the possibilities and impossibilities of

Recursive Ramsey Theory

Chris Heunen

The illustration on the cover is a stylized portrait of Frank Ramsey.

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CHAPTER 1 Preface

Alice laughed. "There's no use trying," she said. "One can't believe impossible things."
"I dare say you haven't had much practice" said the queen.

– Lewis Carroll, 'Through the Looking Glass'

Loosely speaking, Ramsey theory states that any large enough structure will necessarily contain an orderly substructure. The heart of the argument was perhaps best put in words by David Kleitman in the phrase

"Of three ordinary people, two must have the same sex."

However, the real insight is in dimension two, where the relation between two persons cannot be traced back to a property of each individual. In the above example, the 'orderly substructure' is either a pair of men or a pair of women, and the 'large enough structure' is a group of at least three people. The smallest size of the 'large enough structure', in the above case three, is called a Ramsey number. Frank Ramsey's original article already shows how to (recursively) compute an upper bound for the Ramsey number after proving the more interesting infinite version of the theorem. This is often said to be a generalization of the pigeonhole principle, and might be expressed as "Of infinitely many ordinary people, infinitely many must have the same sex". In contrast to the finite version, the classical proof of the infinite version of Ramsey's theorem is non-effective, because one cannot decide in a finite amount of time whether an arbitrary given set is infinite or not. In order to study the effective content of Ramsey theory we use recursion theory, a widely accepted way of thinking about effectivity[1].

This thesis is a study of the apparent non-effectiveness of the infinite version of Ramsey's theorem. We investigate which parts hold effectively and which do not, attempt to repair the non-effective parts or try to understand why that is impossible, and try to see 'how impossible' these impossibilities are.

The classical proof of Ramsey's theorem is one from The Book, as Paul Erdös would say; it does not need elaborate preliminaries. In chapter 2, we discuss the theorem and its classical proof, and illustrate the diversity of applications that are nowadays called Ramsey theory.

Of the other pillar, recursion theory, we presuppose the reader to have some basic knowledge. Chapter 3 summarizes the less basic notions of the arithmetical hierarchy and complete sets, which are our primary tools when assessing 'how impossible' or 'how non-effective' a certain problem is.

When the stage is thus set, the drama begins. The opening scene, chapter 4, introduces the parts of the argument that are certainly possible, even in an effective setting. By chapter 5 the tragedy unfolds: a strongly recursive version of Ramsey's theorem, where we demand the construction of the theorem itself to be recursive, is false, but even the weaker recursive version where we only require the sets and functions under consideration to be recursive does not hold.

In chapter 6, the finale, the audience is instilled with a compelling moral when we prove a deeper result that sheds light on the reason of the impossibilities of the previous chapter.

[1] But by no means the only one, see e.g. [VB].

In the aftermath, chapter 7, various thoughts are mused upon, some left as open questions. The curtain closes on a fleeting high note as we state a beautiful (classical) generalization of Ramsey's theorem.

The meat of this thesis is in chapters 4 to 6. In the first two we largely recreate earlier efforts [Joc, Spe]. As far as we know, the material of the latter chapter is original. However, we strive for a clear and easy to follow exposition without unnecessary ballast throughout the entire text.

It is a pleasure to thank my supervisor Wim Veldman not only for his weekly encouragement in researching this subject by nudging in the right direction and shooting at any weak parts left in my ideas or 'proofs', but also for his uncompromisingly open style of teaching, which played a large part in the development of my (mathematical) critical faculties. I am also grateful to my second reader Wieb Bosma, who found the time to comment on the smallest of things despite his own vacation. Furthermore I would like to thank Lotte Hollands for lovingly enduring all kinds of fundamental mathematical questions in which she was uninterested, and my parents, without whose upbringing and support I would not have been what I am now, my dad also for guarding my English. Then there is mathematics students' association Desda, which brought me many a joyful moment. Finally, my gratitude is with those who ever taught me, as this is a quality I value highly. In particular, Arnoud van Rooij, Ronald Kortram and Frans Clauwens have widened my mathematical awareness greatly.

Notation

To focus attention on the content and to keep sections to the point, we make the stylistic simplification of allowing each section to have at most one definition, at most one theorem and so on. Thus we can refer to a definition or theorem simply by the number of the section it appears in.

Some of the notation we use is specific to Ramsey theory:

$\mathbb{N}$	the set of natural numbers $\{0, 1, 2, \ldots\}$
$[n]$	$\{0, 1, \ldots, n-1\}$
$\#X$	the number of elements of a finite set X
$\mathcal{P}(X)$	the power set $\{Y \subset X\}$ of a set X
$[X]^d$	$\{Y \in \mathcal{P}(X) \ : \ \#Y = d\}$
	we often identify $[X]^d$ with $\{(x_1, \ldots, x_d) \in X^d \mid x_1 < \cdots < x_d\}$

Recursion theoretic notation varies from author to author. This is the notation we use:

ψ_e	the eth (partial) recursive function, the (partial) recursive function with index e
$\mathrm{Dom}(f)$	the domain of a (partial) function f
$\mathrm{Ran}(f)$	the range $\{f(x) \ : \ x \in \mathrm{Dom}(f)\}$ of a function f
W_e	$\mathrm{Dom}(\psi_e)$

The subject is all about natural numbers. Therefore, unless otherwise noted, the lowercase roman letters except f, g, h denote natural numbers, as well as K and N, and the lowercase greek letters except χ, φ. Some letters will consistently have a specific meaning:

c	the number of colors (we obviously mean c to be at least 1)
d	the dimension
e	the index of a (partial) recursive function
k	'time of approximation' in a limit construction
r	a particular color

Uppercase roman letters except K and N denote sets, and f, g, h and χ are used for functions. The greek letter φ is reserved for formulae.

CHAPTER

2 Ramsey theory

Ramsey theory originated in a theorem that is now commonly known as Ramsey's theorem (section 2.1) and first appeared in [Ram]. In the original article this theorem was used to solve a problem of formal logic (section 2.2). However, since it is a very general yet sharp statement, it has led to all sorts of interesting results in combinatorics, number theory and game theory (section 2.3) as well as proof theory (section 2.4). This fertility is the reason we speak of Ramsey theory instead of 'Ramseyian theorems'. In this chapter we prove Ramsey's theorem and illustrate some of the riches of this field of study. Most of Ramsey theory can be expressed in terms of colorings.

Definition For any c and d, we call a function $\chi : [\mathbb{N}]^d \to [c]$ a *coloring* of $[\mathbb{N}]^d$, c the *number of colors*, and d the *dimension*. We say a set $X \subset \mathbb{N}$ is χ*-monochromatic* if χ is constant on $[X]^d$.

We will also speak of a c*-coloring* instead of a coloring with c colors. With the *color* of a χ-monochromatic subset X, we mean the value χ takes on X. When the context renders the coloring clear, we will simply speak of *monochromatic* instead of χ-monochromatic.

2.1 Ramsey's theorem

Ramsey's theorem is an interesting type of theorem. For any coloring of $[\mathbb{N}]^d$ it promises the existence of a large (infinite) set that is simultaneously small enough to be monochromatic.

Theorem *For any d and any coloring of $[\mathbb{N}]^d$, there exists an infinite monochromatic $X \subset \mathbb{N}$.*

Proof by induction on d. First, if χ is a c-coloring of $\mathbb{N}$, then one of $\{x \in \mathbb{N} \mid \chi(x) = r\}$, where $r \in [c]$, is infinite by the Pigeonhole principle (and monochromatic by definition).

Now suppose the theorem has already been established for dimension d, and let χ be a c-coloring of $[\mathbb{N}]^{d+1}$. We will define a sequence of infinite sets $X_0, X_1, \ldots \subset \mathbb{N}$, a sequence of natural numbers $x_1 < x_2 < \ldots$, and a sequence of colors $r_1, r_2, \ldots \in [c]$ as follows. Start with

$$X_0 = \mathbb{N}, \quad x_1 = 0.$$

Now suppose that $X_1, \ldots, X_k$, $x_1 < \cdots < x_k$ and $r_1, \ldots, r_k$ have already been defined, that $\forall_{i \le k} \forall_{y \in [X_i]^d} [x_i \notin y \to \chi(y \cup \{x_i\}) = r_i]$, and that X_k is infinite, say $X_k = \{a_0, a_1, \ldots\}$ with $a_0 < a_1 < \cdots$. Because X_k is infinite, we can define

$$x_{k+1} = \min X_k \backslash \{x_1, \ldots, x_k\},$$

so that $x_k < x_{k+1}$. To define X_{k+1}, we make a c-coloring χ' of $[\{y \in X_k \mid y > x_{k+1}\}]^d$ by $\chi'(\{i_1, \ldots, i_d\}) = \chi(\{x_{k+1}, a_{i_1}, \ldots, a_{i_d}\})$. By the induction hypothesis there exists an infinite χ'-monochromatic $X' \subset \{y \in X_k \mid y > x_{k+1}\}$. Put r_{k+1} to be the color of X', and define

$$X_{k+1} = \{a_i \ : \ i \in X'\}.$$

Then $\forall_{i \le k+1} \forall_{y \in [X_i]^d} [x_i \notin y \to \chi(y \cup \{x_i\}) = r_i]$. Finally, let r be the least color that appears infinitely often in the sequence $r_1, r_2, \ldots$. Then $X = \{x_k \ : \ k \in \mathbb{N} \mid r_k = r\}$ is infinite and

χ-monochromatic. □

A few things about this proof attract attention. First, although the proof seems to construct the promised monochromatic set, the construction is non-effective: we cannot decide in a finite amount of time whether a given (decidable) subset of $\mathbb{N}$ is infinite.

Secondly, by induction, it is enough to prove the case $c = 2$. If $c > 2$, 'go color-blind' and pretend that $c-1$ and c are the same color. By the induction hypothesis there exists a monochromatic set for the $(c-1)$-coloring thus found. If the color of this monochromatic set is less than $c-1$ we are finished. But if the monochromatic set is of the 'blurred color', we recover our sight again and find a set that is monochromatic for the original coloring anyway. (See the proof of proposition 4.1 for more details.)

Finally, it seems sufficient to prove the cases $d \leq 2$, since the inductive step of the proof is nothing more than the two-dimensional case in which one coordinate is a d-tuple in disguise.

This phenomenon appears in most of the theorems we will study. We could have sufficed by confining their proofs to the cases $c = 2$ and $d \leq 2$ because of 'color-blindness' and 'disguised-dimensions' arguments. However, mostly it is not much harder to take all cases into account in one sweep.

2.2 Original application

In fact, Ramsey proved theorem 2.1 only as an appetizer for the finite case, of which the following lemma is an equivalent in modern language.

Lemma *For any c, $l_1, \ldots, l_c$ and d, there exists an N such that for any $n \geq N$ and any c-coloring of $[n]^d$, there exists a monochromatic $X \subset [n]$, say of color r, such that $\#X \geq l_r$.*

We omit the proof, which is elegantly inductive because of the clever formulation above with more than one l (Ramsey himself used $l_1 = \cdots = l_c$), and whose inductive structure automatically gives rise to a suitable upper bound N, which is called a *Ramsey number* of $c, l_1, \ldots, l_c$ and d.

It is also possible to derive lemma 2.2 quite easily from theorem 2.1 by a compactness argument (see [GRS]). In a sense, theorem 2.1 is much more elegant than lemma 2.2 in that we do not need to keep track of such complications as how large N needs to be.

To give a taste of the original application of lemma 2.2, we paraphrase the original article. We say that a relation on a subset of $\mathbb{N}$ is *canonical* if its truth-value only depends on the ordering (derived from the natural numbers[1]) of its arguments, and that a structure, with domain a subset of $\mathbb{N}$, with only relations is canonical if all its relations are. The main theorem in Ramsey's original article is the following.

