on to the results, we recall two basic matroid notions. Firstly, define nullity with regards to matroids [79, Equation (5.1)] and strong maps [79, Section 8.2.8]:

Definition 82. The *nullity* $\mathrm{null}_M(A)$ of a subset A of a matroid M (with regards to M) is defined by

$$\mathrm{null}_M(A) = \#A - \mathrm{rk}_M(A).$$

The *nullity* $\mathrm{null}_f(A)$ of $A \subseteq X$ with regards to a bijective strong map $f : M \to N$ where $|M| = |N| = X$ is defined by

$$\mathrm{null}_f(A) = \mathrm{rk}_M(A) - \mathrm{rk}_N(f(A)).$$

The *nullity* $\mathrm{null}(f)$ of the map f itself is defined by

$$\mathrm{null}(f) = \mathrm{null}_f(|M|).$$

Secondly, define Higgs lifts [79, Exercise 7.20]:

Definition 83. For N a nullity-k quotient of M on the same ground set, define the *ith*

Higgs lift M^i $(0 \leq i \leq k)$ of the map $f : M \mapsto N$ as follows:

$$|M^i| = |M|$$
$$\mathcal{I}_{M^i} = \{I : I \in \mathcal{I}_M, \mathrm{null}_N(f(I)) \leq i\}$$

We now give the results.

Lemma 84. *The category* **GLat** *has a weak factorisation system where $\mathcal{L}$ consists of embeddings and $\mathcal{R}$ consists of contractions; it is not an orthogonal factorisation system.*

Proof. Every surjective map $f\colon M \to N$ in $\mathbf{Matr}_\bullet$ factorises as $f = \tau \circ r \circ l$, where l is an embedding, r is a contraction and τ is a lattice-preserving map. Then $L_\bullet(f) = L_\bullet(\tau \circ r \circ l) = L_\bullet(\tau) \circ L_\bullet(r) \circ L_\bullet(l) = L_\bullet(r) \circ L_\bullet(l)$ with $L_\bullet(l) \in \mathcal{L}$ and $L_\bullet(r) \in \mathcal{R}$. The middle object I is the lattice of the nth Higgs lift of N towards $M + F_n$ along f_+, where $n = \mathrm{null}(f)$, where F_n is the free matroid on n elements, and where $L_\bullet(f_+)$ coincides with $L_\bullet(f)$ on the atoms of $L_\bullet(M)$ and sends atoms of $L_\bullet(F_n)$ to the bottom of $L_\bullet(N)$. Explicitly, $L_\bullet(I)$ is the sublattice $\{X \mid X = \bigvee f^{-1}(f(X))\} \cup \{X \mid \mathrm{rk}(f(X)) = \mathrm{rk}(X)\}$ of $L_\bullet(M)$, the canonical embedding $L_\bullet(r)\colon L_\bullet(M) \to L_\bullet(I)$ corestricts to the identity, and the canonical contraction $L_\bullet(l)\colon L_\bullet(I) \to L_\bullet(N)$ acts as $L_\bullet(f_+)$. All of this extends to maps that are not surjective on flats [34]. For the fill-in (2.7), define $h\colon I \to I'$ as the morphism that sends every atom a of I that is also in $L_\bullet(M)$ to $L_\bullet(f(a))$, and all other atoms to the bottom of I'. Since we have some freedom in defining the action of h on atoms of I without affecting commutativity of the diagram, this fill-in is not unique, hence the factorisation system is not orthogonal. $\square$

The rest of this subsection shows that if **GLat** has an orthogonal factorisation system, then it must induce an orthogonal factorisation system in $\mathbf{Matr}_\bullet$.

Proposition 85. *For every $N \in \mathbf{Matr}_\bullet$, the functor $L_\bullet^N \colon \mathbf{Matr}_\bullet/N \to \mathbf{GLat}/L_\bullet(N)$ between slice categories has a right adjoint R^N that is full.*

Proof. We shall employ the same construction as in Theorem 62.

For an object $l \colon G \to L_\bullet(N)$ in $\mathbf{GLat}/L_\bullet(N)$, where G has atoms a_i, define $R_l^N(G)$ to be the matroid with the following properties: $L_\bullet(R_l^N(G)) = G$; the rank-0 flat of $R_l^N(G)$ is the rank-0 flat of N; the nonloop elements of each rank-1 flat F_i of $R_l^N(G)$ with $L_\bullet(F_i) = a_i$ are copies, indexed by i, of the elements of $L_\bullet^{-1}(l(a_i))$, including extra copies of the loops. Then $R^N(l) \colon R_l^N(G) \to N$ is the map with $L_\bullet(R^N(l)) = l$ that sends each element in $R_l^N(G)$ to the unindexed version of the element in N.

For a morphism $f \colon l_1 \to l_2$ in $\mathbf{GLat}/L_\bullet(N)$, define $R^N(f) \colon R_{l_1}^N(G) \to R_{l_2}^N(G)$ as the map with $L_\bullet(R^N(f)) = f$ that acts as the identity on elements. Since $R^N(l_1) \circ R^N(f) = R^N(l_2)$, this is a morphism in $\mathbf{Matr}_\bullet/N$, and it clearly respects identities and composition, so R^N is a well-defined functor.

We now show the existence of a universal arrow for the adjunction. Concretely, we show that the there exist unique morphisms η_f and $\hat{h}$ making the following diagrams commute for every $f \colon M \to N$ in $\mathbf{Matr}_\bullet$, every $g \colon K \to L_\bullet(N)$ in $\mathbf{GLat}$, and every strong map $h \colon M \to \mathrm{source}(R^N(g))$ with $R^N(g) \circ h = f$:

$$
\begin{tikzcd}
M \arrow[rr, "\eta_f"] \arrow[dr, "f"'] \arrow[ddr, "h"'] & & M'' \arrow[dl, "R^N(L_\bullet(f))"'] \arrow[ddl, dashed, "R^N(h)"] \\
& N & \\
& M' \arrow[u, "R^N(g)"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
L_\bullet(M) \arrow[dr, "L_\bullet^N(f)=L_\bullet(f)"] \arrow[dd, dashed, "\hat{h}"'] & \\
& L_\bullet(N) \\
K \arrow[ur, "g"'] &
\end{tikzcd}
$$

Take η_f to be the lattice-preserving map that acts as f on elements when ignoring indices. In the left diagram, the upper and right triangles then commute by construction. By the left triangle, the map h acts as f on elements when ignoring indices. Because both paths along the outer triangle act as f on elements, and both act as h on the lattice, the outer triangle commutes. Since η_f is lattice-preserving, there can exist at most one $\hat{h}$ that makes the large triangle commute, and it is given by $L_\bullet(h)$. This completes the proof that R^N is right adjoint to $L_\bullet^N$.

Finally, we show that R^N is full: within each rank-1 flat, the strong maps forming the objects of $\mathbf{Matr}_\bullet/N$ are one-to-one. Therefore, fixing the lattice maps that form the objects of $\mathbf{GLat}/L_\bullet(N)$ constrains the morphisms of $\mathbf{Matr}_\bullet/N$ to identities on elements. $\square$

A result by Zangurashvili [81] states that when a functor between slice categories has a right adjoint that is full for every slice then factorisation systems are reflected through the preimage of $\mathcal{L}$. Specifically, Proposition 85 yields the following corollary:

Corollary 86. *Any orthogonal factorisation system* $(\mathcal{L}, \mathcal{R})$ *in* $\mathbf{GLat}$ *induces an orthogonal factorisation system* $(L_\bullet^{-1}(\mathcal{L}), \mathcal{R}')$ *in* $\mathbf{Matr}_\bullet$.

2.7.2 Vector spaces

Next we extend Example 12 to a functor.

Definition 87. Let $\mathbf{FVect}_k$ denote the category of finite vector spaces over k and linear maps.

Proposition 88. *There is a functor* $M_k \colon \mathbf{FVect}_k \to \mathbf{Matr}_\bullet$ *that sends a vector space* V *to the matroid with ground set* V *whose independent sets are the linearly independent subsets, acting on morphisms as* $M_k(f) = f$.

Proof. Since flats correspond to vector subspaces, $M_k(f)$ is indeed a strong map as the inverse image of a vector subspace is a vector subspace. $\square$

From this point on, we shall normally omit the subscript k from M_k, except when the discussion involves more than one underlying field.

The functor M_k is faithful, but not full. As a counterexample to fullness, take $V = k = \mathbb{Z}_4$; the function $f\colon V \to V$ given by $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $f(3) = 3$ is a strong map $M(V) \to M(V)$ but not linear. The counterexample is essentially the same for $k > 4$; as for $k = 2$ and $k = 3$, a counterexample can be found for each by considering mappings $\mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ and $\mathbb{Z}_3^2 \to \mathbb{Z}_3^2$ respectively.

Lemma 89. *The functor M does not preserve coproducts, so has no right adjoint.*

Proof. As we saw in Section 2.3 above, coproducts of (pointed) matroids have to satisfy $|M + N| = |M| + |N|$, where the latter coproduct is in the category of (pointed) sets. But $|M(V \oplus W)| = V \oplus W \neq V \sqcup W = |M(V) + M(W)|$. □

A matroid M is *representable* over k if there is a strong map $f\colon M \to M(V)$ for some vector space V over k such that a subset $X \subseteq |M|$ is independent if and only if $f(X)$ is independent. In particular, given a matrix with entries in k, we can construct a matroid whose ground set consists of the columns of the matrix, and where a subset is independent precisely when the corresponding columns are linearly independent. The rank function of the matroid counts the rank of the matrix of selected columns. Every representable matroid over k arises in this way. Not all matroids are representable over some field, so the functor M is not surjective on objects. Nor is it injective on objects: swapping the role of two collinear elements in a vector space results in the same matroid.

We now examine a variation of the functor M. Intuitively, we consider the above way to turn a matrix into a matroid, and remove some structure from the matrix. From this point on, when the matroid N is represented by a matrix A, and $B \subseteq |M|$, we write $A_{[B]}$ for the ordered set of columns of A labelled by elements of B.

Lemma 90. *If matrices A, B, C satisfy $A = CB$, and $I \subseteq J$ are sequences of columns* *of B, and $I' \subseteq J'$ are the corresponding sequences of columns of A, then*

$$\mathrm{rk}(A_{[J']}) - \mathrm{rk}(A_{[I']}) \leq \mathrm{rk}(B_{[J]}) - \mathrm{rk}(B_{[I]}).$$

Proof. Take any column $j \in J \setminus I$ and the corresponding $j' \in J' \setminus I'$. Let $r = \mathrm{rk}(B_{[I]}) - \mathrm{rk}(A_{[I']})$ and $r' = \mathrm{rk}(B_{[I\cup\{j\}]}) - \mathrm{rk}(A_{[I'\cup\{j'\}]})$. There are three possible cases:

- $\mathrm{rk}(B_{[I\cup\{j\}]}) = \mathrm{rk}(B_{[I]})$ and $\mathrm{rk}(A_{[I'\cup\{j'\}]}) = \mathrm{rk}(A_{[I']})$, so $r' = r$.
- $\mathrm{rk}(B_{[I\cup\{j\}]}) = \mathrm{rk}(B_{[I]}) + 1$ and $\mathrm{rk}(A_{[I'\cup\{j'\}]}) = \mathrm{rk}(A_{[I']})$, so $r' = r + 1$.
- $\mathrm{rk}(B_{[I\cup\{j\}]}) = \mathrm{rk}(B_{[I]}) + 1$ and $\mathrm{rk}(A_{[I'\cup\{j'\}]}) = \mathrm{rk}(A_{[I']}) + 1$, so $r' = r$.

In all cases $r \leq r'$, and so $\mathrm{rk}(A_{[I'\cup\{j'\}]}) - \mathrm{rk}(A_{[I']}) \leq \mathrm{rk}(B_{[I\cup\{j\}]}) - \mathrm{rk}(B_{[I]})$. The proof is completed by repeating for the other elements of $J \setminus I$. □

We will consider matrices as multisets of vectors. Recall [74]:

Definition 91. A *multisubset* of a set S is a *multiplicity function* $j \colon S \to \mathbb{N}$, with *support* $\mathrm{supp}(j) = j^{-1}(\mathbb{N} \setminus \{0\})$. A map between multisubsets $j \to j'$ is a function $\mathrm{supp}(j) \to \mathrm{supp}(j')$.

A multisubset is finite when its support is finite.

Definition 92. Write $\mathbf{MVect}_k$ for the category whose objects are finite multisubsets $j \colon V \to \mathbb{N}$ of some vector space V over k, and whose morphisms $(V, j) \to (V', j')$ are linear maps $V \to V'$ that restrict to $\mathrm{supp}(j) \to \mathrm{supp}(j')$.

There is a canonical inclusion $i : \mathbf{Vect}_k \to \mathbf{MVect}_k$ mapping each element V to the constant multiplicity function $j : V \to \{1\}$ and each linear map to itself.

Theorem 93. *There is a functor $\overline{M} \colon \mathbf{MVect}_k \to \mathbf{Matr}$ sending (V, j) to the matroid with ground set having $j(x)$ elements for each $x \in V$ where a subset is independent* *if and only if the corresponding multisubset of vectors in V is (a subset and) linearly independent. It makes the following diagram commute:*

$$\begin{tikzcd} \mathbf{FVect}_k \arrow[d, "i"] \arrow[dr, "M"] & \\ \mathbf{MVect}_k \arrow[r, "\overline{M}"'] & \mathbf{Matr} \end{tikzcd}$$

It is the left Kan extension of $M\colon \mathbf{FVect}_k \to \mathbf{Matr}$ along $i : \mathbf{FVect}_k \to \mathbf{MVect}_k$.

Proof. It is immediate that $\overline{M}\colon \mathbf{MVect}_k \to \mathbf{Matr}$ is well-defined on objects and functorial, and the diagram commutes by construction. It remains to show that it is well-defined on morphisms.

Consider a morphism $(V, j) \to (V', j')$ of $\mathbf{MVect}_k$. Write $S = \operatorname{supp}(j)$, $S' = \operatorname{supp}(j')$, and denote the restriction by $f\colon S \to S'$. The matroid $M(f(S))$ has the same rank function as $M(S')$, because each counts the rank of submatrices of $f(S)$ and S' respectively, and these are identical apart from possibly repeated columns. Lemma 17 implies that this map $M(f(S)) \to M(S')$ is strong. Therefore it suffices to prove that $M(S) \to M(f(S))$ is strong.

Choose bases for V, V', so we may regard all of f, V, V', S, S' as matrices. Let $I \subseteq J \subseteq |M(S)|$. By Lemma 90 then

$$\begin{aligned}
&\operatorname{rk}_{M(S')}\big(M(f)(J)\big) - \operatorname{rk}_{M(S')}\big(M(f)(I)\big)\\
&= \operatorname{rk}_{M(f(S))}\big(M(f)(J)\big) - \operatorname{rk}_{M(f(S))}\big(M(f)(I)\big)\\
&= \operatorname{rk}\big((f(S))_{[M(f)(J)]}\big) - \operatorname{rk}\big((f(S))_{[M(f)(I)]}\big)\\
&\le \operatorname{rk}(S_{[J]}) - \operatorname{rk}(S_{[I]})\\
&= \operatorname{rk}_{M(S)}(J) - \operatorname{rk}_{M(S)}(I).
\end{aligned}$$

Hence $\operatorname{rk}_{M(S')}\big(M(f)(J)\big) - \operatorname{rk}_{M(S')}\big(M(f)(I)\big) \le \operatorname{rk}_{M(S)}(J) - \operatorname{rk}_{M(S)}(I)$, which in conjunction with Lemma 17 implies that $M(f)$ is strong.

Finally, we show that the functor $\overline{M}\colon \mathbf{MVect}_k \to \mathbf{Matr}$ is a left Kan extension of $M\colon \mathbf{FVect}_k \to \mathbf{Matr}$ along $i\colon \mathbf{FVect}_k \to \mathbf{MVect}_k$: Observe that for every $M'\colon \mathbf{MVect}_k \to \mathbf{Matr}$ and natural transformation $\alpha\colon M \Rightarrow M' \circ i$ there is a unique natural transformation $\beta\colon \overline{M} \Rightarrow M'$ such that $\beta \circ i = \alpha$. Uniqueness follows from the fact that if β satisfies $\beta_B \circ \overline{M}(g) = M'(g) \circ \beta_A$ for all $g\colon A \to B$ in $\mathbf{MVect}_k$ then β must act as the identity on elements (regarding repetitions as the same element). To show existence, note that β is a strong map. □

The rest of this subsection considers the functor $L \circ M$ that turns a vector space into its lattice of vector subspaces, which is of interest to *e.g.* quantum logic.

Proposition 94. *The functor* $L \circ M\colon \mathbf{FVect}_k \to \mathbf{GLat}$ *is nearly faithful.*

Proof. Suppose that two linear maps $f, g\colon V \to W$ give rise to the same lattice morphism $L(M(f)) = L(M(g))$; we show that f and g are multiples of each other. For all flats X of $M(V)$ we have $\mathrm{cl}\big(M(f)(X)\big) = \mathrm{cl}\big(M(g)(X)\big)$. It follows from faithfulness of M that $f(X) = g(X)$ for every subspace X of V. Therefore $f(v) = w$ implies $g(v) = \beta_v w$ for some $\beta_v \in k$. For any $v, v' \in V$, $f(v - v') = g(\beta v - \beta' v')$ for some $\beta, \beta' \in k$. So $f(v - v') = \beta' g(v - v') + (\beta - \beta')g(v)$. Taking v, v' noncollinear, $v - v'$ cannot be a multiple of v, so we must have $\beta = \beta'$, hence $f = \beta g$. □

The functor $L \circ M_k$ is not full. For a counterexample, consider the $\mathbb{Z}_2$-vector space $V = \mathbb{Z}_2^2$, which has three 1-dimensional subspaces. Let $A = \{(0,0),(1,0)\}$, $B = \{(0,0),(0,1)\}$ and $C = \{(0,0),(1,1)\}$, and consider the lattice map $g\colon L(M(V)) \to L(M(V))$ that sends all atoms to B. There is only one $f\colon V \to V$ with $L(f) = g$ (namely $f(0,0) = (0,0)$, $f(1,0) = (0,1)$, $f(0,1) = (0,1)$ and $f(1,1) = (0,1)$), but this function is not linear. This counterexample applies essentially the same to all finite fields. Even if we allow infinite base fields, $L \circ M_k$ is not full. For example, take

$V = \mathbb{R}^2$, map all the lines through the origin to one of the axes, and the origin to itself. This preserves the loop, atoms, and joins, because everything is mapped to the same subspace. But there is no linear map implementing this assignment.

2.7.3 Graphs

This work would not be complete without examining the relationship between matroids and graphs, which are their main application. We regard the construction of Example 13 (that is, turning an undirected graph into a matroid) as a functor and briefly discuss some basic properties that such a functor would have. We simplify the definition of undirected graph [19], as we do not distinguish between bands and loops.

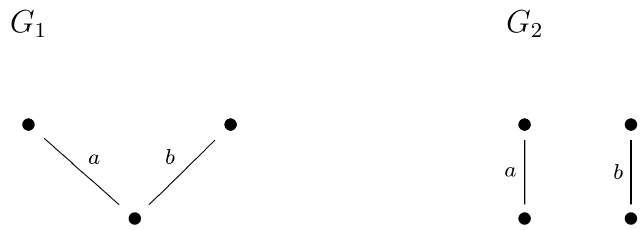
Definition 95. Write **Graph** for the following category. Objects are undirected multigraphs: a set V of vertices, a set E of edges and a boundary map θ from E to the class of singleton and two-element subsets of V. A morphism $(V, E, \theta) \to (V', E', \theta')$ is a pair of maps $f\colon V \to V'$ and $g\colon E \to E'$ satisfying $f \circ \theta = \theta' \circ g$.

To extend Example 13 to a functor $\mathbf{Graph} \to \mathbf{Matr}$, we could restrict the category of graphs to only permit 'strong' morphisms of graphs, whose preimage preserves closed sets (here a set of edges is closed if the addition of an edge does not change the size of a spanning tree in the corresponding subgraph). There is some evidence that this choice of morphisms is useful for some applications of graph theory [66]. Alternatively, we could allow more functions than strong maps as morphisms between matroids. We must at least keep the restriction that loops map to loops: note that in Definition 95, the condition imposed on morphisms implies $\#(\theta'(g(e))) \leq \#(\theta(e))$. Another alternative would be to restrict both the domain and codomain, in which case we write $\mathbf{Graph}^*$ and $\mathbf{Matr}^*$ for the chosen domain and codomain.

For a functor $M\colon \mathbf{Graph}^* \to \mathbf{Matr}^*$ to be of any practical use, it should act as the identity on morphisms. It must then have the following properties:

- It cannot be surjective on objects. A matroid is of the form $M(G)$ precisely when it is *graphic*, and there exist non-graphic matroids.

- It cannot be injective on objects[13], even by identifying isomorphic graphs. Here are graphs $G_1 \not\cong G_2$ with $M(G_1) = M(G_2)$:



- It cannot be full. There are no maps $G_1 \to G_2$ that are surjective on edges, whereas $M(G_1)$ must have at least one morphism to itself (namely the identity).

- It cannot be faithful. Functions $G_2 \to G_1$ corresponding to the identity matroid map may act differently on vertices.

One could define functors from the category of graphs and strong maps to the category of matroids that assign more obscure matroids to graphs, but none of them is surjective or injective on objects, nor full or faithful. We briefly list a few such matroids in use.

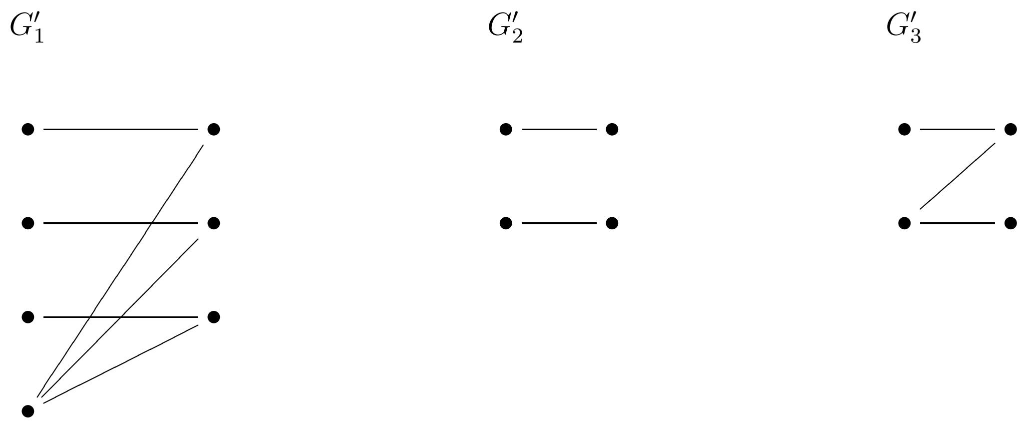
Bicircular matroids[14], as well as Zaslavsky's [82] *frame matroids* and *lift matroids*[15] reduce to the cycle matroid for the graphs G_1 and G_2 given above, and *bond matroids* [62, Section 2.3] in the case of planar graphs reduce to the cycle matroid of the

[13]We may actually make the functor injective by altering the graph G: Form the graph G' by adding to G an extra vertex v as well as one edge vw for every pre-existing vertex w; take $M(G)$ to be the cycle matroid of G'.

[14]The flats of the bicircular matroid [62, Section 12.1] of a graph G are the forests F of G such that in the induced subgraph of $V(G) - V(F)$, every connected component has a cycle.

[15]Frame matroids and lift matroids are defined on bias graphs; we shall not go into bias graphs here.

dual graph[16]; hence for all these classes of matroids the associated functor would inherit the properties listed for the cycle matroid. A *transversal matroid* [62, Section 1.6] is defined on the vertices of one side of a bipartite graph; concretely, the elements are vertices on one side of the bipartition, and the independent subsets are sets of endpoints of matchings of the graph. This type of matroid has similar properties, as demonstrated by the graphs G'_i below (where we define the transversal matroid on the left-hand vertices): for a counterexample to fullness, consider $\mathrm{Hom}(M(G'_1), M(G'_2))$, and specifically the map sending exactly half of the elements of $M(G'_1)$ to each element of $M(G'_2)$; for a counterexample to injectivity on objects, consider G'_2 versus G'_3; for a counterexample to faithfulness, consider the maps in $\mathrm{Hom}(G'_2, G'_3)$.



Finally, a note on (co)limits. The coproduct in **Graph*** is the disjoint union; this seems to be a well-known fact for **Graph**, but a reference could not be found, so we provide a proof for the sake of completeness.

Lemma 96. *Let* **G** *be the category of (un)directed (multi)graphs with (strong) graph homomorphisms (that is,* **G** *is one of eight possible categories corresponding to these different combinations). Then the coproduct in* **G** *is the disjoint union.*

[16] Bond matroids are defined as the *dual matroids* (see Definition 97) of graphic matroids.

Proof. As before, we treat the objects as triples $G = (V, E, \theta)$, where the elements of the codomain of the boundary map θ may or may not be ordered, depending on whether $\mathbf{G}$ is directed or undirected.

Consider the graphs $G_1 = (V_1, E_1, \theta_1)$ and $G_2 = (V_2, E_2, \theta_2)$. Observe that the disjoint union $G_1 \sqcup G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2, \theta_1 \sqcup \theta_2)$ is simple if and only if G_1 and G_2 are. The coproduct injections $p_1 = (f_1, g_1) : G_1 \to G_1 \sqcup G_2$, $p_2 = (f_2, g_2) : G_2 \to G_1 \sqcup G_2$ are the pairs whose underlying functions are coproduct injections $f_i : V_i \to V_1 \sqcup V_2$ and $g_i : E_i \to E_1 \sqcup E_2$ in $\mathbf{Set}$. Check that p_1 is a graph homomorphism: Take $e \in E_1$, then

$$(\theta_1 \sqcup \theta_2) \circ g_1(e) = \theta_1 \circ g_1(e) = \theta_1(e) = \mathrm{id}(\theta_1(e)) = f_1 \circ \theta_1(e).$$

Similarly for p_2. Note that these graph homomorphisms are trivially strong, as preimages are identities.

Now take graph $G' = (V', E', \theta')$ and graph homomorphisms $h_1 = (f_1', g_1') : G' \to G_1$, $h_2 = (f_2', g_2') : G' \to G_2$ and set $[h_1, h_2] = ([f_1', f_2']_{\mathbf{Set}}, [g_1', g_2']_{\mathbf{Set}})$. Coproduct factorisation in $\mathbf{G}$ follows from coproduct factorisation in $\mathbf{Set}$, whence $[h_1, h_2]$ is a graph homomorphism: For $e \in E_1$,

$$\theta' \circ (g_1' \sqcup g_2')(e) = \theta' \circ g_1'(e) = f_1' \circ \theta_1(e) = [f_1', f_2']_{\mathbf{Set}} \circ (\theta_1 \sqcup \theta_2)(e).$$

Similarly for E_2. Note that $[h_1, h_2]$ is strong if h_1 and h_2 are, as preimages are disjoint unions of preimages. Finally, to see uniqueness, consider the factorisation of the identity, which is trivially a strong graph homomorphism. □

Therefore the functor $M \colon \mathbf{Graph}^* \to \mathbf{Matr}$ must preserve coproducts.

2.8 Constructions

This section examines the functoriality of various operations of matroid theory.

2.8.1 Single-matroid operations

We start with one of the most fundamental operations: the dual of a matroid.

Definition 97. The *dual* M^* of a matroid M has ground set $|M|$ and the bases of M^* are complements of bases of M.

This operation is not a functor from/to **Matr** or any immediately obvious variations of this category.

Remark 98. There are matroids M, N with strong maps $M \to N$ but no strong maps $M^* \to N^*$ or $N^* \to M^*$, so taking duals is not functorial on **Matr**. (For an example, take $\mathcal{F}_M = \big\{\emptyset, \{a, b, c\}\big\}$ and $\mathcal{F}_N = \big\{\{*\}\big\}$; then there is only one map $M \to N$ and no maps $N \to M$, but there are maps both $M^* \to N^*$ and $N^* \to M^*$.)

Taking duals is functorial on the subcategory of **Matr** of surjective strong maps between matroids of equal cardinality, since a matroid quotient $q\colon M \to N$ does induce a strong map $q^*\colon N^* \to M^*$.

We now look at addition of elements. There are many ways in which one can add an element to a matroid without changing the inclusion relations of the existing flats: one can make the new element into a loop, an isthmus, or choose to add it into an existing rank-1 flat. Adding a loop or an isthmus are the only ways one can *freely* add an element. We already saw in Remark 33 that adding loops is a functorial process; we now prove that the same holds for adding isthmuses.

Proposition 99. *There is an endofunctor $A\colon$ **Matr** $\to$ **Matr** that acts on objects as $\mathcal{B}_{A(M)} = \{B \cup \{*\} \mid B \in \mathcal{B}_M\}$, and on morphisms as $A(f)(*) = *$ and $A(f)(x) = f(x)$.*

Proof. By construction $*$ is an isthmus in $A(M)$. Since $A = (-) + D$, where D is the free matroid on $\{*\}$, the assignment A is clearly functorial. $\square$

Next we consider the operations of deletion and contraction. These operations are not functorial on **Matr**. Specifically, suppose $f\colon M \to N$ maps $m \mapsto n$ and $g\colon L \to M$ maps $l \mapsto m$, where l, m, n are all nonloops; if m is among the elements by which we contract or delete but the elements l and n are not, then the composite morphism $f \circ g$ cannot canonically be mapped to any strong map, either covariant or contravariant. However, these operations become functorial when we change the category to ensure the deleted/contracted elements in M are exactly those that map to deleted/contracted elements of N.

Proposition 100. *Write* $\mathbf{Matr}_*$ *for the category whose objects are pairs* (M, Z) *of* $M \in \mathbf{Matr}$ *and* $Z \subseteq |M|$*, and whose morphisms* $(M, Z) \to (M', Z')$ *are strong maps* $M \to M'$ *where* Z *is the preimage of* Z'*. There are functors:*

$$\begin{aligned} C\colon \mathbf{Matr}_* &\to \mathbf{Matr} \\ (M, Z) &\mapsto M/Z \\ f &\mapsto f \end{aligned} \qquad\qquad \begin{aligned} D\colon \mathbf{Matr}_* &\to \mathbf{Matr} \\ (M, Z) &\mapsto M\backslash Z \\ f &\mapsto f \end{aligned}$$

Proof. The operations on objects are clearly well-defined, and identity and composition are clearly respected. We need only show that the operations on morphisms are well-defined. For $X \subseteq Y \subseteq |M| \setminus Z$:

$$\begin{aligned} \mathrm{rk}_{C(M')}(f(Y)) - \mathrm{rk}_{C(M')}(f(X)) &= \mathrm{rk}_{M'}(f(Y) \cup Z') - \mathrm{rk}_{M'}(f(X) \cup Z') \\ &= \mathrm{rk}_{M'}(f(Y \cup Z)) - \mathrm{rk}_{M'}(f(X \cup Z)) \\ &\leq \mathrm{rk}_M(Y \cup Z) - \mathrm{rk}_M(X \cup Z) \end{aligned}$$

$$= \mathrm{rk}_{C(M)}(Y) - \mathrm{rk}_{C(M)}(X),$$

and

$$\begin{aligned}
\mathrm{rk}_{D(M')}(f(Y)) - \mathrm{rk}_{D(M')}(f(X)) &= \mathrm{rk}_{M'}(f(Y)) - \mathrm{rk}_{M'}(f(X)) \\
&\leq \mathrm{rk}_M(Y) - \mathrm{rk}_M(X) \\
&= \mathrm{rk}_{D(M)}(Y) - \mathrm{rk}_{D(M)}(X).
\end{aligned}$$

The result now follows from Lemma 17. □

We may implement a series of n deletions and contractions by employing the category $\mathbf{Matr}_{*n}$, whose objects are $(M, Z_1, \ldots, Z_n)$ where the sets $Z_i \subseteq |M|$ are disjoint, and whose morphisms $(M, Z_1, \ldots, Z_n) \to (M', Z'_1, \ldots, Z'_n)$ are strong maps $f\colon M \to M'$ such that Z_i is the preimage of Z'_i for all i. Then we can define contraction and deletion functors $C, D\colon \mathbf{Matr}_{*n+1} \to \mathbf{Matr}_{*n}$. The composition of all these functors produces a minor, so taking minors in $\mathbf{Matr}$ is functorial.

Theorem 101. *Deletion $D\colon \mathbf{Matr}_{*n+1} \to \mathbf{Matr}_{*n}$ is right adjoint to the inclusion $i : \mathbf{Matr}_{*n} \to \mathbf{Matr}_{*n+1}$ given by $(M, Z_1, \ldots, Z_n) \mapsto (M, Z_1, \ldots, Z_n, \emptyset)$ and $f \mapsto f$.*

Proof. Take the universal morphism η to act as the identity on the matroid; then the transpose $\hat{f}\colon (M, Z_1, \ldots, Z_n, \emptyset) \to (M', Z'_1, \ldots, Z'_{n+1})$ is the morphism with the same underlying function as $f\colon (M, Z_1, \ldots, Z_n) \to (M', Z'_1, \ldots, Z'_n)$. □

The following matroid operation turns out not to be functorial.

Definition 102 ([79], Proposition 7.3.3)**.** The *free extension* of a matroid M by p is defined as the matroid $X(M)$ with $|X(M)| = |M| \cup \{p\}$ and flats

$$\big\{K \in \mathcal{F}_M \setminus \{|M|\}\big\} \cup \big\{K \cup \{p\} \mid K \in \mathcal{F}_M \setminus \mathcal{H}(M)\big\}.$$

Remark 103. Let M be the free matroid on $\{a,b\}$ and N the free matroid on $\{a,b,c,d\}$. Let $f\colon M \to N$ be the strong map $f = \{a \mapsto a, b \mapsto b\}$. Then

$$\mathcal{F}_{X(M)} = \big\{\emptyset, \{a\}, \{b\}, \{p\}, \{a,b,p\}\big\}$$

and

$$\begin{aligned}\mathcal{F}_{X(N)} = \big\{&\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{p\},\\ &\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,p\}, \{b,p\}, \{c,p\}, \{d,p\},\\ &\{a,b,c\}, \{a,b,d\}, \{a,b,p\}, \{a,c,d\}, \{a,c,p\},\\ &\{a,d,p\}, \{b,c,d\}, \{b,c,p\}, \{b,d,p\}, \{c,d,p\},\\ &\{a,b,c,d,p\}\big\}.\end{aligned}$$

There are no strong maps $X(M) \to X(N)$ that agree with f. Hence $X(f)$ cannot be canonically defined in a way that respects identities, and the free extension cannot be functorial.

It follows that the matroid operation of *truncation* (contraction by p after free extension by p) cannot be functorial, because free extension is equivalent to truncation by p after the addition of an isthmus p, and addition of isthmuses is functorial by Proposition 99. Two other basic matroid operations are the *free coextension*, defined dually to the free extension as $(X(M^*))^*$, and the *Higgs lift*[17], defined by deletion of p after free coextension by p; a similar counterexample shows that the Higgs lift is not functorial either, whence the free coextension is likewise not functorial (because, as we have shown, deletion can be cast as a functor).

[17]We have seen the (ith) Higgs lift of a map (Definition 83); this is a related notion, where we set $\mathcal{I}_{\mathrm{Lift}(M)} = \{A : A \in E_M, r_M(A) \geq |A| - 1\}$.

Remark 104. Set

$$\begin{aligned}
\mathcal{F}_M = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\\
&\{0,1,3\},\{0,2,5\},\{0,4\},\{1,2,4\},\{1,5\},\{2,3\},\{3,4\},\{3,5\},\{4,5\},\\
&\{0,1,2,3,4,5\}\big\}\\
\mathcal{F}_N = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\\
&\{0,1,2,3\}\big\}
\end{aligned}$$

and let $f : M \to N$ be the strong map that maps $0 \to 0, 1 \to 1, 2 \to 2, 3 \to 3, 4 \to 0, 5 \to 1$. Then

$$\begin{aligned}
\mathcal{F}_{\mathrm{Lift}(M)} = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\\
&\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\\
&\ \{2,5\},\{3,4\},\{3,5\},\{4,5\},\\
&\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,1,5\},\{0,2,3\},\{0,2,4\},\{0,2,5\},\{0,3,4\},\\
&\ \{0,3,5\},\{0,4,5\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\\
&\ \{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\},\\
&\{0,1,2,3,4,5\}\big\}\\
\mathcal{F}_{\mathrm{Lift}(N)} = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\\
&\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\},
\end{aligned}$$

$$\{0,1,2,3\}\big\}$$

There are no strong maps $\mathrm{Lift}(M) \to \mathrm{Lift}(N)$ that agree with f. Hence $\mathrm{Lift}(f)$ cannot be canonically defined in a way that respects identities, and the Higgs lift cannot be functorial.

Extensions generalise free extensions, coextensions generalise free coextensions, quotients generalise truncations and lifts generalise Higgs lifts. Based on our discussion so far, none of these operations is functorial.

Finally we consider *erection*, the inverse matroid operation of truncation. This can obviously be done in many ways. The erections of a matroid M form a lattice based on the following ordering [79, Chapter 7.5]: $M_2 \leq M_1$ if and only if every hyperplane of M_2 is contained in some hyperplane of M_1. The top element $E(M)$ of this lattice is the so-called *free erection* of M and the bottom element is M itself. By convention, $E(M) = M$ if M has no erections; otherwise M is called *erectible.* Erection is not functorial:

Remark 105. Let M, N be the erectible matroids with the following sets of flats:

$$\begin{aligned}
\mathcal{F}_M = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\\
&\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\},\{2,3,4\},\\
&\{0,1,2,3,4\}\big\}\\
\mathcal{F}_N = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\\
&\{0,1,2,3,4\}\big\}.
\end{aligned}$$

Then one can compute[18] their respective free erections:

$$\begin{aligned}
\mathcal{F}_{E(M)} = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\\
&\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\},\{2,3,4\},\\
&\{0,1,2\},\{0,1,3\},\{0,1,4\},\{0,2,3,4\},\{1,2,3,4\},\\
&\{0,1,2,3,4\}\big\}\\
\mathcal{F}_{E(N)} = \big\{&\emptyset,\\
&\{0\},\{1\},\{2\},\{3\},\{4\},\\
&\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\
&\{0,1,2,3,4\}\big\}.
\end{aligned}$$

Then the identity function is a strong map $f\colon M \to N$, but there are no strong maps $E(M) \to E(N)$ that agree with f. This operation is therefore not functorial.

2.8.2 Two-matroid operations

By Proposition 22, the coproduct is a monoidal structure on **Matr**. We now examine which other matroid operations constitute a monoidal structure on **Matr**. These operations mostly come from graph theory, where $\mathbf{Matr}_{\bullet}$ would not be useful, so we shall not examine monoidal structures in $\mathbf{Matr}_{\bullet}$. However, towards the end of this subsection we include an observation on a similar category, which is more appropriate in relation to graphs.

The following operations from matroid theory do not have monoidal structure: the *sum* or *union* $M \cup N$ is the matroid whose independent sets are the unions of the

[18] In this case, this was done using the matroid package of the Python computer language; algorithms that compute the free erection of a matroid have been invented by Las Vergnas [48] and Knuth [42].

independent sets of its constituents; the *product* or *intersection* of $M \cap N$ is the matroid $(M^* \cup N^*)^*$; the *half-dual sum* is the matroid $M^* \cup N^*$.

Remark 106. To see that the union is not a monoidal product, consider the following matroids:

$$
\begin{aligned}
\mathcal{F}_A = \mathcal{F}_B = \big\{&\{\bullet\}, \{\bullet, 0, 1, 2\}\big\}, \\
\mathcal{F}_C = \big\{&\{f\}, \{f, a\}, \{f, b\}, \{f, c\}, \{f, d\}, \{f, e\}, \\
&\{f, a, b\}, \{f, a, c\}, \{f, a, d\}, \{f, a, e\}, \{f, e, b\}, \{f, e, c\}, \{f, e, d\}, \\
&\{f, a, e, b\}, \{f, a, e, c\}, \{f, a, e, d\}, \{f, a, e, b, c, d\}\big\}, \\
\mathcal{F}_D = \big\{&\{e\}, \{e, a\}, \{e, b\}, \{e, c\}, \{e, d\}, \{e, f\}, \\
&\{e, a, b\}, \{e, a, c\}, \{e, a, d\}, \{e, a, f\}, \{e, b, c\}, \\
&\{e, b, d\}, \{e, b, f\}, \{e, c, d\}, \{e, c, f\}, \{e, d, f\}, \\
&\{e, f, a, b\}, \{e, f, a, c\}, \{e, f, a, d\}, \{e, f, b, c\}, \{e, f, b, d\}, \{e, f, c, d\}, \{e, a, b, c, d\}, \\
&\{e, a, b, c, d, f\}\big\}
\end{aligned}
$$

Then $\mathcal{F}_{A \cup B} = \big\{\{\bullet\}, \{\bullet, 0\}, \{\bullet, 1\}, \{\bullet, 2\}, \{\bullet, 0, 1, 2\}\big\}$ and $C \cup D$ is the free matroid on $\{a, b, c, d, e, f\}$. Observe that $A \cup B$ has a loop but $C \cup D$ does not, so there can be no maps $A \cup B \to C \cup D$, whereas there are maps $A \to C$ and $B \to D$. Therefore, union is not a monoidal product in **Matr**.

Remark 107. To see that the intersection is not a monoidal product, consider the following matroids:

$$\mathcal{F}_A = \mathcal{F}_B = \big\{\emptyset, \{1, 2\}\big\}, \qquad \mathcal{F}_C = \mathcal{F}_D = \big\{\emptyset, \{x\}\big\}.$$

Then $\mathcal{F}_{A \cap B} = \big\{\{1, 2\}\big\}$ and $\mathcal{F}_{C \cap D} = \big\{\emptyset, \{x\}\big\}$. There is a map $1, 2 \mapsto x$ in each homset

$\mathbf{Matr}(A, C)$ and $\mathbf{Matr}(B, D)$, but there can be no map $A \cap B \to C \cap D$ because there are loops in $A \cap B$ but no loop in $C \cap D$. Therefore, intersection is not a monoidal product in **Matr**.

Remark 108. To see that the half-dual union is not a monoidal product, consider the following matroids:

$$\mathcal{F}_A = \big\{\{0\}\big\}, \qquad \mathcal{F}_B = \big\{\emptyset, \{0\}\big\}, \qquad \mathcal{F}_C = \mathcal{F}_D = \big\{\{*\}\big\}.$$

Then $\mathcal{F}_{A \cup B^*} = \big\{\{0\}\big\}$ and $\mathcal{F}_{C \cup D^*} = \big\{\emptyset, \{*\}\big\}$. Observe that the homsets $\mathbf{Matr}(A, C)$ and $\mathbf{Matr}(B, D)$ each contain the map sending everything to the loop, whereas there are no maps in $\mathbf{Matr}(A \cup B^*, C \cup D^*)$, because $A \cup B^*$ has a loop and $C \cup D^*$ does not. Therefore, half-dual union is not a monoidal product in **Matr**.

