Quantifiers for quantum logic

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Abstract

We consider categorical logic on the category of Hilbert spaces. More generally, in fact, any pre-Hilbert category suffices. We characterise closed subobjects, and prove that they form orthomodular lattices. This shows that quantum logic is just an incarnation of categorical logic, enabling us to establish an existential quantifier for quantum logic, and conclude that there cannot be a universal quantifier.

1 Introduction

Quantum logic is the study of closed subspaces of a Hilbert space [BV]. Intriguingly, this 'logic' is not distributive, but only satisfies the weaker axiom of orthomodularity. One of the shortcomings that has kept it from wide adoption is the lack of quantifiers. In fact, it has been called a 'non-logic' [Abr].

On the other hand, categorical logic [LS] can be seen as a unified framework for any kind of logic that deserves the name. It is concerned with interpreting (syntactical) logical formulae in categories with enough structure to accommodate this. An important part of it is the study of subobjects of a given object in the category at hand. Perhaps its most gratifying feature is that it gives a canonical prescription of what quantifiers should be.

The aim of this paper is to show that quantum logic is just an incarnation of categorical logic in categories like that of Hilbert spaces. In particular, we will establish an existential quantifier, and conclude that there cannot be a universal quantifier.

Section 2 first abstracts the properties of the category of Hilbert spaces that we need. This results in an axiomatisation of *(pre-)Hilbert categories* greatly resembling that of monoidal Abelian categories. In fact, any (pre-)Hilbert category embeds into the category of (pre-)Hilbert spaces itself [Heu]. Next, Section 3 starts the investigation of subobjects in Hilbert categories. It turns out that the natural objects of study are not the subobjects, but the *closed subobjects* or *†-subobjects*. Section 4 then derives a functor that behaves as an existential quantifier according to categorical logic. Section 5 studies the emergent concept of orthogonality in Hilbert categories. First, it proves that †-subobjects form orthomodular lattices. Second, it exhibits a tight connection between adjoint morphisms in the base category and adjoint functors between the lattices of subobjects, the latter being important in connection to quantifiers.

Related work

The present article should not be confused with the 'categorical quantum logic' of [Dun]. That work develops a type theory. Of course this is related: "every logic is a logic over a type theory" [Jac]. This paper develops the logic over 'the type theory of Hilbert spaces'.

This paper also differs from [Har], in that the aim is explicitly a categorical logic. Another difference is that that paper restricts to those projections that have an orthocomplement, whereas we derive orthomodularity from prior assumptions (namely †-kernels).

Acknowledgement

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2 Pre-Hilbert categories

This section introduces the categories in which our study takes place, somewhat concisely. For more information we refer to [Heu].

A functor $\dagger : \mathbf{H}^{\mathrm{op}} \to \mathbf{H}$ with $X^\dagger = X$ on objects and $f^{\dagger\dagger} = f$ on morphisms is called a †-*functor*; the pair $(\mathbf{H}, \dagger)$ is then called a †-*category*. Such categories are automatically isomorphic to their opposite, and the †-functor witnesses this self-duality. We can consider coherence of the †-functor with all sorts of structures. A morphism m in such a category that satisfies $m^\dagger m = \mathrm{id}$ is called a †-*mono* and denoted $\triangleright\!\!\longrightarrow$. Likewise, e is a †-*epi*, denoted $\longrightarrow\!\!\triangleright$, when $ee^\dagger = \mathrm{id}$. A morphism is called a †-*iso* when it is both †-epi and †-mono. Similarly, a biproduct on such a category is called a †-*biproduct* when $\pi^\dagger = \kappa$, where π is a projection and κ an injection. This is equivalent to demanding $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$. Also, an equaliser is called a †-*equaliser* when it is a †-mono, and a kernel is called a †-*kernel* when it is a †-mono. Finally, a †-category $\mathbf{H}$ is called †-*monoidal* when it is equipped with monoidal structure $(\otimes, C)$ that cooperates with the †-functor, in the sense that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and the coherence isomorphisms are †-isomorphisms.

Definition 1 *A category is called a* pre-Hilbert category *when*

- *it has a †-functor;*
- *it has finite †-biproducts;*
- *it has (finite) †-equalisers;*
- *every †-mono is a †-kernel; and*

- *it is symmetric †-monoidal.*

Notice that a Hilbert category is self-dual (by the †-functor), and therefore that it automatically has all finite colimits, too.

The category **preHilb** itself is a pre-Hilbert category whose monoidal unit is a simple generator, and so are its full subcategories **Hilb**, and **fdHilb** of finite-dimensional Hilbert spaces. Also, if **C** is a small category and **H** a pre-Hilbert category, then $[\mathbf{C}, \mathbf{H}]$ is again a pre-Hilbert category. Working in pre-Hilbert categories can be thought of as 'natural' or 'baseless' (pre-)Hilbert space theory.

3 Subobjects

This section characterises closed subobjects categorically. But let us start with some easy properties of †-mono's.