Theorem *For any c, $l_1, \ldots, l_c$ and d, there exists an N such that for any $n \geq N$, any axiom system with only l_i i-ary relations $(1 \leq i \leq c)$ and d variables, and only universally quantified formulae, has a model of size n if and only it has a canonical model of size d.*

Recall that Hilbert's famous Decision Problem asks whether there exists an algorithm to decide whether a given first order formula is logically valid or not. In effect, the previous theorem solves the Decision Problem for the class of formulae of the form $\forall \cdots \forall[\varphi]$, where φ is a first order formula involving only equality and relations, since it is easy to check whether canonical models of a given size exist (see [DG]). At the end of the original article Ramsey extends his result to the class of formulae of the form $\exists \cdots \exists \forall \cdots \forall[\varphi]$.

[1] A c-ary relation R is canonical if and only if $(l_i < l_j \leftrightarrows l'_i < l'_j)$, $(l_i = l_j \leftrightarrows l'_i = l'_j)$ and $(l_i > l_j \leftrightarrows l'_i > l'_j)$ for $1 \leq i, j \leq c$ imply $R(l_1, \ldots, l_c) \leftrightarrows R(l'_1, \ldots, l'_c)$.

2.3 Ramsey-like theorems

The structure of the statement of theorem 2.1 inspired a whole field of study, encouraged by Paul Erdös, which also drew from independent results of the same type. To illustrate the range of this type of theorem, we state without proof some important Ramsey-like theorems in number theory.

Theorem (Van der Waerden) *For any c and l, there exists an N such that for any $n \geq N$ and any c-coloring of $[n]$, there exists a monochromatic arithmetic progression in $[n]$ of length l* [vdW].

Theorem (Schur) *For any c, there exists an N such that for any $n \geq N$ and any c-coloring of $[n]$, there exist $x, y, z \in [n]$ of the same color such that $x + y = z$* [Sch].

By a *line* in the n-dimensional cube $\{0, 1, \ldots, d-1\}^n$ we mean a set of points $x_0, \ldots, x_{d-1}$, $x_i = (x_{i1}, \ldots, x_{in})$, such that in each coordinate $1 \leq j \leq n$ either $x_{0j} = x_{1j} = \cdots = x_{d-1,j}$ or $x_{sj} = s$ $(0 \leq s < d)$.

Theorem (Hales-Jewett) *For any c and d, there exists an N such that for any $n \geq N$ and any c-coloring of $\{0, 1, \ldots, d-1\}^n$, there exists a monochromatic line* [HJ].

The Hales-Jewett theorem illustrates a connection to game theory: using it, one can prove that for large enough n, the first player has a winning strategy in n-dimensional tic-tac-toe [SSV].

2.4 Paris-Harrington

One particularly interesting area of study where Ramsey's theorem is of use is proof theory [Tak]. The prime example is perhaps the Paris-Harrington theorem [PH]. Like Gödel's incompleteness theorem, it produces a statement that can be formulated but not proved within Peano arithmetic. In contrast to the statement given by Gödel, the statement offered by the Paris-Harrington theorem 'occurs naturally' in mathematics.

We call a set $S \subset \mathbb{N}$ *large* if $\#S > \min S$. Consider the following statement:

For any c, d and n, there exists an m such that for any c-coloring of $[m]^d$ there exists a large monochromatic set $S \subset [m]$ such that $\#S \geq n$. (PH)

The statement (PH) can be formulated in Peano arithmetic.

Theorem (Paris-Harrington) *In Peano arithmetic, (PH) is unprovable.*

However, (PH) is true. Let c, d, n, and a c-coloring of $[\{n, n+1, n+2, \ldots\}]^d$ be given. By theorem 2.1, there exists an infinite monochromatic set $T \subset \{n, n+1, n+2, \ldots\}$. Let S denote the first $\min T$ elements of T. Then S is large and monochromatic. The existence of a finite m follows from a compactness argument, analogous to the derivation of lemma 2.2 from theorem 2.1.

On the whole, we hope to have illustrated that Ramsey theory is a fruitful area of research, without frustrating the reader too much by not undertaking a detailed study. For more on general Ramsey theory, see [GRS], and for more applications of Ramsey theory in set theory, geometry and theoretical computing science, see [Ros].

CHAPTER 3

Recursion theory

Loosely speaking, recursion theory deals with 'effectiveness' on the natural numbers. Especially the arithmetical hierarchy (sections 3.1-3.3) is interesting in the context of non-effective proofs like that of Ramsey's theorem 2.1, since it is a way to 'measure effectiveness'. Among the arithmetical sets, complete sets (section 3.4) are of particular interest (being the pinnacle of an arithmetical class). In this chapter we summarize the notions concerning the arithmetical hierarchy needed in later chapters and fix some notation.

Recursion-theoretic notation varies. To simplify notation, we will freely identify $\mathbb{N}^* = \bigcup_d \mathbb{N}^d$ with $\mathbb{N}$ via a (primitive recursive) coding denoted by $\mathbb{N}^d \ni (n_1, \ldots, n_d) \mapsto \langle (n_1, \ldots, n_d) \rangle \in \mathbb{N}$ and accompanying decoding $\mathbb{N} \ni \langle (n_1, \ldots, n_d) \rangle \mapsto ((n)_1, \ldots, (n)_d) \in \mathbb{N}^d$. We will also use the (primitive recursive) function length : $\mathbb{N} \to \mathbb{N}$ defined by length$(\langle (n_1, \ldots, n_d) \rangle) = d$. Finally, we use the (primitive recursive) concatenation function $\star : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $\langle (m_1, \ldots, m_d) \rangle \star \langle (n_1, \ldots, n_{d'}) \rangle = \langle (m_1, \ldots, m_d, n_1, \ldots, n_{d'}) \rangle$. When dealing with sequences, we abbreviate the constant d-tuple $(n, n, \ldots, n)$ by n^d to speak of $\langle n^d \rangle$.

We already agreed upon some notations on page 3. Furthermore, we denote by IM_d the set of indices of (partial) recursive functions with d arguments, $\mathrm{IM} = \bigcup_d \mathrm{IM}_d$, $\mathrm{TOT}_d = \{e \in \mathrm{IM}_d \mid W_e = \mathbb{N}^d\}$ is the set of indices of total recursive functions with d arguments, and $\mathrm{TOT} = \bigcup_d \mathrm{TOT}_d$. We denote Kleene's (primitive recursive) T-predicate by $T(e, n, z)$, expressing that z is an encoding of the computation upon input n of the output $\psi_e(n)$, which we denote by $\mathrm{OUTP}(z)$.

For basic recursion theory we refer to [Rog]. For example, we often use the S-m-n-theorem [Rog, section 1.8 on page 21].

3.1 Arithmetical sets

We consider sets of natural numbers that are defined by first order formulae in the structure $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1)$. A good practice in descriptive set theory is to classify these sets according to the complexity of their defining formulae [Mos]. We classify formulae according to their quantifier structure: we say a formula is Σ_n or Π_n if it is equivalent in $\mathfrak{N}$ to a formula in prenex form with n alternating quantifiers, of which the first is existential or universal, respectively. More precisely:

- $\varphi = \varphi(x_1, \ldots, x_l)$ is called Σ_0 and Π_0 if it is recursive, i.e. if there is an $e \in \mathrm{TOT}$ such that for every $x_1, \ldots, x_l$, we have $\mathfrak{N} \models \varphi[x_1, \ldots, x_l]$ if and only if $\psi_e(\langle (x_1, \ldots, x_l) \rangle) \neq 0$.
- For $n > 0$, we say that $\varphi = \varphi(x_1, \ldots, x_l)$ is Σ_{n+1} if there is a Π_n formula $\varphi' = \varphi'(x_1, \ldots, x_l, y)$ such that
$$\mathfrak{N} \models \forall_{x_1,\ldots,x_l}[\varphi(x_1, \ldots, x_l) \leftrightarrows \exists_y[\varphi'(x_1, \ldots, x_l, y)]].$$
- Likewise, $\varphi = \varphi(x_1, \ldots, x_l)$ is Π_{n+1} if there is a Σ_n formula $\varphi' = \varphi(x_1, \ldots, x_l, y)$ such that
$$\mathfrak{N} \models \forall_{x_1,\ldots,x_l}[\varphi(x_1, \ldots, x_l) \leftrightarrows \forall_y[\varphi'(x_1, \ldots, x_l, y)]].$$

With abuse of notation we also classify sets in this fashion.

Definition For any n we define:
$\Sigma_n = \{X \subset \mathbb{N}^* \mid$ there is a Σ_n formula φ such that $x \in X$ if and only if $\mathcal{N} \models \varphi[x]\}$
$\Pi_n = \{X \subset \mathbb{N}^* \mid$ there is a Π_n formula φ such that $x \in X$ if and only if $\mathcal{N} \models \varphi[x]\}$
$\Delta_n = \Sigma_n \cap \Pi_n$.
A set X is called *arithmetical* if there is an n such that $X \in \Delta_n$. The sets Σ_n, Π_n and Δ_n are called *arithmetical classes*.

From the definition it is clear that every arithmetical set has a *normal form*. For example, $X \in \Pi_3$ if and only if there exists an $e \in \mathrm{TOT}$ such that

$$X = \{x \in \mathbb{N} \mid \forall_{x_1}\exists_{x_2}\forall_{x_3}[\psi_e(x, x_1, x_2, x_3) \neq 0]\}.$$

In this case, we call e a Π_3-*index of* X. The quantifier-free part of a defining predicate of X in prenex form is called its *matrix*.

3.2 Computing with relations

It is helpful to have some rules of conduct when handling arithmetical sets. This section establishes some results that will frequently help us to prove that a particular set is in Σ_n or Π_n. For example, the arithmetical classes are closed under conjunction, disjunction, and bounded quantification of their predicates.

Theorem
(i) *For all n and $X \subset \mathbb{N}$: $X \in \Sigma_n$ if and only if $\mathbb{N}\backslash X \in \Pi_n$*
(ii) *For all n and $X, Y \subset \mathbb{N}$: if $X, Y \in \Sigma_n$, then also $X \cup Y \in \Sigma_n$ and $X \cap Y \in \Sigma_n$*
(iii) *For all n and $X, Y \subset \mathbb{N}$: if $X, Y \in \Pi_n$, then also $X \cup Y \in \Pi_n$ and $X \cap Y \in \Pi_n$*
(iv) *For all $n > 0$ and $X \subset \mathbb{N}$: if $X \in \Sigma_n$, then $\{x \in \mathbb{N} \mid \exists_y[\langle(x, y)\rangle \in X]\} \in \Sigma_n$*
(v) *For all n and $X \subset \mathbb{N}$: if $X \in \Sigma_n$, then $\{(x, y) \in \mathbb{N}^2 \mid \forall_{y' \leq y}[\langle(x, y, y')\rangle \in X]\} \in \Sigma_n$*
(vi) *For all n and $X \subset \mathbb{N}$: if $X \in \Sigma_n$, then $\{(x, y) \in \mathbb{N}^2 \mid \exists_{y' \leq y}[\langle(x, y, y')\rangle \in X]\} \in \Sigma_n$*