The *intertwining* of two matroids, defined as a minor-minimal matroid that contains them both as minors, is not a monoidal product either, as it is not always unique up to isomorphism.

Altering the category slightly allows for two monoidal structures instead of one. Write $\mathbf{Matr}_\times$ for the category of matroids with a distinguished element and strong maps preserving the distinguished element. The *parallel connection* $M||N$ is the coproduct in this category [79]. Explicitly, the ground set of $M||N$ is the disjoint union of $|M|$ and $|N|$, the distinguished elements are identified, and the flats of $M||N$ are the unions of flats in M and in N. This is similar to the coproduct in $\mathbf{Matr}_\bullet$, except that the distinguished element need not be a loop. Now we get one more monoidal product: the *series connection* MN is defined dually to the parallel connection by $MN = (M^*||N^*)^*$, and is another monoidal structure on $\mathbf{Matr}_\times$. This monoidal structure is not a categorical product, but remarkably enough it is naturally affine, in the sense that there are always natural transformations $MN \to M$ and $MN \to N$. The parallel connection and the series connection do not distribute over each other.

2.9 The greedy algorithm

There exists a well-known characterisation of matroids which, intriguingly, is algorithmic in nature and exemplifies the connection between matroids and problems in combinatorics [62, 60].

Definition 109. Let $\mathcal{I}$ be a collection of subsets of a finite set E that satisfies the nontrivial and downward closed conditions from Definition 3. Given a function $w\colon E \to \mathbb{R}$ define the associated *weight* function $w\colon 2^E \to \mathbb{R}$ by

$$w(X) = \sum_{x \in X} w(x).$$

The *optimisation problem* for the pair $(\mathcal{I}, w)$ is to find a maximal member B of $\mathcal{I}$ of maximum weight.

Definition 110. The *greedy algorithm* for a pair $(\mathcal{I}, w)$ as in Definition 109 is:

(i) Set $X_0 = \emptyset$ and $j = 0$.

(ii) If $E - X_j$ contains an element e such that $X_j \cup \{e\} \in \mathcal{I}$, choose such an element e_{j+1} of maximum weight, let $X_{j+1} = X_j \cup \{e_{j+1}\}$, and go to (iii); otherwise, let $B_G = X_j$ and go to (iv).

(iii) Add 1 to j and go to (ii).

(iv) Stop.

Theorem 111. *Let $\mathcal{I}$ be a nontrivial and downward closed collection of subsets of a finite set E. Then $\mathcal{I}$ is the collection of independent sets of a matroid on E if and* *only if the greedy algorithm for* $(\mathcal{I}, w)$ *solves the optimisation problem for* $(\mathcal{I}, w)$ *for all possible weight functions* $w\colon E \to \mathbb{R}$ *(generalised to* $w\colon 2^E \to \mathbb{R}$ *as in Definition 109).*

Proof. This is a well-known result in matroid theory [62, Theorem 1.8.5]. □

Crucially, this theorem is equivalent to the following statement: "The greedy algorithm solves the optimisation problem if and only if all maximal independent sets have the same cardinality". It is easy to show that the latter condition, given nontriviality and downwards closure, is equivalent to the independence augmentation axiom of Definition 3.

This leads us to a categorical characterisation of matroids. Write $\mathbf{Vect}_{\mathbf{k}}^{\mathbf{b}}$ for the category of vector spaces over k with a chosen basis b and linear transformations between them.

Lemma 112. *Every run of the greedy algorithm produces a maximal chain of epimorphisms in a subcategory of* $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{b}}$.

Proof. Let the list $B = (b_1, b_2, \ldots, b_r)$ denote the final output of the algorithm. Observe that the vector $(w(b_1), w(b_2), \ldots, w(b_r))$ is an element of $\mathbb{R}^r$. At the nth step of the algorithm, the candidate output corresponds to a vector in $\mathbb{R}^n$. Then the nth step of the algorithm corresponds to an epimorphism $e_n\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ which projects out the largest element of the vector. The algorithm continues as long as there are candidate elements, hence forms a maximal chain in the subcategory formed by all such epimorphisms. □

This is the categorical equivalent of the fact that the greedy algorithm produces a maximal independent set; the length of the chain equals the cardinality of that set. The following definition makes precise when a partially ordered set is 'as wide as it is tall'.

Definition 113. Define a *square poset* to be a finite partially ordered set P with a least element, such that any element $A \in P$ covers exactly $n_A + 1$ elements, where n_A is the maximum length of a maximal chain from the least element to A. A *square functor* is a functor $I\colon \mathbf{P} \to \mathbf{Sub}$ from a square poset $\mathbf{P}$, seen as a category, to the category $\mathbf{Sub}$ of sets and inclusions that is injective on objects and preserves chain lengths.

Every square functor $I\colon \mathbf{P} \to \mathbf{Sub}$ induces a pair $(\mathcal{I}, E)$ where $\mathcal{I}$ is a nontrivial downwards closed collection of subsets of a set E. Namely, define E to be the union of all the sets S_i in the image of I, and set $\mathcal{I} = \{S_i \setminus I(0)\}$, where 0 is the least element of P. This is evidently a collection of subsets of E, and it contains the empty set. Because I is injective on objects, the number of inclusions into each object S_i is maximal, guaranteeing that all subsets of each member of $\mathcal{I}$ are in the image of I.

Lemma 114. *Given a square functor $I\colon \mathbf{P} \to \mathbf{Sub}$, the induced pair $(\mathcal{I}, E)$ is the collection of independent sets and ground set of a matroid if and only if every maximal chain in the image of I has the same length; equivalently, if and only if the following holds: for any maximal chain $C \subseteq P$, the colimit of the diagram $I\colon \mathbf{C} \to \mathbf{Sub}$ is independent of C (up to isomorphism).*

Proof. Each nonidentity inclusion in C adds one element to the domain, therefore the length of the chain equals the cardinality of the final codomain. This statement is therefore equivalent to "all maximal elements of $\mathcal{I}$ have the same cardinality", which together with the first two axioms defines a matroid. The colimit formulation now follows from the fact that sets with the same cardinality are isomorphic. □

Lemma 115. *Given a square functor $I\colon P \to \mathbf{Sub}$, the induced pair $(\mathcal{I}, E)$ is the collection of independent sets and ground set of a matroid if and only if for every contravariant functor $W\colon P \to \mathbf{Vect}_{\mathbb{R}}^{\mathbf{b}}$ such that I factors through W, and for every maximal chain $C \subseteq P$, the limit of the diagram $W\colon \mathbf{C} \to \mathbf{Vect}_{\mathbb{R}}^{\mathbf{b}}$ is independent of C.*

Proof. This is equivalent to the above lemma, by the definition of a limit and the fact that vector spaces of the same dimension are isomorphic. □

The following theorem summarises the main result of this section.

Theorem 116. *The greedy algorithm solves the optimisation problem if and only if the chains in* $\mathbf{Vect}^{\mathbf{b}}_{\mathbb{R}}$ *induced by all runs have the same limit.*

Proof. Combine Theorem 111 and Lemma 115. □

2.10 Matroids and Entropy

Before moving on to the next chapter, which concerns itself with Entropy in Classical Thermodynamics, we discuss some links between matroids and other notions of Entropy. Here we focus on Statistical Mechanics.

The properties of matroids are the properties of dependence, a notion that appears in virtually any physical process. Consider the following physical scenario. Suppose that we have two substances A and B, separated from each other and not interacting, respectively comprising N_A and N_B identical particles. Each substance may be in a specific *microstate*, that is, a combination of the *states* of all its particles. Suppose that each particle of A (respectively B) can be in one of s_A (respectively s_B) different states; then we have $s_A^{N_A} s_B^{N_B}$ possible microstates for the whole system. Writing out the elements of each component as a $s_A^{N_A} \times 1$ (respectively $s_B^{N_B} \times 1$) matrix, the whole is the Kronecker product of these two matrices. Performing a process on the system means applying a stochastic matrix to this product, during which the two components may interact; performing a process each of the two components separately then corresponds to a Kronecker-separable stochastic matrix. Recovering the effect of the process on one of the components is tantamount to a projection on a subspace. Taking the matroid of the matrix one may apply our results above in order to isolate the dependence properties of the system and reason about them categorically.

Another angle (further discussed in Chapter 4) comes from the fact that Shannon Entropy (and by extension Entropy in Statistical Mechanics) induces a structure which generalises matroids and to which many results from matroid theory can be extended. In that sense, some properties of matroids *are* properties of Entropy. A possible application of our results on matroids to Entropy is reserved for future work.

Chapter 3

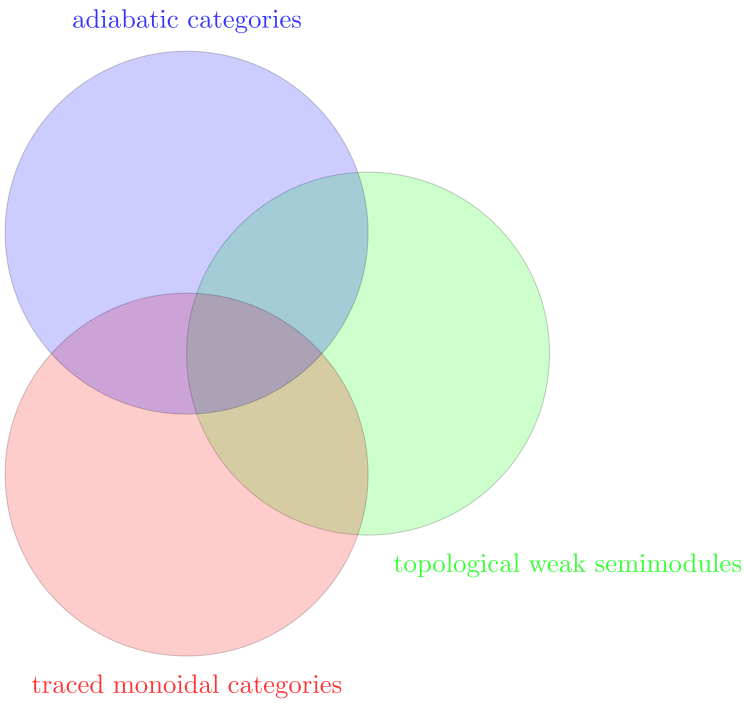
Categorical perspectives on Entropy

3.1 Introduction

3.1.1 Overview

The aim of this work is to explore concepts that may serve as a mathematical foundation for various notions of Entropy. We use Category Theory to capture the mathematical essence of Entropy.

As discussed in Section 1.3, our starting point is an embellished version of Lieb and Yngvason's framework [51]. A large proportion of this project is devoted to showing connections of this expanded model with two other categorical notions, each unique and significant in its own way. These are topological weak semimodules and traced monoidal categories.

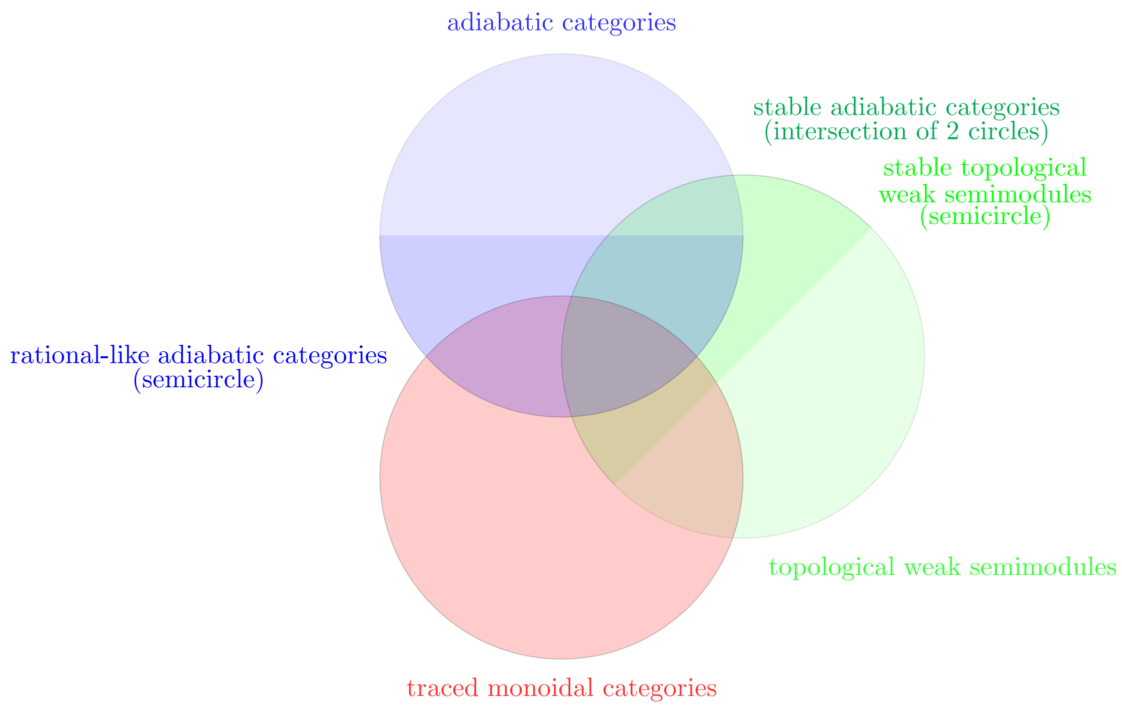


We briefly introduce the notions in the above diagram, without yet giving their proper definitions. Adiabatic categories (introduced in Subsection 3.3.1, here depicted in blue) have a "weakly linear" structure that cooperates with a "convergence" functor; similarly, topological weak semimodules (introduced in Subsection 3.3.2, here depicted in green) have a "weakly linear" structure that cooperates with a topological functor. Finally, recall [38, 31] that a traced monoidal category (here depicted in red) is a braided monoidal category with a natural operation called a trace, generalising a matrix trace, which takes a morphism in $\mathrm{Hom}(A \otimes X, B \otimes X)$ to a morphism in $\mathrm{Hom}(A, B)$ and satisfies certain properties that arise from the case of a linear trace.

We expect physical systems that display Entropy to lie within the intersection of all three circles. We shall now describe our results on the pairwise intersections of the above diagram.

Both adiabatic categories and topological weak semimodules are defined over a topological semiring. Out of all possible underlying semirings, we distinguish adiabatic categories over what we shall call "rational-like" semirings (we call these rational-like adiabatic categories) and topological weak semimodules over what we shall call "stable" semifields. A rational-like topological semiring is a topological semiring that behaves like $\mathbb{Q}$ or $\mathbb{Q}^{\geq 0}$ in some ways; without giving a technical definition here, we note that a rational-like semiring need not be a semifield. We call a topological semifield "stable" if the topology cooperates with the linear structure in a certain intuitive manner (we shall not go into more detail in the context of this introduction).

Refining the above diagram, we shall depict these distinguished adiabatic categories and topological weak semimodules with a semicircle of darker hue.



Observe in the above diagram that the stable topological weak semimodules contain the intersection between weak semimodules and adiabatic categories. We show that the intersection of adiabatic categories and topological weak semimodules contains precisely the adiabatic categories over "stable" topological semifields; we furthermore show that, fixing a specific "stable" topological semifield Λ, adiabatic categories over Λ are a subset of topological weak semimodules over Λ and every topological weak semimodule over Λ can be canonically mapped to an adiabatic category over Λ (Theorem 204). We shall later see an example of a stable topological weak semimodule that is not an adiabatic category, along with its mapping.

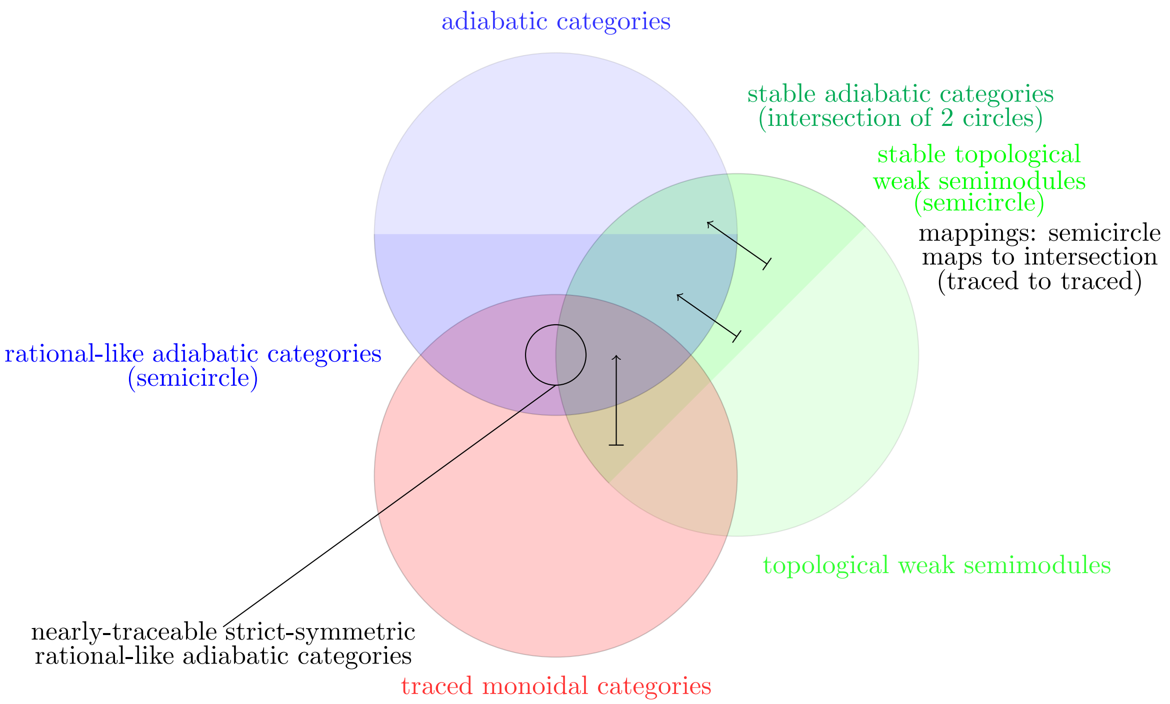
We now point out an intentional omission in the diagram. The intersection of traced monoidal categories with the other two sets is restricted to those categories whose trace is definable in terms of their topological structure; for adiabatic categories in particular, this is done by means of a Cancellation Law, which restricts them to what we shall call rational-like adiabatic categories. That is, we have not depicted adiabatic categories or topological weak semimodules that may admit an arbitrary trace (for the sake of completeness we mention here that we have shown all adiabatic categories over a topological ring to be traced monoidal in the more general sense – see Remark 238). In this context, it almost goes without saying that there exist traced monoidal categories that are neither adiabatic categories nor topological weak semimodules; the reader can just pick their favourite example from the literature on traced monoidal categories, and chances are that it will have no such structure.

Bearing in mind this discussion, note in the above diagram that the rational-like adiabatic categories contain the intersection between adiabatic categories and traced monoidal categories. We show (Theorem 247) that adiabatic categories over certain "rational-like" topological semirings are traced monoidal if they satisfy a set of physically plausible constraints (in which case we call them nearly-traceable[1]) and their sym-

[1] There is a reason that we distinguish strict symmetry among the properties we require. Specifically, only one of the properties of trace directly involves the braiding (here a symmetry) and this is the one

metry is strict. Any topological weak semimodule that maps to such an adiabatic category is also traced monoidal, whether or not it is itself an adiabatic category; this is because the possible extra topological structure of the topological weak semimodule does not interfere with the axioms of a traced monoidal category.

In short, these main results may be depicted as follows.



We note that any physically relevant model is both stable and traced:

- In a thermodynamical setting, the topological semiring corresponds to a set of scaling factors, and specifically to mass[2]. Therefore, we are only interested in $\mathbb{R}^{\geq 0}$ with the standard topology, which is stable and rational-like.

that is often not satisfied in nature. We use the term near-trace for a natural operation in a braided monoidal category with the same type as the trace that satisfies all properties of trace except for the one involving the braiding, and the term nearly-traced monoidal category accordingly. We show that nearly-traceable categories are nearly-traced.

[2]As we shall see, the idea is that as these scaling factors tend to 0, the system "disappears"; hence, mass is the quantity affected. This is the crucial property that enables us to recover an additive extensive function later.

- The constraints that an adiabatic category has to satisfy in order to be traced monoidal are physically plausible. For instance, the monoidal structure of the adiabatic category must have a strict symmetry; since the symmetry is physically interpreted as a way of putting two systems together (eg mixing, or reaching thermal equilibrium), we can make that assumption without losing generality from a thermodynamical point of view. The other relevant constraints are similarly benign.

What do these notions have to do with Entropy? As mentioned in Chapter 1, when choosing the topological semiring to be $\mathbb{R}^{\geq 0}$ with the standard topology we recover Entropy as a unique (up to some factor) subadditive nondecreasing function. In this abstract setting, we provide some categorical examples of a macroscopic treatment. Moreover, as noted in Chapter 1, the function we recover satisfies the properties of Faddeev's theorem on an abstract level, which provides a basis for future work (see Chapters 1 and 4).

Without yet giving proper definitions, we give a brief description of rational-like adiabatic categories in terms of physical intuition:

Remark 117. As per Lieb and Yngvason's work [51], a rational-like adiabatic category (more specifically, an adiabatic category over $\mathbb{R}^{\geq 0}$ with the standard topology) models thermodynamical systems at various states (objects) and adiabatic processes between these states (morphisms). Composition then means performing one process after another, and identity means doing nothing to a system.

- The category is symmetric monoidal. The monoidal product of two systems X_1 and X_2 is the compound system consisting of X_1 and X_2. Symmetry means switching the two systems, which is an adiabatic process. The unit is the empty system. This monoidal product is strict.

- The category is equipped with a family of endofunctors, which can be thought of as mass. They correspond to scaling of a system by a factor $\lambda \in \mathbb{R}^{\geq 0}$, without changing its thermodynamical state; for instance, if X is 100gr of hydrogen at temperature T and pressure P, then $2X$ is 200gr of hydrogen at temperature T and pressure P. These scaling operations have to be functorial, as any notion of "scaling" a multistep process must apply to scaling each step of this process individually (and, of course, doing nothing to a system of mass λm should, in any sensible scaling model, be a scaled version of doing nothing to a system of mass m). They are strict monoidal, as scaling a compound system is the same as scaling each of its component subsystems.[3] They are symmetric, because switching components around has the same effect no matter what the mass of the system is. Lastly, anything with 0 mass is the empty system.

- The category has a "splitting and recombination" property, which means that every system can be split into two parts with the same thermodynamical state (inserting a partition) and then recombined into a single system (removing the partition), or vice versa, with no net effect (invertibility); this is an adiabatic process. Moreover, it makes no difference whether an adiabatic process takes place before or after the splitting/recombination process (naturality). In the end, we see that this is a natural isomorphism.

- Stability for Lieb and Yngvason means that if there exist adiabatic processes $X \boxplus \epsilon Z_1 \to Y \boxplus \epsilon Z_2$ (where $\epsilon \to 0$ in this homset sequence), essentially making the components Z_1 and Z_2 infinitely small, then there must exist an adiabatic process $f : X \to Y$. In our framework, we have a notion of "basic stability" which is stronger than the stability required by Lieb and Yngvason. The difference is that

[3]For an example of an adiabatic category where the scaling functors are not strict, see section on topological weak semimodules.

we further require f to arise as a function on an infinite sequence of "converging" processes. In other words we posit that, given such an f and a homset sequence of a specific type, there must exist specific processes $f_\epsilon : X \boxplus \epsilon Z_1 \to Y \boxplus \epsilon Z_2$ that "approximate" f, essentially "tracing out" the "vanishing" component.[4]

- Since we have required the existence of such a function, we must ensure that it acts as a limit of a sequence in an appropriate space and that it has sensible physical properties. These are "convergence" properties, which are sufficient and necessary for the stability function to correspond exactly to topological convergence [76], and "linearity" properties, which ensure that the topological space in question cooperates with composition, identity, the monoidal structure and the scaling endofunctors in a physically plausible way; in essence, this captures the expectation that certain physical processes must be continuous in certain variables.

It is important to understand the usefulness of having the three different perspectives. The Lieb-Yngvason perspective (adiabatic categories) is straightforward and intuitive (after all, it arose from physical modelling) but offers little insight into the properties of the category. The construction of topological weak semimodules, on the other hand, while on the surface not bearing much resemblance to the underlying physical reality, is much more powerful and elegant, and opens the door to discovery of possible hidden mathematical properties of the physical setup. There are two reasons for this:

1. Even though our treatment restricts topological weak semimodules to the same possible topologies as adiabatic categories, topological weak semimodules capture more topological properties of the physical system than adiabatic categories.

[4]In subsection 3.4.2 we shall see a concrete example of a physically relevant rational-like adiabatic category as a traced monoidal category, where the stability property gives rise to a trace.

This is because adiabatic categories arose from a generalisation of thin categories, which limits the forms of convergent sequences (of morphisms) considered. This fact is reflected in the fact that the relationship between the categories of adiabatic categories and topological weak semimodules is not an equivalence, but an adjunction.

It is important to note here that rational-like adiabatic categories model continuity in terms of one quantity: mass. That is to say, supposing we repeat a process on a composite system, each time removing some mass from one of the components but keeping all intensive parameters equal, if each time the process remains adiabatic no matter how much mass we remove, then we can eventually remove the whole component and the process will remain adiabatic. This is precisely the convergence property that yields Entropy, as it implies the existence of a subadditive function (see Theorem 225). By contrast, topological weak semimodules make no distinction between the topological properties of mass and those of any other quantity; this makes the connection to Entropy far less clear, but also allows us to model changes in any variable of the system state.

2. Topological weak semimodules combine two fundamental categorical notions:

    - Topological categories are a sort of internal categories that not only has existed for a long time, but has also been extensively studied. As a result, topological weak semimodules are more elegant from the topological standpoint and we can use known properties of topological categories to study them.

    - On the other hand, weak semimodules are an intuitive categorification of semimodules (equivalently, a weakening of 2-semimodules), another sort of internal category. While weak semimodules are not themselves internal categories, weak semimodules are similar enough to 2-semimodules that they are easy to work with and in some ways can be thought of as semimodule-like structures. We have a lot of theoretical tools at our disposal to deal with such structures.

Moreover, the category **WSM** of weak semimodules has the property that it has free constructions over the category **SMC** of symmetric monoidal categories; more research into the properties of **WSM** is needed in order to explore possible ways to exploit this.[5] This partly motivates our decision to frame our constructions and theorems in terms of more general semirings rather than just $\mathbb{R}^{\geq 0}$.

Lastly, we establish a connection with traced monoidal categories. These categories have an intuitive interpretation from the physical viewpoint ("erasing" part of a system by means of an operation called the trace) but they offer nothing new in this respect, as rational-like adiabatic categories (which, we remind, include adiabatic categories over $\mathbb{R}^{\geq 0}$) have their own version of this operation (what Lieb and Yngvason call the Cancellation Law); in fact, the Cancellation Law is more general than the trace. The real usefulness here lies in the fact that the properties of traced monoidal categories have been studied extensively, which again gives us yet another toolset to treat a physically relevant subset of adiabatic categories.

3.1.2 Outline

This chapter comprises a lot of definitions and constructions, along with proofs of associated properties. This outline aims to offer some insight into the underlying structure:

[5]For instance, it would be fruitful to find out if **WSM** has internal tensor products and is injectively complete like the category of semimodules, in which case the injective envelope in **WSM** of a free construction over **SMC** may tell us something about the symmetric monoidal category underlying the free construction. This is one possible direction for a future project.

save for a few tangential remarks and minor lemmas, this chapter actually follows a focussed and directed path that the reader should be aware of.

Situated after some essential definitions in Section 3.2 and before presenting some physically-motivated applications in Section 3.4, Section 3.3 is the crux of this chapter, containing all the relevant constructions and theorems. Therein we show the relationship between three different notions.

Subsection 3.3.1 introduces adiabatic categories, which are a direct generalisation of a mathematical construction (introduced by Lieb and Yngvason [51]) that arose naturally from Thermodynamics. These "adiabatic categories" (see Definitions 139 and 148, which are equivalent by Theorem 151) are categories combining something akin to a linear structure (which we call *weak linearity*; see Definition 136) and something resembling a topological structure, defined in terms of a "convergence function" (in fact a functor, as we shall see; see Definition 147) mapping infinite sequences to their limit; these two structures are required to cooperate with each other. We show that the convergence functor indeed gives rise to a topology on the set of objects and a topology on the set of morphisms (see Lemma 157). We also show that some adiabatic categories retain a property that Lieb and Yngvason call a Cancellation Law (see Theorem 161).

At this point, the reader may be reminded of topological vector spaces (where a linear structure and a topological structure coexist cooperatively), of enriched categories, and of internal categories (specifically, topological categories). These intuitions are validated in the following Subsections.

Subsection 3.3.2 introduces topological weak semimodules (see Definitions 166 and 176). These are much more general, fundamental and powerful constructions; specifically, weak semimodules are a categorification of semimodules (in fact, they generalise 2-semimodules; see Remark 169) and their topological counterparts are weak semimodules that are also topological categories. We mention some interesting properties of the categories of weak semimodules and of topological weak semimodules, as well as some nice examples that hint at their versatility and usefulness (see, for instance, Remark 187).

These two constructions (adiabatic categories and topological weak semimodules) have very different origins (with the former coming from modelling physical systems and the latter from a categorification of a familiar concept) and look nothing like each other. However, Subsection 3.3.3 shows a definitive connection between the two (see Theorem 204); in fact, an adjunction (see Theorem 217).

Subsection 3.3.4 restates some of Lieb and Yngvason's results in this more general setting.

Finally, Subsection 3.3.5 reveals certain links between adiabatic categories and traced monoidal categories, as there are two classes of adiabatic categories that are traced monoidal (see Remark 238 and Theorem 247). In one of these classes, Lieb and Yngvason's Cancellation Law is the trace operation.

In summary, the reader is advised to refer to the following diagram for Section 3.3.

$$
\begin{tikzcd}
 & \text{(3.3.1)} & & \text{(3.3.2)} \\
\text{Entropy} \arrow[r, leftrightarrow, "\text{(3.3.4)}"] & \text{adiabatic categories} \arrow[rr, shift right, "T"'] \arrow[d, shift right, "\text{Cancellation Law}"'] \arrow[d, shift left, "\text{Ring structure}"] & & \text{topological weak semimodules} \arrow[ll, shift right, "\text{(3.3.3)}"'] \\
 & \text{traced monoidal categories} & & \\
 & \text{(3.3.5)} & &
\end{tikzcd}
$$

3.2 Preliminary definitions

3.2.1 Algebraic structures

Recall the standard definitions of semiring and commutative semiring:

Definition 118. A *semiring* $R(0, 1)$ is a set R equipped with two binary operations $+$

and $\cdot$, called addition and multiplication, such that the following hold.

- $(R, +)$ is a commutative monoid with identity element 0 (additive unit).
- $(R, \cdot)$ is a monoid with identity element 1 (multiplication unit).
- Multiplication (left and right) distributes over addition.
- $0x = 0$ for all $x \in R$.

A *ring* is a semiring that is an abelian group under addition.

Definition 119. The *trivial* semiring is the semiring with one element, 0=1.

Every semiring in this work will be assumed nontrivial; note that in any nontrivial semiring we always have $0 \neq 1$.

Definition 120. A *commutative* semiring is a semiring where multiplication is commutative.

From here on, we always denote semiring addition with + and multiplication with concatenation. Furthermore, unless otherwise noted, the additive unit of the semiring will always be denoted 0 or *zero* and the multiplicative unit will be denoted 1 or *one*; this is to keep consistency with the usual notation on the semirings $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}^{\geq 0}$, $\mathbb{R}^{\geq 0}$, $\mathbb{Z}$, $\mathbb{N}$, etc. with the usual multiplication and addition operations.

Definition 121. A semiring R is *cancellative* if for any $x, y, z \in R$ we have $x + y = x + z \Rightarrow y = z$.

Any ring is cancellative; also, the semirings $\mathbb{N}$, $\mathbb{Q}^{\geq 0}$ and $\mathbb{R}^{\geq 0}$ are cancellative.

In any cancellative semiring we define a partial subtraction function (written as $-$) as the inverse of addition. Similarly, when multiplicative inverses are unique, we define a partial division function (written as a fraction $\frac{1}{r}$) as the inverse of multiplication; if the semiring is also commutative, fractions are written in the usual form $\frac{r_1}{r_2}$. Positive and negative integer exponents will be used to denote repeated multiplication and (when the operation is defined) repeated division.

In this work we are interested exclusively in commutative cancellative semirings, so we shall use a shorthand.

Definition 122. Call a commutative cancellative semiring a *CC-semiring*.

(Semi)fields also play a crucial role in this work.

Definition 123. A *semifield* is a commutative semiring in which all nonzero elements have a multiplicative inverse. A *field* is a ring that is also a semifield.

Lastly, we define algebraic substructures.

Definition 124. Given a semiring R, a *subsemiring* (*subring*, *subfield*, *subsemifield*) R' of R is a semiring (ring, field, semifield) on a subset of R with the same addition and multiplication operations. We equivalently say that R *extends/is an extension of* R'.

3.2.2 Topology

Firstly, a general note on terminology: In this work, whenever we use the adjective "continuous" to describe a functor, we shall always mean a topological functor, *i.e.* a functor whose underlying functions on objects and on morphisms are continuous in the topological sense (precise definitions to follow in Subsection 3.3.2). We do **not** use this term to refer to a functor that preserves small limits. This use of terminology is consistent with the relevant paper by Baez, Fritz and Leinster [13] discussed in Subsection 1.3.

Here we define a shorthand for a specific type of topological space that we shall use throughout this work.

Definition 125. A *nice* topological space (equivalently, a space equipped with a *nice* topology) in this work will be understood as a topological space with the following properties:

- Every convergent sequence has a unique limit (*i.e.* the space is a *US-space*).
- Every set X that contains the limits of all the convergent sequences contained in X is closed (*i.e.* the topology of the space is *sequential*).

Recall [59, 2.17] that a topology satisfies the T1 separation axiom precisely when singletons are closed. Note that a nice topology in the above sense is always T1.

Definition 126. We define **Top** as the category of topological spaces and continuous maps.

Definition 127. We define **NiceTop** as the category of nice topological spaces and continuous maps.

The category **SeqTop** of sequential spaces is a coreflective subcategory of **Top** [3, Example 4.26A(2)]. Seeing as **Top** is complete and cocomplete [3, Examples 12.6], and because a coreflective subcategory of a small complete and cocomplete accessible category is itself complete and cocomplete [4, Corollaries 2.47, 6.18, 6.29], **SeqTop** is also complete and cocomplete. We now offer a similar result for **NiceTop**.

Lemma 128. **NiceTop** *is a reflective subcategory of the category* **SeqTop** *of sequential spaces.*

Proof. Let F : **SeqTop** $\to$ **NiceTop** be the functor that acts on a space X in the following way:

- For every sequence S in X with a nonempty set of limits L_S, $F(L_S) = l_s$, where l_s is a single point; in other words, we quotient by the equivalence relation where points x and y are equivalent if there exists a sequence that converges to both x and y. Note that this accounts for all points in X, as each is the limit of a constant sequence.

- A set $Q \subseteq F(X)$ is closed if and only if $F^{-1}(Q)$ is closed.

We readily see that $F(X)$ is indeed a topological space: the sets $\emptyset$ and $F(X)$ are closed; intersections of closed sets are closed; finite unions of closed sets are closed.

For a morphism f, set $F(f)(x) = F(f(F^{-1}(x)))$; this is a well-defined function because continuous maps map limits to limits and therefore $F(f(F^{-1}(x)))$ is a single point. Furthermore, $F(f)$ is continuous by construction.

Note that F maps nice spaces to themselves and that a morphism whose target is nice sends all limits of a given sequence to the same point. The latter means that for every sequential space X and every nice space X' there is a 1-1 correspondence between continuous maps $X \to X'$ and continuous maps $F(X) \to X'$; furthermore, the mapping $X \to X'$ factors through the universal morphism $X \to F(X)$ sending $x \mapsto F(x)$ for $x \in X$, which is continuous. Therefore, F is left adjoint to the inclusion **NiceTop** $\to$ **SeqTop**. $\square$

Seeing as **SeqTop** is complete and cocomplete as noted above, and because a reflective subcategory of a small complete and cocomplete accessible category is itself complete and cocomplete [4, Corollaries 2.47, 6.18, 6.24], the above lemma yields the following corollary.

Corollary 129. **NiceTop** *is complete and cocomplete.*

We shall also make use of the following fact:

Remark 130. The discrete topology on a set X is an initial object in the full subcategory of **NiceTop** of topologies on X.

We shall also be using the following shorthand:

Definition 131. A topology τ on a semiring R will be called *structure-respecting* if semiring addition and multiplication are continuous operations $(R, \tau) \times (R, \tau) \to (R, \tau)$.

Note that our structure-respecting condition is stronger than the one used in a lot of the literature, which often only requires pointwise continuity.

Next we define topological semirings. We draw attention to the fact that we use the term "topological semiring" only in conjunction with topologies we have termed "nice". Unlike most literature on topological ring-like structures, we do not require the topology to be Hausdorff (recall [59, 2.17] that a topology satisfies the Hausdorff separation axiom precisely when distinct points have disjoint neighbourhoods).

Definition 132. A *topological semiring* (R, τ_R) is defined to be a semiring R equipped with a nice structure-respecting topology τ_R.

Sometimes we need to restrict to more convenient topological semirings, so we introduce rational-like semirings (we shall explain the terminology shortly). In the following, we slightly abuse our fraction notation, since we do not generally require inverses to be unique; however, the fraction $\frac{1}{x}$ can be thought of as the "canonical" inverse of x.

Definition 133. A topological semiring (Λ, τ) will be called *rational-like* when the following hold:

1. There exists an element $\frac{1}{2} \in \Lambda$ such that[6] $\frac{1}{2} + \frac{1}{2} = 1$.

2. In τ, the sequence $\{\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots\}$ converges to 0.

The notation $\frac{1}{2}$ is used because an element satisfying $\frac{1}{2} + \frac{1}{2} = 1$ is precisely a multiplicative inverse to $2 := 1 + 1$. Moreover, setting[7] $n := \underbrace{1 + \ldots + 1}_{n}$, the following

[6] In the context of commutative (cancellative) semirings, this makes Λ a (cancellative) *midpoint algebra* [27] with respect to the operation $m(a, b) := \frac{1}{2}(a + b)$.

[7] Note that the notation 2^n is unambiguous in this sense.

lemma shows that all simple fractions whose denominators are powers of 2 (exist and)[8] are distinct elements of Λ.

Lemma 134. *For any rational-like topological semiring* (Λ, τ)*, the semiring* Λ *extends* $\mathbb{N}$.

Proof. Together with nontriviality, condition 1 implies that $2 \neq 0$ and condition 2 implies that $2 \neq 1$. Note that each element in the sequence of condition 2 must be different, making the semiring necessarily infinite. Each element $\left(\frac{1}{2}\right)^n$ is an inverse of 2^n [Proof: Base case for $\frac{1}{2}$; suppose $x \cdot 2^n = 1$, then $2^{n+1} \cdot (x \cdot \frac{1}{2}) = (2^n + 2^n) \cdot (x \cdot \frac{1}{2}) = 2^n \cdot x \cdot \frac{1}{2} + 2^n \cdot x \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$; same for multiplication on the other side]. It follows that $2, 2^2, 2^3, \ldots$ are all different [Proof: Suppose $2^{m+n} = 2^n$ for some $n > 0$, then $2^{m+n} \cdot \frac{1}{2^m} = 1$; but $2^{m+n} = 2^m \cdot 2^n$ so we get $2^n = 1$, whence $\frac{1}{2^n} = 1$, which we have rejected], whence each $n \in \mathbb{N}^*$ corresponds to a distinct element of Λ. Thus $\mathbb{N} \subseteq \Lambda$ and furthermore $\mathbb{N}$ is a subsemiring of Λ. $\square$

Finally, we define topological subsemirings.

Definition 135. A topological semiring (R, τ) is a *topological subsemiring* of a topological semiring (R', τ') if R is a subsemiring of R' and τ is a subspace of τ'. We then say that (R', τ') *extends* (R, τ).

Note that any topological semiring extending $(\mathbb{Q}^{\geq 0}, \tau)$ (where τ denotes the inherited standard topology from $\mathbb{R}$) is a rational-like semiring, hence the term. The converse does not hold; a rational-like semiring need not contain, for instance, $\frac{1}{3}$.

[8] Multiplication with natural numbers in this sense is always commutative: $x \cdot (1 + \ldots + 1) = x + \ldots + x = (1 + \ldots + 1) \cdot x$.

3.3 Adiabatic categories, topological weak semimodules and traced categories

Note on notation

Throughout this chapter, the term *source* will be used interchangeably with *domain* and the term *target* will be used interchangeably with *codomain*. We caution the reader that this differs from the use of the term *source* elsewhere in the literature [3].

3.3.1 Adiabatic categories: Definition and properties

In this section, we shall mainly work with what we shall call rational-like adiabatic categories. However, a few of the theorems that follow hold for a more general class of categories, so we first define a generalisation called adiabatic categories. This generalisation will become more important in Subsection 3.3.2, where we introduce topological weak semimodules.

For the definitions in the beginning of this section, we shall use the symbol $x \doteq y$ to denote that either x and y are both undefined or they are both defined and equal. We shall also use the same notation for either a list or a set of indexed elements, *i.e.* $\{s\}_i$ can denote either a set or a list of elements s_i, depending on context that will be made clear. For the sake of brevity, a list is always assumed to be countably infinite unless it is nonterminal in a concatenation operation (or unless otherwise noted), indices are assumed to be the positive integers, and a list is assumed to be sorted by ascending order of indices. We sometimes use $\{s_i\}$ instead of $\{s\}_i$; this is not meant to denote a one-element list. Lastly, S_1S_2 will denote concatenation of lists S_1 and S_2 and $S' \subseteq S$ will denote that S' is an infinite subsequence of S.

Firstly we define a generalisation of adiabatic categories which embodies their linear structure.

Definition 136. Given a topological CC-semiring (Λ, τ), a *weakly linear category* over Λ is a small category $\mathbf{C}$ equipped with the following:

- A monoidal product $\boxplus$ with unit object Θ, associator a, left unit map l and right unit map r. It is also equipped with a symmetry s.[9]

- A family of covariant symmetric monoidal endofunctors λ (with associated natural transformations J_λ) indexed by elements of Λ such that $\lambda_1 \cdot \lambda_2 = \lambda_1 \circ \lambda_2$ (where for clarity we have used $\cdot$ to denote semiring multiplication). Every object in the image of 0 is isomorphic.

- The *splitting and recombination* property (SR): The λ functors are such that for every $X \in \mathrm{Ob}(\mathbf{C})$ and every $\lambda_1, \lambda_2 \in \Lambda$, there exists a natural isomorphism $c_{X,\lambda_1,\lambda_2} : (\lambda_1 + \lambda_2)X \to \lambda_1 X \boxplus \lambda_2 X$.

We shall denote such a weakly linear category as $(\mathbf{C}, \boxplus, \Theta, a, l, r, \lambda, \Lambda, \tau, \{J\}_\lambda, s, c)$.

Note that the above definition differs from the use of the term "weakly linear category" elsewhere in the literature [72].