Lemma 2 *In any †-category:*

(a) *A †-mono which is epi is a †-iso.*

(b) *The composite of †-epi's is again a †-epi.*

(c) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are such that both* gf *and* f *are †-epi, so is* g*.*

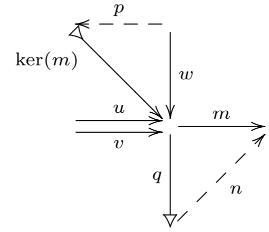
(d) *If* m *and* n *are †-monos, and* f *is an iso with* $nf = m$*, then* f *is a †-iso.*

Proof For (a), notice that $ff^\dagger = \mathrm{id}$ implies $ff^\dagger f = f$, from which $f^\dagger f = \mathrm{id}$ follows from the assumption that f is epi. For (b): $gf(gf)^\dagger = gff^\dagger g^\dagger = g^\dagger g = \mathrm{id}$. And for (c): $gg^\dagger = gff^\dagger g = gf(gf)^\dagger = \mathrm{id}$. Finally, consider (d). If f is iso, in particular it is epi. If both nf and n are †-mono, then so is f, by (c). Hence by (a), f is †-iso. □

From now on, we work in an arbitrary pre-Hilbert category **H**.

Lemma 3 *A morphism* m *is mono iff* $\ker(m) = 0$*. Consequently, if* $mf = 0$ *implies* $f = 0$ *for all* f*, then* m *is mono.*

Proof Suppose $\ker(m) = 0$. Let u, v satisfy $mu = mv$. Put q to be the †-coequaliser of u and v. Since q is a †-epi, $q = \mathrm{coker}(w)$ for some w. As $mu = mv$, m factors through q as $m = nq$. Then $mw = nqw = n0 = 0$, so w factors through $\ker(m)$ as $w = \ker(m) \circ p$ for some p. But since $\ker(m) = 0$, $w = 0$. So q is a †-iso, and in particular mono. Hence, from $qu = qv$ follows $u = v$. Thus m is mono.



Conversely, if m is mono, it follows from $m \circ \ker(m) = 0 = m \circ 0$ that $\ker(m) = 0$.

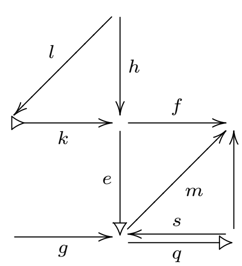
If $f = 0$ whenever $mf = 0$, then $\ker(m) = 0$, so that m is mono. □

3.1 Factorisation

This subsection proves that any morphism $f : X \to Y$ in a pre-Hilbert category can be factorised as an epi $e : X \to I$ followed by a †-mono $m : I \to Y$. (In **Hilb**, this is very easily proved concretely: e is simply the restriction of f to I, the closure of its range, and m is the isometric inclusion of I into Y.) Recall that since a pre-Hilbert category has †-kernels, it automatically also has †-cokernels by $\mathrm{coker}(f) = \ker(f^\dagger)^\dagger$.

Lemma 4 *Any pre-Hilbert category has a factorisation system consisting of mono's and †-epi's. The factorisation is unique up to a unique †-iso. Consequently, every †-epi is a †-cokernel of its †-kernel.*

Proof Let a morphism f be given. Put $k = \ker(f)$ and $e = \mathrm{coker}(k)$. Since $fk = 0$ (as $k = \ker(f)$), f factors through $e(= \mathrm{coker}(k))$ as $f = me$.



We have to show that m is mono. Let g be such that $mg = 0$. By Lemma 3 it suffices to show that $g = 0$. Since $mg = 0$, m factors through $q = \mathrm{coker}(g)$ as $m = rq$. Now qe is a †-epi, being the composite of two †-epi's. So $qe = \mathrm{coker}(h)$ for some h. Since $fh = rqeh = r0 = 0$, h factors through $k(= \ker(f))$ as $h = kl$. Finally $eh = ekl = 0l = 0$, so e factors through $qe = \mathrm{coker}(h)$ as $q = sqe$. But since e is a (†-)epi, this means $sq = \mathrm{id}$, whence q is mono. It follows from $qg = 0$ that $g = 0$, and the factorisation is established.

Since †-epi's are regular epi's, and hence strong epi's, functoriality of the factorisation follows from [Bor, 4.4.5]. By Lemma 2d, the factorisation is unique up to a †-iso.

Finally, suppose that f is a †-epi. Then both the above $f = m \circ e$ and $f = f \circ \mathrm{id}$ are mono-†-epi factorisations of f. Hence $f = e$ up to the unique mediating †-iso m, showing that $f = \mathrm{coker}(\ker(f))$. □

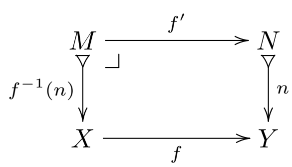
We just showed that any pre-Hilbert category has a factorisation system consisting of mono's and †-epi's. Equivalently, it has a factorisation system of epi's and †-mono's. Indeed, if we can factor $f^\dagger$ as an †-epi followed by a mono, then taking the daggers of those, we find that $f^{\dagger\dagger} = f$ factors as an epi followed by a †-mono. The combination of both factorisations yields that every morphism can be written as a †-epi, followed by a monic epimorphism, followed by a †-mono; this can be thought of generalising *polar decomposition*.

3.2 Closed subobjects, pullbacks

A *subobject* of an object X in a †-category is an equivalence class of mono's $m : M \rightarrowtail X$, where m is equivalent to $n : N \rightarrowtail X$ if there is an isomorphism $f : M \to N$ satisfying $nf = m$. The class of subobjects of X is denoted $\mathrm{Sub}(X)$. It is partially ordered by $M \leq N$ iff there is a morphism $f : M \to N$ with $nf = m$. It also has a largest element, represented by $\mathrm{id}_X : X \to X$. Because a pre-Hilbert category has pullbacks, $\mathrm{Sub}(X)$ is in fact a meet-semilattice[1], the meet of M and N being represented by the pullback of m and n. Moreover, for each $f : X \to Y$, pullback along f induces a meet-preserving map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. Thus we have a functor $\mathrm{Sub} : \mathbf{H}^{\mathrm{op}} \to \mathbf{MeetSLat}$, the *inverse image functor*.