PROOF
(i) Suppose $X \in \Sigma_n$ and determine a Σ_n-index e of X. By the laws of De Morgan, $\neg\exists_{x_1}\forall_{x_2}\cdots_{x_n}[\psi_e(x, x_1, \ldots, x_n) \neq 0]$ if and only if $\forall_{x_1}\exists_{x_2}\cdots_{x_n}[\psi_e(x, x_1, \ldots, x_n) = 0]$. Since the last formula has a recursive matrix, $\mathbb{N}\backslash X \in \Pi_n$. The other implication is analogous.
(ii,iii) We prove (ii) and (iii) simultaneously by induction on n. The case $n = 0$ is trivial. Suppose we have (ii) and (iii) for n, and let $X, Y \in \Sigma_{n+1}$. Determine Σ_{n+1}-indices e of X and e' of Y. Then $X \cup Y$ equals
$\{x \in \mathbb{N} \mid \exists_{x_0}\forall_{x_1}\cdots_{x_n}[\psi_e(x_0, \ldots, x_n) \neq 0] \vee \exists_{x_0}\forall_{x_1}\cdots_{x_n}[\psi_{e'}(x_0, \ldots, x_n) \neq 0]\}$
$= \{x \in \mathbb{N} \mid \exists_x[\forall_{x_1}\cdots_{x_n}[\psi_e((x)_0, x_1, \ldots, x_n) \neq 0] \vee \forall_{x_1}\cdots_{x_n}[\psi_{e'}((x)_1, x_1, \ldots, x_n) \neq 0]]\}$
$= \{x \in \mathbb{N} \mid \exists_x\forall_{x_1}\cdots_{x_n}[\psi_{e''}(x, x_1, \ldots, x_n) \neq 0]\} \in \Sigma_{n+1}$.
Likewise for $X \cup Y$ and $X, Y \in \Pi_{n+1}$.
(iv) Suppose $X \in \Sigma_n$ and $n > 0$, and determine a Σ_n-index e of X. Then $\exists_y[(x, y) \in X]$ if and only if $\exists_y\exists_{x_1}\forall_{x_2}\cdots_{x_n}[\psi_e(x, y, x_1, \ldots, x_n) \neq 0]$ if and only if $\exists_{x_1}\forall_{x_2}\cdots_{x_n}[\psi_e(x, (x_1)_1, (x_1)_2, x_2, \ldots, x_n) \neq 0]$. Since the last formula has a recursive matrix, indeed $\{x \in \mathbb{N} \mid \exists_y[(x, y) \in X] \in \Sigma_n$. This is called *quantifier contraction*.
(v) Suppose $X \in \Sigma_n$ and determine a Σ_n-index e of X. Then $\forall_{y' \leq y}[(x, y, y') \in X]$ if and only if $((x, y, 0) \in X) \wedge ((x, y, 1) \in X) \ldots \wedge ((x, y, y)) \in X)$, which in turn is equivalent to $\exists_z[((x, y, (z)_0) \in X) \wedge \ldots \wedge (x, y, (z)_y) \in X)) \wedge (((z)_0 = 0) \wedge \ldots \wedge ((z)_y = y) \wedge \mathrm{length}(z) = y)]$ and hence a Σ_n formula by (ii) and (iv).
(vi) Analogous to (v).

□

Notice that an arithmetical class is not closed under implication of defining formulae. When given a defining formula of the form $\varphi \to \varphi'$, we need to consider the equivalent $\neg\varphi \vee \varphi'$, and the set defined by $\neg\varphi$ usually does not belong to the same arithmetical class as the set defined by φ by (i) and the Hierarchy theorem, which we will discuss shortly.

For example, let us use these rules to prove that $\mathrm{FIN} = \{e \in \mathrm{IM} \mid W_e \text{ is finite}\} \in \Sigma_2$:

$$\begin{aligned}
\mathrm{FIN} &= \{e \in \mathbb{N} \mid e \in \mathrm{IM} \;\wedge\; \exists_n[W_e \subset [n]]\} \\
&= \{e \in \mathbb{N} \mid e \in \mathrm{IM} \;\wedge\; \exists_n \forall_x[x \in W_e \to x < n]\} \\
&= \{e \in \mathbb{N} \mid \underbrace{\underbrace{e \in \mathrm{IM}}_{\Delta_1} \;\wedge\; \underbrace{\exists_n \underbrace{\forall_x[\underbrace{\underbrace{\neg \underbrace{\exists_z[\underbrace{T(e,x,z)}_{\Delta_1}]}_{\Sigma_1}}_{\Pi_1 \text{ (by (i))}} \vee \underbrace{x < n}_{\Delta_1}}_{\Pi_1 \text{ (by (iii))}}]}_{\Pi_1 \text{ (by (v) and (i))}}}_{\Sigma_2}}_{\Sigma_2 \text{ (by (ii))}}\}
\end{aligned}$$

3.3 The hierarchy theorem

The following theorem is the raison d'être of arithmetical sets: arithmetical sets form a hierarchy. It justifies the classification of arithmetical sets according to predicate complexity.

Theorem

$$\begin{tikzcd}
 & \Sigma_1 & & \Sigma_2 & & \Sigma_3 & & \Sigma_4 & \\
\Delta_1 \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta_2 \arrow[ul, phantom, sloped, "\supsetneq"] \arrow[dl, phantom, sloped, "\supsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta_3 \arrow[ul, phantom, sloped, "\supsetneq"] \arrow[dl, phantom, sloped, "\supsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \Delta_4 \arrow[ul, phantom, sloped, "\supsetneq"] \arrow[dl, phantom, sloped, "\supsetneq"] \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \cdots \arrow[ul, phantom, sloped, "\supsetneq"] \arrow[dl, phantom, sloped, "\supsetneq"] \\
 & \Pi_1 & & \Pi_2 & & \Pi_3 & & \Pi_4 &
\end{tikzcd}$$

Proof Let $n > 0$. The inclusions $\Delta_n \subset \Sigma_n$ and $\Delta_n \subset \Pi_n$ hold by definition of Δ_n. The inclusions $\Sigma_n \subset \Pi_{n+1}$ and $\Sigma_n \subset \Sigma_{n+1}$ hold because adding quantifiers that are not used in the matrix of a defining formula do not change the set. Hence $\Sigma_n \subset \Delta_{n+1}$, and likewise $\Pi_n \subset \Delta_{n+1}$.

We now show that the inclusions $\Delta_n \subset \Pi_n$ are proper, by constructing elements $P_n \in \Pi_n \backslash \Delta_n$. First, define

$$\begin{aligned}
S_1' &= \{(e,x) \in \mathbb{N}^2 \mid \exists_z[T(e,x,z)]\} \\
P_1 &= \{x \in \mathbb{N} \mid (x,x) \notin S_1'\}.
\end{aligned}$$

If $P_1 \in \Sigma_1$, we could determine e such that $P_1 = \{x \in \mathbb{N} \mid (e,x) \in S_1'\}$, which yields a contradiction. So $P_1 \in \Pi_1 \backslash \Sigma_1$, and hence $P_1 \in \Pi_1 \backslash \Delta_1$. Next, define

$$\begin{aligned}
S_{n+1}' &= \{(e,x) \in \mathbb{N}^2 \mid \exists_z[(e, x \star z) \notin S_n']\} \\
P_{n+1} &= \{x \in \mathbb{N} \mid (x,x) \notin S_{n+1}'\}.
\end{aligned}$$

We show by induction that if $X \in \Sigma_n$, then there is an e such that $X = \{x \in \mathbb{N} \mid (e,x) \in S_n'\}$. Suppose this holds for n, let $X \in \Sigma_{n+1}$, and find $Y \in \Pi_n$ such that $X = \{x \in \mathbb{N} \mid \exists_z[\langle(x,z)\rangle \in Y]\}$. Since $\mathbb{N} \backslash Y \in \Sigma_n$ by theorem 3.2(i), the induction hypothesis guarantees we can determine e such that $Y = \{y \in \mathbb{N} \mid (e,y) \notin S_n'\}$. Then X can be written as $\{x \in \mathbb{N} \mid \exists_z[(e, x \star z) \notin S_n']\}$ and hence as $\{x \in \mathbb{N} \mid (e,x) \in S_{n+1}']\}$.

Thus, if $P_n \in \Sigma_n$, we could determine e such that $P_n = \{x \in \mathbb{N} \mid (e,x) \in S_n'\}$, which contradicts the definition of P_n. So $P_n \in \Pi_n \backslash \Sigma_n$, and hence $P_n \in \Pi_n \backslash \Delta_n$.

Furthermore, define $S_n = \mathbb{N} \backslash P_n$. Since $P_n \in \Pi_n \backslash \Sigma_n$ we have by theorem 3.2(i) that $S_n \in \Sigma_n \backslash \Pi_n$, and hence $S_n \in \Sigma_n \backslash \Delta_n$. So the inclusions $\Delta_n \subset \Sigma_n$ are also proper.

Finally, define

$$D_{n+1} = \{2n \;:\; n \in \mathbb{N} \mid n \in S_n\} \cup \{2n+1 \;:\; n \in \mathbb{N} \mid n \in P_n\}.$$

Then $D_{n+1} \in \Delta_{n+1} \backslash (\Sigma_n \cup \Pi_n)$, which establishes the last proper inclusions. $\square$

3.4 Complete sets

Among the arithmetical sets of a certain class, the complete sets are of special interest. They are the sets of 'maximum complexity' within their class. If one could 'solve' a complete set, then one would also be able to 'solve' every set in that class.

Definition A set X is said to be *reducible*[1] to a set Y, denoted by $X \prec Y$, if there is an $e \in \mathrm{TOT}$ such that $x \in X$ if and only if $\psi_e(x) \in Y$. In this case, we call ψ_e a *reducing function*. A set X is called Σ_n*-complete* if $Y \prec X$ for every $Y \in \Sigma_n$, and Π_n*-complete* if $Y \prec X$ for every $Y \in \Pi_n$.

If there is one reducing function for both $X_1 \prec Y_1$ and $X_2 \prec Y_2$, we write $(X_1, X_2) \prec (Y_1, Y_2)$.

The usual approach when trying to prove that some set X is Σ_n-complete is to find a reduction from a set which is already known to be Σ_n-complete to X. Therefore, it is convenient to have a few 'standard' complete sets for various arithmetical classes. Recall that $W_e = \mathrm{Dom}(\psi_e)$.

Theorem
(i) $\mathrm{FIN} = \{e \in \mathrm{IM} \mid W_e$ *is finite* $\}$ *is* Σ_2*-complete.*
(ii) $\mathrm{TOT} = \{e \in \mathrm{IM} \mid W_e = \mathbb{N}\}$ *is* Π_2*-complete.*
(iii) $\mathrm{INF} = \{e \in \mathrm{IM} \mid W_e$ *is infinite* $\}$ *is* Π_2*-complete.*
(iv) $\mathrm{COF} = \{e \in \mathrm{IM} \mid \mathbb{N}\backslash W_e$ *is finite* $\}$ *is* Σ_3*-complete.*
(v) $\mathbb{N}\backslash\mathrm{COF} = \{e \in \mathrm{IM} \mid \mathbb{N}\backslash W_e$ *is infinite* $\}$ *is* Π_3*-complete.*

Proof Let $X \in \Sigma_2$. We will construct a recursive reducing function $f : (X, \mathbb{N}\backslash X) \prec (\mathrm{FIN}, \mathrm{TOT})$, thereby establishing (i) and (ii). Part (iii) follows immediately from (i). For the proof of (iv) we refer to [Soa, corollary 3.5], from which (v) follows immediately.

Determine a Π_2-index e of $\mathbb{N}\backslash X$, so that

$$x \in \mathbb{N}\backslash X \leftrightarrows \forall_y \exists_z [\psi_e(x, y, z) \neq 0].$$

Using the S-m-n-theorem, we define a total recursive function $f : \mathbb{N} \to \mathbb{N}$ by

$$\psi_{f(x)}(y) = \begin{cases} 0 & \text{if } \exists_z[\psi_e(x, y, z) \neq 0], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now, if $x \in \mathbb{N}\backslash X$, then $W_{f(x)} = \mathbb{N}$, so $f(x) \in \mathrm{TOT}$. But if $x \in X$, then $W_{f(x)}$ is finite, so $f(x) \in \mathrm{FIN}$. □

Example We fix the function $\mathrm{Set} : \mathrm{TOT}_1 \to \mathcal{P}(\mathbb{N})$ by $\mathrm{Set}(e) = \{n \in \mathbb{N} \mid \psi_e(n) \neq 0\}$. We define $\mathrm{FINSET} = \{e \in \mathrm{TOT}_1 \mid \mathrm{Set}(e) \text{ is finite}\}$ and $\mathrm{INFSET} = \{e \in \mathrm{TOT}_1 \mid \mathrm{Set}(e) \text{ is infinite}\}$. We can then use the theorem to show that FINSET is Σ_2-complete and INFSET is Π_2-complete by proving that $(\mathrm{FIN}, \mathrm{INF}) \prec (\mathrm{FINSET}, \mathrm{INFSET})$.