The reason we require a topology on the semiring is in order to define primary sequences as below.

Definition 137. Let L be a list of n objects in a weakly linear category $\mathbf{C}$ and let I be a subset of $\{1, 2, \ldots, n\}$. Let $M(L, I, \lambda)$ be the monoidal product $((F_1(L_1) \boxplus F_2(L_2)) \boxplus$

[9]Symmetry is necessary for the Cancellation Law below, as well as Theorem 225 that depends on it; however, dropping symmetry retains most of the properties of these categories as well as a weaker order-preserving version of the Cancellation Law. A version of weakly linear categories without the symmetry may thus be useful and appropriate for some physical models.

$\ldots) \boxplus F_n(L_n)$, composed with any combination of associators, where

$$F_i = \begin{cases} \lambda, & i \in I \\ 1, & \text{otherwise} \end{cases}.$$

Given an infinite sequence $\lambda_i \to 0$, define the *primary sequence* $\mathrm{PS}(L, I, \{\lambda_i\})$ in $\mathbf{C}$ as the infinite sequence of objects $\{M(L, I, \lambda_i)\}_i$.

Given a list of morphisms f_j between lists of objects x_j and y_j of the same length, we define $M(f, I, \lambda)$ in the same way as for objects. We can then construct a primary sequence of morphisms in the same way.

Primary sequences are a straightforward generalisation of sequences of the form $a \boxplus \lambda_i b$, with $\lambda_i \to 0$. We allow more "constant" terms like a, more "vanishing" terms like b and in any order. This is in line with Lieb and Yngvason's setup, as the corresponding category (in our framework) is a strictly symmetric strict monoidal preorder where the λ endofunctors are strict.

Before giving the definition of an adiabatic category, we define an auxiliary construction.

Definition 138. Denote by **AllInfiniteSequences**[$\mathbf{C}$] the category whose objects are the infinite sequences $\{X\}_i$ of objects X_i in $\mathbf{C}$, whose morphisms are the infinite sequences $\{f\}_i : \{X\}_i \to \{Y\}_i$ of morphisms $f_i : X_i \to Y_i$ in $\mathbf{C}$, and where composition is defined pointwise by composition in $\mathbf{C}$.

We are now ready to define an adiabatic category. We shall provide two equivalent definitions. We start with the more explicit one, as it exemplifies the connection with Lieb and Yngvason's work and simplifies some of the proofs that follow, for instance by using a familiar construction of a sequential space.

Definition 139. Let (Λ, τ) be a topological CC-semiring. An adiabatic category $\mathbf{C}$ is a weakly linear category $\mathbf{C}$ equipped with a partial function

$$\mathrm{conv} : \mathrm{Mor}(\mathbf{AllInfiniteSequences}[\mathbf{C}]) \to \mathrm{Mor}(\mathbf{C}),$$

which satisfies the following *stability/continuity* properties:

- The *basic stability/continuity* property (Cont): The λ functors are such that the following holds: Let $S_\Lambda := \{\lambda\}_i$ be a sequence tending to 0 and let $X = \mathrm{PS}(x, I, S_\Lambda)$ and $Y = \mathrm{PS}(y, I, S_\Lambda)$ be primary sequences, where x and y have the same length. If for all i we have $\mathrm{Hom}(X_i, Y_i) \neq \emptyset$ then there exists at least one sequence of morphisms $f_i \in \mathrm{Hom}(X_i, Y_i)$ that uniquely define a morphism $\mathrm{conv}(\{f\}_i) : M(x, I, 0) \to M(y, I, 0)$.

- The following "convergence" properties:

  1. The function conv maps any constant sequence to the sole distinct element of the sequence.
  2. If $\mathrm{conv}(S)$ is defined and $S' \subseteq S$ then $\mathrm{conv}(S')$ is defined and $\mathrm{conv}(S') = \mathrm{conv}(S)$.
  3. If $\mathrm{conv}(S)$ is either undefined or $\mathrm{conv}(S) \neq f$ then there exists $S' \subseteq S$ such that there does not exist $S'' \subseteq S'$ with $\mathrm{conv}(S'') = f$.

- The following "linearity" properties:

  (i) Given sequences $S_1 = \{g_i : B_i \to C_i\}$ and $S_2 = \{h_i : A_i \to B_i\}$, suppose that $\mathrm{conv}(S_1)$ and $\mathrm{conv}(S_2)$ are defined and construct the sequence $S = \{f_i = g_i \circ h_i\}$. Then $\mathrm{conv}(S)$ is defined and $\mathrm{conv}(S) = \mathrm{conv}(S_1) \circ \mathrm{conv}(S_2)$.

  (ii) $\mathrm{conv}(\{\mathrm{id}_{X_i}\}) = \mathrm{id}_{M(L_X, I, 0)}$.

(iii) (a) Given sequences $S_1 = \{g_i : A_i \to B_i\}$ and $S_2 = \{h_i : C_i \to D_i\}$, suppose that $\mathrm{conv}(S_1)$ and $\mathrm{conv}(S_2)$ are defined and construct the sequence $S = \{f_i = g_i \boxplus h_i\}$. Then $\mathrm{conv}(S)$ is defined and $\mathrm{conv}(S) = \mathrm{conv}(S_1) \boxplus \mathrm{conv}(S_2)$.

(b) If A is a sequence of associators between primary sequences, $\mathrm{conv}(A) = a$ where a is the appropriate associator.

(iv) $\mathrm{conv}(\{\lambda x_i\}) \doteq \lambda \mathrm{conv}(\{x\}_i)$.

(v) $\mathrm{conv}(\{f_i \boxplus g_i\}_i) \doteq s_{A,B} \circ \mathrm{conv}(\{g_i \boxplus f_i\}_i) \circ s_{B,A}$ for the appropriate objects B and A.

We shall collectively refer to the "convergence" and "linearity" properties of conv as the *extended stability/continuity* properties.

Note that the notations $\lambda_1\lambda_2$ and λ^n for $n \in \mathbb{Z}$ are unambiguous in the categorical context. Specifically, multiplication of the elements $\lambda \in \Lambda$ is composition of the functors λ; for $n \geq 0$, raising the element $\lambda \in \Lambda$ to the nth power is the same as applying the λ functor n times; for $n < 0$, raising the element $\lambda \in \Lambda$ to the nth power is the same as applying the inverse of the λ functor n times (the latter may only be used if Λ is a semifield).

We shall often refer to the left/right unitor associated to an object $0X$ as a unit. This is defined by the composition of l (respectively r) with the unique isomorphism $u_X : 0X \to \Theta$.

The stability function conv is to be linked to convergent sequences in a certain topological space (see Lemma 157).

Before giving the alternative definition, we advise the reader to revisit Remark 117 in light of Definition 139, and keep in mind the physical motivation behind each element of the definition. The physical intuition applies to the following subcass of adiabatic categories.

Definition 140. A rational-like adiabatic category is an adiabatic category over a rational-like topological semiring.

We now proceed to give an equivalent definition of adiabatic categories that is more elegant and succinct. We first need some auxiliary constructions.

Definition 141. For any category $\mathbf{C}$, a category $\mathbf{S}$ is defined to be a *sequence category* of $\mathbf{C}$ if it is a subcategory of the category $\mathbf{AllInfiniteSequences}[\mathbf{C}]$.

Sequence categories can help define convergent sequences. We are interested in convergence related to the linear structure, so we shall focus on the following class of sequence categories.

Lemma 142. *Given a weakly linear category $\mathbf{C}$, one can construct a weakly linear category $\overline{\mathbf{C}}$ as follows:*

Let $X = (X_1, X_2, \ldots)$ and $Y = (Y_1, Y_2, \ldots)$ denote primary sequences in $\mathbf{C}$.

- *The objects of $\overline{\mathbf{C}}$ are the primary sequences in $\mathbf{C}$ as well as all sequences of the form $(X_1 \boxplus Y_1 \boxplus \cdots \boxplus Z_1, X_2 \boxplus Y_2 \boxplus \cdots \boxplus Z_2, \ldots)$ where each element of the infinite sequence has the same number of terms.*
- *$\overline{\mathbf{C}}$ is a full subcategory of $\mathbf{AllInfiniteSequences}[\mathbf{C}]$.*
- *For X and Y as above, $X \overline{\boxplus} Y = (X_1 \boxplus Y_1, X_2 \boxplus Y_2, \ldots)$.*
- *For X as above, $\overline{\lambda} X = (\lambda X_1, \lambda X_2, \ldots)$ for $\lambda \in \Lambda$.*

Proof. We check that $\overline{\mathbf{C}}$ is indeed a weakly linear category. This can be done by inspection; we readily observe the following:

- $\overline{\mathbf{C}}$ is a category where identities are sequences of identities in $\mathbf{C}$.

- $\overline{\boxplus}$ is indeed a symmetric monoidal product with $\overline{\Theta} = (\Theta, \Theta, \ldots)$. Concretely, for $X = (X_1, X_2, \ldots)$, $Y = (Y_1, Y_2, \ldots)$ and $Z = (Z_1, Z_2, \ldots)$ we have $\overline{l}_X = (l_{X_1}, l_{X_2}, \ldots)$, $\overline{r}_X = (r_{X_1}, r_{X_2}, \ldots)$, $\overline{a}_{X,Y,Z} = (a_{X_1,Y_1,Z_1}, a_{X_2,Y_2,Z_2}, \ldots)$ and $\overline{s}_{X,Y} = (s_{X_1,Y_1}, s_{X_2,Y_2}, \ldots)$.
- Any $\overline{\lambda}$ preserves composition and identity because λ does and is therefore also a covariant functor. Similarly, the functors $\overline{\lambda}$ are symmetric monoidal, multiplication coincides with composition and $\overline{0}$ maps any object to some object isomorphic to the unit; all these properties follow immediately form the fact that the functors λ have the same properties. We also have $\overline{J}_{\lambda,X,Y} = (J_{\lambda,X_1,Y_1}, J_{\lambda,X_2,Y_2}, \ldots)$.
- We have that $\overline{c}_{X,\lambda_1,\lambda_2} = (c_{X_1,\lambda_1,\lambda_2}, c_{X_2,\lambda_1,\lambda_2}, \ldots)$; it is natural because c is natural.

□

Definition 143. Call $\overline{\mathbf{C}}$ as constructed in Lemma 142 the *primary sequence category* of $\mathbf{C}$.

The idea here is to use sequence categories in order to define a notion of convergence, and then use the primary sequence category as a "distinguished" sequence category that links the convergence structure with the weakly linear structure and gives rise to adiabatic categories.

Definition 144. We shall refer to monoidal products in the primary sequence category of some weakly linear category $\mathbf{C}$ as *monoidal products of primary sequences* in $\mathbf{C}$.

Note that if S is a primary sequence (of morphisms or of objects), then every subsequence of S is also primary; if S is a monoidal product of primary sequences, then every subsequence of S is also a monoidal product of primary sequences. More generally, taking subsequences is functorial in $\overline{\mathbf{C}}$; we shall use this fact to recover a categorical description of convergence.

Definition 145. Let $\mathbf{S}$ be a sequence category of some category $\mathbf{C}$. Let I be an infinite list of strictly increasing positive integers; a *subsequence endofunctor* F_I of $\mathbf{S}$ is a functor acting on morphisms as follows: an infinite sequence S of morphisms of $\mathbf{C}$, with $S \in \mathrm{Mor}(\mathbf{S})$, is mapped to the infinite subsequence of S corresponding to the elements with indices in I.

In the definition above, given a category $\mathbf{C}$, note that for an arbitrary sequence category $\mathbf{S}$ of $\mathbf{C}$ and for arbitrary I the functor $F_I : \mathbf{S} \to \mathbf{S}$ may be undefined. Of course, all subsequence endofunctors are defined when $\mathbf{S} = \mathbf{AllInfiniteSequences}[\mathbf{C}]$. Crucially, we observe the following:

Remark 146. In the primary sequence category of a weakly linear category, the subsequence endofunctors F_I are defined for all infinite lists I of strictly increasing positive integers.

We now define a class of categories with a convergence structure.

Definition 147. Given a category $\mathbf{C}$, let $\mathbf{S}$ be a sequence category of $\mathbf{C}$ such that subsequence endofunctors F_I are defined for all infinite lists I of strictly increasing positive integers. A *convergence category* $(\mathbf{S}', \mathrm{conv})$ (with respect to $\mathbf{S}$ and $\mathbf{C}$) is a subcategory $\mathbf{S}'$ of $\mathbf{S}$ (defined by the injection $i : \mathbf{S}' \to \mathbf{S}$) that is closed under all subsequence endofunctors F and is equipped with a functor $\mathrm{conv} : \mathbf{S}' \to \mathbf{C}$ satisfying the following properties:

- conv maps any constant sequence of morphisms to the sole distinct element of the sequence.
- The following triangle commutes:

$$\begin{tikzcd}
\mathbf{S}' \arrow[r, "F"] \arrow[dr, "\mathrm{conv}"'] & \mathbf{S}' \arrow[d, "\mathrm{conv}"] \\
& \mathbf{C}
\end{tikzcd}$$

- Let f be a morphism in $\mathbf{S}$. If for every subsequence endofunctor F_k there exists a subsequence endofunctor F_j such that $(F_j \circ F_k)(f)$ is in the image of i, then f is also in the image of i and moreover $\operatorname{conv}((i^{-1} \circ F_j \circ F_k)(f)) = \operatorname{conv}(i^{-1}(f))$.

So at this point we have categorical notions of both weak linearity and convergence. Adiabatic categories integrate these two notions, as shown in their alternative definition:

Definition 148. An *adiabatic category* $\mathbf{C}$ is a weakly linear category $\mathbf{C}$ where the following hold.

There exists a monoidal subcategory $\overline{\mathbf{C}}'$ of $\overline{\mathbf{C}}$, called the *stability category* of $\mathbf{C}$, which is surjective on objects and closed under the λ endofunctors, such that $\overline{\mathbf{C}}'$ has at least one morphism in each homset that is nonempty in $\overline{\mathbf{C}}$ and such that there exists a symmetric strict monoidal functor $\operatorname{conv} : \overline{\mathbf{C}}' \to \mathbf{C}$, called the *stability functor* of $\mathbf{C}$, making the following square commute for each λ:

$$\begin{tikzcd} \overline{\mathbf{C}}' \arrow[r, "\mathrm{conv}"] \arrow[d, "\lambda"] & \mathbf{C} \arrow[d, "\lambda"] \\ \overline{\mathbf{C}}' \arrow[r, "\mathrm{conv}"] & \mathbf{C} \end{tikzcd}$$

Furthermore, $(\overline{\mathbf{C}}', \operatorname{conv})$ is a convergence category with respect to $\overline{\mathbf{C}}$ and $\mathbf{C}$.

Before showing the equivalence of the two definitions for adiabatic categories, we note that we shall be using an extension of conv. This extension is based on a basic property of convergence, encapsulated in the following two categories:

Definition 149. Define the *extended primary sequence category* of $\mathbf{C}$ as the category where objects are sequences of the form QS, where S is an object in the primary sequence category of $\mathbf{C}$, and morphisms are sequences of the form QS, where S is a morphism in the primary sequence category of $\mathbf{C}$.

Define the *extended stability category* $\overline{\mathbf{CC}}'$ of $\mathbf{C}$ as the category where objects are sequences of the form QS, where S is an object in the stability category of $\mathbf{C}$, and morphisms are sequences of the form QS, where S is a morphism in the stability category of $\mathbf{C}$.

Consider the following "category of convergence categories".

Definition 150. Let $\mathbf{S}$ be a sequence category of a category $\mathbf{C}$. By $\mathbf{ConvCat}[\mathbf{S}, \mathbf{C}]$ denote the following category: An object of $\mathbf{ConvCat}[\mathbf{S}, \mathbf{C}]$ is a convergence category $(\mathbf{C_i}, \mathrm{conv}_i)$ of $\mathbf{C}$ and a morphism $F : (\mathbf{C_1}, \mathrm{conv}_1) \to (\mathbf{C_2}, \mathrm{conv}_2)$ in $\mathbf{ConvCat}[\mathbf{S}, \mathbf{C}]$ is a functor $F : \mathbf{C_1} \to \mathbf{C_2}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C_1} \arrow[r, "\mathrm{conv}_1"] \arrow[d, "F"'] & \mathbf{C} \\
\mathbf{C_2} \arrow[ur, "\mathrm{conv}_2"'] &
\end{tikzcd}$$

For an adiabatic category $\mathbf{C}$ we define an extension $\mathrm{conv} : \overline{\mathbf{CC}}' \to \mathbf{C}$ of $\mathrm{conv} : \overline{\mathbf{C}}' \to \mathbf{C}$ via the canonical embedding $\overline{\mathbf{C}}' \to \overline{\mathbf{CC}}'$, which induces a unique embedding $(\overline{\mathbf{C}}', \mathrm{conv}) \to (\overline{\mathbf{CC}}', \mathrm{conv})$ in $\mathbf{ConvCat}[\mathbf{S}, \mathbf{C}]$. In practice, we shall not distinguish between $\mathrm{conv} : \overline{\mathbf{C}}' \to \mathbf{C}$ and $\mathrm{conv} : \overline{\mathbf{CC}}' \to \mathbf{C}$.

We now show that the two definitions of adiabatic categories are equivalent.

Lemma 151. *Definitions 139 and 148 of an adiabatic category are equivalent.*

Proof. Functoriality of conv*:* In Definition 139, conv is a partial function on infinite sequences of morphisms. These morphisms are in the primary homsets and products thereof (property (iii)), as well as the sequences that are eventually of that form (property 3); these are precisely the homsets of an extended stability category. Furthermore, conv preserves composition (property (i)) and identity (property (ii)).

Stability category surjective on objects of primary sequence category: By Cont property.

Symmetric strict monoidality of conv*:* Symmetry corresponds to property (v) and strict monoidality corresponds to property (iii).

Commutative diagram: The commutative square in Definition 148 corresponds to property (iv).

Convergence category: Property 1 is the same in Definitions 139 and 147, the commutative triangle corresponds to property 2 and the last property in Definition 147 corresponds to property 3. $\square$

A sensible choice of functors between adiabatic categories is one that preserves the monoidal structure and the action of the λ endofunctors.

Definition 152. Let (Λ_1, τ_1) and (Λ_2, τ_2) be topological CC-semirings with a continuous semiring homomorphism $h : (\Lambda_1, \tau_1) \to (\Lambda_2, \tau_2)$. Define an *adiabatic functor* $f : \mathbf{C_1} \to \mathbf{C_2}$ between adiabatic categories $\mathbf{C_1}$ and $\mathbf{C_2}$ over (Λ_1, τ_1) and (Λ_2, τ_2) respectively as a strict monoidal functor such that the following squares commute in $\mathbf{Cat}$ for all $\lambda \in \Lambda$, where $\overline{\mathbf{C_1}}'$ and $\overline{\mathbf{C_2}}'$ are the respective stability categories of $\mathbf{C_1}$ and $\mathbf{C_2}$:

$$\begin{tikzcd} \mathbf{C_1} \arrow[r, "\lambda"] \arrow[d, "f"] & \mathbf{C_1} \arrow[d, "f"] \\ \mathbf{C_2} \arrow[r, "h(\lambda)"] & \mathbf{C_2} \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathbf{C_1}}' \arrow[r, "\mathrm{conv}_1"] \arrow[d, "f\times f\times\cdots"'] & \mathbf{C_1} \arrow[d, "f"] \\ \overline{\mathbf{C_2}}' \arrow[r, "\mathrm{conv}_2"] & \mathbf{C_2} \end{tikzcd}$$

Definition 153. Define $\mathbf{ad}$ as the category of adiabatic categories and adiabatic functors. Define $\mathbf{ad_\Lambda}$ as its subcategory of adiabatic categories over the topological semiring Λ where all the semiring homomorphisms are identities.

To retrieve an exact analogue to Lieb and Yngvason's framework, we introduce the special case of LY-adiabatic categories, defined below.

Definition 154. A category is *thin* if there is at most one morphism in each homset.

Definition 155. An LY-adiabatic category is a thin rational-like adiabatic category.

We may also define a generalised LY-adiabatic category as a thin adiabatic category, which would be a different way to generalise Lieb and Yngvason's work.

A special case that will later be linked to 2-semivector spaces is when the rational-like adiabatic category has a strict monoidal structure with a strict symmetry, the structural functors λ are strict monoidal and the isomorphism c is an identity.

Definition 156. A *tyrannical* adiabatic category is a strict symmetric strict monoidal adiabatic category where the structural endofunctors λ are strict and the isomorphism c is an identity.[10]

We now present some results about adiabatic categories, starting with a lemma that offers some insight into the role of conv.

Lemma 157. *Let* $\mathbf{C}$ *be any category admitting a convergence category* $(\mathbf{G}, \mathrm{conv})$*. Then it is possible to define appropriate T1 topologies on* $\mathrm{Ob} := \mathrm{Ob}(\mathbf{C})$ *and* $\mathrm{Mor} := \mathrm{Mor}(\mathbf{C})$ *such that in each the function* conv *maps each convergent sequence to its limit and is undefined on divergent sequences.*

In particular, given an adiabatic category with monoidal unit $0X$ *(for some* $X \in \mathrm{Ob}(\mathbf{C})$*), these are the finest topologies such that the topology on* Ob *agrees with* τ *on the* λ *functors.*

Proof. By Lemma 151, conv satisfies stability properties 1, 2 and 3 of Definition 139. The fact that these specific properties of the conv function give rise to a topology where conv yields the limit of a sequence, as well as the definition of said topology, can be found in earlier literature [41]; here we also provide a brief proof of correctness for the sake of completeness.

[10]The term "tyrannical" was chosen as a reminder that every aspect of these categories is strict.

Firstly, we define a closed-set topology on Ob × Ob.

Let $\mathcal{S}$ denote the set of all pairs of objects in **G**; for the specific application to adiabatic categories, this is the set of all pairs $X = \{M(L_X, I, \lambda_i)\}, Y = \{M(L_Y, I, \lambda_i)\}$ of primary sequences for which $\emptyset \notin \{\mathrm{Hom}(X_i, Y_i)\}$. For $Q \subseteq \mathrm{Ob} \times \mathrm{Ob}$, let $\mathcal{S}_Q$ denote the set of all sequences in $\mathcal{S}$ contained in Q. Define $\pi : \mathcal{S} \to \mathrm{Ob} \times \mathrm{Ob}$ as $\pi : (X, Y) \mapsto (\mathrm{conv}(X), \mathrm{conv}(Y))$[11] and let the closed sets of the topology be those sets $Q \subseteq \mathrm{Ob} \times \mathrm{Ob}$ where $\pi(\mathcal{S}_Q) = Q$.

We check that this family of sets is indeed a valid closed-set topology:

- The sets Ob × Ob and $\emptyset$ are closed:

  This is immediate for $\emptyset$; for Ob × Ob, $\pi(\mathcal{S}_{\mathrm{Ob}\times\mathrm{Ob}}) \subseteq \mathrm{Ob} \times \mathrm{Ob}$ is immediate and $\mathrm{Ob} \times \mathrm{Ob} \subseteq \pi(\mathcal{S}_{\mathrm{Ob}\times\mathrm{Ob}})$ is seen to hold by functoriality of conv and stability property 1 (in adiabatic categories, set $\lambda_i = 0$ for all i).

- Arbitrary intersections of closed sets are closed:

  Let P_i denote a possibly infinite family of closed sets. $\mathcal{S}_{\bigcap P_i} \subseteq \bigcap \mathcal{S}_{P_i}$, therefore

  $\pi(\mathcal{S}_{\bigcap P_i}) \subseteq \pi(\bigcap \mathcal{S}_{P_i}) = \bigcap \pi(\mathcal{S}_{P_i}) \subseteq \bigcap P_i$,

  where the last relation follows form the hypothesis. So $\bigcap P_i$ is closed.

- Finite unions of closed sets are closed:

  Let P and Q be closed sets; then $\pi(\mathcal{S}_Q) \cup \pi(\mathcal{S}_P) \subseteq Q \cup P$, so it suffices to show that $\pi(\mathcal{S}_{Q\cup P}) \subseteq \pi(\mathcal{S}_Q) \cup \pi(\mathcal{S}_P)$.

  For any sequence $S \in \mathcal{S}_{Q\cup P} - \mathcal{S}_Q - \mathcal{S}_P$, each element of S is either in Q or in P. Partition the elements of S into a subsequence S_Q containing elements of Q and a subsequence $\bar{S_Q}$ containing elements of $P - Q$. If $\bar{S_Q}$ is finite, it must be that $\pi(S) = \pi(S_Q) \in \pi(\mathcal{S}_Q)$ and we are done; similarly if S_Q is finite.

[11] Specifically for an adiabatic category, we note that $\pi : (X, Y) \mapsto (M(L_X, I, 0), M(L_Y, I, 0))$.

Otherwise, observe that $S_Q \in \mathcal{S}_Q$ and $\bar{S_Q} \in \mathcal{S}_P$ by functoriality of conv and stability property 2 (in adiabatic categories, this is owing to the fact that λ_i form a convergent sequence, and therefore its subsequences must converge to the same limit); it follows that $\pi(S) = \pi(S_Q) = \pi(\bar{S_Q}) \in \pi(\mathcal{S}_Q) \cap \pi(\mathcal{S}_P)$.

For adiabatic categories it is obvious that the function π coincides with the limit of sequences in this space; in the general case, the argument is similar to Mor (see below). Also note that the topology is T1, as singletons are closed.

We can now define the required topology on Ob as the one inherited from the diagonal of Ob $\times$ Ob. This topology will also have the required properties (it is T1, π is a limit function and sets containing their limits are closed).

Similarly to the above construction, define a closed-set topology on Mor as follows. Let $\mathcal{S}$ denote the set of all infinite sequences $\{f_i\}$ with $f_i \in \mathrm{Hom}(X_i, Y_i)$, for which $\mathrm{conv}(\{f_i\})$ is defined. For $R \subseteq$ Mor, let $\mathcal{S}_R$ denote the set of all sequences in $\mathcal{S}$ contained in R. Let the closed sets of the topology be those sets $R \subseteq$ Mor for which $\mathrm{conv}(\mathcal{S}_R) = R$.

We check that this family of sets is indeed a valid closed-set topology:

- The sets Mor and $\emptyset$ are closed:

  This is immediate for $\emptyset$; for Mor, $\mathrm{conv}(\mathcal{S}_{\mathrm{Mor}}) \subseteq \mathrm{Mor}$ is immediate and $\mathrm{Mor} \subseteq \mathrm{conv}(\mathcal{S}_{\mathrm{Mor}})$ follows from stability property 1; for adiabatic categories, we see this by setting $\lambda_i = 0$ for all i.

- Arbitrary intersections of closed sets are closed:

  Let P_i denote a possibly infinite family of closed sets. $\mathcal{S}_{\bigcap P_i} \subseteq \bigcap \mathcal{S}_{P_i}$, therefore

  $\mathrm{conv}(\mathcal{S}_{\bigcap P_i}) \subseteq \mathrm{conv}(\bigcap \mathcal{S}_{P_i}) = \bigcap \mathrm{conv}(\mathcal{S}_{P_i}) \subseteq \bigcap P_i$,

  where the last relation follows form the hypothesis. So $\bigcap P_i$ is closed.

- Finite unions of closed sets are closed:

  Let P and Q be closed sets; then $\mathrm{conv}(\mathcal{S}_Q) \cup \mathrm{conv}(\mathcal{S}_P) \subseteq Q \cup P$, so it suffices to show that $\mathrm{conv}(\mathcal{S}_{Q \cup P}) \subseteq \mathrm{conv}(\mathcal{S}_Q) \cup \mathrm{conv}(\mathcal{S}_P)$.

  For any sequence $S \in \mathcal{S}_{R_1 \cup R_2} - \mathcal{S}_{R_1} - \mathcal{S}_{R_2}$, each element of S is either in R_1 or in R_2. Let S_1 be the largest subsequence of S contained in R_1 and S_2 the largest subsequence of S contained in R_2. If S_1 is infinite, from stability property 2 it follows that $\mathrm{conv}(S) = \mathrm{conv}(S_1) \in \mathrm{conv}(\mathcal{S}_{R_1})$; similarly if S_2 is infinite.

Note that the topology is T1, as singletons are closed.

The function conv *maps a convergent sequence to its limit and is undefined for divergent sequences:*

Open sets are defined as the complements of closed sets. Therefore, a set R is open if and only if the following holds: If $\mathrm{conv}(S) \subseteq R$ for some sequence S then $S \cap R^c$ is finite (where R^c denotes the complement of R). But that is the definition of the limit of the sequence S.

The function conv *is defined on all convergent sequences:*

This was proven by Kisyński [41]; notably, the proof uses stability property 3.

Finally, observe that these topologies are indeed the finest topologies given the convergent sequences in these spaces: In every space, a set contains the limits of the sequences it contains if it is closed; here, the converse also holds, and therefore one cannot add any more closed sets to the topology without violating the convergence. $\square$

Since the spaces constructed above are completely determined by their convergent sequences, each has a *sequential topology* on the objects and morphisms of **C** respectively.

A strong convexity property is introduced as an axiom on an associated topology in Lieb and Yngvasson. However, a weaker property (given in the following lemma) can be proven from first principles independently of topology, and is actually sufficient for the main result.

Lemma 158 (Convexity)**.** *For any objects X, Y_1, Y_2 in an adiabatic category $(\mathbf{C}, \boxplus, \Theta, a, l, r, \lambda, \Lambda, \tau, \{J\}_\lambda, s, c, \mathrm{conv})$ the following hold true:*

1. *If there exist morphisms $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ then there exists a morphism $g : X \to \lambda Y_1 \boxplus (1-\lambda)Y_2$ whenever $1-\lambda$ is defined.*

2. *If there exist morphisms $f_1' : Y_1 \to X$ and $f_2' : Y_2 \to X$ then there exists a morphism $g' : \lambda Y_1 \boxplus (1-\lambda)Y_2 \to X$ whenever $1-\lambda$ is defined.*

Proof. Take $g = (\lambda f_1 \boxplus (1-\lambda) f_2) \circ c_{X,\lambda,1-\lambda}$ and $g' = c^{-1}_{X,\lambda,1-\lambda} \circ (\lambda f_1' \boxplus (1-\lambda) f_2')$. ☐

The lemma below lists equivalent statements whose physical interpretation is "it is impossible to create something out of nothing".

Lemma 159. *Let Λ be a topological semiring extending $(\mathbb{Q}^{\geq 0}, \tau)$ where τ denotes the inherited standard topology from $\mathbb{R}$. For all objects X and Y in an adiabatic category $\mathbf{C}$ over Λ, the following statements are equivalent:*

1. *For every $Y \ncong \Theta$ and for every X, $\mathrm{Hom}(X, X \boxplus Y) = \emptyset$.*

2. *For every $X \ncong \Theta$, $\mathrm{Hom}(\Theta, X) = \emptyset$.*

3. *For every $X \ncong \Theta$ and $\lambda > 1$, $\mathrm{Hom}(X, \lambda X) = \emptyset$.*

Any of the above statements imply that for $X \ncong \Theta$ and $\lambda' \neq 1$, $X \ncong \lambda' X$ (4).

Proof. $3 \Rightarrow 4$ is obvious for $\lambda' > 1$; for $\lambda' < 1$, take $\frac{1}{\lambda'} i$ (where i is the isomorphism $X \to \lambda' X$) for proof by contradiction.

$\neg 1 \Rightarrow \neg 2$: Given $f : X \to X \boxplus Y$, one can construct the morphisms $f_n : \frac{1}{2^n}X \boxplus \Theta \to \frac{1}{2^n}X \boxplus Y$; explicitly, set

$$f_n = (\mathrm{id}_{\frac{1}{2^n}X} \boxplus c^{-1}_{Y,\frac{1}{2},\frac{1}{2}}) \circ \frac{1}{2}(a^{-1}_{\frac{1}{2^{n-1}}X,Y,Y} \circ (f_{n-1} \boxplus \mathrm{id}_Y) \circ f_{n-1} \circ l_{\frac{1}{2^{n-1}}X}) \circ J^{-1}_{\frac{1}{2^n},X,\Theta} \circ (\mathrm{id}_{\frac{1}{2^n}X} \boxplus \epsilon_{\frac{1}{2^n}})$$

where $f_0 = f$, $\epsilon_{\frac{1}{2^n}}$ is the isomorphism $\Theta \to \frac{1}{2^n}\Theta$ and $n \in \mathbb{N}$ tends to infinity. Then by Cont there exists a sequence S of morphisms in $\mathrm{Hom}(\frac{1}{2^n}X \boxplus \Theta, \frac{1}{2^n}X \boxplus Y)$ such that $\mathrm{conv}(S) = g : 0X \boxplus \Theta \to 0X \boxplus Y$. So $l_{0X,Y} \circ g \circ l^{-1}_{0X,\Theta}$ gives us a morphism $\Theta \to Y$, where by l_{0X} we denote the isomorphism associated to $0X$ as a unit.

$\neg 3 \Rightarrow \neg 2$: Given $f : X \to \lambda X$, construct

$$(\epsilon^{-1}_{\frac{1}{\lambda^n}} \boxplus \mathrm{id}_{\frac{1}{\lambda^n}X}) \circ l^{-1}_X \circ \frac{1}{\lambda}f \circ \frac{1}{\lambda^2}f \circ \ldots \circ \frac{1}{\lambda^n}f \circ r_{\frac{1}{\lambda^n}X} : \frac{1}{\lambda^n}X \boxplus \Theta \to \frac{1}{\lambda^n}\Theta \boxplus X$$

where $n \in \mathbb{N}$ tends to infinity. Then by Cont there exists a sequence S of morphisms in $\mathrm{Hom}(\frac{1}{\lambda^n}X \boxplus \Theta, \frac{1}{\lambda^n}\Theta \boxplus X)$ such that $\mathrm{conv}(S) = g : 0X \boxplus \Theta \to 0\Theta \boxplus X$. Then $l_{0\Theta,X} \circ g \circ l^{-1}_{0X,\Theta}$ gives us a morphism $\Theta \to X$.

$\neg 2 \Rightarrow \neg 1$: Given $f : \Theta \to X$, construct $(f \boxplus \mathrm{id}_X) \circ l^{-1}_X : X \to (X, X)$.

$\neg 2 \Rightarrow \neg 3$: Given $f : \Theta \to X$, construct $2(c^{-1}_{X,\frac{1}{2},\frac{1}{2}} \circ \frac{1}{2}((f \boxplus \mathrm{id}_X) \circ l^{-1}_X)) : X \to 2X$. $\square$

Similarly to the above, the corollary below lists equivalent statements whose physical interpretation is "it is impossible to make something disappear". It is given without proof, as it is the dual of Lemma 159.

Lemma 160. *Let Λ be a topological semiring extending $(\mathbb{Q}^{\geq 0}, \tau)$ where τ denotes the inherited standard topology from $\mathbb{R}$. For all objects X and Y in an adiabatic category* **C** *over Λ, the following statements are equivalent:*

1. *For every $Y \ncong \Theta$ and for every X, $\mathrm{Hom}(X \boxplus Y, X) = \emptyset$.*

2. *For every $X \ncong \Theta$, $\mathrm{Hom}(X, \Theta) = \emptyset$.*

3. For every $X \not\cong \Theta$ and $\lambda < 1$, $\mathrm{Hom}(X, \lambda X) = \emptyset$.

Any of the above statements imply that for $X \not\cong \Theta$ and $\lambda' \neq 1$, $X \not\cong \lambda' X$ (4).

Call a set "continuously infinite" if it has the cardinality of the real numbers. A morphism $X \to \lambda X$ with $\lambda \neq 1$ induces a continuously infinite set of countably infinite chains; call such a chain a *stream*. A stream goes *upwards* if $\lambda > 1$ and *downwards* if $\lambda < 1$.

The theorem that follows is a generalisation of an observation by Lieb and Yngvason [51].

Theorem 161 (Cancellation law)**.** *In a rational-like adiabatic category, given a morphism*

$$f : X \boxplus Z \to Y \boxplus Z,$$

there exists a morphism $X \to Y$.

Proof. Construct the morphism f_1 by this chain:

$$\begin{tikzcd}
X \boxplus \frac{1}{2}Z \arrow[r, "c_{X,\frac{1}{2},\frac{1}{2}} \boxplus \mathrm{id}_{\frac{1}{2}Z}"] & (\frac{1}{2}X \boxplus \frac{1}{2}X) \boxplus \frac{1}{2}Z \arrow[r, "a^{-1}_{\frac{1}{2}X,\frac{1}{2}X,\frac{1}{2}Z}"] & \frac{1}{2}X \boxplus (\frac{1}{2}X \boxplus \frac{1}{2}Z) \arrow[d, "\mathrm{id}_{\frac{1}{2}X} \boxplus J^{-1}_{\frac{1}{2},X,Z}"] \\
\frac{1}{2}X \boxplus (\frac{1}{2}Y \boxplus \frac{1}{2}Z) \arrow[d, "a_{\frac{1}{2}X,\frac{1}{2}Y,\frac{1}{2}Z}"'] & \frac{1}{2}X \boxplus \frac{1}{2}(Y \boxplus Z) \arrow[l, "\mathrm{id}_{\frac{1}{2}X} \boxplus J_{\frac{1}{2},Y,Z}"'] & \frac{1}{2}X \boxplus \frac{1}{2}(X \boxplus Z) \arrow[l, "\mathrm{id}_{\frac{1}{2}X} \boxplus \frac{1}{2}f"'] \\
(\frac{1}{2}X \boxplus \frac{1}{2}Y) \boxplus \frac{1}{2}Z \arrow[r, "s_{\frac{1}{2}X,\frac{1}{2}Y} \boxplus \mathrm{id}_{\frac{1}{2}Z}"] & (\frac{1}{2}Y \boxplus \frac{1}{2}X) \boxplus \frac{1}{2}Z \arrow[r, "a^{-1}_{\frac{1}{2}Y,\frac{1}{2}X,\frac{1}{2}Z}"] & \frac{1}{2}Y \boxplus (\frac{1}{2}X \boxplus \frac{1}{2}Z) \arrow[d, "\mathrm{id}_{\frac{1}{2}Y} \boxplus J^{-1}_{\frac{1}{2},X,Z}"] \\
\frac{1}{2}Y \boxplus (\frac{1}{2}Y \boxplus \frac{1}{2}Z) \arrow[d, "a_{\frac{1}{2}Y,\frac{1}{2}Y,\frac{1}{2}Z}"'] & \frac{1}{2}Y \boxplus \frac{1}{2}(Y \boxplus Z) \arrow[l, "\mathrm{id}_{\frac{1}{2}Y} \boxplus J_{\frac{1}{2},Y,Z}"'] & \frac{1}{2}Y \boxplus \frac{1}{2}(X \boxplus Z) \arrow[l, "\mathrm{id}_{\frac{1}{2}Y} \boxplus \frac{1}{2}f"'] \\
(\frac{1}{2}Y \boxplus \frac{1}{2}Y) \boxplus \frac{1}{2}Z \arrow[r, "s_{\frac{1}{2}Y,\frac{1}{2}Y} \boxplus \mathrm{id}_{\frac{1}{2}Z}"] & (\frac{1}{2}Y \boxplus \frac{1}{2}Y) \boxplus \frac{1}{2}Z \arrow[r, "c^{-1}_{Y,\frac{1}{2},\frac{1}{2}} \boxplus \mathrm{id}_{\frac{1}{2}Z}"] & Y \boxplus \frac{1}{2}Z
\end{tikzcd}$$

So $f : X \boxplus Z \to Y \boxplus Z$ implies $f_1 : X \boxplus \frac{1}{2}Z \to Y \boxplus \frac{1}{2}Z$, and by extension the existence of a *cancellation sequence* $f_n : X \boxplus \frac{1}{2^n} Z \to Y \boxplus \frac{1}{2^n} Z$ for n tending to infinity. By Cont there must exist a sequence S of morphisms in $\mathrm{Hom}(X \boxplus \frac{1}{2^n} Z, Y \boxplus \frac{1}{2^n} Z)$ such that $\mathrm{conv}(S) = g : X \boxplus 0Z \to Y \boxplus 0Z$. In the case that $\mathrm{conv}(\{f_n\})$ is defined, we set $S = \{f_n\}$, otherwise there is in general no canonical choice.[12] Define a morphism $\mathrm{CL}_Z(f) : X \to Y$ by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{CL}_Z(f)"] & Y \\
X \boxplus 0Z \arrow[u, "r_{0Z,X}"] \arrow[r, "\mathrm{conv}(S)"] & Y \boxplus 0Z \arrow[u, "r_{0Z,Y}"']
\end{tikzcd}$$

where r_{0Z} is the isomorphism associated to $0Z$ as a unit. □

We shall always use the notation $\mathrm{CL}_Z(f)$ for a morphism constructed as above. Similarly, given a morphism $f : Z \boxplus X \to Z \boxplus Y$, there exists a morphism $X \to Y$; call the morphism $X \to Y$ arising from the application of this law $\mathrm{CL}'_Z(f)$. Lastly, observe that one may similarly construct a cancellation sequence for a morphism $f : (X \boxplus Z) \boxplus X' \to (Y \boxplus Z) \boxplus Y'$ to get a morphism $X \boxplus X' \to Y \boxplus Y'$; call this morphism $\mathrm{CL}''_Z(f)$. When we refer to cancellation sequences, we collectively refer to these three cancellation laws; a right-cancellation sequence of f is the cancellation sequence of $\mathrm{CL}(f)$, a left-cancellation sequence of f is the cancellation sequence of $\mathrm{CL}'(f)$, and a middle-cancellation sequence of f is the cancellation sequence of $\mathrm{CL}''(f)$ (with appropriate indices depending on the type of f). When we refer to "the" cancellation law, we always mean CL.

We close this section with an observation on weakly linear categories over a ring. This will play a role later in Section 3.3.5.

Recall [37] tortile monoidal categories:

[12] This is not a problem. None of the theorems concerning rational-like adiabatic categories require CL to be a well-defined deterministic function, except for the section on traced categories. In that section we shall revisit the cancellation law in this light.

Definition 162. A braided monoidal category (with braiding σ, monoidal structure $(\otimes, a, l, r)$ and unit object I) is called *tortile* when it is equipped with the following structure:

- A *twist*, that is to say, a natural isomorphism $\theta_A : A \to A$ which makes the following square commute

$$\begin{tikzcd} A \otimes B \arrow[r, "\sigma_{AB}"] \arrow[d, "\theta_{A\otimes B}"] & B \otimes A \arrow[d, "\theta_B \otimes \theta_A"] \\ A \otimes B & B \otimes A \arrow[l, "\sigma_{BA}"] \end{tikzcd}$$

  and which satisfies $\theta_I = \mathrm{id}_I$.