A †-*subobject* is a subobject that can be represented by a †-mono. We write $\mathrm{ClSub}(X)$ for the class of †-subobjects of X. It inherits the partial ordering of $\mathrm{Sub}(X)$. It can be characterised precisely when a subobject m is a †-subobject, namely when there is an isomorphism φ such that $m^\dagger m = \varphi^\dagger \varphi$ [Sel, 5.6].

Lemma 5 *†-subobjects are stable under pullbacks. Explicitly, given a †-mono n and map f one obtains a pullback*



as $f^{-1}(n) = \ker(\mathrm{coker}(n) \circ f)$.

Proof For convenience, write $m = f^{-1}(n) = \ker(\mathrm{coker}(n) \circ f)$. By construction, $\mathrm{coker}(n) \circ f \circ m = 0$, so that $f \circ m$ factors through $\ker(\mathrm{coker}(n)) = n$, say via $f' \colon M \to N$ with $n \circ f' = f \circ m$, as in the diagram. This yields a pullback: if $a \colon Z \to X$ and $b \colon Z \to N$ satisfy $f \circ a = n \circ b$, then $\mathrm{coker}(n) \circ f \circ a = \mathrm{coker}(n) \circ n \circ f' = 0 \circ f' = 0$, so that there is a unique map $c \colon Z \to M$ with $m \circ c = a$. Then $f' \circ c = b$ because n is monic. □

Hence every morphism $f : X \to Y$ induces a meet-preserving map $f^{-1} : \mathrm{ClSub}(Y) \to \mathrm{ClSub}(X)$. Thus we have a functor

$$\mathrm{ClSub} : \mathbf{H}^{\mathrm{op}} \to \mathbf{MeetSLat},$$

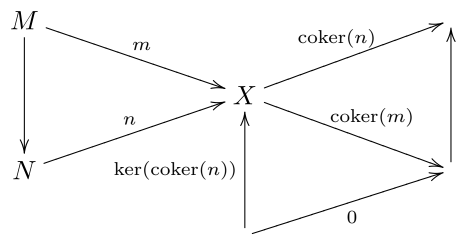
that we also call the *inverse image functor* with abuse of terminology.

[1] We disregard size issues here. A †-category is called †-*well-powered* if $\mathrm{ClSub}(X)$ is a set for all objects X in it. Since $\mathrm{ClSub}(X)$ for $X \in \mathbf{Hilb}$ is the set of closed subspaces of X, $\mathbf{Hilb}$ is †-well-powered.

Recall that a *universal closure operation* [Bor, 5.7.1] consists in giving for every $m \in \mathrm{Sub}(X)$ a $\overline{m} \in \mathrm{Sub}(X)$, satisfying (i) $m \leq \overline{m}$, (ii) $m \leq n \Rightarrow \overline{m} \leq \overline{n}$, (iii) $\overline{\overline{m}} = \overline{m}$, and (iv) $f^{-1}(\overline{m}) = \overline{f^{-1}(m)}$.

Lemma 6 *$m \mapsto \ker(\mathrm{coker}(m))$ is a universal closure operation.*

Proof For (i): $\mathrm{coker}(m) \circ m = 0$, so $m \leq \ker(\mathrm{coker}(m))$. For (ii): if $m \leq n$, then $\mathrm{coker}(m) \circ \ker(\mathrm{coker}(m)) = 0$,



so $\ker(\mathrm{coker}(m)) \leq \ker(\mathrm{coker}(n))$. For (iii): since $\ker(\mathrm{coker}(m)) \in \mathrm{ClSub}(X)$, we have $\ker(\mathrm{coker}(\ker(\mathrm{coker}(m)))) = \ker(\mathrm{coker}(m))$ by Lemma 4. Finally, (iv) is just Lemma 5. $\square$

Lemma 7 *There is a reflection* $\mathrm{Sub}(X) \underset{\perp}{\overset{\ker(\mathrm{coker}(-))}{\rightleftarrows}} \mathrm{ClSub}(X)$.

Proof We have to prove that $\ker(\mathrm{coker}(m)) \leq n$ iff $m \leq n$ for a mono m and a †-mono n. By (i) of Lemma 6 we have $m \leq \ker(\mathrm{coker}(m))$, proving one direction. The converse direction is just (ii) of Lemma 6. $\square$

The previous lemma could be interpreted as a moral justification for studying the (replete) semilattice of closed subobjects instead of that of subobjects.

3.3 Projections

Instead of closed subobjects, it turns out we can also consider projections. A *projection* on X is a morphism $p : X \to X$ satisfying $p \circ p = p = p^\dagger$. We define $\mathrm{Proj}(X)$ as the set of all projections on X. It is partially ordered by defining $p \sqsubseteq q$ iff $p \circ q = p$.

Proposition 8 *There is an order isomorphism* $\mathrm{ClSub}(X) \cong \mathrm{Proj}(X)$.

Proof Any closed subobject m yields a projection $mm^\dagger$. Conversely, any projection p gives a closed subobject $\mathrm{Im}(p)$.