Proof Define a function $f : \mathbb{N}^2 \to \mathbb{N}$ by

$$f(e, N) = \begin{cases} 1 + (N)_0 & \text{if } T(e, \langle (N)_0 \rangle, (N)_1) \text{ and } \forall_{n \in [N]}[1 + (N)_0 \neq f(e, n)], \\ 0 & \text{otherwise.} \end{cases}$$

Then f is recursive and total. Moreover: $e \in \mathrm{FIN}$ if and only if the function $f_e : n \mapsto f(e, n)$ is non-zero only a finite number of times, and $e \in \mathrm{INF}$ if and only if f_e is non-zero an infinite number of times. Now, using the S-m-n-theorem, define $g : \mathbb{N} \to \mathbb{N}$ as a function that assigns to e an index of the function f_e. Then also g is recursive and total, and furthermore $e \in \mathrm{FIN} \leftrightarrows g(e) \in \mathrm{FINSET}$, and $e \in \mathrm{INF} \leftrightarrows g(e) \in \mathrm{INFSET}$. Hence g is a reducing function for $(\mathrm{FIN}, \mathrm{INF}) \prec (\mathrm{FINSET}, \mathrm{INFSET})$. □

[1] This notion of reducibility is usually called *many-one-reducibility* in the literature, and is denoted by $\preccurlyeq_m$ instead of $\prec$. We choose a simpler notation since we have no use for other notions of reducibility.

CHAPTER

4 The possible

As we have observed in section 2.1, the proof of Ramsey's theorem is non-effective. In this chapter, we discuss some positive results regarding Ramsey's theorem in a recursive setting. It turns out that a weakly recursive version of the theorem holds unabated in dimension one (section 4.1). Our later investigations (section 5.4) will reveal that this is impossible in higher dimensions. However, the higher dimensional case is not entirely a lost cause, as we discover by analysing the monochromatic set 'constructed' in the classical proof (section 4.2).

4.1 Weakly recursive version

In dimension one, Ramsey's theorem comes down to the pigeonhole principle: For any c-coloring χ of $\mathbb{N}$, there exists an $r \in [c]$ such that $\{n \in \mathbb{N} \mid \chi(n) = r\}$ is infinite. This holds true even if we only consider recursive colorings and require the resulting monochromatic set to be recursive.

Proposition *For any recursive coloring of $\mathbb{N}$, there exists an infinite recursive monochromatic set $X \subset \mathbb{N}$.*

PROOF by induction on c. When $c = 1$ and $\chi : \mathbb{N} \to \{0\}$ is a recursive 1-coloring of $\mathbb{N}$, $X = \mathbb{N}$ itself is trivially monochromatic, recursive, and infinite.

Now suppose that the proposition holds for colorings with at most c colors, and let χ be a recursive $(c+1)$-coloring. Construct a new recursive c-coloring χ' by

$$\chi'(n) = \begin{cases} \chi(n) & \text{if } \chi(n) < c-1, \\ c & \text{if } \chi(n) \geq c-1. \end{cases}$$

By the induction hypothesis, there exists an infinite recursive χ'-monochromatic $X \subset \mathbb{N}$. If the color of X is less than $c-1$, then X is also χ-monochromatic, and still infinite and recursive. If X's color is $c-1$ or c, then both

$$Y = \{x \in X \mid \chi(x) = c-1\} \text{ and } Z = \{x \in X \mid \chi(x) = c\}$$

are recursive and monochromatic. Since $Y \cup Z = X$ is infinite, at least one of them must be infinite. □

We say a function $f : \mathbb{N} \to \mathbb{N}$ is Σ_n or Π_n if and only if its graph $\{(x, f(x)) \in \mathbb{N}^2 : x \in \mathrm{Dom}(f)\}$ is, and that it is Δ_n if and only if it is both Σ_n and Π_n. (Notice that this makes sense: if f is Δ_1, then f is computable by Post's theorem.) More generally we can then even say: for any Σ_n coloring of $\mathbb{N}$ there exists an infinite Σ_n monochromatic subset, for any Π_n coloring there exists an infinite Π_n monochromatic subset, and for any Δ_n coloring there exists an infinite Δ_n monochromatic subset.

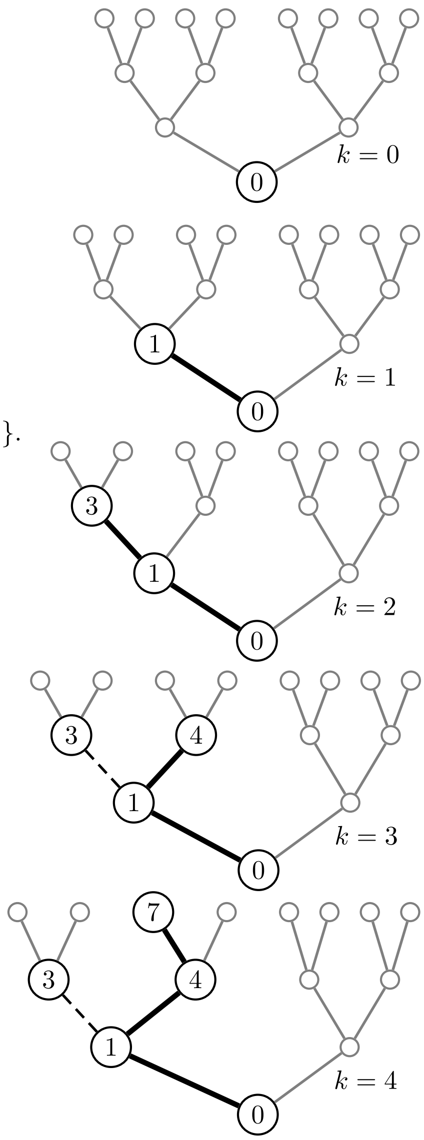
4.2 Merits of the classical proof

For a recursive coloring, the monochromatic set that theorem 2.1 promises might not be recursive. Even so, perhaps we can prove that, in any case, it is in an arithmetical class. After all, we cannot help but feel that the monochromatic set is constructed from the coloring, which is a strong clue that the arithmetical hierarchy should enter the scene. It turns out that for a recursive coloring in dimension d, the set constructed in the classical proof is Π_d.

Theorem *For any recursive coloring of* $[\mathbb{N}]^2$*, there exists an infinite monochromatic* $X \in \Pi_2$*.*
PROOF We alter the construction of the proof of theorem 2.1 slightly, so we can describe the constructed infinite monochromatic set as the result of a search process. Let χ be a recursive c-coloring of $[\mathbb{N}]^2$, and define $n_k \in [c]^*$ and partial functions $f_k : [c]^* \to \mathbb{N}$, $g_k : \mathbb{N} \to [c]^*$ for $k \in \mathbb{N}$ inductively by

$$n_0 = \langle\rangle, \ \ \mathrm{Dom}(f_0) = \{n_0\}, \ \ f_0(n_0) = 0, \ \ \mathrm{Dom}(g_0) = \{0\}, \ \ g_0(0) = \langle\rangle.$$

If n_k, f_k and g_k have already been defined, then



(a) If $\exists_x[x \geq k+1 \ \wedge \ \forall_{i \leq k}[x \notin \mathrm{Dom}(g_i)] \ \wedge \ \forall_{i \leq length(n_k)}[\chi(\{x, f_k((n_k)_i)\}) = (n_k)_i]$ – let us abbreviate this formula to $\varphi(k, n_k) = \exists_x[\varphi'(k, n_k, x)]$ – then determine the least such x, and define

$$\begin{aligned} n_{k+1} &= n_k \star \langle 0\rangle \\ \mathrm{Dom}(f_{k+1}) &= \mathrm{Dom}(f_k) \cup \{n_{k+1}\} \\ f_{k+1}(s) &= f_k(s) \ \text{ if } s \in \mathrm{Dom}(f_k) \\ f_{k+1}(n_{k+1}) &= x \\ \mathrm{Dom}(g_{k+1}) &= \mathrm{Dom}(g_k) \cup \{x\} \\ g_{k+1}(x') &= g_k(x') \ \text{ if } x' \in \mathrm{Dom}(g_k) \\ g_{k+1}(x) &= n_{k+1}. \end{aligned}$$

(b) If $\neg\varphi(k, n_k)$, then determine $l = \max\{l \mid \varphi(k, \langle (n_0)_0, \ldots, (n_k)_l\rangle)\}$. (Since $\neg\varphi(k, n_k)$, then $l < \mathrm{length}(n_k)$.) Define

$$\begin{aligned} n_{k+1} &= \langle (n_k)_0, \ldots, (n_k)_{l-1}, 1 + (n_k)_l\rangle \\ f_{k+1} &= f_k \\ g_{k+1} &= g_k. \end{aligned}$$

(Then $g_k \circ f_k = \mathrm{id}_{\mathrm{Dom}(f_k)}$ and $f_k \circ g_k = \mathrm{id}_{\mathrm{Dom}(g_k)}$.) These definitions capture a search process: f_k represents the (c-ary) search tree. We are looking for an infinite branch, and n_k is the most likely candidate at the moment. Let us illustrate. We start at 0, asking whether there is a 'fresh' number such that the pair is of the first color. Suppose 1 is such a number. We then take leftmost path from the root of the tree, putting 1 at the next node. We then again ask ourselves whether there is a 'fresh' number x such that both $\{x, 0\}$ and $\{x, 1\}$ are of color 0. Suppose 3 is such a number. Take the leftmost path and label the node 3. Suppose at this point there is no suitable candidate. We then track back, discarding 3, asking ourselves whether then there is a 'fresh' number x such that $\{x, 0\}$ is of color 0, but $\{x, 1\}$ is of the next color. Suppose 4 is such a number. Et cetera. In the picture beside, the bold branch in the tree is the 'current branch' n_k, the dashed ones are discarded.

We prove by induction on l that for any l there is a k_l such that $f_K(\langle(n_K)_0, \dots, (n_K)_l\rangle)$ is constant for $K \geq k_l$ (and that n_k, f_k and g_k are well-defined for any k). The case $l = 0$ is trivial. Suppose the claim holds for $l' = 0, \dots, l$. Let

$$k = \min\{k \mid \forall_{l' \leq l}[f_k(\langle(n_K)_0, \dots, (n_K)_{l'}\rangle) \text{ is constant for } K \geq k_{l'}]\}.$$

Then there is an x such that $f_{k+1}(n_{k+1}) = x$ by case (a). Also $f_K(n_{k+1}) = x$ for all $K \geq k$ because of the defining property of k. So k_{l+1} fulfills the conditions and thus the first part of the claim is proven.

This also shows that for $K \geq k$, $\varphi'(K, n, x') \leftrightarrows \varphi'(k, n, x')$. Hence $\{x' \mid \varphi'(K, n, x')\}$ is infinite for any $K \geq k$. If there are infinitely many x' with $\varphi'(K, n, x')$ and $\chi(\{x', x\}) = 0$ then $g_K(x)$ will remain n_{k+1} for all $K \geq k$. Otherwise, the last coordinate of n_{k+1} will change at some $k_1 > k$. If there are infinitely many x' with $\varphi'(K, n, x')$ and $\chi(\{x', x\}) = 1$ then $g_{k_1}(x)$ will remain constant for all $K \geq k_1$, and so on until k_{c-1}. By the pigeonhole principle, n_{k+1} can thus never change color beyond $c - 1$, so n_k and hence f_k and g_k are well-defined.