- *Right duals*, that is to say, for every object A an object A^* (its *right dual*) and morphisms $\epsilon : A^* \otimes A \to I$ (the *co-unit* or *evaluation*) and $\eta : I \to A \otimes A^*$ (the *unit* or *co-evaluation*) making the following diagrams commute:

$$\begin{tikzcd} A^* \otimes (A \otimes A^*) \arrow[dd, "a_{A^*,A,A^*}"] & A^* \otimes I \arrow[l, "\mathrm{id}_{A^*} \otimes \eta_A"'] \arrow[d, "r_{A^*}"] \\ & A^* \arrow[d, "l^{-1}_{A^*}"] \\ (A^* \otimes A) \otimes A^* \arrow[r, "\epsilon_A \otimes \mathrm{id}_{A^*}"] & I \otimes A^* \end{tikzcd} \qquad \begin{tikzcd} (A \otimes A^*) \otimes A \arrow[dd, "a^{-1}_{A,A^*,A}"] & I \otimes A \arrow[l, "\eta_A \otimes \mathrm{id}_A"'] \arrow[d, "l_A"] \\ & A^* \arrow[d, "r^{-1}_A"] \\ A \otimes (A^* \otimes A) \arrow[r, "\mathrm{id}_A \otimes \epsilon_A"] & A \otimes I \end{tikzcd}$$

  If additionally every object A is the right dual of its right dual A^* then the category has *duals*.

  For a morphism $f : A \to B$ define its right dual $f^* : B^* \to A^*$ to be the following chain of morphisms:

$$\begin{tikzcd}
B^* \arrow[r, "r_B"] & B^* \otimes I \arrow[r, "\mathrm{id}_{B^*}\otimes\eta_A"] & B^* \otimes (A \otimes A^*) \arrow[d, "\mathrm{id}_{B^*}\otimes(f\otimes\mathrm{id}_{A^*})"] \\
I \otimes A^* \arrow[d, "l_{A^*}"'] & (B^* \otimes B) \otimes A^* \arrow[l, "\epsilon_B\otimes\mathrm{id}_{A^*}"'] & B^* \otimes (B \otimes A^*) \arrow[l, "a_{B^*,B,A^*}"'] \\
A^*
\end{tikzcd}$$

- The twist satisfies $\theta_{A^*} = \theta_A^*$.

Remark 163. Observe that a weakly linear category over a ring is a tortile monoidal category: any weakly linear category is symmetric, and if it is over a ring then any object X has the dual $-1X$, with $X = -1(-1X)$. Finally, the SR maps of the form $c_{X,\lambda,-\lambda}$ are the co-evaluation maps of the tortile structure.

3.3.2 Topological weak semimodules: Definition and properties

So far we have more or less blindly translated a physical modelling scheme into category-theoretic language. We now define a new class of categories. This time, our starting point is abstract mathematical objects, based on the two fundamental notions that appeared in the previous section: linearity and convergence. We introduce a categorification of a semimodule (definition to follow) in order to reconstruct a weakly linear category, and we assign topological structures to this category in order to reconstruct a convergent category. In the next section, we shall connect the new structure with adiabatic categories.

We begin with the linear-like structure, whose physical meaning is still to be thought of as “scaling”, “putting systems together” and “splitting/recombining”. We then move on to the topological structure, whose physical intuition is continuity of various variables that determine physical states. Lastly, we combine these two structures so that they cooperate.

We shall need some preliminary definitions.

Definition 164. Given a semiring Λ we define the category $\mathbf{\Lambda}$ to be the discrete category whose objects are the semiring elements, equipped with a (primary) monoidal product $\oplus$ corresponding to the addition in Λ. We also define a (secondary) monoidal structure $\otimes$ corresponding to multiplication. Whenever we refer to the monoidal structure of $\mathbf{\Lambda}$ (eg with reference to monoidal functors) we shall always mean the primary monoidal structure, unless we explicitly indicate otherwise.

Observe that the monoidal unit in this category is the 0 of Λ and the secondary monoidal unit is the 1 of Λ. Both monoidal products are strict and are equipped with a strict symmetry.

Recall the standard definition of a (semi)module:

Definition 165. Given a semiring R, a *semimodule* over R consists of a commutative monoid $(M, \boxplus)$ (semimodule addition $\boxplus$) and an operation $\cdot : R \times M \to M$ (semimodule multiplication $\cdot$) such that for all $r, s \in R$ and $x, y \in M$ we have:

1. $r \cdot (x \boxplus y) = (r \cdot x) \boxplus (r \cdot y)$
2. $(r + s) \cdot x = (r \cdot x) \boxplus (s \cdot x)$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1 \cdot x = x$

If R is a ring, the semimodule is called a *module*. If R is a semifield, the semimodule is called a *semivector space*. If R is a field, the semimodule is called a *vector space*.

We now present a weakening of the above notion (which, we note, we shall restrict to CC-semirings):

Definition 166. Let $(\mathbf{C}, a, l, r, s)$ be a symmetric monoidal category and let $\mathbf{\Lambda}$ be the category of a CC-semiring Λ (as per Definition 164). A *weak semimodule* over Λ is (the codomain of) a covariant functor $\cdot : \mathbf{\Lambda} \times \mathbf{C} \to \mathbf{C}$ such that for every $f \in \mathrm{Mor}(\mathbf{C})$ we have $\cdot(1, f) \mapsto f$, for every $\lambda \in \mathrm{Ob}(\Lambda)$ the functor $\mathbf{1} \times \mathbf{C} \xrightarrow{\lambda \times \mathrm{id}_{\mathbf{C}}} \mathbf{\Lambda} \times \mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ is symmetric monoidal, for every $c \in \mathrm{Ob}(\mathbf{C})$ the functor $\mathbf{\Lambda} \times \mathbf{1} \xrightarrow{\mathrm{id}_{\mathbf{\Lambda}} \times c} \mathbf{\Lambda} \times \mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ is monoidal and the following diagram commutes in $\mathbf{Cat}$:

$$
\begin{tikzcd}
\mathbf{\Lambda} \times \mathbf{\Lambda} \times \mathbf{C} \arrow[r, "\otimes \times \mathrm{id}_{\mathbf{C}}"] \arrow[d, "\mathrm{id}_{\mathbf{\Lambda}} \times \cdot"] & \mathbf{\Lambda} \times \mathbf{C} \arrow[d, "\cdot"] \\
\mathbf{\Lambda} \times \mathbf{C} \arrow[r, "\cdot"] & \mathbf{C}
\end{tikzcd}
$$

We may refer to the weak semimodule as $(\mathbf{C}, a, l, r, s, \{J_{\cdot(\lambda,\_)}\}_\lambda, \{c_{\cdot(\_,x)}\}_x)$, where J and c are the natural isomorphisms associated to the corresponding monoidal functors.

If $\mathbf{\Lambda}$ is a ring then $\cdot$ is called a *weak module*. If $\mathbf{\Lambda}$ is a semifield then $\cdot$ is called a *weak semivector space*. If $\mathbf{\Lambda}$ is a field then $\cdot$ is called a *weak vector space*.

In different terms [78, Chapter IV, Definition 4.7] we may say that $(\mathbf{\Lambda}, \otimes)$ *acts* on $\mathbf{C}$. Note that, even though the two functors $\cdot(\lambda, \_)$ and $\cdot(\_, f)$ induced by the action of the secondary structure are monoidal with respect to the primary structure, the action itself is not monoidal in this sense. This sets weak semimodules apart from the so-called module categories [61].

Definition 167. A *linear function* $f : (\cdot_1 : \mathbf{\Lambda} \times \mathbf{C_1} \to \mathbf{C_1}) \to (\cdot_2 : \mathbf{\Lambda} \times \mathbf{C_2} \to \mathbf{C_2})$ between weak semimodules over Λ is defined to be a symmetric strict monoidal functor

$f : \mathbf{C_1} \to \mathbf{C_2}$ that makes the following diagram commute in **Cat**:

$$\begin{tikzcd}
\mathbf{\Lambda} \times \mathbf{C_1} \arrow[r, "\cdot_1"] \arrow[d, "\mathrm{id}_{\mathbf{\Lambda}} \times f"] & \mathbf{C_1} \arrow[d, "f"] \\
\mathbf{\Lambda} \times \mathbf{C_2} \arrow[r, "\cdot_2"] & \mathbf{C_2}
\end{tikzcd}$$

Note that linear functions are defined in the same way as strong module functors [61], but the underlying structure of the action on $\mathbf{C}$ is different (see previous comment). In our context, linear functions generalise 2-semimodule homomorphisms (where we restrict to 2-semimodules over CC-semirings).

Definition 168 ([11])**.** Let $\mathbf{K}$ be a category with pullbacks. We define a category $\mathbf{C}$ *internal* to $\mathbf{K}$ as follows:

- $\mathrm{Ob}(\mathbf{C}) \in \mathrm{Ob}(\mathbf{K})$ and $\mathrm{Mor}(\mathbf{C}) \in \mathrm{Ob}(\mathbf{K})$.
- The following morphisms of $\mathbf{K}$

$$\begin{aligned}
\text{source} &: \mathrm{Mor} \to \mathrm{Ob} & (f : A \to B) &\mapsto A \\
\text{target} &: \mathrm{Mor} \to \mathrm{Ob} & (f : A \to B) &\mapsto B \\
\text{identity} &: \mathrm{Ob} \to \mathrm{Mor} & X &\mapsto \mathrm{id}_X \\
\text{composition} &: P_{\text{source,target}} \to \mathrm{Mor} & (f, g) &\mapsto f \circ g
\end{aligned}$$

  (where $P_{\text{source,target}}$ denotes the pullback of source and target in $\mathbf{K}$) satisfy all the necessary commutative diagrams[13] such that $\mathbf{C}$ is a category.

An *internal functor* $F : \mathbf{C} \to \mathbf{D}$ between internal categories is a pair of morphisms $\mathrm{Mor}(C) \to \mathrm{Mor}(D)$ and $\mathrm{Ob}(C) \to \mathrm{Ob}(D)$ that satisfies all the necessary commutative

[13]For a full list, see original paper [11, Definition 2.1].

diagrams[14] that guarantee functoriality. Similarly for an *internal natural transformation*[15].

Remark 169. A weak semimodule $\cdot : \mathbf{\Lambda}\times\mathbf{C} \to \mathbf{C}$ where $\mathbf{C}$ is strict monoidal with strict symmetry and the functors $\mathbf{1}\times\mathbf{C} \xrightarrow{\lambda\times\mathrm{id}_{\mathbf{C}}} \mathbf{\Lambda}\times\mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ and $\mathbf{\Lambda}\times\mathbf{1} \xrightarrow{\mathrm{id}_{\mathbf{\Lambda}}\times x} \mathbf{\Lambda}\times\mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ are strict monoidal is precisely a category $\mathbf{C}$ internal to the category of (semi)modules or (semi)vector spaces over Λ; the linear functions coincide with the usual definition of a linear function. Call such a weak semimodule a 2-semimodule in accordance with the notion of a 2-vector space; the latter has been explored by Baez and Crans [11]. Define 2-modules and 2-semivector spaces in the obvious way.

Definition 170. Define a *topological category* $\mathbf{C}(\tau_{\mathrm{Ob}}, \tau_{\mathrm{Mor}})$ as a category internal to **NiceTop**. The set Ob of objects of $\mathbf{C}$ is endowed with a nice topology τ_{Ob} and the set Mor of morphisms of $\mathbf{C}$ is endowed with a nice topology τ_{Mor}.

Remark 171. By Corollary 129, seeing as the category $\mathbf{Cat}[\mathbf{C}]$ of internal categories to a complete category $\mathbf{C}$ is finitely complete [17, Exercise 8.4.1], the category $\mathbf{Cat}[\mathbf{NiceTop}]$ of topological categories is finitely complete.

Note that in a topological category $\mathbf{C}(\tau_{\mathrm{Ob}}, \tau_{\mathrm{Mor}})$, the set $I = \{\mathrm{id}_X\}_{X\in\mathrm{Ob}}$ endowed with the topology τ_{Mor} is isomorphic to $(\mathrm{Ob}, \tau_{\mathrm{Ob}})$ with

$$\mathrm{source}|_I = \mathrm{target}|_I = \mathrm{identity}^{-1} : I \to \mathrm{Ob}.$$

Therefore, in all that follows, we may assume that we are always working within τ_{Mor}; just substitute I for Ob and pre- or post-compose with identity accordingly.

With regards to a topological semiring (Λ, τ_Λ), we may speak of the associated topological semiring category, which is the semiring category $\mathbf{\Lambda}$ that is topological with

[14]For a full list, see original paper [11, Definition 2.2].
[15]For a full list, see original paper [11, Definition 2.3].

the topology τ_Λ. We may often omit τ_Λ for brevity, especially when it is understood from context.

Lemma 172. *In a topological category* $\mathbf{C}(\tau_{\text{Ob}}, \tau_{\text{Mor}})$*, the homsets are closed sets in* τ_{Mor}*.*

Proof. Observe that any singleton set $\{(X, Y)\}$ in $\text{Ob} \times \text{Ob}$ is closed in $\tau_{\text{Ob}} \times \tau_{\text{Ob}}$ because the topology is T1. Moreover, $\langle \text{source}, \text{target} \rangle : f \mapsto (\text{source}(f), \text{target}(f))$ is continuous because the functions source and target are continuous. Therefore, the preimage of $\{(X, Y)\}$ under $\langle \text{source}, \text{target} \rangle$ must be closed in τ_{Mor}. But that is exactly $\text{Hom}(X, Y)$. $\square$

Lemma 173. *In a topological category* $\mathbf{C}(\tau_{\text{Ob}}, \tau_{\text{Mor}})$*, let* H *be a nonempty set of morphisms together with the inherited topology* τ_{Mor}*. Then* H *is a homset in* $\mathbf{C}$ *if and only if there exists a unique* $r_H : 1 \to \text{Ob} \times \text{Ob}$ *that makes the following a pullback in* **NiceTop***, where the inclusion is canonical and 1 is the unique topology on the one-element set:*

$$\begin{tikzcd}
H \arrow[r] \arrow[d, hook] & 1 \arrow[d, "r_H"] \\
\text{Mor} \arrow[r, "\langle\text{source},\text{target}\rangle"] & \text{Ob} \times \text{Ob}
\end{tikzcd}$$

Proof. Let $r_H = \langle s, t \rangle$ for some objects s and t of $\mathbf{C}$. Substituting $\text{Hom}(s, t)$ for H we can see that the square commutes for unique s and t. Moreover, in order for any such square to commute, it has to be that $H \subseteq \text{Hom}(s, t)$, making $\text{Hom}(s, t)$ a pullback. $\square$

Remark 174. This is reminiscent of the subobject classifier $t : 1 \to \Omega$ in a topos (1 being the terminal object), defined by the property that there exists a unique morphism χ_s making the following a pullback for any monomorphism $s : S \to A$:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook, "s"] & 1 \arrow[d, "t"] \\
A \arrow[r, "\chi_s"] & \Omega
\end{tikzcd}$$

Definition 175. Given topological categories $\mathbf{C}(\tau_{\mathrm{Ob}(\mathbf{C})}, \tau_{\mathrm{Mor}(\mathbf{C})})$ and $\mathbf{D}(\tau_{\mathrm{Ob}(\mathbf{D})}, \tau_{\mathrm{Mor}(\mathbf{D})})$, a functor $F : \mathbf{C} \to \mathbf{D}$ is defined to be a *topological functor* if it is an internal functor in **NiceTop**.

Definition 176. Let $\mathbf{\Lambda}$ be a topological CC-semiring category. A *topological weak semivector space* over Λ is a weak semivector space on a topological category, where the monoidal product, the symmetry and the functors $\mathbf{1} \times \mathbf{C} \xrightarrow{\lambda \times \mathrm{id}_{\mathbf{C}}} \mathbf{\Lambda} \times \mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ and $\mathbf{\Lambda} \times \mathbf{1} \xrightarrow{\mathrm{id}_{\mathbf{\Lambda}} \times x} \mathbf{\Lambda} \times \mathbf{C} \xrightarrow{\cdot} \mathbf{C}$ are continuous with respect to the topologies involved (presuming the product to be endowed with the topology constituting the categorical product in **NiceTop**). We may in general speak of a *topological weak semimodule* when the semiring Λ is not a semifield; similarly for topological weak modules and topological weak vector spaces.

Whenever we refer to a topological semiring category $\mathbf{\Lambda}$ where Λ is a subsemiring of $\mathbb{R}$, we shall always imply the topology to be inherited from the standard topology on $\mathbb{R}$. Unless Λ is dense in some interval, the associated continuity requirement holds trivially.

Note that $\mathbf{\Lambda}$ is a topological 2-semimodule over itself, since its monoidal product is a topological functor.

Remark 177. It is easy to check that (topological) linear functions are the morphisms of a category where the objects are (topological) weak semimodules or weak semivector spaces over Λ, or their restriction to (topological) 2-semimodules or 2-semivector spaces over Λ. In any such category, monomorphisms are preserved and reflected from **Cat**.

This remark hints to a possible generalisation of linear functions between weak semimodules:

Definition 178. A *generalised linear function* $(l, f) : (\cdot_1 : \mathbf{\Lambda} \times \mathbf{C_1} \to \mathbf{C_1}) \to (\cdot_2 : \mathbf{\Lambda'} \times \mathbf{C_2} \to \mathbf{C_2})$ between weak semimodules over semirings Λ and Λ' respectively is defined to be a pair (l, f) comprising a semiring homomorphism $l : \mathbf{\Lambda} \to \mathbf{\Lambda'}$ and a symmetric strict monoidal functor $f : \mathbf{C_1} \to \mathbf{C_2}$, such that makes the following diagram commute in **Cat**:

$$\begin{tikzcd}
\mathbf{\Lambda} \times \mathbf{C_1} \arrow[r, "\cdot_1"] \arrow[d, "l \times f"] & \mathbf{C_1} \arrow[d, "f"] \\
\mathbf{\Lambda'} \times \mathbf{C_2} \arrow[r, "\cdot_2"] & \mathbf{C_2}
\end{tikzcd}$$

When l is known from context and no confusion arises, we may abbreviate the terminology from (l, f) to f as with linear functions.

Remark 179. We note that it is possible to define a category with all (topological) weak semimodules as objects and pick (topological) generalised linear functions as the morphisms of that category. Monomorphisms of that category are still preserved and reflected from **Cat**. The monomorphisms here correspond to a notion of a (continuous) linear extension of the action of a semiring.

Lemma 180. *Let* $(\mathbf{C}, \tau_{\mathrm{Ob}}, \tau_{\mathrm{Mor}}, \boxplus, \Lambda)$ *be a category defined as follows:*

- *It is a topological category with object topology* τ_{Ob} *and morphism topology* τ_{Mor}*.*
- *The set of objects* Ob *forms a semimodule (semimodule addition* $\boxplus_{\mathrm{Ob}}$ *and semimodule multiplication* $\cdot_{\mathrm{Ob}}$*, zero element* Θ*) over a semiring* Λ*, and so does the set of morphisms* Mor *(semimodule addition* $\boxplus_{\mathrm{Mor}}$ *and semimodule multiplication* $\cdot_{\mathrm{Mor}}$*).*
- $0 \cdot f = \mathrm{id}_\Theta$.
- *The semimodule operations are topological functors (viewing the semiring as a topological category).*

In this category, the functions source, target, composition *and* identity *are linear with respect to the semimodule operations.*

Proof. Functoriality of $\boxplus$ and $\lambda\cdot$ means precisely that composition and identity are preserved. That is, if $f \circ h$ and $g \circ j$ are defined, we have:

$$(f \boxplus g) \circ (h \boxplus j) = (f \circ h) \boxplus (g \circ j)$$
$$\mathrm{id}_X \boxplus \mathrm{id}_Y = \mathrm{id}_{X \boxplus Y}$$
$$\lambda \cdot (f \circ h) = (\lambda \cdot f) \circ (\lambda \cdot h)$$
$$\lambda \cdot \mathrm{id}_X = \mathrm{id}_{\lambda \cdot X}$$

which are precisely the relations for linearity of composition and identity. Moreover, composability dictates that if composition is linear then the source and target functions must also be linear. $\square$

Corollary 181. *A category defined as above is precisely a topological 2-semimodule over* Λ*.*

It is easy to check that the following structure is always a weak semimodule over $\mathbb{N}$.

Definition 182. Given a symmetric monoidal category $(\mathbf{C}, \otimes, I, a, l, r, s)$, define the *free* weak semimodule of $\mathbf{C}$ as the semimodule $(\mathbf{C}, \otimes, \mathbb{N}, I, a, l, r, J_n, s, c)$ (using the notation of Definition 166) where the action of $n \in \mathbb{N}^*$ on $f \in \mathrm{Mor}(\mathbf{C})$ is $\underbrace{f \otimes f \otimes \cdots \otimes f}_{n}$ and 0 maps every morphism to id_I.

We need not specify the natural transforms c_X and J_n in the above definition, because they are uniquely defined by the action of $\mathbb{N}$; the former as a composition of associators and the latter as an appropriate sequence of tensor products of n symmetries and identities.

Let **WSM** denote the category of weak semimodules and linear extensions and let **SMC** denote the category of symmetric monoidal categories and symmetric strict monoidal functors.

Remark 183. Note that the free weak semimodule of a symmetric monoidal category $\mathbf{C}$ is an initial object in the full subcategory of **WSM** consisting of weak semimodules with the symmetric monoidal structure of $\mathbf{C}$. This is in view of the fact that $\mathbb{N}$ is an initial object in the category of semirings. [69]

There is of course a reason we named this weak semimodule "free". Observe that taking the free weak semimodule of a symmetric monoidal category extends to a functor $F : \mathbf{SMC} \to \mathbf{WSM}$ (by sending each symmetric strict monoidal functor between SMCs to itself). Further observe that F is left adjoint to the forgetful functor $U : \mathbf{WSM} \to \mathbf{SMC}$, and the monad of the adjunction is the identity functor (hence the comonad of the adjunction is idempotent).

Moreover, the free weak semimodule of a symmetric strict monoidal category has $c = \mathrm{id}$ and the free weak semimodule of a strict symmetric monoidal category has $J = \mathrm{id}$.

Proof. Firstly, we establish that F is indeed a well-defined functor. Observe that if $f : \mathbf{C} \to \mathbf{D}$ is a symmetric strict monoidal functor then Ff is linear, as by strict monoidality $F(m \boxplus_{\mathbf{C}} \ldots \boxplus_{\mathbf{C}} m) = (Fm) \boxplus_{\mathbf{D}} \ldots \boxplus_{\mathbf{D}} (Fm)$, for $m \in \mathrm{Mor}(\mathbf{C})$, which is precisely the semiring action, thereby satisfying the defining property; composition is preserved because commutative squares compose; identity is preserved because the identity functor is trivially linear.

Secondly, we establish the adjunction. Take any weak semimodule $\mathbf{W}$ over an arbitrary semiring Λ and any symmetric monoidal category $\mathbf{C}$. For a natural isomorphism between $\mathrm{Hom}(\mathbf{C}, \mathbf{UW})$ and $\mathrm{Hom}(\mathbf{FC}, \mathbf{W})$, we need only check that the homsets are isomorphic, as naturality would then hold by construction of the functors F and U.

It is immediate that any generalised linear functor $\hat{f} : \mathbf{FC} \to \mathbf{W}$ induces a symmetric strict monoidal functor $f : \mathbf{C} \to \mathbf{UW}$. Because $\mathbb{N}$ is initial in the category of semirings, there exists exactly one homomorphism $l : \mathbb{N} \to \mathbf{\Lambda}$, therefore $\hat{f}$ is the unique generalised linear function that gives f. Conversely, assuming an arbitrary symmetric monoidal functor $f : \mathbf{C} \to \mathbf{UW}$, pick $n \in \mathbb{N}$ and $m \in \mathrm{Mor}(\mathbf{C})$. Then, using $\cdot_1$ for $\mathbf{FC}$ and $\cdot_2$ for $\mathbf{W}$ as in the diagram of Definition 178, we get:

$$(n, m) \xmapsto{\cdot_1} \underbrace{m \boxplus \ldots \boxplus m}_{n} \xmapsto{f} f(m \boxplus \ldots \boxplus m) \;= \underbrace{(fm) \boxplus \ldots \boxplus (fm)}_{n}$$

$$(n, m) \xmapsto{l \times f} (\underbrace{1_\Lambda + \ldots + 1_\Lambda}_{n}, fm) \xmapsto{\cdot_2} (\underbrace{1_\Lambda + \ldots + 1_\Lambda}_{n}) \cdot_2 (fm) =$$

$$\underbrace{(1_\Lambda \cdot_2 (fm)) \boxplus \ldots \boxplus (1_\Lambda \cdot_2 (fm))}_{n} = \underbrace{(fm) \boxplus \ldots \boxplus (fm)}_{n}$$

So in the end the diagram commutes, giving a generalised linear function.

The observation that $U \circ F = \mathrm{id}_{\mathbf{SMC}}$ is immediate. $\square$

We now give an example of a free weak semimodule.

Definition 184. Let $(\mathbf{M_b}, \otimes)$ denote the following category:

- The objects are positive integers k.
- The morphisms are $b^k \times b^k$ matrices.
- Composition is matrix multiplication.
- The monoidal product $\otimes$ is the Kronecker product.

Remark 185. One can easily check that there is a weak semimodule

$$(\mathbf{M_b}, \otimes, \mathbb{N}, 1, \mathrm{id}, \mathrm{id}, \mathrm{id}, J_n, s, \mathrm{id}),$$

with the structure of $(\mathbf{M_b}, \otimes)$, where

- The matrix s is defined by the relation $s \circ (M_1 \otimes M_2) = M_2 \otimes M_1$; this is easily seen to be a permutation [32] defined by the dimensions of M_1 and M_2.

- There is a family of monoidal endofunctors n with natural transforms J_n, indexed by the natural numbers, acting as Kronecker powers. Denote the action of such a functor n on a morphism M by $M^{\otimes n}$. 0 sends every morphism to the 1-by-1 matrix (1). The natural transforms J_n are appropriate permutations, where the required relation $s^{\otimes n} \circ J_n = J_n \circ s$ obviously holds.

This is the free weak semimodule of $(\mathbf{M_b}, \otimes)$.

Let $(\mathbb{N}, \tau)$ be a topological semiring with the discrete topology. It is straightforward to make a free weak semimodule into a topological weak semimodule over $(\mathbb{N}, \tau)$ by endowing both the objects and the morphisms with the discrete topology. We can also pick a more interesting topology for the morphisms; we give such an example using $\mathbf{M_b}$.

Definition 186. Let $(\mathbf{M'_b}, \otimes, |\cdot|)$ denote the category defined as $(\mathbf{M_b}, \otimes)$ but where each matrix M is assigned a norm $|M|$.

Remark 187. The category $\mathbf{M'_b}$ uniquely defines a topological weak semimodule $(\mathbf{M'_b}, \tau'_{\text{Ob}}, \tau'_{\text{Mor}}, \otimes, \mathbb{N}, \tau, 1, \text{id}, \text{id}, \text{id}, J_n, s, \text{id})$, where τ is the discrete topology, in the following manner:

- $\tau'_{\text{Ob}} = \tau$ is the discrete topology (and therefore in τ'_{Mor} the homsets are clopen).

- Within each homset the norm defines a distance $d(M_1, M_2) = |M_1 - M_2|$ that in turn yields a metric space on the homset; this, together with the fact that the homsets are clopen, completes the definition of τ'_{Mor}.

At this point, we pause to introduce some necessary notation.

Definition 188. From this point on, we adopt the following convention:

- Denote by $\mathbf{P}_{\mathbf{\Lambda},\tau_{\mathbf{\Lambda}}}$ the category of topological weak semimodules over a topological semiring (Λ, τ_Λ) and continuous linear functions.

- Denote by $\mathbf{P}$ the category of topological weak semimodules and continuous linear extensions and by $\mathbf{\Phi}$ its full subcategory of weak modules. Denote by $\overline{\mathbf{P}}$ and $\overline{\mathbf{\Phi}}$ the full subcategories of weak semivector spaces and weak vector spaces respectively. When we want to restrict one of these categories to its full subcategory over a single semiring, we use the modifier $[\cdot]$ (as in $\mathbf{P}_{[\mathbf{\Lambda},\tau_{\mathbf{\Lambda}}]}$) to avoid confusion with the non-full subcategory $\mathbf{P}_{\mathbf{\Lambda},\tau_{\mathbf{\Lambda}}}$.

- In accordance with existing literature, denote by the modifier 2 the corresponding subcategories of topological 2-(semi)modules and 2-(semi)vector spaces (eg $\mathbf{2P}_{\mathbf{\Lambda},\tau_{\mathbf{\Lambda}}}$). These are restricted to CC-semirings.

Continuing our discussion on free constructions, we note that our observation on weak semimodules extends to their topological counterparts.

Definition 189. Given a symmetric monoidal category $(\mathbf{C}, \otimes, I, a, l, r, s)$, define the *free* topological weak semimodule of $\mathbf{C}$ as the topological weak semimodule with the structure of the free weak semimodule of $\mathbf{C}$ where $\tau_{\mathbb{N}} = \tau_{\mathrm{Ob}} = \tau_{\mathrm{Mor}}$ is the discrete topology.

Remark 190. Note that the free topological weak semimodule of a symmetric monoidal category $\mathbf{C}$ is an initial object in the full subcategory of $\mathbf{P}$ consisting of topological weak semimodules with the symmetric monoidal structure of $\mathbf{C}$. This is in view of Remarks 130 and 183.

As before, we note that taking the free topological weak semimodule of a symmetric monoidal category extends (in the obvious way) to a functor $F' : \mathbf{SMC} \to \mathbf{P}$, and moreover that F' is left adjoint to the forgetful functor $U' : \mathbf{P} \to \mathbf{SMC}$.

The monad of the adjunction is idempotent.

We end this section with some examples of topological 2-semimodules:

Definition 191. For $K \in \{\mathbb{Q}, \mathbb{Q}^{\geq 0}\}$, let the category $\mathbf{U'_K}$ be the discrete category defined as follows.

- Objects of $\mathbf{U'_K}$ are nonnegative real numbers.
- The monoidal product $\boxplus$ is multiplication.
- There is a family of symmetric monoidal commutative endofunctors $\lambda \in K$ equipped with the topology τ_Λ, which is defined to be $\tau_\mathbb{R}$ restricted to K; these endofunctors act as exponentiation, where $\frac{1}{n}$ takes the principal nth root.
- We set $\tau_{\mathrm{Ob}} = \tau_{\mathrm{Mor}}$ as $\tau_\mathbb{R}$ restricted to $\mathbb{R}^{\geq 0}$.

Definition 192. For $K \in \{\mathbb{Z}, \mathbb{N}\}$, let the category $\mathbf{L'_K}$ be the discrete category defined as follows.

- Objects of $\mathbf{L'_K}$ are real numbers.
- The monoidal product $\boxplus$ is multiplication.
- There is a family of symmetric monoidal commutative endofunctors $\lambda \in K$ (equipped with the trivial topology) corresponding to exponentiation.
- We set $\tau_{\mathrm{Ob}} = \tau_{\mathrm{Mor}} = \tau_\mathbb{R}$.

Definition 193. Let the category $\mathbf{Z'}$ be the discrete category defined as follows.

- An object (z, n) of $\mathbf{Z'}$ is the n-element set of the nth roots of the complex number z.

- The monoidal product $(z_1, n_1) \boxplus (z_2, n_2)$ is the $n_1 n_2$-element set of products of elements of (z_1, n_1) and (z_2, n_2).
- There is a family of symmetric monoidal commutative endofunctors $\lambda \in \mathbb{Q}$ equipped with the topology $\tau_{\mathbb{R}}$ restricted to $\mathbb{Q}$ such that $\frac{l}{p}\{x_1, x_2, \ldots, x_m\} \mapsto \bigcup_{k=1}^{m}(x_k^l, p)$, where l and p are coprime.
- Let $\tau_{\mathbb{C}}$ be the standard topology on the complex numbers; this defines a metric space with distance function d. The topology $\tau_{\mathrm{Ob}} = \tau_{\mathrm{Mor}}$ is defined by the metric space where the distance between two sets S and Q of points on the complex plane is defined to be $\max_{s \in S} \min_{q \in Q} d(s, q)$.

3.3.3 Adiabatic categories versus topological weak semimodules

At this point we are going to relate topological weak semivector spaces and adiabatic categories. This connection is important from both a conceptual and a pragmatical standpoint. Conceptually, topological weak semivector spaces are more elegant and intuitive, which serves to make Lieb and Yngvason's work more accessible. Pragmatically, topological weak semivector spaces are more general and allow for more structure, which has potential use in applications.

The connection we are about to show only exists for certain semifields, defined thus:

Definition 194. Let Λ be a topological semifield. Call Λ *stable* if the following is true for all topological weak semivector spaces χ and ψ over Λ: if $f : \chi \to \psi$ is a linear continuous function surjective on a sequence $S = \{y_i = y \boxplus \lambda_i y'\}$ with $\lambda_i \to 0$ then there exists a sequence $\{x_i\}$ in χ, with $f(x_i) \cong y_i$, converging to some point $x \in f^{-1}(y)$.

We shall later use the fact that, for discrete categories, the isomorphism in the above definition becomes an equality. Before our main result we offer two preliminary results, the first about topological weak modules and the second about a more general class of topological weak semimodules, which show a range of topological semifields to be stable.

In the following theorem, given a linear function, we construct a convergent sequence in the preimage of a convergent sequence by taking advantage of the map's linearity. The geometrical intuition is that we construct a line of points by fixing two of those points, essentially treating the mapping as a usual linear function.

Theorem 195. *Topological fields are stable.*

Proof. Let $f : \chi(a_\chi, l_\chi, r_\chi, s_\chi, J_\chi, c_\chi) \to \psi(a_\psi, l_\psi, r_\psi, s_\psi, J_\psi, c_\psi)$ be a linear continuous function between topological weak vector spaces over a topological field Λ. We show that if f is surjective on a sequence $S = \{y_i = y \boxplus \lambda_i y'\}$ with $\lambda_i \to 0$ then there exists a sequence $\{x_i\}$ in χ, with $f(x_i) \cong y_i$, converging to some point $x \in f^{-1}(y)$.

Note that, by our definition of linearity, f is such that for $\phi \in \{a, l, r, s, J, c\}$ and for every $x \in \chi$ we have $\phi_\psi(f(x)) \doteq f(\phi_\chi(x))$.

Pick some $x_1 \in f^{-1}(y_1)$ and some $x_2 \in f^{-1}(y_2)$ and set

$$x_i = \frac{\lambda_2 - \lambda_i}{\lambda_2 - \lambda_1} x_1 \boxplus \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_2} x_2.$$

We show that this sequence has the desired property:

$$\begin{aligned}
f(x_i) &= f\left(\frac{\lambda_2 - \lambda_i}{\lambda_2 - \lambda_1} x_1 \boxplus \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_2} x_2\right) \\
&= \frac{\lambda_2 - \lambda_i}{\lambda_2 - \lambda_1} f(x_1) \boxplus \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_2} f(x_2) && \text{(linearity of } f\text{)} \\
&= \frac{\lambda_2 - \lambda_i}{\lambda_2 - \lambda_1} y_i \boxplus \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_2} y_2 \\
&= \frac{\lambda_2 - \lambda_i}{\lambda_2 - \lambda_1} (y \boxplus \lambda_1 y') \boxplus \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_2} (y \boxplus \lambda_2 y')
\end{aligned}$$

$$
\begin{aligned}
&\cong \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y'\right) \boxplus \left(\frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y'\right) && (J \boxplus J)\\
&\cong \left(\left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y'\right) \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y\right) \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y' && (a)\\
&\cong \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y' \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y\right)\right) \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y' && (a^{-1} \boxplus \mathrm{id})\\
&\cong \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \left(\frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y \boxplus \frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y'\right)\right) \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y' && ((\mathrm{id} \boxplus s) \boxplus \mathrm{id})\\
&\cong \left(\left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y\right) \boxplus \frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y'\right) \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y' && (a \boxplus \mathrm{id})\\
&\cong \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}y \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}y\right) \boxplus \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 y' \boxplus \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2 y'\right) && (a^{-1})\\
&\cong \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1} + \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\right) y \boxplus \left(\frac{\lambda_2-\lambda_i}{\lambda_2-\lambda_1}\lambda_1 + \frac{\lambda_1-\lambda_i}{\lambda_1-\lambda_2}\lambda_2\right) y' && (c^{-1} \boxplus c^{-1})\\
&= 1y \boxplus \lambda_i y' = y \boxplus \lambda_i y' = y_i
\end{aligned}
$$

□

The linear combination above is not convex, so it cannot be used for general weak semivector spaces. To deal with the latter, we first define an auxiliary notion.

Definition 196. Let $S = \{x\}_i$ be an infinite sequence in a cancellative semiring. Call S a *downwards* sequence if for every i there exists x'_i such that $x_i = x_{i+1} + x'_i$.

The following proposition uses the construction of the previous theorem to show that there must exist a convex combination, for which we also give the formula. The intuition behind the formula is exactly the same as in the previous case, except that instead of two sequence points we define the line by the first sequence point and the limit point.

Proposition 197. *Let $f : \chi(a_\chi, l_\chi, r_\chi, s_\chi, J_\chi, c_\chi) \to \psi(a_\psi, l_\psi, r_\psi, s_\psi, J_\psi, c_\psi)$ be a linear continuous function between topological weak semivector spaces over a topological cancellative semifield Λ. If f is surjective on a sequence $S = \{y_i = y \boxplus \lambda_i y'\}$ with $\lambda_i \to 0$ a* *downwards sequence then there exists a sequence* $\{x_i\}$ *in* χ*, with* $f(x_i) \cong y_i$*, converging to some point* $x \in f^{-1}(y)$*.*

Proof. Note that, by our definition of linearity, f is such that for $\phi \in \{a, l, r, s, J, c\}$ and for every $x \in \chi$ we have that $\phi_\psi(f(x)) \doteq f(\phi_\chi(x))$.

We treat a simple case first. If we are guaranteed from the beginning that $f^{-1}(y) \neq \emptyset$ then there is a constructive proof as above: We can construct such a sequence by picking some $x \in f^{-1}(y)$ and some $x_1 \in f^{-1}(y_1)$ and setting

$$x_i = \frac{\lambda_1 - \lambda_i}{\lambda_1} x \boxplus \frac{\lambda_i}{\lambda_1} x_1. \tag{*}$$

We show that this sequence has the desired property:

$$\begin{aligned}
f(x_i) &= f\left(\frac{\lambda_1 - \lambda_i}{\lambda_1} x \boxplus \frac{\lambda_i}{\lambda_1} x_1\right) \\
&= \frac{\lambda_1 - \lambda_i}{\lambda_1} f(x) \boxplus \frac{\lambda_i}{\lambda_1} f(x_1) && \text{(linearity of } f) \\
&= \frac{\lambda_1 - \lambda_i}{\lambda_1} y \boxplus \frac{\lambda_i}{\lambda_1} y_1 \\
&= \frac{\lambda_1 - \lambda_i}{\lambda_1} y \boxplus \frac{\lambda_i}{\lambda_1} (y \boxplus \lambda_1 y') \\
&\cong \frac{\lambda_1 - \lambda_i}{\lambda_1} y \boxplus \left(\frac{\lambda_i}{\lambda_1} y \boxplus \frac{\lambda_i}{\lambda_1} \lambda_1 y'\right) && (\mathrm{id} \boxplus J) \\
&\cong \left(\frac{\lambda_1 - \lambda_i}{\lambda_1} y \boxplus \frac{\lambda_i}{\lambda_1} y\right) \boxplus \frac{\lambda_i}{\lambda_1} \lambda_1 y' && (a) \\
&\cong \left(\frac{\lambda_1 - \lambda_i}{\lambda_1} + \frac{\lambda_i}{\lambda_1}\right) y \boxplus \frac{\lambda_i}{\lambda_1} \lambda_1 y' && (c^{-1} \boxplus \mathrm{id}) \\
&= 1y \boxplus \lambda_i y' = y \boxplus \lambda_i y' = y_i
\end{aligned}$$

In the general case where the fact that $f^{-1}(y) \neq \emptyset$ is not given, we proceed as follows.

The set $S \cup \{y\}$ is closed, as every convergent subsequence of S necessarily converges to y. Therefore, the preimage of $S \cup \{y\}$ under f must also be closed; so the set $X := f^{-1}(y) \cup \bigcup_i f^{-1}(y_i)$ is closed. Let $x'_i \in f^{-1}(y_i)$ be such that the sequence $\{x'_i\}$ does not diverge; by construction it cannot converge on a point in any of the $f^{-1}(y_i)$, and because X is closed it must converge on a point of X, so it has to converge on a point of $f^{-1}(y)$. We conclude that any convergent sequence of $x'_i \in f^{-1}(y_i)$ must converge on a point in $f^{-1}(y)$. So we only have to show that there exists a sequence of $x'_i \in f^{-1}(y_i)$ that does not diverge.

Denote by ξ the chain of isomorphisms described in Theorem 195. Choose integers $k > j > 1$ and let $X_{j,k}$ be the subset of $f^{-1}(\xi(y_j))$ for which there exists $x_k \in f^{-1}(y_k)$ such that $x_j = \frac{\lambda_j - \lambda_k}{\lambda_1 - \lambda_k} x_1 \boxplus \frac{\lambda_1 - \lambda_j}{\lambda_1 - \lambda_k} x_k$. Set $X_j := \bigcap_k X_{j,k}$. Let $\mathcal{S}_k$ denote the set of sequences of $x_j \in X_j$ for $j \leq k$ such that $x_j = \frac{\lambda_j - \lambda_k}{\lambda_1 - \lambda_k} x_1 \boxplus \frac{\lambda_1 - \lambda_j}{\lambda_1 - \lambda_k} x_k$. Note that there is a canonical injection $\iota_{k_2,k_1} : \mathcal{S}_{k_2} \to \mathcal{S}_{k_1}$ for $k_2 \geq k_1$, as every element of $\mathcal{S}_{k_1}$ can have at most one possible extension $S \in \mathcal{S}_{k_2}$. Set

$$\mathcal{S} := \bigcap_{k \geq 2} \iota_{k,2}(\mathcal{S}_k).$$

This set is nonempty as an intersection of nonempty sets and as we have shown in Theorem 195 each of its elements defines a convergent sequence.

The elements of any such sequence can be obtained by (*). $\square$

Finally, we generalise to the case where $\lambda_i \to 0$ is not downwards.

Definition 198. Let (R, τ_R) be a topological cancellative semiring; (R, τ_R) is *well-behaved* if every infinite sequence tending to 0 in τ_R can be partitioned into downwards sequences (apart from a finite number of points).

Remark 199. Many topological semirings of interest are well-behaved in the above sense. Notably, each of $\mathbb{N}$, $\mathbb{Q}^{\geq 0}$, $\mathbb{R}^{\geq 0}$ with the topology inherited from $\tau_{\mathbb{R}}$ is well-behaved.

Proof. For $\mathbb{N}$ the fact is rather obvious: Every sequence S that converges to 0 is eventually 0 and is therefore downwards apart from a finite number of points.

For either $\mathbb{Q}^{\geq 0}$ or $\mathbb{R}^{\geq 0}$: For any sequence S that converges to 0 it suffices to show that every element of S belongs to a downwards subsequence of S; then forming a partition is just a matter of partitioning the elements of each subsequence. If there is an infinite number of elements of S equal to 0, form a downwards sequence S^0 with these elements, else ignore them. Take the (nonzero) element S_n; by the defining property of convergence in a metric space, there exists an element $S_{f(n)}$ such that $S_i \leq S_n$ for $i \geq f(n)$; form the downwards sequence $\{S_n, S_{f(n)}, S_{f(f(n))} \ldots\}$; obviously, every nonzero element of S is contained in such a sequence. $\square$

Corollary 200. *Well-behaved topological semifields are stable.*

Now that we have established that the stability prerequisite is satisfied by many physically relevant semifields, we go on to show the connection between topological weak semimodules and adiabatic categories.