Let us verify that these maps are each others inverses. Starting with a closed subobject represented by m, we end up with $\mathrm{Im}(mm^\dagger)$. Since m is †-mono and $m^\dagger$ is †-epi, this is already a factorisation in the sense of Lemma 4, and hence $\mathrm{Im}(mm^\dagger) = m$ as closed subobjects. Conversely, a projection p maps to $ii^\dagger$, where p factors as $p = ie$ for an epi $e : X \to I$ and †-mono $i = \mathrm{Im}(p)$. By functoriality of the factorisation it follows from $pp = p$ that $pi = i$. Now

$$i = pi = p^\dagger i = (ie)^\dagger i = e^\dagger i^\dagger i = e^\dagger,$$

so indeed $ii^\dagger = ie = p$.

Finally let us consider the order. If $m \leq n$ as subobjects, say $m = n\varphi$ for a †-mono φ, then $mm^\dagger nn^\dagger = n\varphi\varphi^\dagger n^\dagger nn^\dagger = nn^\dagger nn^\dagger = nn^\dagger$, so indeed $mm^\dagger \sqsubseteq nn^\dagger$. Conversely, if $p \sqsubseteq q$, then $pq = p$, whence $\mathrm{Im}(pq) = \mathrm{Im}(p)$, so that indeed $\mathrm{Im}(p) \leq \mathrm{Im}(q)$ by functoriality of the factorisation. □

Consequently, every result we derive about the partial order of closed subobjects holds for the projections and vice versa.

4 Existential quantifier

This section establishes an existential quantifier, *i.e.* a left adjoint to the inverse image functor that satisfies the Beck-Chevalley condition.

Proposition 9 $\mathrm{ClSub}(X)$ *is a lattice.*

Proof Since we already know that $\mathrm{ClSub}(X)$ is a meet-semilattice, it suffices to show that it has joins and a least element. Joins follow from *e.g.* [Bor, 4.2.6]. Explicitly, $M \vee N \xrightarrow{m \vee n} X$ is given by $\mathrm{Im}(s)$, where $s = [m, n] : M \oplus N \to X$. The closed smallest subobject, the bottom element of $\mathrm{ClSub}(X)$, is given by $0 \xrightarrow{0} X$. □

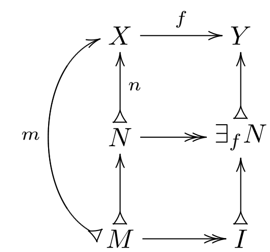
The †-mono $m : M \rightarrowtail Y$ arising in the factorisation of a morphism $f : X \to Y$ of $\mathbf{H}$ is called the *(direct) image* of f, denoted $\mathrm{Im}(f)$. Notice that $\mathrm{Im}(f)$ defines a unique †-subobject, although the representing †-mono is only unique up to a †-iso. This †-subobject is denoted $\exists_f$. More generally, for $n : N \rightarrowtail X$ in $\mathrm{ClSub}(X)$, we define

$$\exists_f(N) = \mathrm{Im}(fn),$$

which gives a well-defined map $\exists_f : \mathrm{ClSub}(X) \to \mathrm{ClSub}(Y)$ for any morphism $f : X \to Y$ of $\mathbf{H}$.

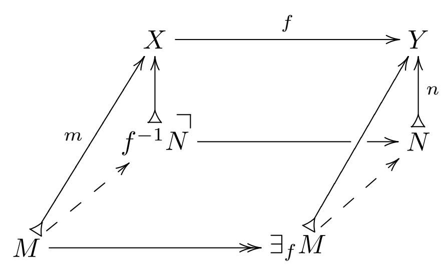
Theorem 10 *Let* $f : X \to Y$ *be a morphism of* $\mathbf{H}$. *The map* $\exists_f : \mathrm{ClSub}(X) \to \mathrm{ClSub}(Y)$ *is monotone and left-adjoint to* $f^{-1} : \mathrm{ClSub}(Y) \to \mathrm{ClSub}(X)$. *If* $g : Y \to Z$ *is another morphism then* $\exists_g \circ \exists_f = \exists_{g \circ f} : \mathrm{ClSub}(X) \to \mathrm{ClSub}(Z)$. *Also* $\exists_{\mathrm{id}} = \mathrm{id}$.

Proof We follow the proof of [But, Lemma 2.5]. For monotonicity of $\exists_f$ let $M \leq N$ in $\mathrm{ClSub}(X)$. First factorise n and then $M \to \exists_f N$ to get the following diagram.



Now $M \twoheadrightarrow I \rightarrowtail Y$ is an epi-†-mono factorisation of fm, so I represents $\exists_f M$, and $\exists_f M \leq \exists_f N$.

To show the adjunction, let $M \in \mathrm{ClSub}(X)$ and $N \in \mathrm{ClSub}(Y)$, and consider the solid arrows in the following diagram.

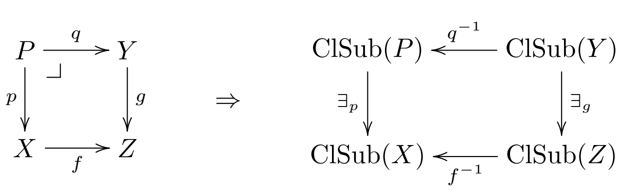


If $\exists_f M \leq N$ then the right dashed map $\exists_f M \to N$ exists and the outer square commutes. Hence, since $f^{-1}N$ is a pullback, the left dashed map $M \to f^{-1}N$ exists, and $M \leq f^{-1}N$. Conversely, if $M \leq f^{-1}N$, factorise the map $M \to N$ to get the image of M under f. In particular, this image then factors through N, whence $\exists_f M \leq N$.