Thus we can define $f = \cup_k f_k$ and $g = \cup_k g_k$ and $\alpha = \lim_{k\to\infty} n_k \in \mathbb{N}^{\mathbb{N}}$. (Then α is the unique infinite branch of the tree f.) By construction, for $k, k' \in \mathrm{Dom}(\alpha), k < k'$, then $\chi(\{k, k'\})$ equals the eventual color $(g(k))_{\mathrm{length}(g(k))-1}$. So if we define $X' = \{x \mid \forall_{i<\mathrm{length}(g(x))}[(g(x))_i = (\alpha)_i]\}$ and $X_r = \{x \in X' \mid (g(x))_{\mathrm{length}(g(x))-1}\} = r\}$ for $r \in [c]$, then X_r is χ-monochromatic for any $r \in [c]$, and at least one of them must be infinite.

Moreover, notice that given a trace (an encoding of choices (a) along with x or (b) along with l) of the first k steps in the construction, to check whether the trace is correct is a Π_1-problem: in case (a), checking whether $\varphi'(k, n_k, x)$ is Δ_1, and checking whether x is that smallest such number is also Δ_1, and in case (b), checking whether $\neg\varphi$ is Π_1, as is checking whether l works and $l + 1$ doesn't. Hence

$$\begin{aligned}
x \notin X' &\leftrightarrows \exists_{k\geq x}\forall_{l\leq x}[f_k(\langle(n_k)_0, \dots, (n_k)_l\rangle) \neq x] \\
&\leftrightarrows x \notin \mathrm{Dom}(g) \vee \exists_{k\geq x}[\neg\varphi(k, n_k)] \\
&\leftrightarrows \underbrace{\exists_{k\geq x}\exists_t \underbrace{[\underbrace{t \text{ is a trace of the first } k \text{ steps}}_{\Pi_1} \wedge \underbrace{x \notin \mathrm{Dom}(g)}_{\Delta_1(\text{from } t)} \wedge \underbrace{\neg\varphi(k, n_k)}_{\Pi_1}]}_{\Pi_1}}_{\Sigma_2}
\end{aligned}$$

So $X' \in \Pi_2$. Also

$$x \notin X_0 \leftrightarrows \underbrace{\underbrace{x \notin X'}_{\Sigma_2} \vee \exists_k\exists_t[\underbrace{t \text{ is a trace of the first } k \text{ steps}}_{\Pi_1} \wedge \underbrace{\neg\varphi(k, n_k) \wedge (n_k)_l = 1}_{\Delta_1(\text{from } t)}]}_{\Sigma_2}.$$

So $X_0 \in \Pi_2$. Similarly $X_r \in \Pi_2$ for any $r \in [c]$. Hence surely the first infinite one of the X_r is Π_2. $\square$

More generally, for any $d \geq 2$, $n \geq 1$ and any Δ_n coloring of $[\mathbb{N}]^d$, there exists an infinite set in Π_{n+d-1}[Joc]. The classical proof retains some merits even in a recursive setting.

CHAPTER

5 The impossible

The proof we gave of Ramsey's theorem in section 2.1 is non-effective. This chapter strenghtens this observation to the fact that Ramsey's theorem is false in a recursive setting. First of all, Ramsey's theorem is false when interpreted strongly recursively (section 5.1). But even reasoning classically cannot save Ramsey's theorem in a recursive setting in dimensions higher than one: there are recursive colorings such that no recursive set can be monochromatic (section 5.4). To find such a coloring we discuss bi-immune sets (section 5.2), and to show that such colorings can even be recursive, we use a limit construction (section 5.3).

5.1 Recursive point of view

The strongest possible recursive setting is when we not only require all 'input' and 'output' of a theorem to be recursive, but also require every construction to be recursive.

Definition We set $\mathrm{SET} = \mathrm{TOT}_1$, and fix the function $\mathrm{Set} : \mathrm{SET} \to \mathcal{P}(\mathbb{N})$ by $\mathrm{Set}(e) = \{n \in \mathbb{N} \mid \psi_e(n) \neq 0\}$. We define $\mathrm{FINSET} = \{e \in \mathrm{SET} \mid \mathrm{Set}(e) \text{ is finite}\}$ and $\mathrm{INFSET} = \{e \in \mathrm{SET} \mid \mathrm{Set}(e) \text{ is infinite}\}$. For c and d, we define $\mathrm{COL}_c^d = \{e \in \mathrm{IM}_d \cap \mathrm{TOT} \mid \mathrm{Ran}(\psi_e) \subset [c]\}$, the set of indices of recursive c-colorings of $[\mathbb{N}]^d$.

After having agreed upon a suitable adaptation of Ramsey's theorem, we find that it is false. It does not even hold in dimension one.

Lemma *For no $c > 1$ does there exist a total recursive function $f : \mathrm{COL}_c^1 \to \mathrm{INFSET}$ such that $\mathrm{Set}(f(e))$ is ψ_e-monochromatic for every $e \in \mathrm{COL}_c^d$.*
PROOF by contradiction. Suppose f were such a function. Define

$$H = \{e \in \mathbb{N} \mid \exists_z[T(e, \langle e\rangle, z)]\}$$
$$I_e = \{z \in \mathbb{N} \mid \forall_{z' \leq z}[\neg T(e, \langle e\rangle, z')]\}.$$

Notice that exactly one of I_n and $\mathbb{N}\backslash I_n$ is infinite, the other finite. Also, I_n is recursive, and by using for each n the S-m-n-theorem we can even determine a total recursive function $g : \mathbb{N} \to \mathbb{N}$ such that

$$\psi_{g(n)}(m) = \begin{cases} 0 & \text{if } m \notin I_n, \\ 1 & \text{if } m \in I_n. \end{cases}$$

Moreover, $n \notin H$ if and only if $I_n = \mathbb{N}$ is infinite. So $n \in H$ if and only if $\mathrm{Set}((f \circ g)(n)) \subset \mathbb{N}\backslash I_n$. Because $\mathrm{Set}((f \circ g)(n))$ is infinite and $\psi_{g(n)}$-monochromatic, $n \in H$ if and only if $\min \mathrm{Set}((f \circ g)(n)) \in \mathbb{N}\backslash I_n$. But then H would be recursive, contradicting the unsolvability of the halting problem. Hence such a function f cannot exist. □

The higher-dimensional case is now easily reduced to the one-dimensional case.

Theorem *For no $c > 1$ and $d > 0$ does there exist a total recursive function $f : \mathrm{COL}_c^d \to \mathrm{INFSET}$ such that $\mathrm{Set}(f(e))$ is ψ_e-monochromatic for every $e \in \mathrm{COL}_c^d$.*

PROOF by contradiction. Suppose f were such a function. Define a function $g : \mathrm{COL}_c^1 \to \mathrm{INFSET}$ as follows:

Let e be in COL_c^1.
Make $\chi' : [\mathbb{N}]^d \to [c]$ by $\chi'(n_1, \ldots, n_d) = \psi_e(n_1)$.
Then χ' is recursive: determine $e' \in \mathrm{COL}_c^d$ such that $\chi' = \psi_{e'}$.
Define $g(e) = f(e')$.

Then g would be recursive and $\mathrm{Set}(g(e))$ would be ψ_e-monochromatic for every $e \in \mathrm{COL}_c^1$, which contradicts the previous lemma. Hence such a function f cannot exist. □

5.2 Bi-immune sets

Bi-immune sets are named after Post's immune sets, which are defined by the fact that they contain no infinite Σ_1 sets. A set is bi-immune if both the set itself and its complement are immune, i.e. if the set itself nor its complement contains an infinite Σ_1 set. Stated otherwise, for a bi-immune set seen as a 2-coloring there are no infinite monochromatic Σ_1 sets.

Definition A set $X \subset \mathbb{N}$ is called *bi-immune* if for any $Y \in \Sigma_1$ both $Y \not\subset X$ and $Y \not\subset \mathbb{N}\backslash X$.

More generally, we call X *bi-n-immune* if for any $Y \in \Sigma_n$ both $Y \not\subset X$ and $Y \not\subset \mathbb{N}\backslash X$. Let us now show that bi-immune sets actually exist. The idea is simply to 'construct' a bi-immune set by making sure one element of every infinite Σ_1 set is in, and one element is out. The key is that there are only a countable number of Σ_1 sets.

Theorem *There exists a bi-immune set.*

PROOF Define $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$ and $X \subseteq \mathbb{N}$ by recursion, such that, for each n,

$$\begin{aligned}
U_n &= W_n \backslash \{f(0), \ldots, f(n-1), g(0), \ldots, g(n-1)\}, \\
f(n) &= \begin{cases} 0 & \text{if } U_n = \emptyset, \\ \min U_n & \text{if } U_n \neq \emptyset, \end{cases} \\
V_n &= W_n \backslash \{f(0), \ldots, f(n), g(0), \ldots, g(n-1)\}, \\
g(n) &= \begin{cases} 0 & \text{if } V_n = \emptyset, \\ \min V_n & \text{if } V_n \neq \emptyset, \end{cases} \\
X &= \mathrm{Ran}(f).
\end{aligned}$$

Let $n \in N$, and assume $n \in \mathrm{IM}_1$ and that W_n is infinite. Then $U_n \neq \emptyset$, so that $f(n) \in X \cap W_n$. So $W_n \cap X \neq \emptyset$, or in other words, $W_n \not\subset \mathbb{N}\backslash X$. Hence $\mathbb{N}\backslash X$ contains no (infinite) Σ_1 set.

Also $V_n \neq \emptyset$, so that $g(n) \in W_n$. Now suppose that $g(n) \in X$, say $g(n) = f(m)$. On the one hand we must then have $n > m$, because $g(n) = f(m) \notin U_{m+1}$ by definition of U_{m+1}. But on the other hand we must also have $n \leq m$, because $f(m) = g(n) \notin V_{n+1}$ by definition of V_{n+1}. So $g(n) \in \mathbb{N}\backslash X$, and hence $W_n \cap \mathbb{N}\backslash X \neq \emptyset$. Hence X contains no (infinite) Σ_1 set.

Conclusion: X is bi-immune. □

More generally, there exist bi-n-immune sets, because there are only a countable number of Σ_n sets.

5.3 Limits

Approximations and limit constructions are powerful tools in any setting, and the arithmetical hierarchy is no exception. Describing a set as a limit construction often provides insight in its (minimum) arithmetical complexity. We say a sequence $X_0, X_1, X_2, \ldots$ of recursive sets is *uniformly recursive* if $\{(x,k) \in \mathbb{N}^2 \mid x \in X_k\}$ is recursive.

Definition For any uniformly recursive sequence $X_0, X_1, X_2, \ldots$ of sets we define

$$\liminf_{k\to\infty} X_k = \{x \in \mathbb{N} \mid \exists_K \forall_{k \geq K}[x \in X_k]\}$$
$$\limsup_{k\to\infty} X_k = \{x \in \mathbb{N} \mid \forall_K \exists_{k \geq K}[x \in X_k]\}.$$

We call $X_0, X_1, X_2, \ldots$ *convergent* if $\liminf_{k\to\infty} X_k = \limsup_{k\to\infty} X_k$, and in that case we define

$$\lim_{k\to\infty} X_k = \liminf_{k\to\infty} X_k.$$

A few facts are clear. First, $\liminf_{k\to\infty} X_k \subset \limsup_{k\to\infty} X_k$. Secondly, if $X_0 \subset X_1 \subset X_2 \subset \cdots$ then the sequence is convergent with limit $\bigcup_k X_k$. And finally, if $X_0 \supset X_1 \supset X_2 \supset \cdots$ then the sequence is convergent with limit $\bigcap_k X_k$. The following theorem is what makes these limits so powerful.