Firstly, we observe that weak semimodules and weakly linear categories are the same entity.

Remark 201. It is easy to check that a weak semimodule is precisely a weakly linear category, as their definitions contain exactly the same data: they both have a symmetric monoidal structure, the structural endofunctors λ of the weakly linear category coincide with the functors $(\lambda, \_)$ of the weak semimodule and the natural isomorphism c of the weakly linear category coincides with the natural isomorphisms associated to the monoidal functors $(\_, x)$ of the weak semimodule.

Secondly, we observe that topological categories and convergence categories have the same data.

Lemma 202. *There is a one-to-one correspondence between topological categories and categories* $\mathbf{C}$ *together with a convergence category* $(\mathbf{G}, \mathrm{conv})$ *on* $\mathbf{C}$.

Proof. Given $\mathbf{C}$, Lemma 157 guarantees the existence of nice topologies on the objects and morphisms; these are unique because they are sequential. Conversely, it is easy to see that the partial function induced by the convergence in any nice topology is a function satisfying the 3 relevant stability properties (again refer to Lemma 157). Continuity of identity, composition, source and target implies functoriality of conv and vice versa, as in sequential spaces sequential continuity implies continuity. $\square$

Corollary 203. *In an adiabatic category endowed with topologies* τ_{Mor} *and* τ_{Ob} *as in Lemma 157, homsets are open sets in* τ_{Mor} *if and only if* τ_{Ob} *is the discrete topology.*

We are now ready to show a correspondence between topological weak semimodules and adiabatic categories.

Theorem 204. *Let* Λ *be a stable topological semifield. From a topological weak semivector space* $\mathbf{C}$ *over* Λ *one can construct an adiabatic category* $\mathbf{D}$ *over* Λ *such that the underlying weak semivector space and weakly linear category coincide and such that the* conv *function associated to* $\mathbf{D}$ *agrees with the convergence in the topologies* τ_{Ob} *and* τ_{Mor} *associated to* $\mathbf{C}$*. Moreover, an adiabatic category* $\mathbf{D}$ *over* Λ *yields a topological weak semivector space* $\mathbf{C}'$ *over* Λ *where the linear structures agree and* conv *specifies* τ'_{Ob} *and* τ'_{Mor}.

Proof. We already know from Remark 201 and Lemma 202 that an adiabatic category is both a weak semimodule and a topological category, where conv gives the limit of a convergent sequence in Mor. It remains to show that the topological and linear structures cooperate in the same way in $\mathbf{C}$ and $\mathbf{D}$.

Part 1:

We first construct an adiabatic category from a topological weak semivector space.

To get the Cont property: The functions source and target are continuous and linear, and therefore so is $\langle \text{source}, \text{target} \rangle$. Take a convergent sequence S of elements $(x, y) \boxplus \lambda_i(x', y')$ in $\mathrm{Ob} \times \mathrm{Ob}$ such that $\langle \text{source}, \text{target} \rangle$ is surjective on S. This mapping is topologically isomorphic to a linear function $\mathbf{Dis}(\mathrm{Mor}) \to \mathbf{Dis}(\mathrm{Ob} \times \mathrm{Ob})$ between the discrete topological weak semivector space categories whose objects are respectively the morphisms and the pairs of objects of $\mathbf{C}$ with the same topology. We know that conv gives us exactly the limits of sequences of the form $f_i \in \mathrm{Hom}(X_i, Y_i)$, with X_i, Y_i primary sequences. Stability of Λ hence ensures the basic stability property (taking into account the fact that the stability isomorphism becomes an identity in the case of discrete categories).

We now check that the properties of the continuity function hold for this construction. Property (iii) holds by continuity of $\boxplus$. Property (iv) holds by continuity of $\lambda\cdot$. Property (v) holds by continuity of symmetry. The rest hold by Lemma 202.

Part 2:

We now construct a topological weak semimodule from an adiabatic category. Let τ'_{Ob} and τ'_{Mor} be the sequential topologies as constructed in Lemma 157; we check that the weak semimodule operations are continuous. Using the closed-set criterion and the stability properties of Definition 139, $\boxplus$ is continuous by stability property (iii), λ are continuous by stability property (iv) and symmetry is continuous by stability property (v). $\square$

Remark 205. Note that if the semifield is not stable, then the above correspondence necessarily fails. Therefore, stability of the semifield *characterises* the topological weak semimodules that give rise to an adiabatic category.

In view of Lemma 181, which states that the source and target functions in a topological 2-semivector space are continuous and linear, we get the following corollary.

Corollary 206. *Let* Λ *be a stable topological semifield. From a topological 2-semivector space* $\mathbf{C}$ *over* Λ *one can construct a tyrannical adiabatic category* $\mathbf{D}$ *over* Λ *such that the underlying 2-semivector space and weakly linear category coincide and such that the* conv *function associated to* $\mathbf{D}$ *agrees with the convergence in the topologies* τ_{Ob} *and* τ_{Mor} *associated to* $\mathbf{C}$*. Moreover, a tyrannical adiabatic category* $\mathbf{D}$ *over* Λ *yields a topological 2-semivector space* $\mathbf{C}'$ *over* Λ *where the linear structures agree and* conv *specifies* τ'_{Ob} *and* τ'_{Mor}*.*

There is more to be said about this correspondence. First of all, we give these constructions a name.

Definition 207. Given a stable topological semifield $\Lambda := (\Lambda, \tau_\Lambda)$, the *primary* topological weak semivector space of an adiabatic category $\mathbf{D}$ over Λ is defined as the topological weak semivector space arising from $\mathbf{D}$ as in Theorem 204. A topological weak semivector space arising in this way will be called *O-primary.*

The prefix "O" refers to objects and will be expanded upon later.

One might wonder which objects of $\overline{\mathbf{P}}$ are O-primary. We have shown in Lemma 157 that, given a topology on the λ functors, the adiabatic category defines the convergent sequences of a specific form in Ob and Mor. Since the topologies of the weak semivector space are taken to be sequential, they are completely determined by their convergent sequences. So a weak semivector space is O-primary if and only if all its convergent sequences are of this form. This is an important class of sequences, so we give it a name.

Definition 208. In a topological weak semivector space, define an *O-primary sequence* as follows: An O-primary sequence in Ob is a sequence that is eventually a monoidal product of (0, 1 or more) primary sequences of objects (where "monoidal product of primary sequences" is defined in the same way as for adiabatic categories); an O-primary sequence in Mor is a sequence in Homsets between O-primary sequences of objects.

Remark 209. Subsequences of O-primary sequences are O-primary.

The O-primary convergent sequences are those convergent sequences that arise as part of the linear structure on the objects of the topological weak semivector space. In particular, O-primary convergent sequences in Ob are precisely the necessary convergent sequences imposed by the linear structure and the topology of the semifield; they capture, in a sense, the "minimal convergence" that the topological weak semivector space must have, hence the finest possible topology.[16] In Mor they describe a topological weak semivector space's behaviour in the necessarily convergent homsets; this behaviour is encapsulated in the conv function of the associated adiabatic category. Every sequence that is not O-primary corresponds to additional structure.

From the above discussion it is evident that there is an even more basic notion than O-primary sequences and O-primary spaces.

Definition 210. In a topological weak semivector space, define an *M-primary sequence* (in Mor or in Ob) as a sequence that is eventually a monoidal product of (0, 1 or more) primary sequences of the form $\{M(L_f, I, \lambda_i)\}_i$ (where "monoidal product of primary sequences" is defined in the same way as for adiabatic categories).

Definition 211. Define an *M-primary* topological weak semivector space as a topological weak semivector space where all the convergent sequences are M-primary.

Remark 212. Linear functions and linear extensions map M-primary sequences to M-primary sequences and O-primary sequences to O-primary sequences.

[16]A characterisation of finest topological vector spaces can be found online [35]; in the case where the field is $\mathbb{R}$ or $\mathbb{C}$, a characterisation in terms of F-seminorms is known [68, 26.31].

Note that every M-primary sequence is O-primary; likewise, every M-primary topological weak semivector space is O-primary. These spaces correspond to the adiabatic categories where the conv function is only defined for M-primary sequences of morphisms; in other words, these are the adiabatic categories where the convergence for both objects and morphisms is completely defined by the linear structure, so that conv contains no additional information.

Definition 213. Define an *M-adiabatic category* as an adiabatic category equipped with a trivial conv function that is only defined for M-primary sequences of morphisms.

Explicitly, in the case of an M-adiabatic category, the conv functor maps $\{f \boxplus \lambda_i g\}_i \mapsto f$ (with $\lambda_i \to 0$) and is undefined everywhere else.

In some physical contexts, M-adiabatic categories may be a more appropriate model than adiabatic categories. For processes on a compound system $A \boxplus B$, observe that any process that requires the components A and B to interact nontrivially with each other (thereby altering each other's state) is not a monoidal product of a process on A and a process on B. Depending on which classifier we use for the processes in the category (*i.e.* which parameter determines which physical processes are represented by the same morphism), it may be the case that such "nonseparable" processes on compound systems never form a convergent sequence in the corresponding primary homset sequence.

Observe that the construction of an adiabatic category from a topological weak semivector space and vice versa as in Theorem 204 is functorial. Therefore, we may proceed with the following definitions, which encapsulate the correspondence between adiabatic categories and weak semivector spaces.

Definition 214. Define the following categories and functors:

- Let $\widehat{\mathbf{P}}$ be the subcategory of $\overline{\mathbf{P}}$ restricted to topological weak semivector spaces over stable topological semifields. Let $\widehat{\mathbf{ad}}$ denote the subcategory of $\mathbf{ad}$ restricted to adiabatic categories over stable topological semifields.

- Let the functor $\mathrm{Seq}' : \widehat{\mathbf{P}} \to \mathbf{ad}$ send each topological weak semivector space to the unique adiabatic category where the conv function gives exactly the limits of primary convergent sequences in Mor. Functors are mapped onto themselves. Denote by Seq the corestriction of Seq' to $\widehat{\mathbf{ad}}$.

- Let the functor $\mathrm{Prim}' : \widehat{\mathbf{ad}} \to \overline{\mathbf{P}}$ send each adiabatic category to its primary topological weak semivector space. Functors are mapped onto themselves. Denote by Prim the corestriction of Prim' to $\widehat{\mathbf{P}}$.

It is easy to see that $\mathrm{Prim}(')$ is an embedding and that $\mathrm{Seq} \circ \mathrm{Prim} = \mathrm{id}$. In other words, $\widehat{\mathbf{ad}}$ is precisely the full subcategory of $\widehat{\mathbf{P}}$ where the objects are the O-primary topological weak semivector spaces.

We now offer a simple example of an M-primary and a nonprimary topological weak semivector space that map to the same adiabatic category via Seq.

Example 215. Let $\tau_{\mathbb{R}}$ be the standard topology on $\mathbb{R}$. Let $\tau_{\mathbb{Q}}$ be the restriction of $\tau_{\mathbb{R}}$ to $\mathbb{Q}$. Let $\mathbf{V}$ be the discrete topological 2-vector space $(\mathbf{V}, \tau_{\mathbb{R}}, \tau_{\mathbb{R}}, +, \mathbb{Q}, \tau_{\mathbb{Q}})$ with $\mathrm{Ob} = \mathbb{R}$ where elements of $\mathbb{Q}$ act as multiplication. Then $\mathbf{W} := \mathrm{Prim}(\mathrm{Seq}(\mathbf{V}))$ is given by the discrete topological 2-vector space $(\mathbf{W}, \tau, \tau, +, \mathbb{Q}, \tau_{\mathbb{Q}})$, with $\mathrm{Ob} = \mathbb{R}$ and where elements of $\mathbb{Q}$ act as multiplication, where the topology τ on $\mathbb{R}$ is defined as follows.

Call a countable set of real numbers $\{r\}_i$ linearly dependent if its elements are linearly dependent in $\mathbb{R}$ as a vector space over $\mathbb{Q}$; that is, if there exist $q_i \in \mathbb{Q}$ such that $\sum_i q_i r_i = 0$ with $q_k \neq 0$ for some k. Call an uncountable set of real numbers linearly dependent if it has a linearly dependent countable subset. Let $I \subset \mathbb{R} - \mathbb{Q}$ be a maximal set of linearly independent numbers. We define the basis topology τ_I as follows. Let $\mathcal{R}$ be a collection of sets of real numbers such that each possible finite subset of I is contained in a member of $\mathcal{R}$. Every set of the form

$$X_{\mathcal{R},a,b} = \left(\bigcup_{R \in \mathcal{R}} [\mathbb{Q}(R) - F_R] \right) \cap (a, b) \tag{3.1}$$

is a basis of τ_I, where $\mathbb{Q}(R)$ denotes the field extension of $\mathbb{Q}$ by the elements of R and F_R is a finite subset of $\mathbb{Q}(R) - \mathbb{Q}$. Notice that for any set I' with the defining property, $\tau_{I'} = \tau_I$ so the topology is actually independent of the choice of I. We set $\tau = \tau_I$.

The adiabatic category $\mathbf{R} := \mathrm{Seq}(\mathbf{V}) = \mathrm{Seq}(\mathbf{W})$ is the discrete tyrannical adiabatic category given by $(\mathbf{R}, +, 0, q, \mathbb{Q}, \tau_{\mathbb{Q}}, \lim)$, with $\mathrm{Ob}(\mathbf{R}) = \mathbb{R}$ and with multiplication as the action of q.

Proof. Firstly, observe that $\mathbf{V}$ and $\mathbf{W}$ are indeed topological 2-vector spaces, as addition and multiplication are continuous operations, and their topologies are nice.

Secondly, observe that $\mathbf{V}$ is not O-primary: denote by $\mathcal{S}$ the set of all primary sequences on $\mathrm{Ob}(\mathbf{V})$; explicitly, all sequences $S_{r_1,\{r_k\}_k,j}$ of the form $\{r_1 + \sum_{k=2}^{n} q_{i,k} r_k\}_i$, for $r_1, r_k \in \mathbb{R}$ and $q_{i,k} \in \mathbb{Q}$, with $q_{i,k} \to 0$ as $i \to \infty$ for each k (where we index by j the different possible sequences of $q_{i,k}$). In $\mathbf{V}$, every sequence $S_{r_1,\{r_k\}_k,j} \in \mathcal{S}$ converges to r_1, but there also exist convergent sequences $S \notin \mathcal{S}$, for instance any convergent sequence of algebraically independent real numbers. So this is not an O-primary topological 2-vector space.

Thirdly, observe that for $\mathbf{W}$ to be O-primary, τ must have the property that all its convergent sequences are in $\mathcal{S}$. Since the category is discrete, this also makes it M-primary.

We now provide a proof that τ has the desired property. Denote by $\mathcal{S}_X$ the set of all sequences in $\mathcal{S}$ contained in a set $X \subseteq \mathbb{R}$ and define $\lim(\{r_1 + \sum_{k=2}^{n} q_{i,k} r_k\}_i) := r_1$ for $q_{i,k} \to 0$ as $i \to \infty$ for each k. Call a set $X \subseteq \mathbb{R}$ $\mathcal{S}$-open if for all $S \in \mathcal{S}$ with $\lim(S) \in X$, all of the elements of S apart from a finite number belong to X. We claim that the open sets of τ (*i.e.* unions of basis sets) are precisely the $\mathcal{S}$-open sets.

Open sets in τ are $\mathcal{S}$-open: Note that elements of any convergent sequence $S_{r_1,\{r_k\}_k,j}$ are in $\mathbb{Q}(r_1, \{r_k\}_k)$ and therefore all but a finite number of these elements are in any set of the form (3.1) containing r_1, so sets of the form (3.1) are also $\mathcal{S}$-open. It is immediate that all unions of basis sets are also $\mathcal{S}$-open.

$\mathcal{S}$-open sets are open in τ: We shall show that any set "smaller" than those of the form (3.1) is not $\mathcal{S}$-open. Consider a set of the form

$$X'_{\mathcal{R},a,b} = \left(\bigcup_{R \in \mathcal{R}} [\mathbb{Q}(R) - F_R] - G_{\{r'_1\} \cup \{r'_k\}_k} \right) \cap (a,b) \tag{3.2}$$

for some fixed set $\{r'_1\} \cup \{r'_k\}_k \subseteq I$ with $r'_1 \in (a,b)$; the set $G_{\{r'_1\} \cup \{r'_k\}_k} \subseteq \mathbb{Q}(\{r'_1\}, \{r'_k\}_k) - \mathbb{Q}$ is this time infinite with $r'_1 \notin G_{\{r'_1\} \cup \{r'_k\}_k}$. Now consider a convergent sequence $S_{r'_1,\{r'_k\}_k,j}$ containing $G_{\{r'_1\} \cup \{r'_k\}_k}$. Then $\lim(S_{r'_1,\{r'_k\}_k,j}) = r'_1 \in X'_{a,b}$ and $G_{\{r'_1\} \cup \{r'_k\}_k}$ is an infinite set of elements of $S_{r'_1,\{r'_k\}_k,j}$ that $X'_{a,b}$ does not contain, hence $X'_{a,b}$ is not $\mathcal{S}$-open. $\square$

We now move on to the major result for this section: that the relation between adiabatic categories and topological weak semimodules is in fact an adjunction. For this we shall need the following lemma.

Lemma 216. *Let X and Y be sequential spaces over the same underlying set. Let l_S be the function mapping every convergent sequence of a sequential space S to its limit. The map $i : X \to Y$ sending each element to itself is continuous if and only if l_X is a restriction of l_Y.*

Proof. The "only if" is a basic property of convergence and holds in all spaces. For the "if", it suffices to show that the set of closed sets of Y is a subset of the set of closed sets of X. Let Q be a set that is not closed in X. Then there must exist a convergent sequence S in $Q \subseteq X$ whose limit is not in Q. Then S would also be a convergent sequence in $Q \subseteq Y$ with the same limit, making Q not closed in Y. $\square$

We now give the main result.

Theorem 217. *There is an adjunction* $\mathrm{Prim} \dashv \mathrm{Seq}$.

Proof. We shall show the existence of a couniversal morphism $\epsilon_{\mathbf{V}} : \mathrm{Prim}(\mathrm{Seq}(\mathbf{V})) \to \mathbf{V}$ for every $\mathbf{V} \in \mathrm{Ob}(\hat{\mathbf{P}})$.

Let $\epsilon_{\mathbf{V}}$ be the functor mapping every morphism to itself. Note that the linear structure in $\mathbf{V}$ and $\mathrm{Prim}(\mathrm{Seq}(\mathbf{V}))$ is the same, as both Seq and Prim respect the linear structure; this makes $\epsilon_{\mathbf{V}}$ trivially linear. For the topological structure, note that the set of convergent sequences of $\mathrm{Mor}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$ (respectively of $\mathrm{Ob}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$) is a subset of the set of convergent sequences of $\mathrm{Mor}(\mathbf{V})$ (respectively of $\mathrm{Ob}(\mathbf{V})$) and the limits agree. Therefore, as the topologies are sequential, by Lemma 216 we get that $\epsilon_{\mathbf{V}}$ a topological functor. In conclusion, the functor $\epsilon_{\mathbf{V}}$ is always a morphism in $\hat{\mathbf{P}}$.

Now we show the couniversal property. Pick $\mathbf{D} \in \mathrm{Ob}(\widehat{\mathbf{ad}})$ such that there exists $f : \mathrm{Prim}(\mathbf{D}) \to \mathbf{V}$ in $\hat{\mathbf{P}}$. Clearly, there is a unique factorisation $f = \epsilon_{\mathbf{V}} \circ \hat{f}$ that respects the linear structure, namely the factorisation where the underlying linear extension maps of f and $\hat{f}$ are the same. It remains to show that $\hat{f}$ is a topological functor.

Let X be a set that is not closed in $\mathrm{Mor}(\mathrm{Prim}(\mathbf{D}))$ (respectively in $\mathrm{Ob}(\mathrm{Prim}(\mathbf{D}))$) and such that $f(X)$ not closed in $\mathrm{Mor}(\mathbf{V})$ (respectively in $\mathrm{Ob}(\mathbf{V})$); we shall show that $\hat{f}(X)$ cannot be closed in $\mathrm{Mor}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$ (respectively in $\mathrm{Ob}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$). There must exist a convergent sequence Q in $X \subseteq \mathrm{Mor}(\mathrm{Prim}(\mathbf{D}))$ (respectively in $X \subseteq \mathrm{Ob}(\mathrm{Prim}(\mathbf{D}))$) with limit $q \notin X$; Q must be an O-primary sequence because $\mathrm{Prim}(\mathbf{D})$ is an O-primary topological weak semivector space. Because f is continuous, $f(Q)$ must converge to $f(q) \notin f(X)$ for $f(X) \subseteq \mathrm{Mor}(\mathbf{V})$ (respectively for $f(X) \subseteq \mathrm{Ob}(\mathbf{V})$); because f is linear, by Remark 212 $f(Q)$ must also be an O-primary sequence. But all O-primary convergent sequences have limits the same in $\mathrm{Mor}(\mathbf{V})$ as in $\mathrm{Mor}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$ (respectively in $\mathrm{Ob}(\mathbf{V})$ as in $\mathrm{Ob}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$), so $\hat{f}(X)$ is not closed in $\mathrm{Mor}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$ (respectively in $\mathrm{Ob}(\mathrm{Prim}(\mathrm{Seq}(\mathbf{V})))$). $\square$

Corollary 218. *Let $\Lambda := (\Lambda, \tau)$ be a stable topological semifield. Let $\mathrm{Prim}_\Lambda : \mathbf{ad}_{\mathbf{\Lambda}} \to \mathbf{P}_{\mathbf{\Lambda}}$ be the restriction of* Prim *and $\mathrm{Seq}_\Lambda : \mathbf{P}_{\mathbf{\Lambda}} \to \mathbf{ad}_{\mathbf{\Lambda}}$ be the restriction of* Seq. *Then* $\mathrm{Prim}_\Lambda \dashv \mathrm{Seq}_\Lambda$.

Example 219. Consider the topological 2-vector spaces $\mathbf{V}$ and $\mathbf{W}$ from Example 215 and the functor $i : \mathbf{W} \to \mathbf{V}$ mapping each real number to itself. This functor is trivially linear; to see that it is a topological functor, set $\mathcal{R} = \{I\}$ and $F_R = \emptyset$ for all $R \subseteq I$ in equation (3.1) to retrieve the family of bases of $\mathbf{V}$ as a subset of the bases of $\mathbf{W}$. This functor i is the couniversal arrow of $\mathbf{V}$ with respect to the adjunction.

Remark 220. Since Seq is the right adjoint to an injection, $\widehat{\mathbf{ad}}$ is a coreflective subcategory of $\widehat{\mathbf{P}}$.

Finally, we note that the above discussion on O-primary spaces is also valid for M-primary spaces.

Definition 221. Define the following categories and functors:

- Let $\mathbf{Mad}$ denote the full subcategory of $\mathbf{ad}$ where the objects are M-adiabatic categories. Define $\widehat{\mathbf{Mad}}$ and $\mathbf{Mad}_{\mathbf{\Lambda}}$ analogously to $\widehat{\mathbf{ad}}$ and $\mathbf{ad}_{\mathbf{\Lambda}}$.
- Let the functor $\mathrm{MSeq}' : \widehat{\mathbf{P}} \to \mathbf{Mad}$ send each topological weak semivector space to the unique M-adiabatic category with the same linear structure. Functors are mapped onto themselves. Denote by MSeq the corestriction of MSeq' to $\widehat{\mathbf{Mad}}$.
- Let the functor $\mathrm{MPrim}' : \widehat{\mathbf{Mad}} \to \overline{\mathbf{P}}$ send each M-adiabatic category to the M-primary topological weak semivector space with the same linear structure.

Functors are mapped onto themselves. Denote by MPrim the corestriction of MPrim' to $\widehat{\mathbf{P}}$.

We then obtain the following results for M-primary spaces, identically to the ones for O-primary spaces.

Corollary 222.

- $\mathrm{MPrim} \dashv \mathrm{MSeq}$.
- *Let* $\Lambda := (\Lambda, \tau)$ *be a stable topological semifield. Let* $\mathrm{MPrim}_\Lambda : \mathbf{Mad}_{\boldsymbol{\Lambda}} \to \mathbf{P}_{\boldsymbol{\Lambda}}$ *be the restriction of* MPrim *and* $\mathrm{MSeq}_\Lambda : \mathbf{P}_{\boldsymbol{\Lambda}} \to \mathbf{Mad}_{\boldsymbol{\Lambda}}$ *be the restriction of* MSeq. *Then* $\mathrm{MPrim}_\Lambda \dashv \mathrm{MSeq}_\Lambda$.
- $\widehat{\mathbf{Mad}}$ *is a coreflective subcategory of* $\widehat{\mathbf{P}}$.

Observe that the comonads of the adjunctions $\mathrm{Prim} \dashv \mathrm{Seq}$ and $\mathrm{MPrim} \dashv \mathrm{MSeq}$ are idempotent.

3.3.4 Further insights on Lieb and Yngvason

Morphisms $\mathbf{C} \to \boldsymbol{\Lambda}$ in $\mathbf{P}_{\boldsymbol{\Lambda}}$ map all the connected parts of $\mathbf{C}$ to the same object of $\boldsymbol{\Lambda}$. Note that, in a setting where the morphisms represent physical processes, this is tantamount to a conservation law (such as mass in a closed system). In a streamed category, every object within (or connected to) a stream would have 0 of the conserved quantity.

Let us now consider morphisms $\mathbf{C} \to \mathbb{R}^{\geq \mathbf{0}}_{\mathbf{ord}}$ in $\mathbf{P}_{\mathbb{R}^{\geq \mathbf{0}}}$, where $\mathbb{R}^{\geq \mathbf{0}}_{\mathbf{ord}}$ is a thin category with the objects and monoidal structures of $\mathbb{R}^{\geq \mathbf{0}}$, with a morphism $a \to b$ if and only if $a \leq b$. Then morphisms $S : \mathbf{C} \to \mathbb{R}^{\geq \mathbf{0}}_{\mathbf{ord}}$ correspond to a nondecreasing quantity (entropy) in the sense that, for connected parts of $\mathbf{C}$, S satisfies the following: For

$\sum_i \lambda_i = \sum_j \lambda_j$ there exists a morphism

$$\lambda_{i_1} X_{i_1} \boxplus \lambda_{i_2} X_{i_2} \boxplus \cdots \boxplus \lambda_{i_n} X_{i_n} \to \lambda_{j_1} X_{j_1} \boxplus \lambda_{j_2} X_{j_2} \boxplus \cdots \boxplus \lambda_{j_m} X_{j_m}$$

if and only if $\sum_i \lambda_i S(X_i) \leq \sum_j \lambda_j S(X_j)$.

It turns out that this can always be done, *i.e.* $\mathrm{Hom}(\mathbf{C}, \mathbb{R}^{\geq \mathbf{0}}_{\mathbf{ord}}) \neq \emptyset$, and furthermore that S is unique up to multiplication by a factor[17]; this was the main achievement of Lieb and Yngvason's work, to which we now return.

Definition 223. Call a category *strongly connected* if $\mathrm{Hom}(X, Y) \neq \emptyset$ or $\mathrm{Hom}(Y, X) \neq \emptyset$ for every pair of objects X, Y.

Remark 224. Given an entropy function S as described above, call the condition $S(\mathrm{source}(f)) \leq S(\mathrm{target}(f))$ for $f \in \mathrm{Mor}(\mathbf{C})$ the "adiabaticity rule". Let $\mathbf{C}$ be a connected rational-like adiabatic category; let us add one morphism that does not follow the adiabaticity rule, then add in all necessary morphisms to make the new category a rational-like adiabatic category. The result of Lieb and Yngvason implies that this process would create morphisms in every homset, making $\mathbf{C}$ strongly connected. In the case of LY-adiabatic categories, this would make the category trivial. In conclusion, the LY-adiabatic structure ensures compliance with such an adiabaticity rule; hence the physical connection to the class of adiabatic processes as opposed to all thermodynamical processes.

All that follows in this subsection is a categorical reformulation of Lieb and Yngvason's results. We name the theorems in the same way as Lieb and Yngvason, except where indicated otherwise in a footnote.

[17] Lieb and Yngvason did not have a concept of a unit, or any other special object that must map to 0; therefore, their function is unique only up to affine equivalence, *i.e.* multiplication by a factor and addition of a constant.

Let $(\mathbf{C}, \boxplus, \Theta, a, l, r, \lambda, \{J\}_\lambda, s, c, \mathrm{conv})$ be a rational-like adiabatic category. Let $\{X_i\} \subseteq \mathrm{Ob}(\mathbf{C})$. Write $\Gamma_i^{\lambda_1} \times \Gamma_j^{\lambda_2}$ for the full subcategory of $\mathbf{C}$ involving objects $\lambda_1 X_i \boxplus \lambda_2 X_j$; write Γ_i^λ for $\Gamma_i^\lambda \times \Gamma_j^0$ and Γ_i or Γ for Γ_i^1.

Theorem 225 (Equivalence of entropy and strongly connected rational-like adiabatic categories[18])**.** *Let X_i be objects in a rational-like adiabatic category. Then the following are equivalent:*

- *Each induced subcategory $\Gamma_i^{1-\lambda} \times \Gamma_j^\lambda$ is strongly connected for all $\lambda \leq 1$, i, j.*
- *There exists a function $S : \mathrm{Ob}(\Gamma) \to \mathbb{R}$, unique up to affine equivalence, such that for $\sum_i \lambda_i = \sum_j \lambda_j$ there exists a morphism*

$$\lambda_{i_1} X_{i_1} \boxplus \lambda_{i_2} X_{i_2} \boxplus \cdots \boxplus \lambda_{i_n} X_{i_n} \to \lambda_{j_1} X_{j_1} \boxplus \lambda_{j_2} X_{j_2} \boxplus \cdots \boxplus \lambda_{j_m} X_{j_m}$$

  if and only if $\sum_i \lambda_i S(X_i) \leq \sum_j \lambda_j S(X_j)$.

Proof. That the function S induces the strong-connectedness of the diagrams is obvious. We need only prove that the connectivity of the diagrams induces the function S.

If all objects X_i are connected with a pair of morphisms going both ways then we may choose a constant function $S(X) = w$ for all X. Suppose they do not all have morphisms between them both ways; pick objects $X_0 \ncong X_1$. By hypothesis, there exists a morphism $f : X_0 \to X_1$. Consider a factorisation of f through an object X, that is, $f = f_1 \circ f_0$, where $f_0 : X_0 \to X$ and $f_1 : X \to X_1$. For at most one of f_0 and f_1 can we get a morphism with the target and source reversed.

Uniqueness up to affine transformation: If S exists, then $S(X_0) < S(X_1)$ and $S(X_0) \leq S(X) \leq S(X_1)$ hence there exists a unique λ that satisfies the relation $S(X) = (1-\lambda)S(X_0) + \lambda S(X_1)$. This implies that they have morphisms between

[18] Original reference: "Equivalence of entropy and A1-A5, given CH".

them both ways, so by SR$'$we get composite morphisms between them both ways. Clearly, any function S' that satisfies $S(X) = (1-\lambda)S(X_0) + \lambda S(X_1)$ must be an affine transformation of S.

Existence: Produce unique λ such that they have morphisms between them both ways.

- Uniqueness of λ: It suffices to show that there exists a morphism

$$g : (1-\lambda)X_0 \boxplus \lambda X_1 \to (1-\lambda')X_0 \boxplus \lambda' X_1$$

  if and only if $\lambda \leq \lambda'$.

  - For the "if": Construct g as the morphism that makes the following pentagon commute:

$$\begin{tikzcd}
(1-\lambda)X_0 \boxplus \lambda X_1 \arrow[r, "g"] \arrow[d, "c_{(1-\lambda)X_0, \frac{1-\lambda'}{1-\lambda}, \frac{\lambda'-\lambda}{1-\lambda}} \boxplus \lambda \mathrm{id}_{X_1}"] & (1-\lambda')X_0 \boxplus \lambda' X_1 \\
((1-\lambda')X_0 \boxplus (\lambda'-\lambda)X_0) \boxplus \lambda X_1 \arrow[dr, "((1-\lambda')\mathrm{id}_{X_0} \boxplus (\lambda-\lambda')f) \boxplus \lambda \mathrm{id}_{X_1}"'] & (1-\lambda')X_0 \boxplus ((\lambda'-\lambda)X_1 \boxplus \lambda X_1) \arrow[u, "(1-\lambda')\mathrm{id}_{X_0} \boxplus c^{-1}_{\lambda' X_1, \frac{\lambda'-\lambda}{\lambda'}, \frac{\lambda}{\lambda'}}"'] \\
 & ((1-\lambda')X_0 \boxplus (\lambda'-\lambda)X_1) \boxplus \lambda X_1 \arrow[u, "a_{(1-\lambda')X_0, (\lambda'-\lambda)X_1, \lambda X_1}"']
\end{tikzcd}$$

  - For the "only if" we need the cancellation law. If $\lambda > \lambda'$ then construct

$$h = c_{(1-\lambda')X_0, \frac{1-\lambda}{1-\lambda'}, \frac{\lambda-\lambda'}{1-\lambda'}} \boxplus g \boxplus c_{\lambda X_1, \frac{\lambda'}{\lambda}, \frac{\lambda-\lambda'}{\lambda}}.$$

    Then applying the cancellation law twice yields

$$g' = \frac{1}{\lambda - \lambda'} \mathrm{CL}'_{(1-\lambda)X_0}(\mathrm{CL}_{\lambda' X_1}(h))$$

    which by hypothesis cannot exist; therefore $\lambda \leq \lambda'$.

- Existence of λ: This proof only holds if the proof for the uniqueness of λ holds. Set

  $$\lambda_{\max} = \sup\{\lambda : \exists a_\lambda : (1-\lambda)X_0 \boxplus \lambda X_1 \to X\}$$

  and

  $$\lambda_{\min} = \inf\{\lambda : \exists b_\lambda : X \to (1-\lambda)X_0 \boxplus \lambda X_1\}.$$

  Because of Cont, these are achieved, so we get the morphisms

  $$a_{\lambda_{\max}} : (1-\lambda_{\max})X_0 \boxplus \lambda_{\max}X_1 \to X$$

  and

  $$b_{\lambda_{\min}} : X \to (1-\lambda_{\min})X_0 \boxplus \lambda_{\min}X_1.$$

  The morphism

  $$b_{\lambda_{\min}} \circ a_{\lambda_{\max}} : (1-\lambda_{\max})X_0 \boxplus \lambda_{\max}X_1 \to (1-\lambda_{\min})X_0 \boxplus \lambda_{\min}X_1$$

  implies that $\lambda_{\max} \leq \lambda_{\min}$ by the proof for uniqueness of λ. For every λ with $\lambda_{\max} \leq \lambda \leq \lambda_{\min}$ the required relation holds (because by connectedness since there is no morphism $(1-\lambda)X_0 \boxplus \lambda X_1 \to X$ there must be a morphism $X \to (1-\lambda)X_0 \boxplus \lambda X_1$, and likewise for the other way).

$\square$

By Theorem 225, one can define a functor $S : \mathbf{\Delta} \to \mathbf{T}$ from a full strongly connected subcategory $\mathbf{\Delta}$ of a rational-like adiabatic category $\mathbf{C}$ to a poset category $\mathbf{T}$ where the objects are the nonnegative real numbers and the morphisms are $\leq$.

We give an adjusted definition of the following quantities from Lieb and Yngvason's paper.

Definition 226. Given a partition of the objects of a rational-like adiabatic category $\mathbf{C}$ into strongly connected full subcategories Γ_i, associate a function S_i to each one. Fix Γ and Γ'; for every $A \in \mathrm{Ob}(\Gamma)$ and $B \in \mathrm{Ob}(\Gamma')$, for every morphism $f : A \to B$, consider all chains c of n morphisms f_i for which $f = f_n \circ f_{n-1} \circ \ldots \circ f_1$, where every part of the chain $f_i \circ \ldots \circ f_j \neq \mathrm{id}_Z$ for any object Z (nontrivial), such that the morphisms within it have no nontrivial decomposition. This chain passes through N of these Γ_i. Define the following:

$$E(\Gamma, \Gamma') = \inf_c \{\sum_{i=1}^{N-1} \inf_{X,Y} \{S_{i+1}(Y) - S_i(X) : X \in \Gamma_i, Y \in \Gamma_{i+1}, \mathrm{Hom}(X,Y) \neq \emptyset\}\}$$

$$F(\Gamma, \Gamma') = \inf_{\Gamma_0} \{E(\Gamma \times \Gamma_0, \Gamma' \times \Gamma_0)\}$$

where we assume $\inf\{S_{i+1}(Y) - S_i(X) : X \in \Gamma_i, Y \in \Gamma_{i+1}, \mathrm{Hom}(X,Y) \neq \emptyset\} = \infty$ if there are no morphisms between any of the pairs.

Lemma 227 (Constant entropy differences)**.** *Given a partition of the objects of a rational-like adiabatic category* $\mathbf{C}$ *into strongly connected full subcategories ("parts")* Γ_i*, if* Γ *and* Γ' *are two strongly connected parts of* $\mathbf{C}$ *then for every* $X \in \mathrm{Ob}(\Gamma)$ *and* $Y \in \mathrm{Ob}(\Gamma')$ *the following holds:*

$$\mathrm{Hom}(X,Y) \neq \emptyset \Leftrightarrow S_\Gamma(X) + F(\Gamma, \Gamma') \leq S_{\Gamma'}(Y)$$

Proof. $\Rightarrow$ is obvious. We show $\Leftarrow$.

- Special case: All three infima in the definition of F are minima. That is,

$$F(\Gamma, \Gamma') = \min_{X,Y} \{S_1(Y) - S_{\Gamma \times \Gamma_0}(X) : X \in \Gamma \times \Gamma_0, Y \in \Gamma_1, \mathrm{Hom}(X,Y) \neq \emptyset\}$$

$$
\begin{aligned}
&+\min_{X,Y}\{S_{\Gamma'\times\Gamma_0}(Y)-S_N(X):X\in\Gamma_N,Y\in\Gamma'\times\Gamma_0,\mathrm{Hom}(X,Y)\neq\emptyset\}\\
&+\sum_{i=1}^{N-1}\min_{X,Y}\{S_{i+1}(Y)-S_i(X):X\in\Gamma_i,Y\in\Gamma_{i+1},\mathrm{Hom}(X,Y)\neq\emptyset\}\\
=&S_1(Y_1)-S_{\Gamma\times\Gamma_0}(\overline{X}\boxplus X_0)+S_{\Gamma'\times\Gamma_0}(\overline{Y}\boxplus Y_0)-S_N(X_N)\\
&+\sum_{i=1}^{N-1}(S_{i+1}(Y_{i+1})-S_i(X_i))\\
=&S_1(Y_1)-S_\Gamma(\overline{X})-S_0(X_0)+S_{\Gamma'}(\overline{Y})+S_0(Y_0)-S_N(X_N)\\
&+\sum_{i=1}^{N-1}(S_{i+1}(Y_{i+1})-S_i(X_i))=\\
=&S_{\Gamma'}(\overline{Y})+\sum_{i=0}^{N}S_i(Y_i)-S_\Gamma(\overline{X})-\sum_{i=0}^{N}S_i(X_i)
\end{aligned}
$$

for objects $X_i, Y_i \in \Gamma_i$, $\overline{X}\in\Gamma$ and $\overline{Y}\in\Gamma'$ with morphisms

$f:\overline{X}\boxplus X_0\to Y_1,\ f_i:X_i\to Y_{i+1},\ g:X_N\to\overline{Y}\boxplus Y_0.$

So from the assumption we have

$$S_\Gamma(X)+S_{\Gamma'}(\overline{Y})+\sum_{i=0}^{N}S_i(Y_i)\le S_\Gamma(\overline{X})+S_{\Gamma'}(Y)+\sum_{i=0}^{N}S_i(X_i)$$

or equivalently

$$
\begin{aligned}
&S_{\Gamma\times\Gamma'\times\Gamma_0\times\Gamma_1\times\ldots\times\Gamma_N}(X\boxplus\overline{Y}\boxplus Y_0\boxplus\ldots\boxplus Y_N)\le\\
&\qquad\qquad S_{\Gamma\times\Gamma'\times\Gamma_0\times\Gamma_1\times\ldots\times\Gamma_N}(\overline{X}\boxplus Y\boxplus X_0\boxplus\ldots\boxplus X_N)
\end{aligned}
$$

so there exists a morphism

$$h:X\boxplus\overline{Y}\boxplus Y_0\boxplus\ldots\boxplus Y_N\to\overline{X}\boxplus Y\boxplus X_0\boxplus\ldots\boxplus X_N$$

But there must also exist a morphism

$$h' : \overline{X} \boxplus X_0 \boxplus \ldots \boxplus X_N \to \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N$$

which arises from $f \boxplus f_1 \boxplus f_2 \boxplus \ldots \boxplus f_{N-1} \boxplus g$ composed with a chain of morphisms of the form $\mathrm{id} \boxplus \mathrm{id} \boxplus \ldots \boxplus s \boxplus \ldots \boxplus \mathrm{id} \boxplus \mathrm{id}$. We then have

$$s_{\overline{Y},Y} \circ (\mathrm{id}_Y \boxplus h') \circ s_{\overline{X},Y} : \overline{X} \boxplus Y \boxplus X_0 \boxplus \ldots \boxplus X_N \to Y \boxplus \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N$$

hence the composition

$$s_{\overline{Y},Y} \circ (\mathrm{id}_Y \boxplus h') \circ s_{\overline{X},Y} \circ h : X \boxplus \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N \to Y \boxplus \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N$$

Apply the cancellation law to get

$$\mathrm{CL}_{\overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N}(s_{\overline{Y},Y} \circ (\mathrm{id}_Y \boxplus h') \circ s_{\overline{X},Y} \circ h) : X \to Y$$

- General case: We can choose X_i, Y_i, $\overline{X}$ and $\overline{Y}$ arbitrarily close to achieving the infima, so that to arbitrary precision ϵ we have

$$S_{\Gamma \times \Gamma' \times \Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_N}(X \boxplus \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N) - \epsilon \leq$$
$$S_{\Gamma \times \Gamma' \times \Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_N}(\overline{X} \boxplus Y \boxplus X_0 \boxplus \ldots \boxplus X_N)$$

therefore by Cont we have

$$S_{\Gamma \times \Gamma' \times \Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_N}(X \boxplus \overline{Y} \boxplus Y_0 \boxplus \ldots \boxplus Y_N) <$$
$$S_{\Gamma \times \Gamma' \times \Gamma_0 \times \Gamma_1 \times \ldots \times \Gamma_N}(\overline{X} \boxplus Y \boxplus X_0 \boxplus \ldots \boxplus X_N)$$

so then we can proceed according to the previous case.