Finally, the identity $\exists_g \circ \exists_f = \exists_{g \circ f}$ just states how left adjoints compose. □

4.1 The Beck-Chevalley condition

Recall the *Beck-Chevalley condition*: if the left square below is a pullback, then the right one must commute.



(BC)

It ensures that the semantics of the existential quantifier is sound with respect to substitution. To show that our $\exists_f$ satisfies (BC), we will assume that the monoidal unit C of our pre-Hilbert category $\mathbf{H}$ is a simple generator. Recall that an object C is called a *generator* when $fx = gx$ for all $x : C \to X$ implies $f = g : X \to Y$. It is called *simple* when $\mathrm{Sub}(C) = \{0, C\}$. In this case, [Heu, Theorem 4.6] shows that **H** is enriched over Abelian groups, so that we can talk of adding and subtracting morphisms.

Lemma 11 *In a pre-Hilbert category whose monoidal unit is a simple generator, epi's are stable under pullback.*

Proof The proof of [Bor, Proposition 1.7.6] works verbatim. □

The previous lemma entails that **H** is a *regular category*, and hence that all results of [But] apply. Thus, in such a category **H** one can soundly interpret regular logic, in particular the existential quantifier.

Theorem 12 *In a pre-Hilbert category whose monoidal unit is a simple generator,* (BC) *holds.*

Proof The proof of [But, Lemma 2.9] works verbatim. □

Also the *Frobenius identity* holds. Let $f : X \to Y$ be a morphism of **Hilb**. Let $M \in \mathrm{ClSub}(X)$ and $N \in \mathrm{ClSub}(Y)$. Then $\exists_f(M \wedge f^{-1}N) = \exists_f M \wedge N$ as †-subobjects of Y. For a proof, we refer to [But, Lemma 2.6].

5 Orthogonality

We will now recover the orthogonal subspace construction from the †-functor in any pre-Hilbert category. The idea is to mimick the fact that $\ker(f)^\perp = \mathrm{Im}(f^\dagger)$ in **Hilb**.

Proposition 13 *There is an involutive functor* $(-)^\perp : \mathrm{ClSub}(X)^{\mathrm{op}} \to \mathrm{ClSub}(X)$ *determined by* $m^\perp = \ker(m^\dagger)$ *for* $m \in \mathrm{ClSub}(X)$.

Proof To show that the above definition extends functorially, let $m, n \in \mathrm{ClSub}(X)$ be such that $m \leq n$. Say that m factors through n by $m = ni$ for $i : M \to N$. Then

$$m^\dagger \circ \ker(n^\dagger) = i^\dagger \circ n^\dagger \circ \ker(n^\dagger) = i^\dagger \circ 0 = 0.$$

Hence $\ker(n^\dagger)$ factors through $\ker(m^\dagger)$, that is, $n^\perp \leq m^\perp$.

We finish the proof by showing that $\perp$ is involutive:

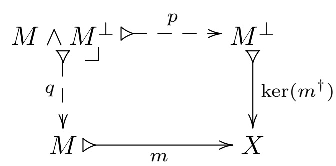
$$m^{\perp\perp} = (\ker(m^\dagger))^\perp = \ker(\ker(m^\dagger)^\dagger) = \ker(\mathrm{coker}(m)) = m.$$

Here, the last equation follows from Lemma 4. □

The functor $(-)^\perp$ cooperates with $\wedge$ and $\vee$ as expected.

Lemma 14 $\mathrm{ClSub}(X)$ *is an orthocomplemented lattice, that is,* $m \wedge m^\perp = 0$ *and* $m \vee m^\perp = 1$ *for all* $m \in \mathrm{ClSub}(X)$*. (A forteriori, the cotuple* $[m, m^\perp]$ *is a †-iso.)*

Proof Recall that $m \wedge m^{\perp}$ is defined as the †-pullback



Because m is a †-mono, we have $q = m^{\dagger} \circ m \circ q = m^{\dagger} \circ \ker(m^{\dagger}) \circ p = 0 \circ p = 0$. Hence $m \wedge m^{\perp} = m \circ q = m \circ 0 = 0$.

To prove the second claim, let f satisfy $f \circ [m, m^{\perp}] = 0$. Then $f \circ m = 0$, so f factors through $\mathrm{coker}(m)$ as $f = g \circ \mathrm{coker}(m)$. Also $f \circ m^{\perp} = 0$, so

$$g = g \circ \ker(m^{\dagger})^{\dagger} \circ \ker(m^{\dagger}) = f \circ \ker(m^{\dagger}) = 0,$$

whence $f = 0$. So, by Lemma 3, $[m, m^{\perp}]$ is epi. Hence $[m, m^{\perp}]$ factors as $\mathrm{id} \circ [m, m^{\perp}]$, but also as $(m \vee m^{\perp}) \circ p$. So $m \vee m^{\perp}$ must be a †-iso. That is, $m \vee m^{\perp} = 1$.