Theorem *$X \in \Delta_2$ if and only if there exists a uniformly recursive sequence $X_0, X_1, X_2, \ldots$ of sets such that $X = \lim_{k\to\infty} X_k$.*

Proof First, suppose that $X_1, X_2, X_3, \ldots$ is a convergent uniformly recursive sequence of sets such that $X = \lim_{k\to\infty} X_k$. On the one hand then

$$X = \liminf_{k\to\infty} X_k = \{x \in \mathbb{N} \mid \exists_K \forall_{k \geq K}[x \in X_k]\},$$

so that $X \in \Sigma_2$. On the other hand $X = \limsup_{k\to\infty} X_k$, so that likewise $X \in \Pi_2$. Hence $X \in \Delta_2$.

Now suppose that $X \in \Delta_2$. Determine a Π_2-index a of X and a Π_2-index b of $\mathbb{N}\backslash X$ such that

$$x \in X \leftrightarrows \forall_m \exists_n[\psi_a(x,m,n) \neq 0],$$
$$x \notin X \leftrightarrows \forall_m \exists_n[\psi_b(x,m,n) \neq 0].$$

Determine indices $a'' \in \mathrm{IM}_3$ and $a' \in \mathrm{IM}_2$ such that

$$\psi_{a''}(x,i,n) = \sum_{j=0}^{n} \psi_a(x,i,j),$$
$$\psi_{a'}(x,n) = \max\{I \in [n-1] \mid \forall_{i<I}[\psi_{a''}(x,i,n) \neq 0]\}.$$

Then $\psi_{a'}(x,0) \leq \psi_{a'}(x,1) \leq \psi_{a'}(x,2) \leq \cdots$, and moreover

$$\lim_{n\to\infty} \psi_{a'}(x,n) = \infty \leftrightarrows \forall_m \exists_n[\psi_a(x,m,n) \neq 0] \leftrightarrows x \in X.$$

Also determine $b'' \in \mathrm{IM}_3$ and $b' \in \mathrm{IM}_2$ such that

$$\psi_{b''}(x,i,n) = \sum_{j=0}^{n} \psi_b(x,i,j),$$
$$\psi_{b'}(x,n) = \max\{I \in [n-1] \mid \forall_{i<I}[\psi_{b''}(x,i,n) \neq 0]\}.$$

Then $\psi_{b'}(x,0) \leq \psi_{b'}(x,1) \leq \psi_{b'}(x,2) \leq \cdots$, and moreover

$$\lim_{n\to\infty} \psi_{b'}(x,n) = \infty \leftrightarrows \forall_m \exists_n [\psi_b(x,m,n) \neq 0] \leftrightarrows x \notin X.$$

Because $X \cup (\mathbb{N}\backslash X) = \mathbb{N}$ we have $\lim_{n\to\infty} \psi_{a'}(x,n) = \infty$ or $\lim_{n\to\infty} \psi_{b'}(x,n) = \infty$. And because $X \cap (\mathbb{N}\backslash X) = \emptyset$, precisely one of them holds. Define

$$X_k = \{x \in N \mid \psi_{a'}(x, \min\{n \in \mathbb{N} \mid \psi_{a'}(x,n) \geq k \vee \psi_{b'}(x,n) \geq k\}) \geq k\}.$$

Then the sequence $X_1, X_2, X_3, \ldots$ is uniformly recursive. Furthermore

$$\begin{aligned}
\liminf_{k\to\infty} X_k &= \{x \in \mathbb{N} \mid \exists_K \forall_{k\geq K} [\psi_{a'}(x, \min\{n \in \mathbb{N} \mid \psi_{a'}(x,n) \geq k \vee \psi_{b'}(x,n) \geq k\}) \geq k]\} \\
&= \{x \in \mathbb{N} \mid \lim_{n\to\infty} \psi_{b'}(x,n) \neq \infty\} = X, \\
\limsup_{k\to\infty} X_k &= \{x \in \mathbb{N} \mid \forall_K \exists_{k\geq K} [\psi_{a'}(x, \min\{n \in \mathbb{N} \mid \psi_{a'}(x,n) \geq k \vee \psi_{b'}(x,n) \geq k\}) \geq k]\} \\
&= \{x \in \mathbb{N} \mid \lim_{n\to\infty} \psi_{a'}(x,n) = \infty\} = X.
\end{aligned}$$

Hence the sequence is convergent and $X = \lim_{k\to\infty} X_k$. □

Examining the proof we see that the Δ_1-index of X_k given is recursively attainable from k, a and b. Hence we can even strengthen the statement to the following.
There exists a recursive function $\mathrm{Lim} : \mathrm{SET}^2 \to \mathrm{TOT}_2$ such that if $a, b \in \mathrm{SET}$ and

$$\forall_x \big[\forall_m \exists_n [\psi_a(x,m,n) \neq 0] \leftrightarrows \exists_m \forall_n [\psi_b(x,m,n) \neq 0]\big],$$

then for every x

$$\begin{aligned}
&\forall_K \exists_{k\geq K} [\psi_{\mathrm{Lim}(a,b)}(x,k) \neq 0] \\
\leftrightarrows\ &\exists_K \forall_{k\geq K} [\psi_{\mathrm{Lim}(a,b)}(x,k) \neq 0] \\
\leftrightarrows\ &\forall_m \exists_n [\psi_a(x,m,n) \neq 0].
\end{aligned}$$

We can now show that the bi-immune set of theorem 5.2 has the lowest possible arithmetical complexity.

Corollary *There exists a bi-immune set in Δ_2, but not in $\Sigma_1 \cup \Pi_1$.*
PROOF If a set is infinite and is in Σ_1, it certainly contains an infinite Σ_1 set, to wit itself, and thus the set cannot be bi-immune. Likewise, if a set has infinite complement and is in Π_1, then its complement contains an infinite Σ_1 set, so the set cannot be bi-immune.

Let X be the bi-immune set given in the proof of theorem 5.2. Define

$$\begin{aligned}
W_{n,k} &= \{p \in \mathbb{N} \mid \exists_{z<k} [T(n, \langle p \rangle, z)]\}, \\
U_{n,k} &= W_{n,k} \backslash \{f_k(0), \ldots, f_k(n-1), g_k(0), \ldots, g_k(n-1)\}, \\
f_k(n) &= \begin{cases} 0 & \text{als } U_{n,k} = \emptyset, \\ \min U_{n,k} & \text{als } U_{n,k} \neq \emptyset, \end{cases} \\
V_{n,k} &= W_{n,k} \backslash \{f_k(0), \ldots, f_k(n), g_k(0), \ldots, g_k(n-1)\}, \\
g_k(n) &= \begin{cases} 0 & \text{als } V_{n,k} = \emptyset, \\ \min V_{n,k} & \text{als } V_{n,k} \neq \emptyset, \end{cases} \\
X_k &= \{f_k(0), \ldots, f_k(k)\}.
\end{aligned}$$

Then $X = \lim_{k\to\infty} X_k$. Hence $X \in \Delta_2$ by the previous theorem. □

More generally: there exists a bi-n-immune set in Δ_{n+1}, but not in $\Sigma_n \cup \Pi_n$.

5.4 The counterexample

Whereas proposition 4.1 showed that in dimension one Ramsey's theorem holds in the recursive setting, we can now use bi-immune sets to show that this cannot be so in higher dimensions.

Theorem *For any $c \geq 2$ and $d \geq 2$ there exists a recursive c-coloring of $[\mathbb{N}]^d$ such that no infinite set in Σ_1 is monochromatic.*

Proof Let X be a bi-immune set in Δ_2, and determine a uniformly recursive sequence $X_0, X_1, X_2, \ldots$ such that $X = \lim_{k\to\infty} X_k$. Define a c-coloring χ of $[\mathbb{N}]^d$ by

$$\chi(\{n_1, \ldots, n_d\}) = \begin{cases} 0 & \text{if } \min\{n_1, \ldots, n_d\} \notin X_{\max\{n_1,\ldots,n_d\}}, \\ 1 & \text{if } \min\{n_1, \ldots, n_d\} \in X_{\max\{n_1,\ldots,n_d\}}. \end{cases}$$

Then χ is recursive. Now suppose that $Y \subset \mathbb{N}$ is infinite and χ-monochromatic. If Y is of color 0, then $\min\{y_1, \ldots, y_d\} \notin X_{\max\{y_1,\ldots,y_d\}}$ for every $\{y_1, \ldots, y_d\} \in [Y]^d$, so that $y \notin X$ for every $y \in Y$, and hence $Y \subset \mathbb{N}\backslash X$. But X is bi-immune, so Y cannot be Σ_1. If Y is of color 1, then likewise $Y \subset X$, so that Y cannot be Σ_1.

Conclusion: infinite Σ_1 sets cannot be χ-monochromatic. □

From theorem 4.2 we can conclude that the previous theorem cannot be improved: there is no recursive coloring of $[\mathbb{N}]^d$ such that no infinite set in Δ_2 is monochromatic.

Theorem 5.4 was first proved in [Spe] by using incomparable degrees. The easier strategy we followed in this chapter, using bi-immune sets to arrive at the counterexample of theorem 5.4, was first set out by [Joc].

CHAPTER

6 Understanding

After learning in the previous chapter that Ramsey's theorem does not hold in a recursive setting, a question that arises naturally is why this is so: can the impossible part of the 'construction' be circumvented? This chapter concludes by providing a clue as to why Ramsey's theorem seems to resist effective approaches: to determine whether a given recursive coloring has any infinite recursive monochromatic sets at all is Σ_3-complete (section 6.1).

6.1 Are there infinite monochromatic sets?

Theorem 5.4 showed that Ramsey's theorem is false in a recursive setting. Apparently, the combination of infinity and monochromaticity is a hard requirement to fulfill effectively. The theorem in this section will help us understand why.

Definition We define the set RECCOL_c^d as

$$\{e \in \mathrm{COL}_c^d \mid \text{ there exists an infinite recursive } \psi_e\text{-monochromatic set } \}.$$

The combination of infinity and monochromaticity is a requirement impossible to fulfill effectively: we prove that RECCOL_c^d is Σ_3-complete.

Theorem *For any $c \geq 2$ and $d \geq 2$, RECCOL_c^d is Σ_3-complete.*

PROOF First of all, we have

$$\mathrm{RECCOL}_c^d = \{e \in \mathbb{N} \mid \underbrace{\underbrace{e \in \mathrm{COL}_c^d}_{\Pi_2} \wedge \underbrace{\exists_{e'}[\underbrace{e' \in \mathrm{INFSET}}_{\Pi_2} \wedge \underbrace{\mathrm{Set}(e') \text{ is } \psi_e\text{-monochromatic}}_{\Pi_2}]}_{\Sigma_3}}_{\Sigma_3}\},$$

because

$$\begin{aligned}
e \in \mathrm{COL}_c^d &\leftrightarrows \forall_{n_1,\ldots,n_d}\exists_z\big[\ (T(e,\langle(n_1,\ldots,n_d)\rangle,z) \wedge \mathrm{OUTP}(z) < c) \\
&\qquad\qquad \vee \neg(n_1 < \cdots < n_d)\big] \ \wedge\ e \in \mathrm{IM}_d, \\
e' \in \mathrm{INFSET} &\leftrightarrows e' \in \mathrm{IM}_1 \ \wedge\ \forall_n \exists_z\big[T(e',\langle n\rangle,z)\big] \\
&\qquad \wedge \forall_N \exists_{n>N}\exists_z\big[T(e',\langle n\rangle,z) \wedge \mathrm{OUTP}(z) \neq 0\big], \\
\mathrm{Set}(e') \text{ is } \psi_e\text{-monochromatic} &\leftrightarrows \forall_{n_1,\ldots,n_d}\exists_r\exists_{z_0,\ldots,z_d}\big[(T(e,\langle(n_1,\ldots,n_d)\rangle,z_0) \wedge \mathrm{OUTP}(z_0) = r) \\
&\qquad \vee \exists_{1\leq i\leq d}[T(e',\langle n_i\rangle,z_i) \wedge \mathrm{OUTP}(z_i) = 0] \\
&\qquad \vee \neg(n_1 < \cdots < n_d)\big].
\end{aligned}$$