□

Lieb and Yngvason note that F satisfies the following properties:

$$-F(\Gamma', \Gamma) \leq F(\Gamma, \Gamma') \tag{3.3}$$

$$F(\Gamma, \Gamma) = 0 \tag{3.4}$$

$$F(t\Gamma, t\Gamma') = tF(\Gamma, \Gamma') \text{ for } t > 0 \tag{3.5}$$

$$F(\Gamma_1 \times \Gamma_2, \Gamma_1' \times \Gamma_2') \leq F(\Gamma_1, \Gamma_1') + F(\Gamma_2, \Gamma_2') \tag{3.6}$$

$$F(\Gamma \times \Gamma_0, \Gamma' \times \Gamma_0) = F(\Gamma, \Gamma') \text{ for all } \Gamma_0 \tag{3.7}$$

We conclude the section with a theorem which asserts the possibility of defining an entropy functor on the entirety of a rational-like adiabatic category $\mathbf{C}$. There are three steps to the process of constructing said functor. If every connected full subcategory of $\mathbf{C}$ is strongly connected, the theorem can be applied as-is. If this is not the case, however, the second step of the theorem requires $\mathbf{C}$ to satisfy an additional axiom.

Definition 228. Partition a rational-like adiabatic category $\mathbf{C}$ into strongly connected full subcategories ("parts") Γ_i. The partition is *sink-free*[19] if it satisfies the following requirement: If there exists a morphism $f : X \boxplus X_0 \to X'$ and a morphism $g : Y' \to Y \boxplus Y_0$ where X' and Y' are weakly connected and $X_0, Y_0 \in \Gamma_0$ then there exist morphisms $f' : X'' \to X \boxplus X_1$ and $g' : Y \boxplus Y_1 \to Y''$ where X'' and Y'' are weakly connected and $X_1, Y_1 \in \Gamma_1$.

Definition 229. A rational-like adiabatic category is sink-free if it admits a sink-free partition.

[19] We caution the reader that this differs from the use of the term *sink* elsewhere in the literature [3].

The requirement that a rational-like adiabatic category be sink-free ensures that $F(\Gamma, \Gamma') \neq -\infty$.

We now give the theorem.

Theorem 230 (Consistent entropy scales - Universal entropy)**.** *Let* $\mathbf{C}$ *be a rational-like adiabatic category that satisfies either of the following properties:*

- $\mathbf{C}$ *is sink-free.*
- *Every connected full subcategory of* $\mathbf{C}$ *is strongly connected.*

Then there is a functor $S : \mathbf{C} \to \mathbf{T}$ *(with* $\mathbf{T}$ *defined as above).*

Proof. Since $\mathbf{T}$ is a poset, it suffices to define S on the objects of $\mathbf{C}$.

Partition the objects of $\mathbf{C}$ into as few strongly connected full subcategories ("parts") Γ_i as possible, as long as the partition is sink-free.

- One can define functions S_Γ for every strongly connected part Γ_i of the category.
- For (weakly) connected parts, we can find an appropriate S as follows. Let $\mathcal{S}$ be the set of pairs (Γ, Γ'). On $\mathcal{S}$ define the equivalence relation $(\Gamma, \Gamma') \cong (\Gamma \times \Gamma_0, \Gamma' \times \Gamma_0)$ for all Γ_0 and denote by $[\Gamma, \Gamma']$ the associated equivalence class. On the set $\mathcal{L}$ of these equivalence classes we define multiplication by $t[\Gamma, \Gamma'] = [|t|\Gamma, |t|\Gamma']$ with $t \in \mathbb{R}$ and addition by $[\Gamma_1, \Gamma'_1] + [\Gamma_2, \Gamma'_2] = [\Gamma_1 \times \Gamma_2, \Gamma'_1 \times \Gamma'_2]$, making it into a vector space. Define the function $H([\Gamma, \Gamma']) = F(\Gamma, \Gamma')$; this is homogeneous and subadditive by the properties of F. Then by the Hahn-Banach theorem there exists a real-valued linear function L on $\mathcal{L}$ with $-F(\Gamma, \Gamma') \leq L([\Gamma, \Gamma']) \leq F(\Gamma, \Gamma')$. Pick one strongly connected part Γ_0 and define $B(\Gamma) = L([\Gamma_0 \times \Gamma, \Gamma_0])$. Set these $B(\Gamma)$ as the additive constants of S_Γ of each strongly connected part Γ. Since the $B(\Gamma)$ satisfy $B(\Gamma_1^{(\lambda_1)} \times \Gamma_2^{(\lambda_2)}) = \lambda_1 B(\Gamma_1) + \lambda_2 B(\Gamma_2)$ and $-F(\Gamma, \Gamma') \leq$

$B(\Gamma) - B(\Gamma') \leq F(\Gamma, \Gamma')$, it follows that this assignment defines a function S for the entire connected part.

- Now consider a different partitioning of $\mathbf{C}$ by joining the strongly connected parts into the largest possible connected full subcategories ("connected parts"). Once functions have been defined on the connected parts of $\mathbf{C}$, pick one connected part Γ'_0 containing objects Z_0 and Z_1 and an associated function S_0 with $S_0(Z_0) = 0$ and $S_0(Z_1) = 1$. Pick objects X_Γ from every connected part Γ. Extend S_0 to the rest of $\mathbf{C}$ as

$$S(X) = \inf\{\lambda : \mathrm{Hom}(X \boxplus \lambda Z_0, X_\Gamma \boxplus \lambda Z_1) \neq \emptyset\}$$

  for every object X of Γ.

$\square$

3.3.5 Adiabatic categories versus traced monoidal categories

In this section we examine connections with an interesting class of categories. Recall [31]:

Definition 231 (Strict traced symmetric monoidal categories)**.** A strict monoidal category $(\mathbf{C}, \otimes, I)$ with a symmetry s is *traced* if it is equipped with a *trace*; that is, a natural family of functions $\mathrm{Tr}^X_{A,B} : \mathrm{Hom}(A \otimes X, B \otimes X) \to \mathrm{Hom}(A, B)$ satisfying three axioms:

1. Vanishing: $\mathrm{Tr}^I_{A,B}(f) = f$ and $\mathrm{Tr}^{X\otimes Y}_{A,B}(g) = \mathrm{Tr}^X_{A,B}(\mathrm{Tr}^Y_{A\otimes X, B\otimes X}(g))$, with $f : A \to B$ and $g : A \otimes X \otimes Y \to B \otimes X \otimes Y$.

2. Superposing: $\mathrm{Tr}^X_{C\otimes A, C\otimes B}(\mathrm{id}_C \otimes f) = \mathrm{id}_C \otimes \mathrm{Tr}^X_{A,B}(f)$, with

$$f : A \otimes X \to B \otimes X.$$

3. Yanking: $\mathrm{Tr}^X_{X,X}(s_{X,X}) = \mathrm{id}_X$.

Traditionally, literature on traced monoidal categories (including in the paper where they were first introduced [38]) uses the strict definition to simplify notation. Since we would like to characterise the nonstrict case, we reintroduce associators and unit maps.[20]

Definition 232 (Traced symmetric monoidal categories)**.** A monoidal category $(\mathbf{C}, \otimes, I, a, l, r)$ with a symmetry s is *traced* if it is equipped with a *trace*; that is, a natural family of functions

$$\mathrm{Tr}^X_{A,B} : \mathrm{Hom}(A \otimes X, B \otimes X) \to \mathrm{Hom}(A, B)$$

satisfying three axioms:

1. Vanishing: $\mathrm{Tr}^I_{A,B}(f) = r_B \circ f \circ r_A^{-1}$ and

   $$\mathrm{Tr}^{X\otimes Y}_{A,B}(a^{-1}_{B,X,Y} \circ g \circ a_{A,X,Y}) = \mathrm{Tr}^X_{A,B}(\mathrm{Tr}^Y_{A\otimes X,B\otimes X}(g)),$$

   with $f : A \otimes I \to B \otimes I$ and $g : (A \otimes X) \otimes Y \to (B \otimes X) \otimes Y$.

2. Superposing: $\mathrm{Tr}^X_{C\otimes A,C\otimes B}(a_{C,B,X} \circ (\mathrm{id}_C \otimes f) \circ a^{-1}_{C,A,X}) = \mathrm{id}_C \otimes \mathrm{Tr}^X_{A,B}(f)$, with $f : A \otimes X \to B \otimes X$.

3. Yanking: $\mathrm{Tr}^X_{X,X}(s_{X,X}) = \mathrm{id}_X$.

We shall have to introduce some additional axioms that will ensure the compatibility of rational-like adiabatic categories with traced categories. We first formulate these axioms in the context of topological weak semimodules, as the intuition behind them

[20] For our proof of the main theorem of this section (Theorem 247), these structural morphisms are essential up to the point where appropriate sequences are constructed as input to the conv function; then they can be safely ignored in light of stability properties (i) and (iii), hence we switch to string diagrams (more on string diagrams to follow).

becomes more clear: as we shall discuss after the definition that follows, these axioms correspond to the commutativity of two "distributivity" structures.

We now define a concept that links topological weak semimodules to traced categories.

Definition 233. A *T-weak semimodule* is a weak semimodule where:

- The following diagram commutes:

$$\begin{tikzcd}
\kappa(\lambda_1+\lambda_2)\cdot X \arrow[r, "\kappa(c_{X,\lambda_1,\lambda_2})"] \arrow[d, "c_{X,\kappa\lambda_1,\kappa\lambda_2}"] & \kappa(\lambda_1\cdot X \boxplus \lambda_2\cdot X) \arrow[dl, "J_{\kappa,\lambda_1\cdot X,\lambda_2\cdot X}"] \\
\kappa\lambda_1\cdot X \boxplus \kappa\lambda_2\cdot X &
\end{tikzcd}$$

- If $1-\lambda$ is defined, the following diagram commutes:

$$\begin{tikzcd}
(\lambda\cdot X\boxplus(1-\lambda)\cdot X)\boxplus(\lambda\cdot Y\boxplus(1-\lambda)\cdot Y) \arrow[dr, "a_{\lambda\cdot X\boxplus(1-\lambda)\cdot X,\lambda\cdot Y,(1-\lambda)\cdot Y}"] & \\
X\boxplus Y \arrow[u, "c_{X,\lambda,1-\lambda}\boxplus c_{Y,\lambda,1-\lambda}"] \arrow[d, "c_{X\boxplus Y,\lambda,1-\lambda}"] & ((\lambda\cdot X\boxplus(1-\lambda)\cdot X)\boxplus\lambda\cdot Y)\boxplus(1-\lambda)\cdot Y \arrow[d, "a^{-1}_{\lambda\cdot X,(1-\lambda)\cdot X,\lambda\cdot Y}\boxplus\mathrm{id}_{(1-\lambda)\cdot Y}"] \\
\lambda\cdot(X\boxplus Y)\boxplus(1-\lambda)\cdot(X\boxplus Y) \arrow[d, "J_{\lambda,X,Y}\boxplus J_{1-\lambda,X,Y}"] & (\lambda\cdot X\boxplus((1-\lambda)\cdot X\boxplus\lambda\cdot Y))\boxplus(1-\lambda)\cdot Y \arrow[d, "(\mathrm{id}_{\lambda\cdot X}\boxplus s_{(1-\lambda)\cdot X,\lambda\cdot Y})\boxplus\mathrm{id}_{(1-\lambda)\cdot Y}"] \\
(\lambda\cdot X\boxplus\lambda\cdot Y)\boxplus((1-\lambda)\cdot X\boxplus(1-\lambda)\cdot Y) & (\lambda\cdot X\boxplus(\lambda\cdot Y\boxplus(1-\lambda)\cdot X))\boxplus(1-\lambda)\cdot Y \arrow[dl, "a_{\lambda\cdot X,\lambda\cdot Y,(1-\lambda)\cdot X}\boxplus\mathrm{id}_{(1-\lambda)\cdot Y}"] \\
((\lambda\cdot X\boxplus\lambda\cdot Y)\boxplus(1-\lambda)\cdot X)\boxplus(1-\lambda)\cdot Y \arrow[u, "a^{-1}_{\lambda\cdot X\boxplus\lambda\cdot Y,(1-\lambda)\cdot X,(1-\lambda)\cdot Y}"] &
\end{tikzcd}$$

We now offer some more insight into the two above diagrams. The first of the above diagrams essentially says that the two distributivity properties (of $\mathbf{\Lambda}$ and $\mathbf{C}$) commute. This becomes apparent when the diagram is rewritten as follows:

$$\begin{tikzcd}
\kappa(\lambda_1+\lambda_2)\cdot X \arrow[r, "\kappa(c_{X,\lambda_1,\lambda_2})"] \arrow[d, "="] & \kappa(\lambda_1\cdot X\boxplus\lambda_2\cdot X) \arrow[d, "J_{\kappa,\lambda_1\cdot X,\lambda_2\cdot X}"] \\
(\kappa\lambda_1+\kappa\lambda_2)\cdot X \arrow[r, "c_{X,\kappa\lambda_1,\kappa\lambda_2}"] & \kappa\lambda_1\cdot X\boxplus\kappa\lambda_2\cdot X
\end{tikzcd}$$

The second of the above diagrams expresses a notion of these structures being "jointly symmetric" in the following sense. We have already required that

$$\mathbf{\Lambda} \times \mathbf{1} \xrightarrow{\mathrm{id}_{\mathbf{\Lambda}} \times x} \mathbf{\Lambda} \times \mathbf{C} \xrightarrow{\cdot} \mathbf{C}$$

be symmetric monoidal; explicitly, we have required that

$$\begin{tikzcd}
\lambda \cdot (X \boxplus Y) \arrow[r, "\lambda \cdot s_{X,Y}"] \arrow[d, "J_{\lambda,X,Y}"] & \lambda \cdot (Y \boxplus X) \arrow[d, "J_{\lambda,Y,X}"] \\
\lambda \cdot X \boxplus \lambda \cdot Y \arrow[r, "s_{\lambda \cdot X, \lambda \cdot Y}"] & \lambda \cdot Y \boxplus \lambda \cdot X
\end{tikzcd}$$

commute. By this additional diagram, ignoring for a moment the associators and assuming that Λ is a semifield, we also have

$$\begin{tikzcd}
 & (\lambda_1 + \lambda_2) \cdot (X \boxplus Y) \arrow[dl, "c_{X \boxplus Y, \lambda_1, \lambda_2}"'] \arrow[dr, "J_{(\lambda_1+\lambda_2),X,Y}"] & \\
\lambda_1 \cdot (X \boxplus Y) \boxplus \lambda_2 \cdot (X \boxplus Y) \arrow[d, "J_{\lambda_1,X,Y} \boxplus J_{\lambda_2,X,Y}"] & & (\lambda_1 + \lambda_2) \cdot X \boxplus (\lambda_1 + \lambda_2) \cdot Y \arrow[d, "c_{X,\lambda_1,\lambda_2} \boxplus c_{Y,\lambda_1,\lambda_2}"] \\
\lambda_1 \cdot X \boxplus \lambda_1 \cdot Y \boxplus \lambda_2 \cdot X \boxplus \lambda_2 \cdot Y \arrow[rr, "\mathrm{id}_{\lambda_1 \cdot X} \boxplus s_{\lambda_1 \cdot Y, \lambda_2 \cdot X} \boxplus \mathrm{id}_{\lambda_2 \cdot Y}"] & & \lambda_1 \cdot X \boxplus \lambda_2 \cdot X \boxplus \lambda_1 \cdot Y \boxplus \lambda_2 \cdot Y
\end{tikzcd}$$

commute.

Remark 234. Any (topological) 2-semimodule over a CC-semiring is a (topological) T-weak semimodule.

Note that the converse does not hold. For example, $\mathbf{M_b}$ is a T-weak semimodule that is not a 2-semimodule.

As another general remark, we note that (topological) T-weak semimodules are closed under (continuous) linear functions and extensions.

We now give the axioms in terms of rational-like adiabatic categories; the correspondence is straightforward:

Definition 235. A *T-adiabatic category* is an adiabatic category where:

- The diagram

$$\begin{tikzcd}
X \arrow[r, "\lambda_1(c_{\lambda_2,X})"] \arrow[d, "c_{\lambda_2,\lambda_1 X}"] & \lambda_1(\lambda_2 x \boxplus (1-\lambda_2)X) \arrow[dl, "J_{\lambda_1,\lambda_2 X,(1-\lambda_2)X}"] \\
\lambda_2\lambda_1 X \boxplus (1-\lambda_2)\lambda_1 X &
\end{tikzcd}$$

  commutes whenever $1-\lambda_2$ is defined.

- The diagram

$$\begin{tikzcd}
(\lambda X \boxplus (1-\lambda)X) \boxplus (\lambda Y \boxplus (1-\lambda)Y) \arrow[dr, "a_{\lambda X \boxplus (1-\lambda)X,\lambda Y,(1-\lambda)Y}"] & \\
X \boxplus Y \arrow[u, "c_{\lambda,X} \boxplus c_{\lambda,Y}"] \arrow[d, "c_{\lambda,X\boxplus Y}"] & ((\lambda X \boxplus (1-\lambda)X) \boxplus \lambda Y) \boxplus (1-\lambda)Y \arrow[d, "a^{-1}_{\lambda X,(1-\lambda)X,\lambda Y} \boxplus \mathrm{id}_{(1-\lambda)Y}"'] \\
\lambda(X \boxplus Y) \boxplus (1-\lambda)(X \boxplus Y) \arrow[d, "J_{\lambda,X,Y} \boxplus J_{1-\lambda,X,Y}"] & (\lambda X \boxplus ((1-\lambda)X \boxplus \lambda Y)) \boxplus (1-\lambda)Y \arrow[d, "(\mathrm{id}_{\lambda X} \boxplus s_{(1-\lambda)X,\lambda Y}) \boxplus \mathrm{id}_{(1-\lambda)Y}"'] \\
(\lambda X \boxplus \lambda Y) \boxplus ((1-\lambda)X \boxplus (1-\lambda)Y) & (\lambda X \boxplus (\lambda Y \boxplus (1-\lambda)X)) \boxplus (1-\lambda)Y \arrow[dl, "a_{\lambda X,\lambda Y,(1-\lambda)X} \boxplus \mathrm{id}_{(1-\lambda)Y}"] \\
((\lambda X \boxplus \lambda Y) \boxplus (1-\lambda)X) \boxplus (1-\lambda)Y \arrow[u, "a^{-1}_{\lambda X \boxplus \lambda Y,(1-\lambda)X,(1-\lambda)Y}"] &
\end{tikzcd}$$

  commutes whenever $1-\lambda$ is defined.

Remark 236. Any tyrannical adiabatic category is a T-adiabatic category.

Based on the above definitions, Theorem 204 yields the following corollary.

Corollary 237. *Given a stable semifield Λ, a topological T-weak semivector space over Λ is a T-adiabatic category. Conversely, a T-adiabatic category over Λ gives rise to a topological T-weak semivector space.*

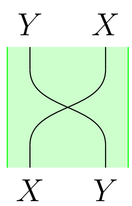
We shall link T-rational-like adiabatic categories (in the case that the semiring is also a stable semifield, the result can be generalised to topological T-weak semivector spaces) to traced categories. We can readily observe one link between adiabatic categories and traced conoidal categories.

Remark 238. By Remark 163, every adiabatic category over a ring is tortile monoidal, hence traced monoidal [37].

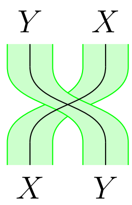
However, these are not the only adiabatic categories that are traced monoidal. We shall present another class of traced adiabatic categories, where the trace is given by the cancellation law. Before proceeding, we pause to introduce the graphical calculus we shall use later.

The reader is assumed to be familiar with standard graphical calculus for monoidal categories, braided monoidal categories and symmetric monoidal categories [71]. McCurdy has extended the graphical calculus to cover monoidal functors and monads [57]. Here we use McCurdy's graphical calculus (where each functor corresponds to a colour) with the further convention that composition of commutative endofunctors is depicted by composition of colours.

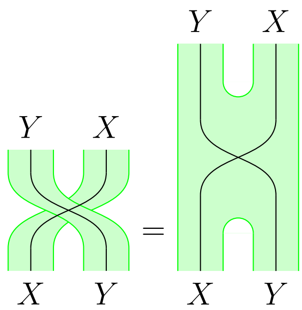
Let $(\mathbf{C}, \boxplus, I, a, l, r, s)$ and $(\mathbf{C}', \boxplus, I', a', l', r', s')$ be symmetric monoidal categories and let $F : \mathbf{C} \to \mathbf{C}'$ be a monoidal functor with associated natural transform J. Representing F by green, we may depict $F(s_{X,Y})$ as



and $s'_{FX,FY}$ as

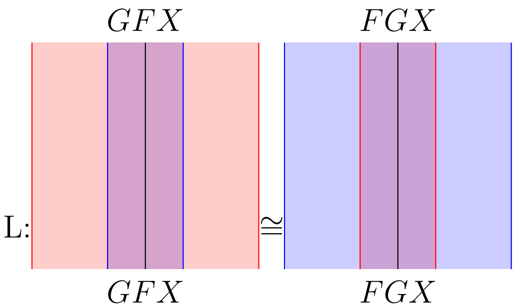


where the direction of the braiding of the coloured areas is meaningless. Then F is a symmetric monoidal functor if and only if it satisfies the equation

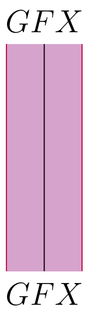


namely $s'_{FX,FY} = J_{Y,X} \circ F(s_{X,Y}) \circ J^{-1}_{X,Y}$.

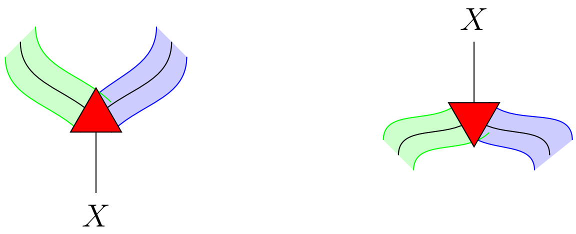
In this work, we often encounter strictly commutative endofunctors. A weak notion of commutativity between G (red) and F (blue) is a natural isomorphism



but when the commutativity is strict, $L = \mathrm{id}$, so we can simply write:



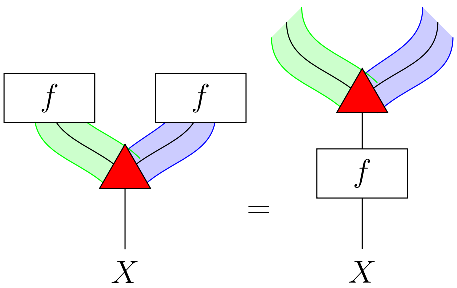
We now move on to notation specific to adiabatic categories. We denote the natural transform c as follows:



depicted: $c_{X,\lambda,1-\lambda}$ depicted: $c^{-1}_{X,\lambda,1-\lambda}$

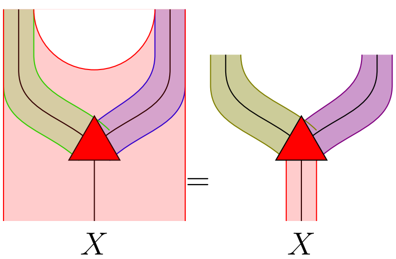
where λ is depicted by green and $1-\lambda$ by blue.

Naturality of c means precisely the following equation.

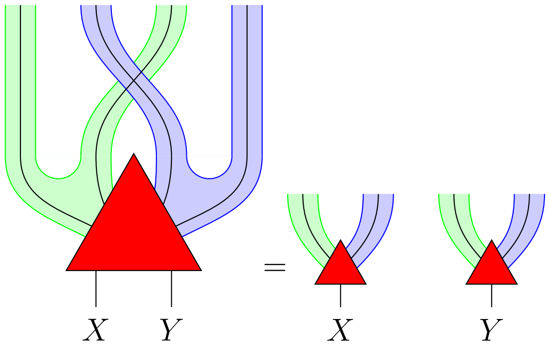


Note that in the left-hand side of the equation we do not use the notation λf and $(1-\lambda)f$, as the functor is implied by the colour of the area underneath the morphism. This is in line with McCurdy's convention.

The T-adiabatic category axioms (depicting λ with green and $1-\lambda$ with blue as above) are as follows:

- 

where red is used for κ.

- 

We note at this point that, while in the graphical calculus for symmetric monoidal categories two morphisms are equal if and only if their string diagrams are equal up to four-dimensional isotopy [71], the new notation introduced by McCurdy and by us has no such interpretation. Therefore in our manipulations of the coloured areas we shall proceed step by stem, explicitly following the axioms. When our manipulations concern only the symmetric monoidal structure, we shall instead use isotopy as usual.

Having defined our graphical calculus, we discuss a few more properties before giving our main theorem.

It is a common experience that the third axiom of trace is the one that usually fails. This time, unfortunately, is no exception; no matter how one constructs a cancellation sequence, the permutations of objects involved will not approach the trivial permutation in the limit. This is not a big problem in our case, because in the physical applications we consider we expect the symmetry to be strict. We therefore only show a weaker property, defined as follows.

Definition 239. Call a symmetric monoidal category *nearly-traced* if it is equipped with a *near-trace*, that is, a natural family of functions of the same type as a trace, satisfying the first two axioms of Definition 232.

Up to this point, the cancellation law has only been used to show existence of a morphism in a homset and did not need to be a well-defined deterministic function. We now require it to be a function, and furthermore, we examine categories where every cancellation sequence is convergent.

Definition 240. Call a rational-like adiabatic category *cancellative* if every cancellation sequence therein is convergent.

We shall also require the conv function to behave "sensibly" in a sense.

Definition 241. Call an adiabatic category *sensible* if the following holds: For object sequences x and y of the same length, let f_j be a list of morphisms $x_j \to y_j$; then $\mathrm{conv}(\{M(f, I, \lambda_i)\}_i) = M(f, I, 0)$ for any sequence $\lambda_i \to 0$.

This is an intuitive principle that we expect to hold true in physical applications. Mathematically, it can follow from simpler axioms, as we shall show. One is what we shall call "near-faithfulness of 0":

Definition 242. In an adiabatic category over a semiring Λ, we say that "0 is nearly faithful" if

$$0f = 0g \Leftrightarrow \lambda_1 f = \lambda_2 g$$

for some nonzero $\lambda_1, \lambda_2 \in \Lambda$.

At first glance, this axiom may seem strange. The intuition behind it is that the morphisms of a category have a property that is shared by "multiples" and retained when they are acted upon by 0. We shall try to motivate it with the following example, which is based on the traditional formulation of 2-dimensional real vectors as a pair of a norm and an angle, with the caveat that the 0 vector can be ascribed *any* angle. Note that the following example is not an adiabatic category, as the requirement that $(\lambda_1 \lambda_2)x = \lambda_1(\lambda_2 x)$ has been dropped; this problem could perhaps be fixed by taking only the upper half of the plane and fixing the morphisms accordingly, but such an alteration would likely make the category too complicated to serve its purpose as an intuitive example.

Example 243. Consider the following category. Objects C are pairs (v_C, a_C) where v_C is a 2-dimensional real vector and a_C is defined as follows:

- For $v_C \neq 0$, $a_C = \frac{v_C}{|v_C|}$.
- For $v_C = 0$, a_C can have any value on the unit circle.

The idea is that the objects are vectors, but the zero vector is not unique, as it has a "direction". In a similar vein, morphisms encapsulate the direction of a vector; concretely, they are 2-dimensional real vectors of norm 1 or 0 such that, for $f : A \to B$:

- If $v_A \neq v_B$, $f = \frac{v_B - v_A}{|v_B - v_A|}$ (*i.e.* there is only one morphism allowed in the homset).
- If $v_A = v_B$, $f = 0$ if and only if $A = B$ and $f = \mathrm{id}_A$. All points of the unit circle are in $\mathrm{Hom}(A, B)$.

Note that, if a morphism starts and ends at the same point on the plane (which, in the case of the 0 point, may or may not be the same object), it can have any direction, but 0 (to be understood as lack of direction) is reserved for the identity. Composition $g \circ f$ of $f : A \to B$ and $g : B \to C$ is defined as follows:

- If $v_A \neq v_C$, there is a unique way to define the composition.
- If $v_A = v_C$, define $g \circ f$ as follows:
  - If $A = C$ and $g = \mathrm{id}_A$, then $g \circ f = f$.
  - Otherwise, $g \circ f$ has the same value as g.

It is easy to see that this is indeed a category, as identity and associativity are satisfied. We now equip it with a monoidal structure that amounts to taking an average; where this average is 0, we pick the direction in a manner that consistently prefers the direction closest to (0,1), so that functoriality and associativity are satisfied. Explicitly, the monoidal structure is defined as follows:

- $v_{A \boxplus B} = \frac{v_A + v_B}{2}$, whence $A \boxplus B$ is fully defined for $v_A + v_B \neq 0$. If $v_A + v_B = 0$ but v_A and v_B are nonzero, for $a_{A \boxplus B}$ we pick whichever of a_A and a_B lies in the upper half of the plane, including (1,0) and excluding (-1,0). If $v_A = v_B = 0$ and only one of a_A and a_B lies in the upper half of the plane, pick that value for $a_{A \boxplus B}$, otherwise pick whichever is closest to (0,1), preferring the first and fourth quadrants in case the distances are equal.

- From the definition of the monoidal product on objects it follows that, when $f + g \neq 0$, $f \boxplus g = \frac{f+g}{|f+g|}$. When f and g are both identities, let $f \boxplus g$ be an identity. Otherwise, let $f \boxplus g$ take the value of whichever of f and g lies in the upper half of the plane, including (1,0) and excluding (-1,0).

It is easy to check that this is indeed a monoidal structure, and furthermore that it is strict with strict symmetry. The monoidal unit is (0,(0,-1)).

Now let the real numbers act on an object C as scalar multiplication on v_C; it is immediate then that then that a number x acts on a_C as multiplication by $\frac{x}{|x|}$, and acts similarly on a morphism f. Then 0 is a nearly faithful functor.

The following lemma then holds.

Lemma 244. *An adiabatic category is sensible if one of the following holds true.*

1. *0 is nearly faithful.*

2. *any morphism g between objects in the image of 0 is of the form $g = 0f$ for some morphism f ("0 is nearly full").*

Proof. Assume that either of the enumerated properties holds. For object sequences x and y of the same length, let f_j be a list of morphisms $x_j \to y_j$; we show that $\mathrm{conv}(\{M(f, I, \lambda_i)\}_i) = M(f, I, 0)$ for any sequence $\lambda_i \to 0$.

In view of stability properties 1 and (iii) it suffices to show that $\mathrm{conv}(\{\lambda_i f\}_i) = 0f$. Since (by properties 1 and (iv)) $0\mathrm{conv}(\{\lambda_i f\}_i) = \mathrm{conv}(\{0f\}) = 0f$, the result follows if 0 is nearly faithful; otherwise, note that it suffices that $\mathrm{conv}(\{\lambda_i f\}_i)$ be in the image of 0, which is true if 0 is nearly full. $\square$

As we have mentioned, the third axiom of trace does not hold for adiabatic categories unless they are strict-symmetric. We shall show that for a certain class of adiabatic categories the cancellation law satisfies the three naturality conditions, the second axiom of trace and the first part of the first axiom of trace. Whether it satisfies the second part of the first axiom remains an open problem for the time being, so for the purposes of this treatment we shall add it as an "extra" requirement.

Definition 245. We shall say that a rational-like adiabatic category satisfies the *extra property* if the following holds: Given a morphism $g : (X \boxplus U) \boxplus W \to (Y \boxplus U) \boxplus W$, the cancellation law satisfies $\mathrm{CL}_{U\otimes W}(a^{-1}_{Y,U,W} \circ g \circ a_{X,U,W}) = \mathrm{CL}_U(\mathrm{CL}_W(g))$.

Definition 246. A sensible cancellative T-rational-like adiabatic category that satisfies the extra property will be called a *nearly-traceable* category.

We now give our main theorem for this section.

Theorem 247. *In a nearly-traceable category* $\mathbf{D}$ *(with monoidal unit* $0A$*), the cancellation law is a near-trace; furthermore, if the symmetry is strict,* $\mathbf{D}$ *is traced monoidal with trace operation* $\mathrm{Tr}^Z_{X,Y}(f) = \mathrm{CL}_Z(f)$ *for* $f : X \boxplus Z \to Y \boxplus Z$.

Proof. Check the three axioms of trace:

1. (a) Recall [58, Proposition 4.1.1] that in any monoidal category with unit Θ for every morphism $f : X \boxplus \Theta \to Y \boxplus \Theta$ we have $f = f' \boxplus \mathrm{id}_\Theta$ for some morphism $f' : X \to Y$, as the endofunctor $\_ \boxplus \mathrm{id}_\Theta$ is full and faithful. So we have $\mathrm{CL}_{0A}(f) = \mathrm{CL}_{0A}(f' \boxplus \mathrm{id}_{0A})$ hence its cancellation law is defined by the commutative diagram

$$\begin{tikzcd} X \boxplus 0A \arrow[r, "\mathrm{conv}(\{f' \boxplus \frac{1}{2^i}\mathrm{id}_{0A}\})"] \arrow[d, "r_X"] & Y \boxplus 0A \arrow[d, "r_Y"] \\ X \arrow[r, "\mathrm{CL}_{0A}(f)"] & Y \end{tikzcd}$$

which simplifies to

$$\begin{tikzcd} X \boxplus 0A \arrow[r, "\mathrm{conv}(\{f' \boxplus \mathrm{id}_{0A}\})"] \arrow[d, "r_X"] & Y \boxplus 0A \arrow[d, "r_Y"] \\ X \arrow[r, "\mathrm{CL}_{0A}(f)"] & Y \end{tikzcd}$$

which by stability property 1 becomes

$$\begin{tikzcd} X \boxplus 0A \arrow[r, "f' \boxplus \mathrm{id}_{0A}"] \arrow[d, "r_X"] & Y \boxplus 0A \arrow[d, "r_Y"] \\ X \arrow[r, "\mathrm{CL}_{0A}(f)"] & Y \end{tikzcd}$$

which by definition is

$$\begin{tikzcd} X \boxplus 0A \arrow[r, "f"] \arrow[d, "r_X"] & Y \boxplus 0A \arrow[d, "r_Y"] \\ X \arrow[r, "\mathrm{CL}_{0A}(f)"] & Y \end{tikzcd}$$

In the end we have $\mathrm{Tr}^{0A}_{X,Y}(f) = \mathrm{CL}_{0A}(f) = r_Y \circ f \circ r_X^{-1}$.

(b) This is true because of the extra property.

2. Let $f : X \boxplus W \to Y \boxplus W$, let F denote the right-cancellation sequence of f and

let S denote the right-cancellation sequence of $a_{Z,Y,W} \circ (\mathrm{id}_Z \boxplus f) \circ a^{-1}_{Z,X,W}$. Then $\mathrm{CL}_W(a_{Z,Y,W} \circ (\mathrm{id}_Z \boxplus f) \circ a^{-1}_{Z,X,W})$ is defined by the commutative diagram

$$\begin{tikzcd}
(Z \boxplus X) \boxplus 0W \arrow[r, "\mathrm{conv}(S)"] \arrow[d, "r_{0W,Z\boxplus X}"] & (Z \boxplus Y) \boxplus 0W \arrow[d, "r_{0W,Z\boxplus Y}"] \\
Z \boxplus X \arrow[r, "\mathrm{CL}_W(a_{Z,Y,W}\circ(\mathrm{id}_Z\boxplus f)\circ a^{-1}_{Z,X,W})"] & Z \boxplus Y
\end{tikzcd}$$

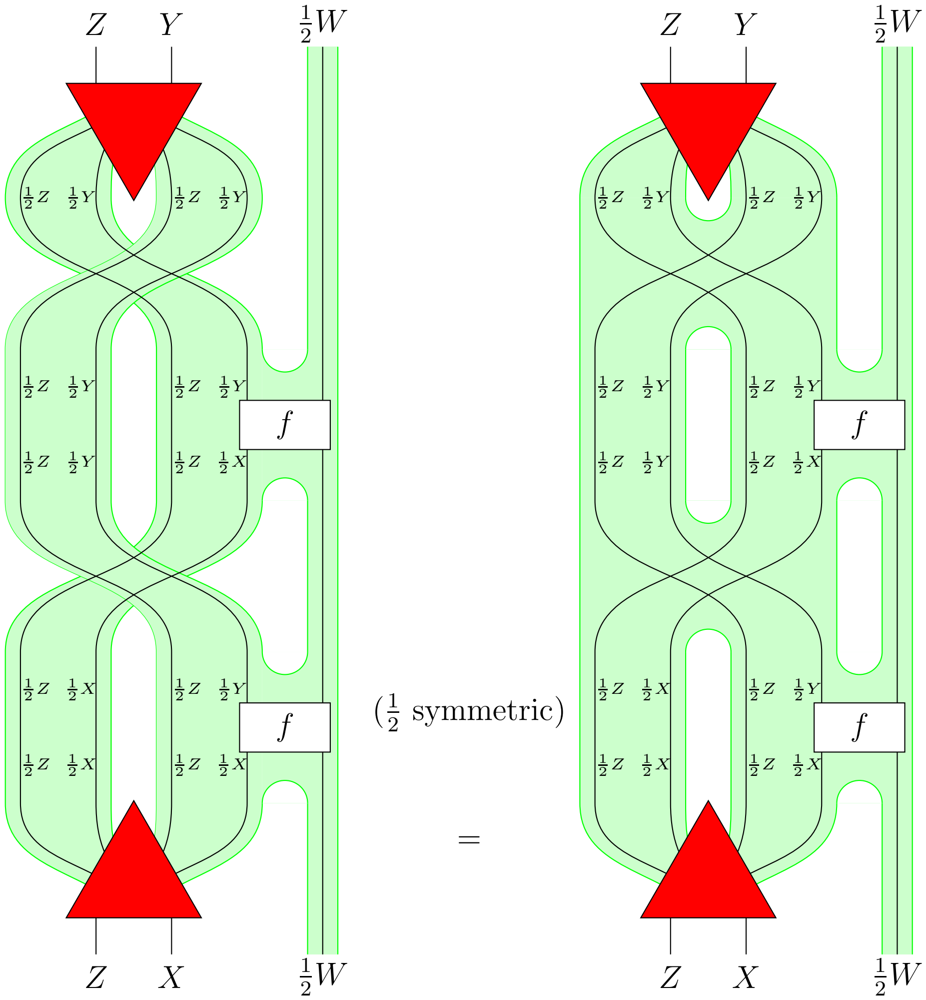
The first term of the right-cancellation sequence equals the chain

$$\begin{tikzcd}
(Z \boxplus X) \boxplus \tfrac12 W \arrow[r, "c_{Z\boxplus X,\frac12,\frac12}\boxplus \mathrm{id}_{\frac12 W}"] & (\tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus X)) \boxplus \tfrac12 W \arrow[r, "a^{-1}_{\frac12(Z\boxplus X),\frac12(Z\boxplus X),\frac12 W}"] & \tfrac12(Z \boxplus X) \boxplus (\tfrac12(Z \boxplus X) \boxplus \tfrac12 W) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus J^{-1}_{\frac12,Z\boxplus X,W}"] \\
\tfrac12(Z \boxplus X) \boxplus (\tfrac12(Z \boxplus Y) \boxplus \tfrac12 W) \arrow[d, "a_{\frac12(Z\boxplus X),\frac12(Z\boxplus Y),\frac12 W}"'] & \tfrac12(Z \boxplus X) \boxplus \tfrac12((Z \boxplus Y) \boxplus W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus J_{\frac12,Z\boxplus Y,W}"'] & \tfrac12(Z \boxplus X) \boxplus \tfrac12((Z \boxplus X) \boxplus W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus \frac12(a_{Z,Y,W}\circ(\mathrm{id}_Z\boxplus f)\circ a^{-1}_{Z,X,W})"'] \\
(\tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[r, "s_{\frac12(Z\boxplus X),\frac12(Z\boxplus Y)}\boxplus \mathrm{id}_{\frac12 W}"] & (\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus X)) \boxplus \tfrac12 W \arrow[r, "a^{-1}_{\frac12(Z\boxplus Y),\frac12(Z\boxplus X),\frac12 W}"] & \tfrac12(Z \boxplus Y) \boxplus (\tfrac12(Z \boxplus X) \boxplus \tfrac12 W) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus J^{-1}_{\frac12,Z\boxplus X,W}"] \\
\tfrac12(Z \boxplus Y) \boxplus (\tfrac12(Z \boxplus Y) \boxplus \tfrac12 W) \arrow[d, "a_{\frac12(Z\boxplus Y),\frac12(Z\boxplus Y),\frac12 W}"'] & \tfrac12(Z \boxplus Y) \boxplus \tfrac12((Z \boxplus Y) \boxplus W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus J_{\frac12,Z\boxplus Y,W}"'] & \tfrac12(Z \boxplus Y) \boxplus \tfrac12((Z \boxplus X) \boxplus W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus \frac12(a_{Z,Y,W}\circ(\mathrm{id}_Z\boxplus f)\circ a^{-1}_{Z,X,W})"'] \\
(\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[r, "s_{\frac12(Z\boxplus Y),\frac12(Z\boxplus Y)}\boxplus \mathrm{id}_{\frac12 W}"] & (\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[r, "c^{-1}_{Z\boxplus Y,\frac12,\frac12}\boxplus \mathrm{id}_{\frac12 W}"] & (Z \boxplus Y) \boxplus \tfrac12 W
\end{tikzcd}$$