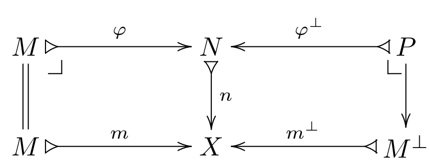
Let us prove that $[m, m^{\perp}]$ is also a †-mono, and hence even a †-iso:

$$\begin{aligned}
[m, m^{\perp}]^{\dagger} \circ [m, m^{\perp}] &= \langle m^{\dagger}, \ker(m^{\dagger})^{\dagger} \rangle \circ [m, \ker(m^{\dagger})] \\
&= \begin{pmatrix} m^{\dagger} \circ m & m^{\dagger} \circ \ker(m^{\dagger}) \\ \ker(m^{\dagger})^{\dagger} \circ m & \ker(m^{\dagger})^{\dagger} \circ \ker(m^{\dagger}) \end{pmatrix} \\
&= \mathrm{id}_{M \oplus M^{\perp}}.
\end{aligned}$$ □

However, $(-)^{\perp}$ has poor 'substitution properties', as it does not commute with pullbacks. For a counterexample in **Hilb**, let $X = \mathbb{C}^2$, $Y = \mathbb{C}$, $f = \pi : X \to Y : (x, y) \mapsto x$ and $m = 0 : 0 \to Y$. Then $f^{-1}(m^{\perp}) = \mathbb{C}^2$, but $(f^{-1}(m))^{\perp} = \{(x, 0) \mid x \in \mathbb{C}\}$.

In spite of this, a special case of "$(-)^{\perp}$ is stable under pullbacks" still holds: we now recover orthomodularity of $\mathrm{ClSub}(X)$ using the previous lemma.

Theorem 15 $\mathrm{ClSub}(X)$ *is an orthomodular lattice: for* $m \leq n \in \mathrm{ClSub}(X)$*, say via* φ *with* $n \circ \varphi = m$*, one has the following pullbacks.*



This means that $m \vee (m^{\perp} \wedge n) = n$.

Proof The square on the left is obviously a pullback. For the one on the right we use a simple calculation, following Lemma 5:

$$\begin{aligned} n^{-1}(m^{\perp}) &= \ker(\mathrm{coker}(m^{\perp}) \circ n) \\ &= \ker(\mathrm{coker}(\ker(m^{\dagger})) \circ n) \\ &= \ker(m^{\dagger} \circ n) \qquad \text{since } m^{\dagger} \text{ is a cokernel} \\ &\overset{(*)}{=} \ker(\varphi^{\dagger}) \\ &= \varphi^{\perp}, \end{aligned}$$

where the marked equation holds because $n \circ \varphi = m$, so that $\varphi = n^{\dagger} \circ n \circ \varphi = n^{\dagger} \circ m$ and thus $\varphi^{\dagger} = m^{\dagger} \circ n$. Then:

$$m \vee (m^{\perp} \wedge n) \;=\; (n \circ \varphi) \vee (n \circ \varphi^{\perp}) \;\overset{(*)}{=}\; n \circ (\varphi \vee \varphi^{\perp}) \;=\; n \circ \mathrm{id} \;=\; n.$$

The marked equation holds because $n \circ (-)$ preserves joins, since it is a left adjoint: $n \circ k \leq m$ iff $k \leq n^{-1}(m)$, for †-subobjects k, m. □

Corollary 16 *There cannot be right adjoints $f^{-1} \dashv \forall_f$ for all morphisms f of* **H***, that satisfy the Beck-Chevalley condition.*

Proof If there would be, then $\wedge$ would have a right adjoint in every $\mathrm{ClSub}(X)$ [AB, 3.4.16]. That is, there would be an implication. But the prime example **Hilb** shows that $\mathrm{ClSub}(X)$ is in general not a Heyting algebra. □

Lemma 17 *The functor $\perp\colon \mathrm{ClSub}(X)^{\mathrm{op}} \to \mathrm{ClSub}(X)$ is an equivalence of categories. In particular, it is both left and right adjoint to its opposite $\perp^{\mathrm{op}}$: $\mathrm{ClSub}(X) \to \mathrm{ClSub}(X)^{\mathrm{op}}$.*

Proof This means precisely that $m^{\perp} \leq n$ iff $n^{\perp} \leq m$, which holds since $\perp$ is involutive. □

The following theorem, inspired by [Pal], provides a connection between adjoint morphisms in a pre-Hilbert category and adjoint functors between lattices of †-subobjects. It explicates the relationship between $\exists_f$ and $\exists_{f^{\dagger}}$.

Theorem 18 *For a morphism $f : X \to Y$, define*

$$f^{\perp} = \perp_Y \circ \exists_f^{\mathrm{op}} : \mathrm{ClSub}(X)^{\mathrm{op}} \to \mathrm{ClSub}(Y).$$

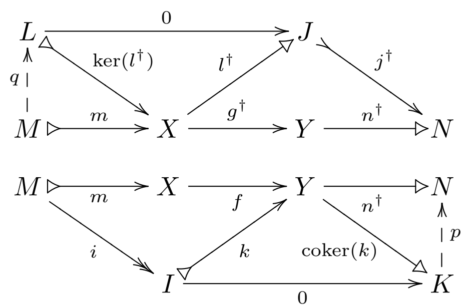
Then

$$(f^{\perp})^{\mathrm{op}} \dashv (f^{\dagger})^{\perp}.$$

Proof In general, for $g : Y \to X$, the adjunction $(f^{\perp})^{\mathrm{op}} \dashv g^{\perp}$ means that for $M \in \mathrm{ClSub}(X)$ and $N \in \mathrm{ClSub}(Y)$,

$$\mathrm{ClSub}(Y)^{\mathrm{op}}(\perp_Y^{\mathrm{op}} \circ \exists_f(M), N) \cong \mathrm{ClSub}(X)(M, \perp_X \circ \exists_{f^{\dagger}}^{\mathrm{op}}(N)).$$