Now, by theorem 3.4 it suffices to show that COF $\prec$ RECCOL^d_c. Let $e \in \mathbb{N}$. We define a sequence $n_{e,0}, n_{e,1}, \ldots$ of natural numbers by

$$n_{e,0} = 0,$$
$$n_{e,k+1} = \begin{cases} n_{e,k} & \text{if } k \in W_e, \\ 1 + n_{e,k} & \text{if } k \notin W_e. \end{cases}$$

Then, if we denote by S_k the (finite) group of permutations of $[k]$,

$$x = n_{e,k} \leftrightarrows \underbrace{\exists_{\pi \in S_k} \underbrace{\big(\underbrace{\forall_{i \in \{0,\ldots,x-1\}} [\underbrace{\pi(i) \notin W_e}_{\Pi_1}]}_{\Pi_1} \wedge \underbrace{\forall_{i \in \{x,\ldots,k-1\}} [\underbrace{\pi(i) \in W_e}_{\Sigma_1}] \big)}_{\Sigma_1}}_{\Delta_2}}_{\Delta_2},$$

so we can determine $\alpha, \beta \in$ TOT such that

$$x = n_{e,k} \leftrightarrows \forall_m \exists_n [\psi_\alpha(m,n,e,k,x) \neq 0] \leftrightarrows \exists_m \forall_n [\psi_\beta(m,n,e,k,x) \neq 0].$$

We define a sequence $X_{e,0}, X_{e,1}, \ldots$ of elements of $\bigcup_d \{0,1\}^d$ by:

$$X_{e,0} = \langle\rangle,$$
$$X_{e,k+1} = \begin{cases} X_{e,k} \star \langle 0 \rangle & \text{if } k \in W_e, \\ X_{e,k} \star \langle 0^{y+1-\text{length}(X_{e,k})} \rangle & \text{if } k \notin W_e \wedge\ n_{e,k} = 2e' \\ & \quad \wedge\ y = \min\{y \in \mathbb{N} \mid y \in W_{e'} \wedge y \geq \text{length}(X_{e,k})\}, \\ X_{e,k} \star \langle 0 \rangle & \text{if } k \notin W_e \wedge\ n_{e,k} = 2e' \\ & \quad \wedge \neg\exists_y [y \in W_{e'} \wedge y \geq \text{length}(X_{e,k})], \\ X_{e,k} \star \langle 0^{y-\text{length}(X_{e,k})} \rangle \star \langle 1 \rangle & \text{if } k \notin W_e \wedge\ n_{e,k} = 2e'+1 \\ & \quad \wedge\ y = \min\{y \in \mathbb{N} \mid y \in W_{e'} \wedge y \geq \text{length}(X_{e,k})\}, \\ X_{e,k} \star \langle 0 \rangle & \text{if } k \notin W_e \wedge\ n_{e,k} = 2e'+1 \\ & \quad \wedge \neg\exists_y [y \in W_{e'} \wedge y \geq \text{length}(X_{e,k})]. \end{cases}$$

Then $(X_{e,k})_x = 1$ if and only if

$$\begin{aligned} &\exists_a \big[(a)_0 + \cdots + (a)_k = x \wedge \\ &\quad \forall_{k'<k} \big[(k' \in W_e \wedge (a)_{k'} = 0) \\ &\qquad \vee (k' \notin W_e \wedge \exists_y [y \in W_{\lfloor n_{e,k'}/2 \rfloor} \wedge y \geq (a)_0 + \cdots + (a)_{k'-1} \\ &\qquad\qquad \wedge \forall_{y'<y} [y' \notin W_{\lfloor n_{e,k'}/2 \rfloor} \vee y' < (a)_0 + \cdots + (a)_{k'-1}] \\ &\qquad\qquad \wedge (a)_{k'} = y + 1 - (a)_0 - \cdots - (a)_{k'-1}]) \\ &\qquad \vee (k' \notin W_e \wedge \neg\exists_y [y \in W_{\lfloor n_{e,k'}/2 \rfloor} \wedge y \geq (a)_0 + \cdots + (a)_{k'-1}] \wedge (a)_{k'} = 1) \big] \\ &\quad \wedge k \notin W_e \wedge \exists_y [y \in W_{\lfloor n_{e,k}/2 \rfloor} \wedge y \geq (a)_0 + \cdots + (a)_{k-1} \\ &\qquad\qquad \wedge \forall_{y'<y} [y' \notin W_{\lfloor n_{e,k}/2 \rfloor} \vee y' < (a)_0 + \cdots + (a)_{k-1}] \\ &\qquad\qquad \wedge (a)_k = y + 1 - (a)_0 - \cdots - (a)_{k-1}] \end{aligned}$$

First, because $p \mapsto \lfloor p/2 \rfloor$ is a total recursive function, we have

$$\begin{aligned} y \in W_{\lfloor n_{e,k}/2 \rfloor} &\leftrightarrows \underbrace{\exists_p \exists_z [\underbrace{p = n_{e,k}}_{\Delta_2} \wedge \underbrace{T(\lfloor p/2 \rfloor, \langle y \rangle, z)}_{\Delta_1}]}_{\Sigma_2} \\ &\leftrightarrows \underbrace{\forall_p [\underbrace{p \neq n_{e,k}}_{\Delta_2} \vee \underbrace{\exists_z [T(\lfloor p/2 \rfloor, \langle y \rangle, z)]}_{\Sigma_1}]}_{\Pi_2}, \end{aligned}$$

so that $y \in W_{\lfloor n_{e,k}/2 \rfloor}$ is a Δ_2 formula. Secondly,

$$\begin{aligned}
\neg\exists_y[y \in W_{\lfloor n_{e,k}/2\rfloor}] &\leftrightarrows \underbrace{\forall_y[\underbrace{y \notin W_{\lfloor n_{e,k}/2\rfloor}}_{\Delta_2}]}_{\Pi_2} \\
&\leftrightarrows \exists_x\forall_y\forall_z[x = n_{e,k} \wedge \neg T(\lfloor x/2\rfloor, \langle y\rangle, z)] \\
&\leftrightarrows \underbrace{\exists_x\exists_m\forall_n\forall_y\forall_z[\psi_\beta(m,n,e,k,x) \neq 0 \wedge \neg T(\lfloor x/2\rfloor, \langle y\rangle, z)]}_{\Sigma_2},
\end{aligned}$$

hence also $\neg\exists_y[y \in W_{\lfloor n_{e,k}/2\rfloor}]$ is a Δ_2 formula. Thirdly, $\exists_y[y \in W_{\lfloor n_{e,k}/2\rfloor} \wedge \forall_{y'<y}[y' \notin W_{\lfloor n_{e,k}/2\rfloor}]]$ if and only if

$$\begin{aligned}
&\exists_x\exists_y\exists_z\forall_{z'}[x = n_{e,k} \wedge T(\lfloor x/2\rfloor, \langle y\rangle, z) \wedge \forall_{y'<y}[\neg T(\lfloor x/2\rfloor, \langle y'\rangle, z')]] \\
\leftrightarrows\, &\underbrace{\exists_x\exists_y\exists_z\exists_m\forall_n\forall_{z'}[\psi_\beta(m,n,e,k,x) \neq 0 \wedge T(\lfloor x/2\rfloor, \langle y\rangle, z) \wedge \forall_{y'<y}[\neg T(\lfloor x/2\rfloor, \langle y'\rangle, z')]]}_{\Sigma_2} \\
\leftrightarrows\, &\forall_x\forall_{z'}\exists_y\exists_z[x \neq n_{e,k} \vee (T(\lfloor x/2\rfloor, \langle y\rangle, z) \wedge \forall_{y'<y}[\neg T(\lfloor x/2\rfloor, \langle y'\rangle, z'))]] \\
\leftrightarrows\, &\underbrace{\forall_x\forall_{z'}\forall_m\exists_n\exists_y\exists_z[\psi_\alpha(m,n,e,k,x) = 0 \vee (T(\lfloor x.2\rfloor, \langle y\rangle, z) \wedge \forall_{y'<y}[\neg T(\lfloor x/2\rfloor, \langle y'\rangle, z'))]]}_{\Pi_2},
\end{aligned}$$

so that this is also a Δ_2 formula, and we can thus determine $\gamma, \delta \in \text{TOT}$ such that

$$(X_{e,k})_x = 1 \leftrightarrows \forall_m\exists_n[\psi_\gamma(m,n,e,k,x) \neq 0] \leftrightarrows \exists_m\forall_n[\psi_\delta(m,n,e,k,x) \neq 0].$$

Notice that $X_{e,k}$ is a prefix of $X_{e,l}$ when $k < l$, and that $\lim_{k\to\infty} \text{length}(X_{e,k}) = \infty$ because $\text{length}(X_{e,k}) \geq k$. So if we define

$$\begin{aligned}
X &= \{(e,x) \in \mathbb{N}^2 \mid \exists_k[(X_{e,k})_x = 1]\} \\
&= \{(e,x) \in \mathbb{N}^2 \mid \forall_k[(X_{e,k})_x = 1 \vee x < k]\},
\end{aligned}$$

then $X \in \Delta_2$. Hence we can determine $\epsilon, \zeta \in \text{TOT}$ such that

$$(e,x) \in X \leftrightarrows \exists_m\forall_n[\psi_\epsilon(m,n,e,x) \neq 0] \leftrightarrows \forall_m\exists_n[\psi_\zeta(m,n,e,x) \neq 0].$$

Define $\eta = \text{Lim}(\epsilon, \zeta) \in \text{TOT}$, and by theorem 5.3 we have

$$X = \lim_{k\to\infty} \{(e,x) \in \mathbb{N}^2 \mid \psi_\eta(e,x,k) \neq 0\}.$$

We define a recursive c-coloring χ_e of $[\mathbb{N}]^d$ (just like in theorem 5.4) by

$$\chi_e(\{n_1,\ldots,n_d\}) = \begin{cases} 0 & \text{if } \psi_\eta(e, \min\{n_1,\ldots,n_d\}, \max\{n_1,\ldots,n_d\}) = 0, \\ 1 & \text{if } \psi_\eta(e, \min\{n_1,\ldots,n_d\}, \max\{n_1,\ldots,n_d\}) \neq 0. \end{cases}$$

Finally we define $f : \mathbb{N} \to \mathbb{N}$ using the S-m-n-theorem by assigning to e an index of χ_e. Then f is a total recursive function.

Now, if $e \notin \text{COF}$, then $X_e = \{x \in \mathbb{N} \mid (e,x) \in X\}$ is bi-immune (by construction). Thus no infinite Σ_1-set is χ_e-monochromatic. Hence $f(e) \notin \text{RECCOL}_c^d$.

If $e \in \text{COF}$, then X_e is finite (by construction). Consider the function $h : \mathbb{N} \to \mathbb{N}$ given by

$$\begin{aligned}
h(0) &= 1 + \max X_e, \\
h(n+1) &= \mu k[\forall_{n' \in \{0,\ldots,n\}}[k > h(n') \wedge \psi_\eta(e, h(n'), k) = 0]].
\end{aligned}$$

Then h is strictly increasing, so $\{h(n) : n \in \mathbb{N}\}$ is recursive, infinite, and χ_e-monochromatic. Hence $f(e) \in \text{RECCOL}_c^d$.

So $e \in \mathrm{COF} \leftrightarrows f(e) \in \mathrm{RECCOL}_c^d$.
So $\mathrm{COF} \prec \mathrm{RECCOL}_c^d$.
Conclusion: RECCOL_c^d is Σ_3-complete. □

In line with our previous experiences, we find the same question for dimension 1 to be silly. We will now prove that RECCOL_c^1 is Π_2-complete. The main reason the complexity of RECCOL_c^1 is relatively high for such a simple question, is that investigating whether a given index represents a coloring is already Π_2(-complete).