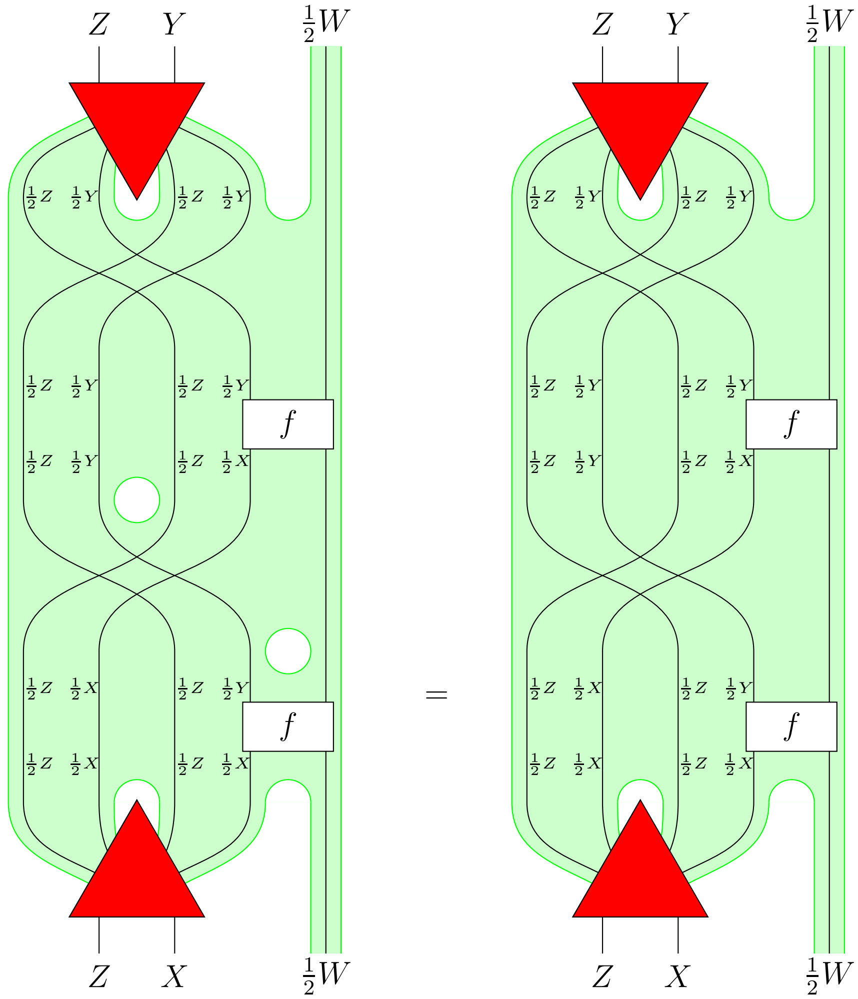
or

$$\begin{tikzcd}
(Z \boxplus X) \boxplus \tfrac12 W \arrow[r, "c_{Z\boxplus X,\frac12,\frac12}\boxplus \mathrm{id}_{\frac12 W}"] & (\tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus X)) \boxplus \tfrac12 W \arrow[r, "a^{-1}_{\frac12(Z\boxplus X),\frac12(Z\boxplus X),\frac12 W}"] & \tfrac12(Z \boxplus X) \boxplus (\tfrac12(Z \boxplus X) \boxplus \tfrac12 W) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus J^{-1}_{\frac12,Z\boxplus X,W}"] \\
\tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus (Y \boxplus W)) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus \frac12 a_{Z,Y,W}"'] & \tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus (X \boxplus W)) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus \frac12(\mathrm{id}_Z\boxplus f)"'] & \tfrac12(Z \boxplus X) \boxplus \tfrac12((Z \boxplus X) \boxplus W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus \frac12 a^{-1}_{Z,X,W}"'] \\
\tfrac12(Z \boxplus X) \boxplus \tfrac12((Z \boxplus Y) \boxplus W) \arrow[r, "\mathrm{id}_{\frac12(Z\boxplus X)}\boxplus J_{\frac12,Z\boxplus Y,W}"] & \tfrac12(Z \boxplus X) \boxplus (\tfrac12(Z \boxplus Y) \boxplus \tfrac12 W) \arrow[r, "a_{\frac12(Z\boxplus X),\frac12(Z\boxplus Y),\frac12 W}"] & (\tfrac12(Z \boxplus X) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[d, "s_{\frac12(Z\boxplus X),\frac12(Z\boxplus Y)}\boxplus \mathrm{id}_{\frac12 W}"] \\
\tfrac12(Z \boxplus Y) \boxplus \tfrac12((Z \boxplus X) \boxplus W) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus \frac12 a^{-1}_{Z,X,W}"'] & \tfrac12(Z \boxplus Y) \boxplus (\tfrac12(Z \boxplus X) \boxplus \tfrac12 W) \arrow[l, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus J^{-1}_{\frac12,Z\boxplus X,W}"'] & (\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus X)) \boxplus \tfrac12 W \arrow[l, "a^{-1}_{\frac12(Z\boxplus Y),\frac12(Z\boxplus X),\frac12 W}"'] \\
\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus (X \boxplus W)) \arrow[r, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus \frac12(\mathrm{id}_Z\boxplus f)"] & \tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus (Y \boxplus W)) \arrow[r, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus \frac12 a_{Z,Y,W}"'] & \tfrac12(Z \boxplus Y) \boxplus \tfrac12((Z \boxplus Y) \boxplus W) \arrow[d, "\mathrm{id}_{\frac12(Z\boxplus Y)}\boxplus J_{\frac12,Z\boxplus Y,W}"] \\
(\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[d, "c^{-1}_{Z\boxplus Y,\frac12,\frac12}\boxplus \mathrm{id}_{\frac12 W}"'] & (\tfrac12(Z \boxplus Y) \boxplus \tfrac12(Z \boxplus Y)) \boxplus \tfrac12 W \arrow[l, "s_{\frac12(Z\boxplus Y),\frac12(Z\boxplus Y)}\boxplus \mathrm{id}_{\frac12 W}"'] & \tfrac12(Z \boxplus Y) \boxplus (\tfrac12(Z \boxplus Y) \boxplus \tfrac12 W) \arrow[l, "a_{\frac12(Z\boxplus Y),\frac12(Z\boxplus Y),\frac12 W}"'] \\
(Z \boxplus Y) \boxplus \tfrac12 W & &
\end{tikzcd}$$

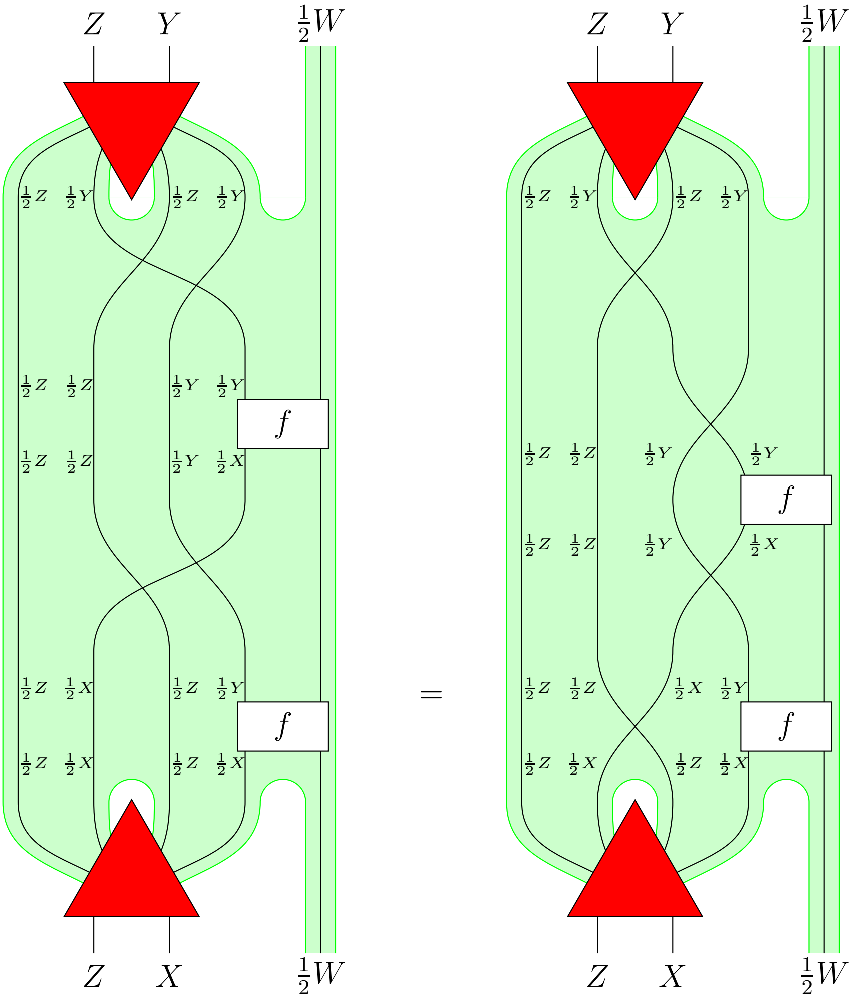
We can convert the above to the language of string diagrams, using our extended graphical calculus. In this calculation, the functor $\frac{1}{2}$ is depicted by green.



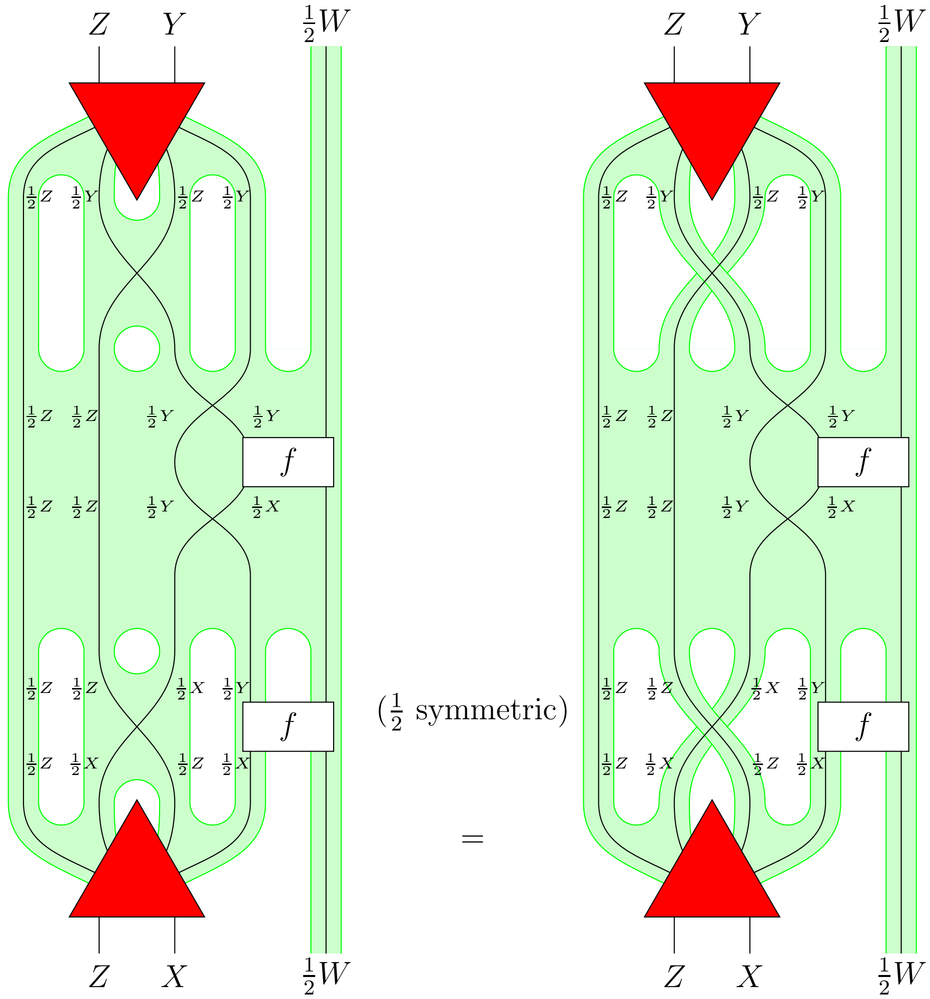
which by naturality of J in 4 operations becomes



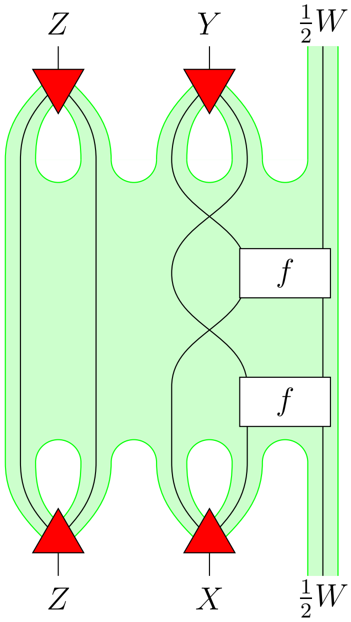
which in turn becomes



which by naturality of J can be rewritten as

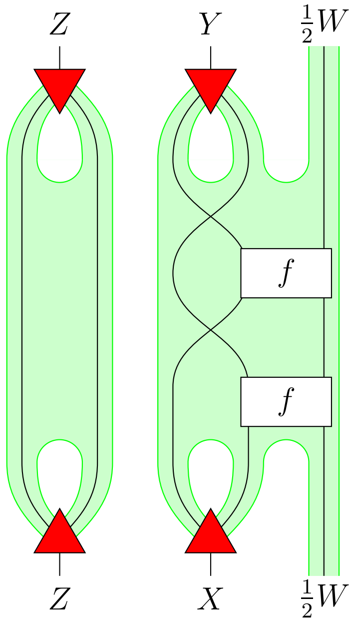


which by the second axiom of T-rational-like adiabatic categories becomes

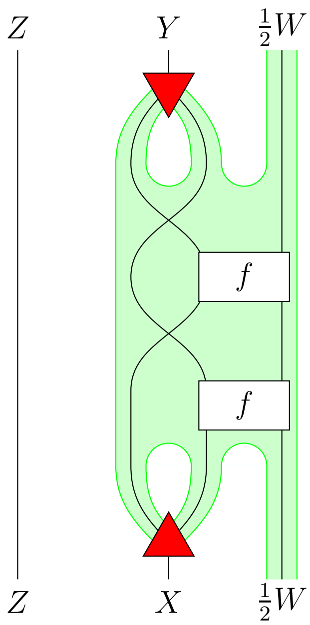


(J natural)

$=$

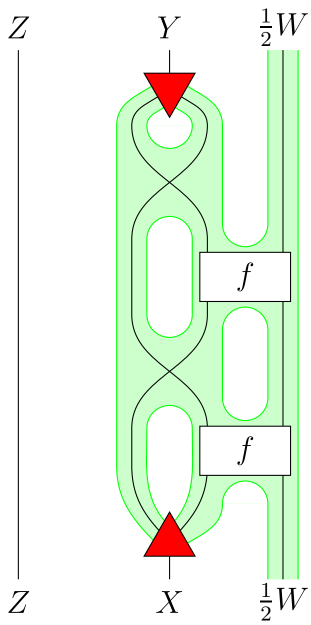


which of course equals

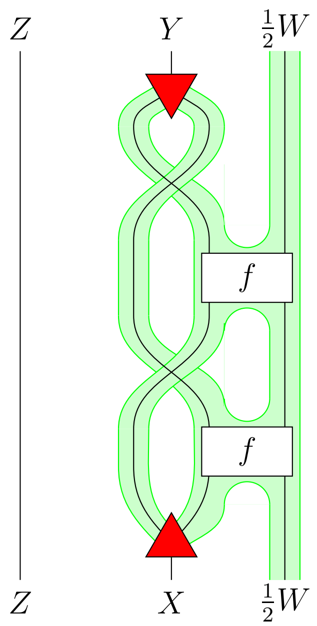


(J natural)

$=$



which because $\frac{1}{2}$ is symmetric equals



So in the end we have that $S_1 = \mathrm{id}_Z \boxplus F_1$ hence by construction of the right-cancellation sequence $S_i = \mathrm{id}_Z \boxplus F_i$. Then by stability properties (iii) and (ii) we get $\mathrm{conv}(S) = \mathrm{id}_Z \boxplus \mathrm{conv}(F)$.

In the end, $\mathrm{Tr}^W_{Z\boxplus X, Z\boxplus Y}(a_{Z,Y,W} \circ (\mathrm{id}_Z \boxplus f) \circ a^{-1}_{Z,X,W}) = \mathrm{id}_Z \boxplus \mathrm{Tr}^W_{X,Y}(f)$.

3. If the symmetry is strict, this condition is immediate from stability property (ii).

Naturality in source of trace (Left Tightening):

We need to show that $\mathrm{Tr}^Z_{X,Y}(f \circ (g \boxplus \mathrm{id}_Z)) = \mathrm{Tr}^Z_{X',Y}(f) \circ g$, where $f : X' \boxplus Z \to Y \boxplus Z$ and $g : X \to X'$; equivalently, that $\mathrm{CL}_Z(f \circ (g \boxplus \mathrm{id}_Z)) = \mathrm{CL}_Z(f) \circ g$. Let S denote the right-cancellation sequence of $f \circ (g \boxplus \mathrm{id}_Z)$ and F the right-cancellation sequence of f. The left-hand side of the equation is defined by the commutative diagram

$$\begin{tikzcd}
X \boxplus 0Z \arrow[r, "\mathrm{conv}(S)"] \arrow[d, "r_{0Z,X}"] & Y \boxplus 0Z \arrow[d, "r_{0Z,Y}"] \\
X \arrow[r, "\mathrm{CL}_Z(f\circ(g\boxplus\mathrm{id}_Z))"] & Y
\end{tikzcd}$$

whereas the right-hand side by the commutative diagram

$$\begin{tikzcd}
 & X' \boxplus 0Z \arrow[r, "\mathrm{conv}(F)"] \arrow[d, "r_{0Z,X'}"] & Y \boxplus 0Z \arrow[d, "r_{0Z,Y}"] \\
X \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{CL}_Z(f)\circ g"'] & X' \arrow[r, "\mathrm{CL}_Z(f)"] & Y
\end{tikzcd}$$

Identifying the bottom edges and simplifying, in the end we want

$$\begin{tikzcd}
X \boxplus 0Z \arrow[r, "\mathrm{conv}(S)"] \arrow[d, "g \boxplus \mathrm{id}_{0Z}"'] & Y \boxplus 0Z \\
X' \boxplus 0Z \arrow[ur, "\mathrm{conv}(F)"'] &
\end{tikzcd}$$

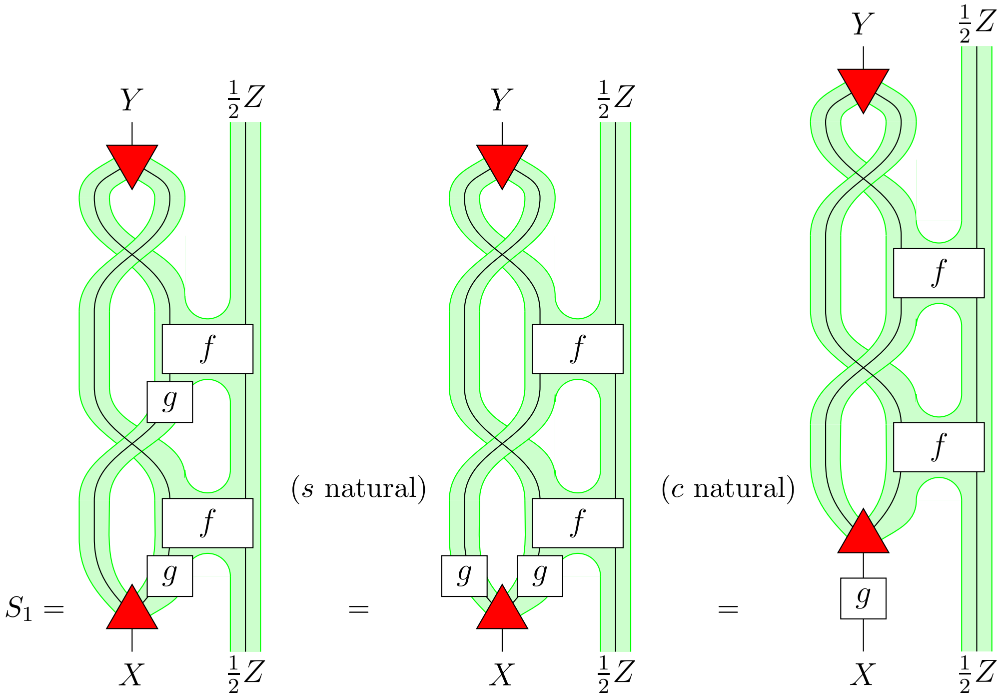
Similarly to the superposing principle, we can rewrite this diagram as

$$\begin{tikzcd}
X \boxplus 0Z \arrow[r, "\mathrm{conv}(S)"] \arrow[d, "\mathrm{conv}(\{g \boxplus \mathrm{id}_{\frac{1}{2^i}Z}\}_i)"'] & Y \boxplus 0Z \\
X' \boxplus 0Z \arrow[ur, "\mathrm{conv}(F)"'] &
\end{tikzcd}$$

which by stability property (i) becomes

$$\begin{tikzcd}
X \boxplus 0Z \arrow[r, shift left, "\mathrm{conv}(S)"] \arrow[r, shift right, "\mathrm{conv}(\{F_i \circ (g \boxplus \mathrm{id}_{\frac{1}{2^i}Z}\}_i))"'] & Y \boxplus 0Z
\end{tikzcd}$$

We need only show that $S_1 = F_1 \circ (g \boxplus \mathrm{id}_{\frac{1}{2}Z})$ and then by construction of the right-cancellation sequence the rest of the terms are also equal, making the diagram commute. In string diagram language,



$= F_1 \circ (g \boxplus \mathrm{id}_{\frac{1}{2}Z})$

Naturality in target of trace (Right Tightening):

This proof is completely analogous to the left tightening.

Naturality in traced object (Sliding):

We need to show that $\mathrm{Tr}^{Z}_{X,Y}((\mathrm{id}_Y \boxplus g) \circ f) = \mathrm{Tr}^{Z'}_{X,Y}(f \circ (\mathrm{id}_X \boxplus g))$, where $f : X \boxplus Z \to Y \boxplus Z'$ and $g : Z' \to Z$; equivalently, that $\mathrm{CL}_Z((\mathrm{id}_Y \boxplus g) \circ f) = \mathrm{CL}_{Z'}(f \circ (\mathrm{id}_X \boxplus g))$; equivalently, that $r_{0Z,Y} \circ \mathrm{conv}(S) \circ r^{-1}_{0Z,X} = r_{0Z',Y} \circ \mathrm{conv}(Q) \circ r^{-1}_{0Z',X}$, where S and Q are the right-cancellation sequences for $(\mathrm{id}_Y \boxplus g) \circ f$ and $f \circ (\mathrm{id}_X \boxplus g)$ respectively.

We rewrite $S_i = (\mathrm{id}_Y \boxplus \frac{1}{2^i} g) \circ H_i$ and $Q_i = H_i \circ (\mathrm{id}_X \boxplus \frac{1}{2^i} g)$, where H_i is defined to be the following chain (with $f_0 = f$):

$$
\begin{tikzcd}
X \boxplus \frac{1}{2^i} Z \arrow[r, "c_{X,\frac{1}{2},\frac{1}{2}} \boxplus \mathrm{id}_{\frac{1}{2^i} Z}"] & (\frac{1}{2} X \boxplus \frac{1}{2} X) \boxplus \frac{1}{2^i} Z \arrow[r, "a^{-1}_{\frac{1}{2} X, \frac{1}{2} X, \frac{1}{2^i} Z}"] & \frac{1}{2} X \boxplus (\frac{1}{2} X \boxplus \frac{1}{2^i} Z) \arrow[d, "\mathrm{id}_{\frac{1}{2} X} \boxplus J^{-1}_{\frac{1}{2}, X, Z}"] \\
\frac{1}{2} X \boxplus (\frac{1}{2} Y \boxplus \frac{1}{2^i} Z') \arrow[d, "a_{\frac{1}{2} X, \frac{1}{2} Y, \frac{1}{2^i} Z'}"'] & \frac{1}{2} X \boxplus \frac{1}{2}(Y \boxplus \frac{1}{2^{i-1}} Z') \arrow[l, "\mathrm{id}_{\frac{1}{2} X} \boxplus J_{\frac{1}{2}, Y, \frac{1}{2^{i-1}} Z'}"'] & \frac{1}{2} X \boxplus \frac{1}{2}(X \boxplus \frac{1}{2^{i-1}} Z) \arrow[l, "\mathrm{id}_{\frac{1}{2} X} \boxplus \frac{1}{2} f_{i-1}"'] \\
(\frac{1}{2} X \boxplus \frac{1}{2} Y) \boxplus \frac{1}{2^i} Z' \arrow[r, "s_{\frac{1}{2} X, \frac{1}{2} Y} \boxplus \frac{1}{2^i} g"] & (\frac{1}{2} Y \boxplus \frac{1}{2} X) \boxplus \frac{1}{2^i} Z \arrow[r, "a^{-1}_{\frac{1}{2} Y, \frac{1}{2} X, \frac{1}{2^i} Z}"] & \frac{1}{2} Y \boxplus (\frac{1}{2} X \boxplus \frac{1}{2^i} Z) \arrow[d, "\mathrm{id}_{\frac{1}{2} Y} \boxplus J^{-1}_{\frac{1}{2}, X, \frac{1}{2^{i-1}} Z}"] \\
\frac{1}{2} Y \boxplus (\frac{1}{2} Y \boxplus \frac{1}{2^i} Z') \arrow[d, "a_{\frac{1}{2} Y, \frac{1}{2} Y, \frac{1}{2^i} Z'}"'] & \frac{1}{2} Y \boxplus \frac{1}{2}(Y \boxplus \frac{1}{2^{i-1}} Z') \arrow[l, "\mathrm{id}_{\frac{1}{2} Y} \boxplus J_{\frac{1}{2}, Y, \frac{1}{2^{i-1}} Z'}"'] & \frac{1}{2} Y \boxplus \frac{1}{2}(X \boxplus \frac{1}{2^{i-1}} Z) \arrow[l, "\mathrm{id}_{\frac{1}{2} Y} \boxplus \frac{1}{2} f_{i-1}"'] \\
(\frac{1}{2} Y \boxplus \frac{1}{2} Y) \boxplus \frac{1}{2^i} Z' \arrow[r, "s_{\frac{1}{2} Y, \frac{1}{2} Y} \boxplus \mathrm{id}_{\frac{1}{2^i} Z'}"] & (\frac{1}{2} Y \boxplus \frac{1}{2} Y) \boxplus \frac{1}{2^i} Z' \arrow[r, "c^{-1}_{Y,\frac{1}{2},\frac{1}{2}} \boxplus \mathrm{id}_{\frac{1}{2^i} Z'}"] & Y \boxplus \frac{1}{2^i} Z'
\end{tikzcd}
$$

Sequence H is similar to the right-cancellation sequence of f, with the crucial difference that $\frac{1}{2^i} g$ acts on the traced object (third row, first morphism). Note that, since the right-cancellation sequence of f converges by hypothesis, so must H; indeed, we can check that in this case each of the component morphisms in H_i converges (by stability properties (ii), (iv) and (iii) combined with the sensibility property) and therefore the composition converges by stability property (i).

So by stability property (i) we have

$$\mathrm{conv}(S) = \mathrm{conv}(\{\mathrm{id}_Y \boxplus \frac{1}{2^i} g\}_i) \circ \mathrm{conv}(H) = (\mathrm{id}_Y \boxplus 0g) \circ \mathrm{conv}(H)$$

and

$$\mathrm{conv}(Q) = \mathrm{conv}(H) \circ \mathrm{conv}(\{\mathrm{id}_X \boxplus \frac{1}{2^i} g\}_i) = \mathrm{conv}(H) \circ (\mathrm{id}_X \boxplus 0g),$$

where the last equalities for each sequence arise from stability property (iii) and the sensibility property. In the end we require that the following diagram commute.

$$\begin{tikzcd}
X \boxplus 0Z \arrow[r, "\mathrm{conv}(H)"] \arrow[d, "r_{0Z,X}"] & Y \boxplus 0Z' \arrow[r, "\mathrm{id}_Y \boxplus 0g"] & Y \boxplus 0Z \arrow[d, "r_{0Z,Y}"] \\
X \arrow[d, "r^{-1}_{0Z',X}"] & & Y \arrow[d, "r^{-1}_{0Z',Y}"] \\
X \boxplus 0Z' \arrow[r, "\mathrm{id}_X \boxplus 0g"] & X \boxplus 0Z \arrow[r, "\mathrm{conv}(H)"] & Y \boxplus 0Z'
\end{tikzcd}$$

Once again using the fact that the functor $\_ \boxplus \mathrm{id}_{0A}$ is full and faithful, and taking into account that there exists a unique morphism between the canonical unit $0A$ and any other unit such as $0Z$ and $0Z'$, we deduce that each of the horizontal compositions of the diagram can be rewritten as follows, for some $m, n : X \to Y$:

$$\mathrm{conv}(H) \circ (\mathrm{id}_X \boxplus 0g) = m \boxplus \mathrm{id}_{0Z'}$$

$$(\mathrm{id}_Y \boxplus 0g) \circ \mathrm{conv}(H) = n \boxplus \mathrm{id}_{0Z}$$

and the lateral compositions of the diagram can be rewritten as $\mathrm{id}_X \boxplus u_{Z,Z'}$ (left) and $\mathrm{id}_Y \boxplus u_{Z,Z'}$ (right), where u is the unique isomorphism between units.

In the end, $n \boxplus u_{Z,Z'} = m \boxplus u_{Z,Z'}$, so we arrive at $m = n$, which makes the diagram commute. □

Corollary 248. *In a strict nearly-traceable category* $\mathbf{D}$*, the cancellation law is a near-trace; furthermore, if the symmetry is strict,* $\mathbf{D}$ *is strict traced monoidal with trace operation* $\mathrm{Tr}^Z_{X,Y}(f) = \mathrm{CL}_Z(f)$ *for* $f : X \boxplus Z \to Y \boxplus Z$*.*

Remark 249. For a nearly-traceable category $\mathbf{D}$, if the underlying semiring is a stable semifield, the properties of Theorem 247 and Corollary 248 hold for any topological T-weak semivector space $\mathbf{C}$ in $\mathrm{Seq}^{-1}(\mathbf{D})$ by replacing the conv function in the cancellation law with the function mapping sequences to their limit in $\mathrm{Mor}(\mathbf{C})$. This is immediate from the fact that conv is a restriction of the function mapping sequences to their limit in $\mathrm{Mor}(\mathbf{C})$, namely restricted to the homsets on which trace is defined, which makes the theorem stated in the adiabatic category context equivalent to its statement in the topological weak semimodule context.

Remark 250. In this section we have discussed trace as an operation that "erases" an object on the right-hand side, and for this we have used right-cancellation. Obviously, we can similarly define an alternative "trace" that "erases" something on the left-hand side using left-cancellation, or even a "trace" that "erases" something in the middle using middle-cancellation. We effectively have three different "trace-like" operations.

3.4 Use of adiabatic categories in Physics

We now apply LY-adiabatic categories in the way Lieb and Yngvason intended, and then do the same with more general rational-like adiabatic categories. The aim of this section is to showcase how both these models are used to describe physical processes. In each application, we first describe a physical setup, then define a category that describes it, and finally show how it fits the framework.

Of course, this framework is contingent on certain assumptions on the system. The main caveat here is continuous scalability. For a setup of a few molecules, this would not be a good approximation, and would be beyond the limit of the model's applicability. Similarly for the continuous behaviour of processes, in the case of thick models; we assume that if a system of mass m undergoes a process f, then a system of mass $m - \epsilon$ (where ϵ is arbitrarily small) at the same state can undergo a process f' such that the results only differ in mass, and such that f' continues a cancellation sequence starting from f. Intuitively, this means that all "similar" processes must be included in the model: for example, assuming that the state is defined by temperature and pressure, if we are allowed to, say, compress a system of mass m at temperature T and pressure P, we must be allowed to compress systems of any mass at temperature

T and pressure P. Other assumptions can be safely assumed to hold (at least within reasonable approximation), such as symmetry or splitting and recombination; it would indeed be very surprising to discover that something fundamentally changes in a box filled with gas just by inserting a partition and then removing it, or that the order of components in a compound thermodynamical system is somehow meaningful.

The reader is cautioned that, in what follows, we take the meaning of "adiabatic" to be the same as Lieb and Yngvason [51]. That is, by "adiabatic process" we do **not** mean a process done on a system isolated by means of an adiabatic barrier, but rather we use Definition 1 as discussed in Subsection 3.1.1. The systems in question are assumed to be finite (*i.e.* the auxiliary system cannot be a heat bath).

3.4.1 Unlabelled adiabatic processes (LY-adiabatic categories)

This subsection describes thin models. That is to say, we follow precisely the setup of Lieb and Yngvason, which is only concerned with whether an adiabatic process with given initial and final state exists. The nuance here is how one should define the way in which systems are allowed to interact to form a combined system, so that the following basic axiom (as posited by Lieb and Yngvason) holds: *If there is an adiabatic process $X \to Y$ and an adiabatic process $X' \to Y'$, then there is an adiabatic process $X \boxplus X' \to Y \boxplus Y'$.* This is the key consideration so that this operation is indeed a monoidal product as required.

In Lieb and Yngvason's setup, the interacting systems remained physically independent. This is conceptually simpler than our examples, but would make for a category whose objects are complicated. This is because each object would have to keep track of its component systems, and thus contain a lot of data. For this reason, in the applications below, we have opted for appropriate mixing regimes such that adiabaticity should[21] be preserved if it is also preserved in Lieb and Yngvason's setup. One may of course define the combining operation in such a category as they see fit, as long as they ensure that adiabaticity is preserved in this way.

A thermodynamical system with only one degree of freedom (mass) would yield a discrete category and would be of no interest. In order to provide examples of simple but nontrivial LY-adiabatic categories, we examine two and three degrees of freedom.

A system with two degrees of freedom

We shall first describe a simple physical setup and then define an associated category that, as we shall show, is an LY-adiabatic category.

We deal with a homogeneous thermal system of mass M at temperature T and energy E, composed of distinguishable particles of the pure substance Hypotheticum that obey statistics specified by a function $\epsilon(T)$ (for example, Maxwell-Boltzmann statistics). The system can be thought of as, for example, a gas in an incompressible insulated vial with a stirrer attached; the role of the stirrer is to do dissipative work on the system. We consider triplets (M, T, E) of positive real numbers, varying two of these variables (the third one being dependent); we also consider empty systems $(0, T, 0)$, where T can be assigned an arbitrary positive real value.[22] Two states are the same either if all three of their properties are equal or if both states refer to an empty system.

We consider adiabatic processes $(M, T, E) \to (M, T', E')$ where the equalities $E = M\epsilon(T)$ and $E' = M\epsilon(T')$ are satisfied. By definition of internal energy and the adiabaticity condition we have that $E' - E = W$, where W denotes work done on the system. In this system there are no work variables, so the total work equals the dis-

[21] The choice of words here is not accidental: this is not something that we have rigorously proven, and we do not assume it. As we shall see below, we have only been able to prove monoidality for a system with two degrees of freedom.

[22] Note that mass, energy and temperature are all finite. As we shall see at the end of this subsection, the fact that we do not allow infinite energy is crucial to **H** being an LY-adiabatic category.

sipative work W_d. Note that, since $W_d \geq 0$, we require that $E' \geq E$ or equivalently that $\epsilon(T') \geq \epsilon(T)$. We shall require ϵ to be a strictly monotone function, so that T is defined by M and E; in other words, M and E give a complete description of the system. Then the adiabaticity condition is equivalent to the condition $T' \geq T$.

We now define the category.

Definition 251. The category $\mathbf{H}_\epsilon$ is defined as follows:

- Objects are triplets (M, T, E), with $M \geq 0$, $T > 0$ and $E \geq 0$, obeying the *statistics rule* for this category: $E = M\epsilon(T)$, where ϵ is a strictly monotone function. These objects represent physical systems of mass M, temperature T and energy E.

- It is thin; morphisms $f : (M, T, E) \to (M', T', E')$ exist if and only if $M = M'$ and $E \leq E'$ (which for nonempty systems is equivalent to $M = M'$ and, depending on whether ϵ increases or decreases, either $T \leq T'$ or $T \geq T'$). Call this condition the *adiabaticity rule* for this category. Morphisms physically correspond to the existence of adiabatic processes.

- It is equipped with a strict monoidal product $\boxplus$, defined thus:

  – For nonempty objects $A = (M_A, T_A, E_A)$ and $B = (M_B, T_B, E_B)$, define $A \boxplus B = (M_A + M_B, T_{A\boxplus B}, E_A + E_B)$, where $T_{A\boxplus B}$ is defined to be the solution to the equation $M_A\epsilon(T_A) + M_B\epsilon(T_B) = (M_A + M_B)\epsilon(T_{A\boxplus B})$; it follows that $\boxplus$ is equipped with a strict symmetry. This monoidal product physically corresponds to a merge of systems A and B.

  – As for morphisms, since the category is thin, for $f : A \to A'$ and $g : B \to B'$, $f \boxplus g$ can only be the unique morphism $A \boxplus B \to A' \boxplus B'$.[23]

[23] The existence of this morphism is immediate from the definition of the monoidal product on objects and from the adiabaticity rule.

– Empty systems $(0, T, 0)$ with arbitrary T are all units; furthermore, we identify them all to be the same object O.

- It is equipped with a family of strict monoidal covariant endofunctors F_λ, $\lambda \in \mathbb{R}^{\geq 0}$, which take (M, T, E) to $(\lambda M, T, \lambda E)$.[24] It is evident that $F_{\lambda_1} F_{\lambda_2} = F_{\lambda_1 \lambda_2}$, that $F_{\lambda_1+\lambda_2} X = F_{\lambda_1} X \boxplus F_{\lambda_2} X$ and that $F_0 X = O$ for any X. We also write λ for F_λ as there is no ambiguity in the notation. Physically, these functors correspond to scaling a system by some factor.

Now we show that $\mathbf{H}_\epsilon$ is an LY-adiabatic category.

Theorem 252. *The category $\mathbf{H}_\epsilon$ is an LY-adiabatic category.*

Proof. It is immediate that $\mathbf{H}_\epsilon$ is thin and weakly linear over $\mathbb{R}^{\geq 0}$. We need only establish the basic stability condition; the extended stability conditions then follow from thinness.

Concretely, what we need to show is that, given a sequence of morphisms $X \boxplus \lambda_i X' \to Y \boxplus \lambda_i Y'$ with $\lambda_i \to 0$, there exists a morphism $X \to Y$. Equivalently, setting $X = (M_X, T_X, E_X)$, $X' = (M_{X'}, T_{X'}, E_{X'})$, $Y = (M_Y, T_Y, E_Y)$ and $Y' = (M_{Y'}, T_{Y'}, E_{Y'})$, we must show that if $M_X + \lambda_i M_{X'} = M_Y + \lambda_i M_{Y'}$ and $E_X + \lambda_i E_{X'} \leq E_Y + \lambda_i E_{Y'}$ for a sequence $\lambda_i \to 0$ then $M_X = M_Y$ and $E_X \leq E_Y$.

If the sequence is eventually 0, the result is immediate. Suppose that the sequence is not eventually 0.

The first part of the adiabaticity rule is straightforward. Pick $\lambda_1 > \lambda_2 > 0$ in the sequence:

$$\left.\begin{array}{c} M_X + \lambda_1 M_{X'} = M_Y + \lambda_1 M_{Y'} \\ M_X + \lambda_2 M_{X'} = M_Y + \lambda_2 M_{Y'} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} M_X = M_Y \\ M_{X'} = M_{Y'} \end{array}\right. .$$

[24] It is immediately apparent that existence of $f : (M, T_X, E_X) \to (M, T_Y, E_Y)$ implies existence of $\lambda f : (\lambda M, T_X, \lambda E_X) \to (\lambda M, T_Y, \lambda E_Y)$ and that $(\lambda M_X, T_X, \lambda E_X) \boxplus (\lambda M_Y, T_Y, \lambda E_Y) = (\lambda(M_X + M_Y), T_{X \boxplus Y}, \lambda(E_X + E_Y))$. Therefore, the F_λ are well-defined strict monoidal functors.

For the second part, suppose that $E_X > E_Y$. Then $E_X = E_Y + E$ with $E > 0$. $E_X + \lambda_i E_{X'} \leq E_Y + \lambda_i E_{Y'} \Leftrightarrow E + \lambda_i E_{X'} \leq \lambda_i E_{Y'} \Leftrightarrow E \leq \lambda_i(E_{Y'} - E_{X'})$ for every λ_i. But this is impossible because $E_{Y'} - E_{X'}$ is finite and $\lambda_i \to 0$. So $E_X \leq E_Y$.

This completes the proof. $\square$

Example 253. For a concrete example, let us consider the category $\mathbf{H_b}$ that corresponds to Maxwell-Boltzmann statistics. Let us assume infinite energy levels ϵ_i, which we fix at $\epsilon_i = i$ (as we have not allowed any degrees of freedom that can affect them). As per Maxwell-Boltzmann statistics, $b(T) = \sum_i \epsilon_i \frac{g_i e^{-\frac{\epsilon_i}{kT}}}{m \sum_j g_j e^{-\frac{\epsilon_j}{kT}}}$, where m is the atomic mass, k is Boltzmann's constant and the degeneracies g_i are known constants; for the LY-adiabatic category approximation, we must assume that m is an infinitely small quantity and treat the total mass as a continuous variable.

Furthermore, we shall define the degeneracies as $g_i = T_{i+1} = \frac{(i+1)(i+2)}{2}$, where T_n is the nth triangular number. This stems from the physical assumption that the energy levels are due to momentum, hence depend on movement in three spatial dimensions, so the states of a given energy level i correspond to the distinct ways to express i as the sum of three nonnegative integers. Using these degeneracies we get $b(T) = \frac{3e^{\frac{3}{kT}}}{m(e^{\frac{1}{kT}}-1)(1-3e^{\frac{1}{kT}}+3e^{\frac{2}{kT}})}$.

Ignoring k and m, set $b(T) = \frac{3e^{\frac{3}{T}}}{(e^{\frac{1}{T}}-1)(1-3e^{\frac{1}{T}}+3e^{\frac{2}{T}})}$. The resulting expressions for $T(M,E)$ and for $T_{A \boxplus B}$ are very complicated, and are only included here for the sake of completeness. What follows was calculated by *Mathematica*; the reader is not expected to go through these formulas in detail.

$$T(M,E) = \frac{1}{\log(a+b+c)}$$

$$a = \frac{4\sqrt[3]{2}EM}{(E-M)\sqrt[3]{27E^3+162E^2M+\sqrt{(27E^3+162E^2M+243EM^2)^2-186624E^3M^3}+243EM^2}}$$

$$b = \frac{\sqrt[3]{27E^3+162E^2M+\sqrt{(27E^3+162E^2M+243EM^2)^2-186624E^3M^3}+243EM^2}}{9\sqrt[3]{2}(E-M)}$$

$$c = \frac{2E}{3(E-M)}$$

$$
\begin{aligned}
T_{A\boxplus B} &= \frac{1}{\log(\frac{n_1}{d_{1,1}d_{1,2}} + \frac{n_2}{d_2} + \frac{n_3}{d_3})} \\
n_1 &= 4\sqrt[3]{2}(M_A+M_B)\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right) \\
d_{1,1} &= \frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}-M_A+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}-M_B \\
d_{1,2} &= \sqrt[3]{k_1+k_2+k_3+k_4} \\
k_1 &= 27\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right)^3 \\
k_2 &= 162(M_A+M_B)\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right)^2 \\
k_3 &= 243(M_A+M_B)^2\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right) \\
k_4 &= \sqrt{(k_1+k_2+k_3)^2-186624(M_A+M_B)^3\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right)^3} \\
n_2 &= 2\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}\right) \\
d_2 &= 3\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}-M_A+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}-M_B\right) \\
n_3 &= \sqrt[3]{k_1+k_2+k_3+k_4} \\
d_3 &= 9\sqrt[3]{2}\left(\frac{3e^{\frac{3}{T_A}}M_A}{\left(-1+e^{\frac{1}{T_A}}\right)\left(1-3e^{\frac{1}{T_A}}+3e^{\frac{2}{T_A}}\right)}-M_A+\frac{3e^{\frac{3}{T_B}}M_B}{\left(-1+e^{\frac{1}{T_B}}\right)\left(1-3e^{\frac{1}{T_B}}+3e^{\frac{2}{T_B}}\right)}-M_B\right)
\end{aligned}
$$

A note on systems with three or more degrees of freedom

For our framework to make sense as a physical model, we would naturally like to show that it applies to arbitrarily complicated adiabatic processes. Unfortunately, the calculations quickly become extremely complicated, even for the case of three degrees of freedom. This section presents the settings for which we would like to prove relevant conjectures; it has been included for context and as an indication of intended future work.