That is, $n \le \ker(\mathrm{Im}(fm)^\dagger)$ iff $m \le \ker(\mathrm{Im}(gn)^\dagger)$. That means that in



(1)

we must show that there is a p making the lower diagram commute iff there is a q making the upper one commute, for the special case $g = f^\dagger$. So, let such a q be given. Then

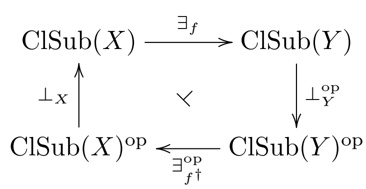
$$n^\dagger \circ k \circ i = n^\dagger \circ f \circ m = j \circ l^\dagger \circ m = j \circ l^\dagger \circ \ker(l^\dagger) \circ q = j \circ 0 \circ q = 0 = 0 \circ i,$$

and since i is epi, $n^\dagger k = 0$. Hence $n^\dagger$ factors through $\mathrm{coker}(k)$ via some p. Conversely, given p, we have

$$j^\dagger \circ l^\dagger \circ m = n^\dagger \circ f \circ m = n^\dagger \circ k \circ i = p \circ \mathrm{coker}(k) \circ k \circ i = p \circ 0 \circ i = 0 = j^\dagger \circ 0,$$

so since $j^\dagger$ is mono, $l^\dagger m = 0$. Hence m factors through $\ker(l^\dagger)$ via some q. $\square$

In a diagram, the adjunction of the previous theorem is the following.



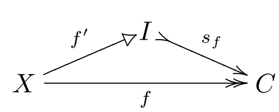
A converse to this theorem needs some preparation, and the assumption that the monoidal unit is a simple generator.

Lemma 19 *Let C be a simple object in a pre-Hilbert category. If $f, g : X \to C$ satisfy $\ker(f) \le \ker(g)$, then $g = sf$ for some $s : C \to C$. Unless $f = 0$, this s is unique.*

Proof Consider $\exists_f X \in \mathrm{ClSub}(C)$. Either $\exists_f X = 0$, or $\exists_f X$ is an iso and hence a †-iso since it is a †-mono.

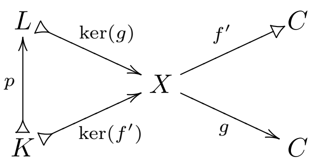
If $\exists_f X = 0$, then $f = 0$. So $\ker(f)$ is a †-iso, and since $\ker(f) \le \ker(g)$, also $\ker(g)$ is †-iso, whence $g = 0$. Thus $g = 0f$.

If $\exists_f X$ is a †-iso, in particular it is epi, and so is f. It can be factorised as a †-epi f' followed by a mono s_f.



Now either $s_f = 0$ or s_f is iso. If $s_f = 0$ then $\exists_f X = 0$ and hence $f = 0$, so that we are done by $g = 0f$. Hence we may assume s_f iso.

Since $\ker(f') \leq \ker(f) \leq \ker(g)$ we are thus left with the following situation.



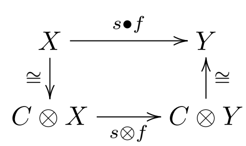
Now $f' = \mathrm{coker}(\ker(f'))$, and

$$g \circ \ker(f') = g \circ \ker(g) \circ p = 0 \circ p = 0.$$

Hence there is a unique s' such that $g = s' \circ f'$. Finally, putting $s = s' s_f^{-1}$ satisfies $g = s' f' = s' s_f^{-1} f = sf$. □

In a monoidal category, morphisms $s : C \to C$ play the role of scalars, and multiplication with them is natural. As mentioned before, if C is a simple generator, then the scalars comprise an involutive field [Heu, Theorem 4.6]. The following lemma summarises some well-known (and easily proved) results.

Lemma 20 *Let* $\mathbf{H}$ *be a monoidal category. Then* $\mathbf{H}(C, C)$ *is an involutive semiring that acts on* $\mathbf{H}$ *by* scalar multiplication *as follows: for* $s : C \to C$ *and* $f : X \to Y$, $s \bullet f$ *is defined by*



Moreover, scalar multiplication is natural, that is, $(s \bullet g) \circ f = g \circ (s \bullet f)$. *Finally,* $s \bullet f = s \circ f$ *for* $s : C \to C$ *and* $f : X \to C$. □

Now we can state and prove a converse to Theorem 18.

Theorem 21 *In a pre-Hilbert category whose monoidal unit is a simple generator, if* $(f^{\perp})^{\mathrm{op}} \dashv g^{\perp}$, *then* $g = s \bullet f^{\dagger}$ *for a scalar* s. *Unless* $f = 0$, *this* s *is unique.*