Proposition *For any c, RECCOL_c^1 is Π_2-complete.*

Proof First of all, $\mathrm{RECCOL}_c^1 = \mathrm{COL}_c^1$ by proposition 4.1! Hence $\mathrm{RECCOL}_c^1 = \{e \in \mathbb{N} \mid \forall_x \exists_z [T(e, \langle x \rangle, z) \wedge \mathrm{OUTP}(z) < c]\} \in \Pi_2$.

Now, by theorem 3.4, it suffices to show that $\mathrm{TOT} \prec \mathrm{COL}_1^c$. Using the S-$m$-$n$-theorem, we define a total recursive function $f : \mathbb{N} \to \mathbb{N}$ by $\psi_{f(x)} = 0 \circ \psi_x$, that is,

$$\psi_{f(x)}(y) = \begin{cases} 0 & \text{if } \psi_x(y) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $x \in \mathrm{TOT} \leftrightarrows f(x) \in \mathrm{COL}_c^1$, so $\mathrm{TOT} \prec \mathrm{RECCOL}_c^1$. Hence RECCOL_c^1 is Π_2-complete. □

CHAPTER 7

Further

This chapter functions as a coda: it contains some miscellaneous subjects inspired by the previous chapters. First, we have a closer look at limits: we show that to determine whether a given set is a limit is Π_3-complete (section 7.1). We then pose the question whether there exists a suitable weakening of the statement of Ramsey's theorem that does hold in a strongly recursive fashion (section 7.2). Next, we formulate an interesting Ramsey-like statement in the language of complete sets which we leave as an open question (section 7.3). By way of conclusion we state without proof a beautiful generalization of Ramsey's theorem (section 7.4).

7.1 Is a set a limit?

We have used the limit construction of section 5.3 in two of our most compelling negative theorems 5.4 and 6.1. In this section we prove that the problem of determining whether a given index is that of a limit set is Π_3-complete, indicating that the limit construction is indeed very powerful.

If $e \in \mathrm{TOT}_2$, we write $\liminf e$ as an abbreviation of the set $\{x \in \mathbb{N} \mid \exists_K \forall_{k \geq K}[\psi_e(x,k) \neq 0]\}$ and $\limsup e$ for the set $\{x \in \mathbb{N} \mid \forall_K \exists_{k \geq K}[\psi_e(x,k) \neq 0]\}$.

Theorem *The set* $\mathrm{LIM} = \{e \in \mathrm{TOT}_2 \mid \liminf e = \limsup e\}$ *is* Π_3*-complete.*

Proof First of all, since $\liminf e \subset \limsup e$ always holds,

$$\begin{aligned}\mathrm{LIM} &= \{e \in \mathbb{N} \mid e \in \mathrm{TOT}_2 \wedge \forall_x\big[\exists_K \forall_{k \geq K}[\psi_e(x,k) \neq 0] \leftarrow \forall_K \exists_{k \geq K}[\psi_e(x,k) \neq 0]\big]\}\\ &= \{e \in \mathbb{N} \mid \underbrace{\underbrace{e \in \mathrm{TOT}_2}_{\Pi_2} \wedge \forall_x\big[\underbrace{\exists_K \forall_{k \geq K}[\psi_e(x,k) \neq 0]}_{\Sigma_2} \vee \underbrace{\neg\forall_K \exists_{k \geq K}[\psi_e(x,k) \neq 0]}_{\Sigma_2}\big]}_{\Pi_3}\},\end{aligned}$$

so that $\mathrm{LIM} \in \Pi_3$.

By theorem 3.4 it suffices to show that $\mathbb{N}\backslash\mathrm{COF} \prec \mathrm{LIM}$. Define

$$X_{e,k} = \{x \in \mathbb{N} \mid k \text{ is even } \vee \exists_{n \leq k}\forall_{z \leq k}[n > x \wedge \neg T(e, \langle n \rangle, z)]\}.$$

Now, if $\mathbb{N}\backslash W_e$ is infinite, then $\forall_x \exists_{n>x} \forall_z[\neg T(e,\langle n \rangle, z)]$, so $\forall_x \exists_K \forall_{k \geq K}[x \in X_{e,k}]$. Hence certainly $\forall_x[\exists_K \forall_{k \geq K}[x \in X_{e,k}] \vee \exists_K \forall_{k \geq K}[x \notin X_{e,k}]]$ in this case.

But if $\mathbb{N}\backslash W_e$ is finite, we can define $x = \max \mathbb{N}\backslash W_e$. Then $\neg\exists_{n>x}\forall_z[\neg T(e,\langle n \rangle, z)]$, so that $\forall_k \neg\exists_{n \geq k}\forall_{z \geq k}[n > x \wedge \neg T(e, \langle n \rangle, z)]$. So $x \in X_{e,k}$ if and only if k is even. Hence in this case we have $\neg\forall_x[\exists_K \forall_{k \geq K}[x \in X_{e,k}] \vee \exists_K \forall_{k \geq K}[x \notin X_{e,k}]]$.

Since the sequence $X_{e,0}, X_{e,1}, X_{e,2}, \ldots$ is uniformly recursive, we can use the S-m-n-theorem to define a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\psi_{f(e)}(x,k) = \begin{cases} 0 & \text{if } x \notin X_{e,k}, \\ 1 & \text{if } x \in X_{e,k}. \end{cases}$$

Then f is a total recursive function. Moreover,

$$\mathbb{N}\backslash W_e \text{ is infinite } \leftrightarrows \forall_x[\exists_K\forall_{k\geq K}[\psi_{f(e)}(x,k)\neq 0]\vee\exists_K\forall_{k\geq K}[\psi_{f(e)}(x,k)=0]].$$

In other words: $e \in \mathbb{N}\backslash\mathrm{COF} \leftrightarrows f(e)\in\mathrm{LIM}$. Conclusion: LIM is Π_3-complete. □

7.2 Almost monochromatic

Since theorem 5.1 we know that there is no recursive function that turns a coloring into an infinite monochromatic set. But perhaps such a function does exist if we weaken the requirements on its output.

Definition Let d and a coloring χ of $[\mathbb{N}]^d$ be given. We call a set $X \subset \mathbb{N}$ *almost-χ-monochromatic* if there exists a finite $Y \subset X$ such that $X\backslash Y$ is χ-monochromatic.

Perhaps there is a recursive function that turns a coloring into an infinite almost-monochromatic set. First of all it is clear that the existence of the finite set Y in the definition must be interpreted weakly (i.e. classically). After all, there can be no recursive function that makes such a set Y out of a coloring, otherwise one could repair that ‘almost-monochromatic function’ into a ‘monochromatic function’. If you know your mistakes up front, you can prevent them.

Conjecture *For no $c > 1$ and $d \geq 1$ does there exist a total recursive function $f : \mathrm{COL}_c^d \to \mathrm{INFSET}$ such that $\mathrm{Set}(f(e))$ is almost-ψ_e-monochromatic for every $e \in \mathrm{COL}_c^d$.*

The problem lies in dimension one; the cases $d > 1$ are easily derived from $d = 1$. For suppose that f were a function as described in the conjecture. Then:

$$\forall_{e\in\mathrm{COL}_c^d}\exists_N\exists_{r\in[c]}\forall_{n_1,\dots,n_d\in\mathrm{Set}(f(e))\backslash[N]}\big[n_1<\cdots<n_d \;\to\; \psi_e(\{n_1,\dots,n_d\})=r\big].$$

Now define a function $g : \mathrm{COL}_c^1 \to \mathrm{INFSET}$ by assigning to $e \in \mathrm{COL}_c^1$ the value $f(e')$ for an $e' \in \mathrm{COL}_c^d$ such that

$$\psi_{e'}(n_1,\dots,n_d) = \psi_e(\min\{n_1,\dots,n_d\}) \text{ for every } \{n_1,\dots,n_d\}\in[\mathbb{N}]^d.$$

Then g is recursive, and for every $e \in \mathrm{SET}$ the set $\mathrm{Set}(g(e))$ is an almost-ψ_e-monochromatic set. But that would contradict the conjecture for dimension one.

To prove the case $d = 1$, a recursive coloring $e \in \mathrm{COL}_c^1$ must somehow be constructed such that we can draw a contradictory conclusion from the almost-ψ_e-monochromaticity of $\mathrm{Set}(f(e))$. The easiest contradiction is probably with the hierarchy theorem 3.3. However, it seems hard to use the almost-monochromaticity to draw conclusions without first ‘converting’ to real monochromaticity, thereby moving from Δ_1 to Δ_2 and losing the contradiction.

The status of the conjecture is highly interesting. Somewhere along the transition of a classical setting (theorem 2.1) to a recursive one (theorem 5.4), the statement of Ramsey’s theorem lost its truth. Solving this conjecture would help to pinpoint the ‘point of no return’, as the conjecture mixes elements of the classical setting (the existential quantor for *almost*) into a recursive setting (the function f must be recursive). Unfortunately, we must leave this problem to any second edition of this thesis.

7.3 Complete version

In the simplest case, with two colors and in dimension one, Ramsey's theorem is about a large enough (infinite) set that meets some given structural requirements (monochromatic with respect to a given coloring). If we transpose that situation to the arithmetical hierarchy, translating 'large enough' by 'complete' and substituting 'intersection with a given recursive set' for the given structural requirements, we arrive at the following interesting question.

Question *Suppose $A \subset \mathbb{N}$ is Σ_1-complete and $C \subset \mathbb{N}$ is recursive. Is either $A \cap C$ or $A \cap (\mathbb{N} \backslash C)$ still Σ_1-complete?*

Intuitively, the answer seems to be affirmative. If C is finite, then $A \cap C$ is obviously Σ_1-complete since A itself reduces to it. Likewise, if C is co-finite, then $A \cap (\mathbb{N} \backslash C)$ is Σ_1-complete. Even if C and $\mathbb{N} \backslash C$ are more balanced, the statement seems to be true. For example, the (Σ_1-complete) halting set H reduces to $\{n \in H \mid n \text{ is even } \}$.

But a general proof, or, for that matter, a counterexample, seems quite hard. Therefore, we leave the more general question open to further investigation: if A is Σ_n- or Π_n-complete and C is beneath A in the arithmetical hierarchy, is either $A \cap C$ or $A \cap (\mathbb{N} \backslash C)$ still Σ_n- or Π_n-complete, respectively?

7.4 Topological version

In a sense, Ramsey's theorem is rather a countably infinite number of theorems, one for each dimension. There is a beautiful generalization, known as the *clopen Ramsey theorem*, that is independent of the dimension [Fra]. We write $\mathcal{N}$ for the set of all functions $\mathbb{N} \to \mathbb{N}$, equipped with the product topology.

For an infinite set $X \subset \mathbb{N}$ we define $\alpha_X \in \mathcal{N}$ by $\alpha_X(n) = \min X \backslash \{\alpha(0), \ldots, \alpha(n-1)\}$. Then $\alpha_X \in \mathcal{N}$ is strictly increasing and $X = \mathrm{Ran}(\alpha_X)$.

Theorem *For any continuous coloring of $\mathcal{N}$, there exists an infinite $X \subset \mathbb{N}$ such that for any infinite $Y \subset X$, the color of α_Y remains the same as that of α_X.*

Let us show that this generalizes theorem 2.1. Let χ be a c-coloring of $[\mathbb{N}]^d$. Define a c-coloring χ' of $\mathcal{N}$ by assigning to f the χ-color of the first d values of f. Then χ' is continuous. Hence there is an infinite $X \subset \mathbb{N}$ such that $\chi'(\alpha_X) = \chi'(\alpha_Y)$ for every infinite $Y \subset X$. Then X is infinite and χ-monochromatic.

A bonus of this generalized version is that the dimension need not be uniformly bounded: χ' need to depend on only the first d values of f as above.

Classical proofs of the clopen Ramsey theorem involve cardinal numbers, which we will not go into. This leaves an entire field of questions untouched. Are there relations similar to those of chapters 4–6 in the Borel hierarchy (see [Mos]) instead of the arithmetical hierarchy?

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