Here we present the case of three degrees of freedom. We begin by describing the physical setup, continue to define an associated category, then conjecture that it is an LY-adiabatic category.

We deal with a homogeneous thermal system of mass M at temperature T, volume V, pressure P and energy E, composed of distinguishable particles of the substance Fantasticum (which is characterised by its specific heat $c_v(T, P)$ at constant volume v as a function of temperature T and pressure P or equivalently[25] its specific heat $c_\pi(T, V)$ at constant pressure π as a function of temperature T and volume V) that obey statistics specified by a function $\epsilon(V, T)$ (for example, Maxwell-Boltzmann statistics). Pressure in nonempty systems is defined by mass, volume and temperature by some equation of state $P = Mp(V, T)$; we require p to be strictly monotone in V and T. We consider quintuples (M, T, V, P, E) of positive real numbers, varying three of these variables (the other two being dependent); we also consider empty systems $(0, T, 0, P, 0)$, where T and P can be assigned an arbitrary positive real value. Two states are the same either if all three of their properties are equal or if they both refer to an empty system.

Since p is monotone in V and T, we may express V (or T) as a function of $\frac{P}{M}$ and T (or V). We write $V = \upsilon(\frac{P}{M}, T)$ and $T = \tau(V, \frac{P}{M})$. In the end, given p, all partial derivatives of the form $\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3}$ (where x_i is P, V or T) are known.

We consider adiabatic (in the same sense as above) processes $(M, T, V, P, E) \to (M, T', V', P', E')$, where the equalities $E = M\epsilon(V, T)$ and $E' = M\epsilon(V', T')$ are satisfied. We require ϵ to be strictly monotone in V and T. Since $\epsilon(V, T)$ is monotone in T, we can express $T = \theta(\frac{E}{M}, V)$ and volume $V = a(\frac{E}{M}, T)$.

[25] These two functions contain the same information in view of the fact that we are also given the equation of state (see below), using which we can connect these two quantities [70, Equation (4-6)]; note that the partial derivative $\left(\frac{\partial u}{\partial v}\right)_T$ appearing in the referenced equation can also be expressed in terms of the equation of state [18, Table II].

Example 254. For a concrete example, let us consider an ideal gas obeying Maxwell-Boltzmann statistics. We set $p(V,T) = \frac{RT}{mV}$, where R is the universal gas constant and m is the atomic mass. There are infinite energy levels $\epsilon_i(V)$; for a fairly typical system, let us assume the relation $\epsilon_i(V) = \frac{(i^2+2)h^2}{8mV^{\frac{2}{3}}}$, where h is Planck's constant. As per Maxwell-Boltzmann statistics, $\epsilon(V,T) = \sum_i \epsilon_i(V) \frac{g_i e^{-\frac{\epsilon_i(V)}{kT}}}{m \sum_j g_j e^{-\frac{\epsilon_j(V)}{kT}}}$, where k is Boltzmann's constant and the degeneracies g_i are known constants; for this system to yield an LY-adiabatic category, we must assume that m is an infinitely small quantity and treat the total mass as a continuous variable. Similarly to the previous case, we shall define the degeneracies as $g_i = T_{i+1} = \frac{(i+1)(i+2)}{2}$, where T_n is the nth triangular number.

By definition of internal energy and the adiabaticity condition we have that $E' - E = W$, where W denotes work done on the system. We further distinguish dissipative work W_d and configurative work W_c, so that $E' - E = W_d + W_c$. Since $W_d \geq 0$ we have

$$W_c \leq E' - E. \tag{3.8}$$

For the purpose of determining if a process $f : (M,T,V,P,E) \to (M,T',V',P',E')$ is adiabatic, the path makes no difference; only the initial and final state matter. We may therefore replace f with a sequence of two processes, a reversible adiabatic process $f_{\text{ad}} : (M,T,V,P,E) \to (M,T',V'',P'',E'')$ (*i.e.* with dissipation $W_{d,\text{ad}} = 0$) followed by an irreversible isothermal process $f_{\text{th}} : (M,T',V'',P'',E'') \to (M,T',V',P',E')$ with dissipation $W_{d,\text{th}} = W_d$. We can then compute the middle state and reformulate the adiabaticity condition in terms of state variables and the specific heat function c_v.

More specifically (momentarily switching to v for volume and t for temperature so as not to confuse the variable with its value in the first state), using $W_c = M \int_{V_2}^{V_1} p(v,t)dv$ we can in principle compute $W_{c,\text{ad}}$, $W_{c,\text{th}}$ and $W_{d,\text{th}}$ [18, Table II]. This can be complicated in the general case, but is straightforward for an ideal gas.

Example 255. For a concrete example, consider an ideal gas, where c_v is just a constant:

$$
\begin{aligned}
E'' &= E + Mc_v(T' - T) \\
V'' &= \left(\frac{T}{T'}\right)^{\frac{c_v}{R}} V \\
P'' &= \left(\frac{T}{T'}\right)^{-\frac{c_\pi}{R}} P
\end{aligned}
$$

The configurative work of the isothermal process is $W_{c,\mathrm{th}} = \frac{M}{m} RT' \ln\left(\frac{V''}{V'}\right)$. So by $W_c = W_{c,\mathrm{ad}} + W_{c,\mathrm{th}}$ Equation (3.8) becomes

$$
Mc_v(T' - T) + \frac{M}{m} RT' \ln\left(\frac{\left(\frac{T}{T'}\right)^{\frac{c_v}{R}} V}{V'}\right) \le E' - E,
$$

which is the explicit form of the adiabaticity condition.

In line with the previous example, we have to assume that $m << M$ in the sense that we treat M as a continuous variable.

Remark 256. We can reformulate the adiabaticity condition in terms of the initial entropy S and final entropy S'. Specifically, we want $S \le S'$. The reversible adiabatic process does not change the entropy, and the isothermal process satisfies $\Delta S = \frac{W_{d,\mathrm{th}}}{T'} = \frac{E' - E'' - W_{c,\mathrm{th}}}{T'} = \frac{E' - E - W_{c,\mathrm{ad}} - W_{c,\mathrm{th}}}{T'}$. This yields the same inequality.

Before defining the category, we need to make the following conjecture, which corresponds to the axiom discussed in the introduction of this subsection.

Conjecture 257. *Suppose that we have two systems of Fantasticum, sitting side by side in a box, separated by a rigid partition such that the two systems do not exchange energy in any form, at respective states* $A = (M_A, T_A, V_A, P_A, E_A)$ *and* $B =$

$(M_B, T_B, V_B, P_B, E_B)$. *Further suppose that each of these systems can undergo respectively adiabatic process* $f_A : A \to A'$ *and adiabatic process* $f_B : B \to B'$, *where* $A' = (M_A, T'_A, V'_A, P'_A, E'_A)$ *and* $B = (M_B, T'_B, V'_B, P'_B, E'_B)$. *Let* $A \boxplus B$ *denote the system resulting from removing the partition from the box (i.e. mixing A and B at constant total volume) and similarly define the system* $A' \boxplus B'$. *Then there exists an adiabatic process* $A \boxplus B \to A' \boxplus B'$.

This is where the calculations become extremely complicated, even for an ideal gas. We intend to return to this calculation in a future project.

We now define the category. Parts of the following definition are contingent on Conjecture 257.

Definition 258. The category $\mathbf{F}_\gamma$, where $\gamma > 1$, is defined as follows:

- Objects are quintuples (M, T, V, P, E), with $M \geq 0$, $T > 0$, $V > 0$, $P \geq 0$ and $E \geq 0$, obeying the *state rule* for this category $P = Mp(V, T) = \frac{MT}{V}$ and the *Maxwell-Boltzmann rule* for this category $E = M\epsilon(V, T)$, where $\epsilon(V, T) = \sum_i \epsilon_i(V) \frac{g_i e^{-\frac{\epsilon_i(V)}{T}}}{\sum_j g_j e^{-\frac{\epsilon_j(V)}{T}}}$, where $g_i = \frac{(i+1)(i+2)}{2}$ and $\epsilon_i(V)$ is a function that grows with i and is proportional to $V^{-\frac{2}{3}}$ (for example, $\epsilon_i(V) = \frac{i^2+2}{8V^{\frac{2}{3}}}$).[26] These objects represent physical systems of mass M, temperature T, volume V, pressure P and energy E.

- It is thin; morphisms $f : (M, T, V, P, E) \to (M', T', V', P', E')$ exist if and only if $M = M'$ and

$$\frac{M}{\gamma - 1}(T' - T) + MT' \ln \left(\frac{\left(\frac{T}{T'}\right)^{\frac{1}{\gamma-1}} V}{V'} \right) \leq E' - E.$$

[26] We ignore R, k and m for this treatment.

Call this condition the *adiabaticity rule* for this category. Morphisms physically correspond to the existence of adiabatic processes.

- It is equipped with a strict monoidal product $\boxplus$, defined thus:
  - For objects $A = (M_A, T_A, V_A, P_A, E_A)$ and $B = (M_B, T_B, V_B, P_B, E_B)$, where A, B are nonempty, define $A \boxplus B = (M_A + M_B, T_{A\boxplus B}, V_A + V_B, P_{A\boxplus B}, E_A + E_B)$, where $T_{A\boxplus B}$ and $P_{A\boxplus B}$ are derived from the state rule and the Maxwell-Boltzmann rule; it follows that $\boxplus$ is equipped with a strict symmetry. This monoidal product physically corresponds to a merge of systems A and B with constant total volume.
  - As for morphisms, since the category is thin, for $f : A \to A'$ and $g : B \to B'$, $f \boxplus g$ can only be the unique morphism $A \boxplus B \to A' \boxplus B'$, which exist if Conjecture 257 holds.
  - Empty systems $(0, T, 0, P, 0)$ with arbitrary T and P are all units; furthermore, we identify them all to be the same object O.
- It is equipped with a family of strict monoidal covariant endofunctors F_λ, $\lambda \in \mathbb{R}^{\geq 0}$, which take (M, T, V, P, E) to $(\lambda M, T, \lambda V, P, \lambda E)$.[27] It is evident that we have $F_{\lambda_1} F_{\lambda_2} = F_{\lambda_1 \lambda_2}$, that $F_{\lambda_1 + \lambda_2} X = F_{\lambda_1} X \boxplus F_{\lambda_2} X$ and that $F_0 X = O$ for any X. We also write λ for F_λ as there is no ambiguity in the notation. Physically, these functors correspond to scaling a system by some factor.

Contingent on Conjecture 257, this is an LY-adiabatic category.

[27]It is immediately apparent that existence of $f : (M, T_X, V_X, P_X, E_X) \to (M, T_Y, V_Y, P_Y, E_Y)$ implies existence of $\lambda f : (\lambda M, T_X, \lambda V_X, P_X, \lambda E_X) \to (\lambda M, T_Y, \lambda V_Y, P_Y, \lambda E_Y)$ and that

$$(\lambda M_X, T_X, \lambda V_X, P_X, \lambda E_X) \boxplus (\lambda M_Y, T_Y, \lambda V_Y, P_Y, \lambda E_Y) =$$
$$(\lambda(M_X + M_Y), T_{X\boxplus Y}, \lambda(V_X + V_Y), P_{X\boxplus Y}, \lambda(E_X + E_Y)).$$

Therefore, the F_λ are well-defined strict monoidal functors.

Theorem 259. *If Conjecture 257 holds, the category* **F** *is an LY-adiabatic category.*

Proof. Provided Conjecture 257, most of the axioms of LY-adiabatic categories are immediate for **F**, or easy to show. It is immediate that **F** is thin and weakly linear over $\mathbb{R}^{\geq 0}$. We need only establish the basic stability condition; the extended stability conditions then follow from thinness.

Concretely, what we need to show is that, given a sequence of morphisms $X \boxplus \lambda_i X' \to Y \boxplus \lambda_i Y'$ with $\lambda_i \to 0$, there exists a morphism $X \to Y$. Equivalently, setting $X = (M_X, T_X, V_X, P_X, E_X)$, $X' = (M_{X'}, T_{X'}, V_{X'}, P_{X'}, E_{X'})$, $Y = (M_Y, T_Y, V_Y, P_Y, E_Y)$ and $Y' = (M_{Y'}, T_{Y'}, V_{Y'}, P_{Y'}, E_{Y'})$, we must show that if

$$M_X + \lambda_i M_{X'} = M_Y + \lambda_i M_{Y'} \tag{3.9}$$

$$\begin{aligned} &\frac{M_X + \lambda_i M_{X'}}{\gamma - 1}(T_{Y\boxplus\lambda_i Y'} - T_{X\boxplus\lambda_i X'}) + \\ &\qquad (M_X + \lambda_i M_{X'}) T_{Y\boxplus\lambda_i Y'} \ln \left(\frac{\left(\frac{T_{X\boxplus\lambda_i X'}}{T_{Y\boxplus\lambda_i Y'}}\right)^{\frac{1}{\gamma-1}} (V_X + \lambda_i V_{X'})}{V_Y + \lambda_i V_{Y'}} \right) \\ &\leq E_Y + \lambda_i E_{Y'} - E_X - \lambda_i E_{X'} \end{aligned} \tag{3.10}$$

for a sequence $\lambda_i \to 0$, then

$$M_X = M_Y \tag{3.11}$$

$$\frac{M_X}{\gamma - 1}(T_Y - T_X) + M_X T_Y \ln \left(\frac{\left(\frac{T_X}{T_Y}\right)^{\frac{1}{\gamma-1}} V_X}{V_Y} \right) \leq E_Y - E_X. \tag{3.12}$$

This is true because all the functions involved are continuous and all the quantities involved are finite.

$\square$

3.4.2 Labelled adiabatic processes (rational-like adiabatic categories)

We now look at a category where the morphisms do not correspond to mere existence of an adiabatic process, but rather to classes of adiabatic processes labelled by a set of parameters. As discussed in Subsection 3.1.1, in our choice of classifier we are constrained to properties that cannot be used to extract work. We repeat our observation that possible contradictions between the model and physical reality would hint at an undiscovered process or hidden variable of the system such that a seemingly unrelated property is somehow used to extract work.

Here we have chosen time (as in, duration of the process) as a classifier. We only examine the case of two degrees of freedom. We see that our generalised model can be used to describe this simple system, and furthermore check that it is a traced category.

A system with two degrees of freedom

We define a category $\mathbf{H}'_\epsilon$, fashioned after $\mathbf{H}_\epsilon$, with the same objects but where each morphism f_t is a class of "strictly nontrivial" processes labelled by their duration t. This category is no longer thin.

We briefly remind the reader of the setup for two degrees of freedom, as described in the previous subsection: We deal with a pure substance at state (M, T, E) obeying statistics as described by the strictly monotone $\epsilon(T)$, undergoing adiabatic processes wherein we do dissipative work on the system. Again, one may think of a gas in an incompressible insulated vial with a stirrer attached, where the stirrer serves as a way to do dissipative work on the system.

Now we embellish the setup with distinct classes of processes. Each morphism f_t will now have the physical interpretation of a set of processes with duration t. This gives each homset a topology inherited from this time variable, namely that of the positive real numbers. Physically, the different morphisms in a homset equivalently correspond to stirrers with different average power, powered by a rising and falling weight subject to a greater or smaller force.

The new category is defined as follows.

Definition 260. The category $\mathbf{H}'_\epsilon$ is defined as follows:

- Objects are triplets (M, T, E), with $M \geq 0$, $T > 0$ and $E \geq 0$, obeying the *statistics rule* for this category: $E = M\epsilon(T)$, where ϵ is strictly monotone. These objects represent physical systems of mass M, temperature T and energy E.

- $\mathrm{Hom}((M, T, E), (M', T', E')) \neq \emptyset$ if and only if $M = M'$ and $E \leq E'$ (which for nonempty systems is equivalent to $M = M'$ and either $T \leq T'$ or $T \geq T'$). Call this condition the *adiabaticity rule* for this category. Each morphism f_t is characterised by an index $t \in \mathbb{R}^{\geq 0}$ (*i.e.* no two morphisms in a given homset share this index) and the index 0 is reserved for identities. Each morphism physically corresponds to a class of processes labelled by their duration t; we only consider strictly nontrivial processes, in the sense that at every stage of the process some variable of the system changes. Identity morphisms correspond to doing nothing, which by strict nontriviality forces their duration to 0.

- Composition works as follows. Let $\mathrm{Hom}(A, B) = \{f\}_t$, $\mathrm{Hom}(B, C) = \{g\}_t$ and $\mathrm{Hom}(A, C) = \{h\}_t$. Then $g_{t_2} \circ f_{t_1} = h_{t_1+t_2}$. Composition physically corresponds to performing one process after the other.

- It is equipped with a strict monoidal product $\boxplus$, defined thus:

  - For nonempty objects $A = (M_A, T_A, E_A)$ and $B = (M_B, T_B, E_B)$, define $A \boxplus B = (M_A + M_B, T_{A \boxplus B}, E_A + E_B)$, where $T_{A \boxplus B}$ is defined to be the solution to the equation $M_A\epsilon(T_A) + M_B\epsilon(T_B) = (M_A + M_B)\epsilon(T_{A\boxplus B})$; it follows that $\boxplus$ is equipped with a strict symmetry. This monoidal product physically corresponds to a merge of systems A and B.

  - For objects $A = (M_A, T_A, E_A)$, $B = (M_B, T_B, E_B)$, $A' = (M'_A, T'_A, E'_A)$ and $B' = (M'_B, T'_B, E'_B)$, where A, B, A', B' nonempty, and morphisms $f_{t_A} : A \to A'$ and $g_{t_B} : B \to B'$, set $f_{t_A} \boxplus g_{t_B} = h_{t_A+t_B} : A \boxplus B \to A' \boxplus B'$. This process has no special physical meaning.[28]

  - Empty systems $(0, T, 0)$ with arbitrary T are all units; furthermore, we identify them all to be the same object O.

- It is equipped with a family of strict monoidal covariant endofunctors F_λ, $\lambda \in \mathbb{R}^{\geq 0}$, which take (M, T, E) to $(\lambda M, T, \lambda E)$ and take each morphism f_t to the morphism $g_{\lambda t}$ in the appropriate homset.[29] It is immediate that $F_{\lambda_1} F_{\lambda_2} = F_{\lambda_1 \lambda_2}$, that $F_{\lambda_1+\lambda_2} f_t = F_{\lambda_1} f_t \boxplus F_{\lambda_2} f_t$ for any morphism f_t, that identities are preserved and that $F_0 X = O$ for any object X. We also write λ for F_λ as there is no ambiguity in the notation. Physically, these functors correspond to scaling the systems by some factor without changing the average power of the heating process between them.

- Let $\{f_{t_i} : X \boxplus \lambda_i X' \to Y \boxplus \lambda_i Y'\}_i$ be a sequence of morphisms where $\lambda_i \to 0$. If the corresponding sequence t_i converges to t, then let $\mathrm{conv}(\{f_{t_i}\}) = f_t : X \to Y$.[30]

Now we show that $\mathbf{H}'_\epsilon$ is a tyrannical rational-like adiabatic category.

Theorem 261. *The category $\mathbf{H}'_\epsilon$ is a tyrannical rational-like adiabatic category with monoidal product $\boxplus$, monoidal unit O and convergence function* conv.

[28] The existence of this morphism is immediate.

[29] These are well-defined strict monoidal functors as in the thin case.

[30] This must exist by the same rationale that the category $\mathbf{H}$ of the previous section is an LY-adiabatic category and because our stirrer can be as powerful as we want.

Proof. It is immediate that $\mathbf{H}'_\epsilon$ is weakly linear over $\mathbb{R}^{\geq 0}$, that the monoidal product is strict with strict symmetry, that the monoidal functors are strict and that the SR isomorphism is strict. The basic stability condition is also satisfied. It remains to show the extended stability conditions.

Properties 1, 2 and 3 must hold because conv reduces to convergence of t_i. Property (i) holds because addition is a continuous operation in $\tau_{\mathbb{R}}$. Property (ii) holds because identities are precisely the processes with zero duration. The first part of property (iii) holds because, like composition, we have defined the monoidal product in terms of added durations; the second part holds because of strictness. Property (iv) holds because the scaling functors act as multiplication on the time indices, which is a continuous operation in $\tau_{\mathbb{R}}$. Finally, property (v) holds because the symmetry is strict. □

Note that cancellation sequences are convergent in this category: the duration of each term of the cancellation sequence is equal to the duration t of the whole process, therefore there is a limit; namely, the morphism with duration t in the appropriate homset.

Theorem 262. *The cancellation operation in $\mathbf{H}'_\epsilon$ is a (strict) trace.*

Proof. We check the axioms of trace (Definition 231):

1. (a) The cancellation of $f_t : A \boxplus O \to B \boxplus O$ produces the constant sequence $\{f_t : A \boxplus O \to B \boxplus O\}$, which converges to f_t itself.

   (b) The cancellation of $f_t : A \boxplus X \boxplus Y \to B \boxplus X \boxplus Y$ with respect to Y yields $f'_t : A \boxplus X \to B \boxplus X$, whose cancellation with respect to X yields $f''_t : A \to B$; as does the cancellation of f_t with respect to $X \boxplus Y$.

2. Given $f_t : A \boxplus X \to B \boxplus X$ and $f_0 = \mathrm{id}_C$, the cancellation of $f_0 \boxplus f_t : C \boxplus A \boxplus X \to C \boxplus B \boxplus X$ with respect to X yields $f''_t : C \boxplus A \to C \boxplus B$; but $f''_t = f_0 \boxplus f'_t$ with $f'_t : A \to B$, where the latter is the cancellation of f_t with respect to X.

3. The symmetry axiom holds because the category has strict symmetry.

□

3.4.3 Discussion on examples given

From these initial applications of our model we have seen that an adiabatic category corresponds to a physical model of adiabatic processes with two degrees of freedom, and is furthermore traced. We have also seen the difficulty of extending this correspondence to larger degrees of freedom. Unsurprisingly, Entropy does appear as a nondecreasing quantity: in the case of two degrees of freedom, Energy is also nondecreasing, so Entropy was not explicitly mentioned; in the case of three degrees of freedom, Entropy is clearly linked to the adiabaticity criterion (Remark 256).

In the examples we have seen, we are mainly working within Classical Thermodynamics, but we have borrowed notions from Statistical Mechanics to simplify the picture; namely, we have assumed that there exists a statistical distribution underpinning the system's behaviour, which gives us a convenient function we can work with in the abstract.

We would like to have a more complete result on applicability of adiabatic categories for arbitrary degrees of freedom. Unfortunately, however, as discussed in the introduction to this section, the choice of an appropriate monoidal product straddles a precarious line between making the objects too complicated to handle and making the proofs too difficult.

Chapter 4

Conclusion

We begin this chapter by making explicit the connection between the two strands of this work.

On the one hand, Chapter 2 is very general in scope. Matroids are about as abstract as any mathematical entity, on par with topological spaces in terms of generality, thus extremely versatile. As we have noted, however, the fact that they capture the essence of dependence connects them to probabilistic conditional dependence. More specifically, it has been pointed out [30] that Shannon Entropy induces a “generalised matroid” structure[1] by virtue of being a nondecreasing submodular function, and that many results from Matroid Theory transfer to that setting. Hence our work on Matroids ultimately provides a foundation for the study of Shannon Entropy, and consequently Entropy in its Statistical Mechanics formulation.

On the other hand, Chapter 3 explores the *categorical* implications of Entropy being a nondecreasing submodular continuous function, and provides a basis for recovering the Statistical Mechanics formulation of Entropy as a limiting case.

In the end, these two projects are set to converge on uniting the various facets

[1]These generalisations are called *polymatroids*.

of Entropy under their shared abstract properties (monotonicity, submodularity and continuity). This could not have happened in the context of this thesis, but rather is reserved for future work.

4.1 Matroids

4.1.1 Our contribution

We have worked out many interesting properties of a useful category of matroids; in the process, we have also corrected some mistakes in the literature (see Propositions 30 and 42, as well as the first footnote of Section 2.5).

Most of our results on matroids are negative; that is, neither the category of all matroids nor any notable subcategories is very well behaved, as it does not have many limits and colimits (Section 2.3), and moreover most of the commonly used matroid constructions are not functorial. This is of course helpful information, but also a rather unfortunate fact, since it constrains the properties of functors one might exploit. On a more positive note, taking minors did turn out to be functorial (Section 2.8). We have also found many adjunctions between matroids and notable subcategories (Section 2.4), and simplification turned out to be monadic (Section 2.7).

We have also examined properties of functors between matroids and other categories, most notably vector spaces and geometric lattices, and discovered that the associated functors $L \circ M$ and $L_\bullet$ exhibit properties "close" to fullness and faithfulness; these properties may perhaps be of use. Shifting momentarily our focus from Physics, we note that these functors also carry implications for matroid representability; a major topic in matroid theory, with longstanding open problems. It would be interesting if Category Theory turned out to be the key to progress towards their solutions.

We have shown that this category features two orthogonal factorisation systems and

(Co)limit	present
Products	✗
Equalisers	✓
Pullbacks	✗
Coproducts	✓
Coequalisers	✗
Pushouts	✗
Exponentials	✗

Construction	functorial
Duality	✗
Free addition	✓
Deletion	✓
Contraction	✓
Minor	✓
(Free) Extension	✗
(Free) Coextension	✗
(Higgs) Lift	✗
Truncation	✗
Quotient	✗
Erection	✗

Construction	monoidal
Union	✗
Intersection	✗
Half-dual union	✗
Intertwining	✗
Parallel connection	✓
Series connection	✓

Figure 4.1: Overview of the categorical nature of matroids

a double factorisation system (Section 2.6); a further notable result of this project is that this category inherits orthogonal factorisation systems from a category of geometric lattices, which is simpler in some sense (Corollary 86).

Finally, we have offered a categorical characterisation of the ubiquitous greedy algorithm (Section 2.9).

Our results are summarised in Figures 4.1 and 4.2.

4.1.2 Future work

Duality is a fundamental aspect of matroid theory that we have not covered. We conjecture that if taking duals can be somehow cast as a functor, it must be in the context of a category of "strong relations"; that is, a category that arises from the category of matroids and strong maps as the category of sets and relations arises from

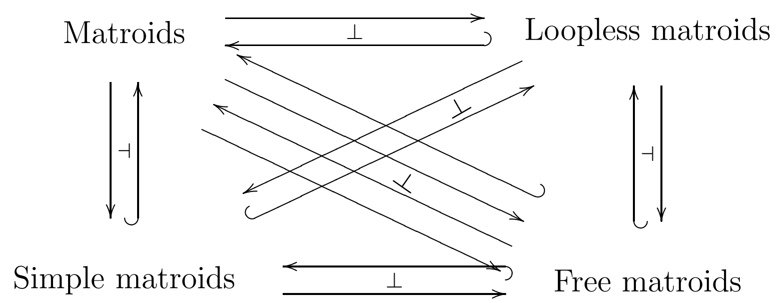


Figure 4.2: Overview of adjunctions involving the category of matroids

the category of sets and functions. This seems a worthwhile pursuit, because duality is so central to matroid theory that, in any model of a physical system involving matroids, duality would almost certainly have some physical meaning.

Another direction of future work is closer to our work on Entropy. Our results on adjunctions, limits and colimits could be extended to subcategories of matroids representable over some field, aiming to provide some categorical insight into matroid representability. Representability has been linked [30] to so-called *probabilistic representability*, which connects matroids to probabilistic structures, providing insight into Statistical Mechanics.

Other possible directions of future research might be determining which of our results extend to infinite matroids, examining generalisations of matroids, or searching further for possible functors between graphs and matroids.

As a final point of interest, we note that there exist matroids with additional structure called bimatroids [47] that form a 2-category (albeit a trivial one).

4.2 Entropy

4.2.1 Our contribution

Chapter 3 offers a categorical framework to model adiabatic processes between thermodynamical states, along with examples of application. We have described a class of categories that arises from physical modelling, based on work by Lieb and Yngvason [51], shown that it is a special case of a more elegant and powerful class of categories (topological weak semimodules), and shown that under certain physically plausible constraints it also exhibits the well-studied trace property. More precisely, our main achievements are the following:

- We have generalised Lieb and Yngvason's result in two ways to conclude the existence of an Entropy functor for rational-like semirings and for categories that are not preorders (Theorems 225 and 230).
- Defining a set of "stable" semifields $\mathcal{L}$, with $\mathbb{R}, \mathbb{R}^{\geq 0} \in \mathcal{L}$, we have proven that for $\Lambda \in \mathcal{L}$ there exists an adjunction between adiabatic categories over Λ and topological weak semimodules over Λ (Theorem 217). This ties into the existence of an Entropy functor (which exists when $\Lambda = \mathbb{R}^{\geq 0}$, as it is rational-like).
- We have shown (Remark 238 and Theorem 247) that two classes of adiabatic categories are traced monoidal; in view of the adjunction mentioned in the previous point, this implies that certain topological weak semimodules are also traced monoidal (Remark 249). Of these two classes of adiabatic categories involved, we are mainly interested in the ones over rational-like semirings, as these are equipped with an Entropy functor.
- We have shown that our model applies to some physical systems.

4.2.2 Future work

As we discuss in Chapter 1 and expand on in Chapter 3, there are strong links between the properties of our category and Entropy in classical Thermodynamics. A way of recovering the familiar Entropy formula from Statistical Thermodynamics is discussed in Section 1.3; since there are complications that require a higher categorical approach, this is one of our future goals.

The long-term goal of this project is to encompass all notions of Entropy through different "theory" functors. The precise formulation of the Entropy function we recover depends on the choice of objects of the codomain of the "theory" functor: a macroscopic description of a thermodynamical system yields the macroscopic definition; discrete probability measures on finite sets of microstates yields the statistical formulation; continuous probability measures on infinite sets of microstates yields the Entropy formula for the continuous case. Another way to phrase this, which offers some insight in light of the above discussion, is that we seek a functor from the general category we construct to a category **System**, such that the functor is a "theory" functor for adiabatic processes (in the sense discussed above); here, **System** is either the category of macroscopic states and all processes, or the category of all finite probability measures and stochastic maps, or the category of all continuous probability measures and continuous stochastic maps. That is to say, there is a general notion of an "Entropy functor" from our category to $\mathbb{R}^{\geq \mathbf{0}}$ that factorises uniquely through **System** via the "theory" functor and the relevant Entropy formula.

The potential usefulness of this approach is that, in this abstract setting, we may consider "theory" functors from our category with a variety of different codomains (in other words, "plugging in" different objects into our category) and examine similar factorisations of the "Entropy functor". This is an objective for future work, with the long-term goal of integrating all notions of Entropy (classical and quantum) into this categorical approach. The basic insight behind this is that subadditivity is a basic property of any notion of Entropy [50], and is precisely the property that is captured by our “Entropy functor”.

The first step is to complete our framework for the macroscopic case by working out a concrete example involving general adiabatic processes (which at this point has only been worked out for the simple case of two degrees of freedom). The second step is to set up a framework for the microscopic case, which is significantly more complicated and necessitates the use of bicategories as successive approximations; here, recovering an adiabatic category in the limiting case immediately yields the statistical formulation of Entropy via Faddeev’s theorem (as discussed in Section 1.3), which would already be a nice result linking the macroscopic and microscopic notions of Entropy. A further step would then be to examine the quantum regime, which for the time being is at the speculative stage.

There is also a possibility of refining our results on the relation between our class of categories and traced categories. We would like to recover a set of minimal sufficient and necessary conditions that turn a certain trace-like operation of our categories into a trace.

Meanwhile, there is another possible direction of future work (mentioned in passing in the Introduction to Chapter 3), which is not directly relevant to Physics but is of mathematical interest. One of the constructions introduced in Chapter 3, called a weak semimodule, is a categorification of a semimodule. Weak semimodules form a category that turns out to have free objects over the category of symmetric monoidal categories and symmetric strict monoidal functors, while at the same time they inherit some useful properties of semimodules. It would be worth exploring whether these properties can be taken advantage of to construct transforms in the service of solving problems involving symmetric monoidal categories.

Bibliography

[1] Samson Abramsky and Bob Coecke. Categorical quantum mechanics. *Handbook of quantum logic and quantum structures: quantum logic*, pages 261–324, 2009.

[2] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.

[3] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and concrete categories*. John Wiley, 1990.

[4] Jiří Adámek and Jiří Rosicky. *Locally presentable and accessible categories*, volume 189. Cambridge University Press, 1994.

[5] T. A. Al-Hawary. *Toward an Elementary Axiomatic Theory of the Category of Loopless Pointed Matroids and Pointed Strong Maps*. PhD thesis, University of Montana, 1997.

[6] T. A. Al-Hawary. Free objects in the category of geometries. *International Journal of Mathematics and Mathematical Sciences*, 26(12):765–770, 2000.

[7] T. A. Al-Hawary and D. G. McRae. Toward an elementary axiomatic theory of the category of LP-matroids. *Applied Categorical Structures*, 11:157–169, 2003.

[8] Laura Anderson. Matroid bundles. *New perspectives in algebraic combinatorics*, pages 1–21, 1999.

[9] John Baez and Mike Stay. Physics, topology, logic and computation: a rosetta stone. In *New structures for physics*, pages 95–172. Springer, 2010.

[10] John C Baez. Spin foam models. *Classical and Quantum Gravity*, 15(7):1827, 1998.

[11] John C Baez and Alissa S Crans. Higher-dimensional algebra vi: Lie 2-algebras. *Theory Appl. Categ*, 12(15):492–528, 2004.

[12] John C Baez and James Dolan. From finite sets to feynman diagrams. In *Mathematics Unlimited - 2001 and beyond*, pages 29–50. Springer, 2017.

[13] John C. Baez, Tobias Fritz, and Tom Leinster. A characterization of entropy in terms of information loss. *Entropy*, 13(11):1945, 2011.

[14] John C Baez and John Huerta. An invitation to higher gauge theory. *General Relativity and Gravitation*, 43(9):2335–2392, 2011.

[15] John C Baez and Aaron Lauda. *A prehistory of n-categorical physics*. Cambridge University Press Cambridge, UK, 2011.

[16] Michael Barr and Charles Wells. *Toposes, triples and theories*, volume 278. Springer-Verlag New York, 1985.

[17] Francis Borceux. *Handbook of categorical algebra*, volume 1 of *Encyclopedia of mathematics and its applications ; v. 50-52*. Cambridge University Press, Cambridge, 1994.

[18] Percy Williams Bridgman. *A condensed collection of thermodynamic formulas*. Harvard University Press, 1925.

[19] Ronald Brown, Ifor Morris, J Shrimpton, and Christopher D Wensley. Graphs of morphisms of graphs. *The electronic journal of combinatorics*, 15(A1):1, 2008.

[20] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, and P. Wollan. Axioms for infinite matroids. *Advances in Mathematics*, 239(1):18–46, 2013.

[21] Thomas H Brylawski. A combinatorial model for series-parallel networks. *Transactions of the American Mathematical Society*, 154:1–22, 1971.

[22] Bob Coecke, Tobias Fritz, and Robert W Spekkens. A mathematical theory of resources. *Information and Computation*, 250:59–86, 2016.

[23] Bob Coecke and Eric Oliver Paquette. Categories for the practising physicist. In *New structures for physics*, pages 173–286. Springer, 2010.

[24] Kevin Costello. Topological conformal field theories and calabi–yau categories. *Advances in Mathematics*, 210(1):165–214, 2007.

[25] H Crapo. Constructions in combinatorial geometries. In *NSF Advanced Science Seminar in Combinatorial Theory)(Notes, Bowdoin College)*, 1971.

[26] Predrag Cvitanovic. *Group theory*. Nordita (Nordisk Institut for teoretisk Atomfysik), 1984.

[27] Martín Hötzel Escardó and Alex K Simpson. A universal characterization of the closed euclidean interval. In *Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science*, pages 115–125. IEEE, 2001.

[28] Dmitrii Konstantinovich Faddeev. On the concept of entropy of a finite probabilistic scheme. *Uspekhi Matematicheskikh Nauk*, 11(1):227–231, 1956.

[29] Tobias Fritz. Resource convertibility and ordered commutative monoids. *Mathematical Structures in Computer Science*, pages 1–89, 2015.

[30] Satoru Fujishige. Polymatroidal dependence structure of a set of random variables. *Information and control*, 39(1):55–72, 1978.

[31] Masahito Hasegawa. Recursion from cyclic sharing: traced monoidal categories and models of cyclic lambda calculi. In *International Conference on Typed Lambda Calculi and Applications*, pages 196–213. Springer, 1997.

[32] Harold V. Henderson and S. R. Searle. The vec-permutation matrix, the vec operator and kronecker products: a review. *Linear and Multilinear Algebra*, 9(4):271–288, 1981.

[33] Chris Heunen and Vaia Patta. The category of matroids. *Applied Categorical Structures*, pages 1–33, 2017.

[34] DA Higgs. Strong maps of geometries. *Journal of Combinatorial Theory*, 5(2):185–191, 1968.

[35] Etienne (https://math.stackexchange.com/users/80469/etienne). Is there finest topology which makes given vector space into a topological vector space? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/446432 (version: 2013-07-18).

[36] Edwin T Jaynes. The gibbs paradox. In *Maximum entropy and bayesian methods*, pages 1–21. Springer, 1992.

[37] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993.

[38] André Joyal, Ross Street, and Dominic Verity. Traced monoidal categories. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 119, pages 447–468. Cambridge Univ Press, 1996.

[39] Jeff Kahn and Joseph PS Kung. Varieties of combinatorial geometries. *Transactions of the American Mathematical Society*, 271(2):485–499, 1982.

[40] Daniel Kennedy. Majors of geometric strong maps. *Discrete Mathematics*, 12(4):309–340, 1975.

[41] J Kisyński. Convergence du type l. In *Colloquium Mathematicae*, volume 2, pages 205–211, 1960.

[42] Donald E Knuth. Random matroids. *Discrete Mathematics*, 12(4):341–358, 1975.

[43] Anders Kock. *Topos theoretic methods in geometry: a collection of articles.* Number 30. Matematisk institut, Aarhus universitet, 1979.

[44] Anders Kock. *Synthetic differential geometry*, volume 333. Cambridge University Press, 2006.

[45] Joachim Kock. *Frobenius algebras and 2-d topological quantum field theories*, volume 59. Cambridge University Press, 2004.

[46] J. P. S. Kung. Twelve views of matroid theory. *Combinatorial and Computational Mathematics: Present and Future*, pages 56–96, 2001.

[47] Joseph PS Kung. Bimatroids and invariants. *Annals of Mathematics*, 30(3):238–249, 1978.

[48] Michel Las Vergnas. On certain constructions for matroids. *Proc. of the 5th British Combinatorial Conference*, pages 395–404, 1975.

[49] René Lavendhomme. *Basic concepts of synthetic differential geometry*, volume 13. Springer Science & Business Media, 2013.

[50] Elliott H Lieb and Mary Beth Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *Journal of Mathematical Physics*, 14(12):1938–1941, 1973.

[51] Elliott H Lieb and Jakob Yngvason. The mathematical structure of the second law of thermodynamics. *Current Developments in Mathematics*, pages 89–129, 2001.

[52] LX Lu and WW Zheng. Categorical relations among matroids, fuzzy matroids and fuzzifying matroids. *Iranian Journal of Fuzzy Systems*, 7(1):81–89, 2010.

[53] Russell Lyons. Determinantal probability measures. *Publications mathématiques de l'IHÉS*, 98(1):167–212, 2003.

[54] S. Mac Lane. Some interpretations of abstract linear independence in terms of projective geometry. *American Journal of Mathematics*, 58(1):236–240, 1936.

[55] Keye Martin and Prakash Panangaden. A domain of spacetime intervals in general relativity. *Communications in mathematical physics*, 267(3):563–586, 2006.

[56] František Matúš. Probabilistic conditional independence structures and matroid theory: Background. *International Journal Of General System*, 22(2):185–196, 1993.

[57] Micah Blake McCurdy. Graphical methods for tannaka duality of weak bialgebras and weak hopf algebras. *Theory and applications of categories*, 26(9):233–280, 2012.

[58] Paul-André Mellies. Categorical semantics of linear logic. *Panoramas et syntheses*, 27:15–215, 2009.

[59] James R. Munkres. *Topology*. Pearson custom library. Harlow, United Kingdom, pearson new international edition, second edition. edition, 2013.

[60] D. L. Neel and N. A. Neudauer. Matroids you have known. *Mathematics Magazine*, 82(1):26–41, 2009.

[61] Victor Ostrik. Module categories, weak hopf algebras and modular invariants. *Transformation Groups*, 8(2):177–206, 2003.

[62] J. Oxley. *Matroid theory.* Oxford University Press, 2nd edition, 2011.

[63] Robert Paré. Simply connected limits. *Canadian Journal of Mathematics*, 42(4):731–746, 1990.

[64] A. Pultr and W. Tholen. Free Quillen factorization systems. *Georgian Mathematical Journal*, 9(2):807–820, 2002.

[65] A Recski. On partitional matroids with applications. *Coll. János Bolyai Mathematical Society*, 10:1169–1179, 1973.

[66] András Recski. Elementary strong maps of graphic matroids. *Graphs and Combinatorics*, 3(1):379–382, 1987.

[67] András Recski. Maps of matroids with applications. *Discrete mathematics*, 303(1):175–185, 2005.

[68] Eric Schechter. *Handbook of Analysis and its Foundations.* Academic Press, 1996.

[69] Niels Schwartz. Positive semifields and their ideals. In *Ordered Algebraic Structures and Related Topics: International Conference on Ordered Algebraic Structures and Related Topics, October 12-16, 2015, Centre International de Rencontres Mathématiques (CIRM), Luminy, France*, volume 697, page 301. American Mathematical Soc., 2017.

[70] Francis Weston Sears and Gerhard L Salinger. *Thermodynamics, kinetic theory, and statistical thermodynamics.* Addison-Wesley principles of physics series. Addison-Wesley, Reading, Mass ; London, 3rd ed. edition, 1975.

[71] Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

[72] Masaru Shirahata. Geometry of interaction explained. *代数　論理　幾何と情報科学短期共同研究報告集*. 京都数理解析研究所講究録, 1318, 2003.

[73] Matthew T Stamps. Topological representations of matroid maps. *Journal of Algebraic Combinatorics*, 37(2):265–287, 2013.

[74] Apostolos Syropoulos. Mathematics of multisets. In *Workshop on Membrane Computing*, pages 347–358. Springer, 2000.

[75] Jos Uffink. Bluff your way in the second law of thermodynamics. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 32(3):305–394, 2001.

[76] Paul Urysohn. Sur les classes (l) de m. fréchet. *L'Enseignement Mathématique*, 25:77–83, 1926.

[77] Jamie Vicary. Topological structure of quantum algorithms. In *Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 93–102. IEEE Computer Society, 2013.

[78] Charles A Weibel. *The K-book*, volume 145. American Mathematical Society Providence, RI, 2013.

[79] N. White. *Theory of matroids*. Cambridge University Press, 1986.

[80] H. Whitney. On the abstract properties of linear independence. *American Journal of Mathematics*, 57(3):509–533, 1935.

[81] Dali Zangurashvili. Several constructions for factorization systems. *Theory and Applications of Categories*, 12(11):326–354, 2004.

[82] T. Zaslavsky. Biased graphs II. The three matroids. *Journal of Combinatorial Theory B*, 51(1):46–72, 1991.