Proof The adjunction of the hypothesis means that there is a q making the upper diagram in (1) commute iff there is a p making the lower one commute. So, if $n^{\dagger} f m = 0$, then $n^{\dagger} k i = 0$, and because i is epi hence $n^{\dagger} k = 0$. So p exists, whence q exists, so that $n^{\dagger} g^{\dagger} m = j^{\dagger} 0 q = 0$. Taking $m = \ker(n^{\dagger} f)$ thus gives that $\ker(n^{\dagger} f) \leq \ker(n^{\dagger} g^{\dagger})$ for all n. Applying Lemma 19 yields that for all $n : C \to Y$, there exists $s_n : C \to C$ such that $n^{\dagger} g^{\dagger} = s_n n^{\dagger} f$. Using Lemma 20 and dualising, this becomes: for all $n : C \to Y$, there is $s_n : C \to C$ with $gn = (s_n^{\dagger} \bullet f^{\dagger}) n$. We will show that all s_n are in fact equal to each other (or zero). If all $y : C \to Y$ would have $y = 0$, then $Y \cong 0$, in which case $g = 0 \bullet f^\dagger$. Otherwise, pick an $y : C \to Y$ with $y \neq 0$. There is an $s : C \to C$ with $gy = (s^\dagger \bullet f^\dagger)y$. Put $n' = yy^\dagger n : C \to Y$ and $n'' = \ker(y^\dagger) \circ \ker(y^\dagger)^\dagger \circ n : C \to Y$. Then

$$\begin{aligned} n' + n'' &= [\mathrm{id}, \mathrm{id}] \circ ((y \circ y^\dagger \circ n) \oplus (\ker(y^\dagger) \circ \ker(y^\dagger)^\dagger \circ n)) \circ \langle \mathrm{id}, \mathrm{id} \rangle \\ &= [y, y^\perp] \circ [y, y^\perp]^\dagger \circ n \\ &= n. \end{aligned}$$

Moreover,

$$(s_{n'}^\dagger \bullet f^\dagger)n' = gn' = gyy^\dagger n = (s^\dagger \bullet f^\dagger)yy^\dagger n = (s^\dagger \bullet f^\dagger)n',$$

so $s_{n'} = s$. Finally

$$\begin{aligned} (s_{n'}^\dagger \bullet f^\dagger)n' + (s_{n''}^\dagger \bullet f^\dagger)n'' &= gn' + gn'' \\ &= gn \\ &= (s_n^\dagger \bullet f^\dagger)n \\ &= (s_n^\dagger \bullet f^\dagger)n' + (s_n^\dagger \bullet f^\dagger)n''. \end{aligned}$$

Hence $s_n = s_{n'} = s$ for all $n : C \to Y$, and we have $gn = (s^\dagger \bullet f^\dagger)n$. But since C is a generator, $g = s^\dagger \bullet f^\dagger$. Reviewing our choice of s in the above proof, we see that it is unique unless $f = 0$. □

As a consequence, we find that, modulo scalars, the passage from morphisms f to functors $\perp \circ \exists_f^{\mathrm{op}}$ is one-to-one.

A Fibred account

We can summarise our results in terms of fibred category theory [Jac]. There are fibrations $\mathrm{Sub}(\mathbf{H}) \to \mathbf{H}$ and $\mathrm{ClSub}(\mathbf{H}) \to \mathbf{H}$. The latter is in fact a fibration of meet-semilattices by Lemma 5. The reflection of Lemma 7 is a fibred reflection. Our functor $\exists$ of Theorem 10 is a fibred coproduct, and hence truely provides a existential quantifier.

The assignments $\mathbf{H} \to \mathrm{Sub}(\mathbf{H})$, $X \mapsto \mathrm{id}_X$ assemble into a fibred terminal object $1 : \mathbf{H} \to \mathrm{Sub}(\mathbf{H})$, also for $\mathbf{H} \to \mathrm{ClSub}(\mathbf{H})$. The fibrations $\mathrm{Sub}(\mathbf{H}) \to \mathbf{H}$ and $\mathrm{ClSub}(\mathbf{H}) \to \mathbf{H}$ admit comprehension. This means that $1 : \mathbf{H} \to \mathrm{Sub}(\mathbf{H})$ has a right adjoint, usually denoted by $\{-\} : \mathrm{Sub}(\mathbf{H}) \to \mathbf{H}$. Indeed, if we take $\{m : M \rightarrowtail X\} = M$, then $\mathrm{Sub}(\mathbf{H})(\mathrm{id}_X, m) \cong \mathbf{H}(X, \{m\})$.

In fact, the fibration $\mathrm{ClSub}(\mathbf{H}) \to \mathbf{H}$ is a bifibration by Theorem 10 and [Jac, 9.1.2] – notice that the Beck-Chevalley condition is not needed for this. Thus, $\mathrm{ClSub}(\mathbf{H})^{\mathrm{op}} \to \mathbf{H}^{\mathrm{op}}$, $(m : M \rhd\!\!\longrightarrow X) \mapsto X$ is also a fibration. The following proposition shows that orthogonality can be extended to a functor between fibrations, but it is not a fibred functor, basically because it does not commute with pullback.

Proposition 22 $(-)^\perp$ *extends to a functor* $\mathrm{ClSub}(\mathbf{H})^{\mathrm{op}} \to \mathrm{ClSub}(\mathbf{H})$ *satisfying*

$$\begin{array}{ccc} \mathrm{ClSub}(\mathbf{H})^{\mathrm{op}} & \xrightarrow{(-)^\perp} & \mathrm{ClSub}(\mathbf{H}) \\ \downarrow & & \downarrow \\ \mathbf{H}^{\mathrm{op}} & \xrightarrow[(-)^\dagger]{} & \mathbf{H} \end{array} \tag{2}$$

However, it is not a fibred functor.

Proof We can understand $(-)^\perp$ as a functor $\mathrm{ClSub}(\mathbf{H})^{\mathrm{op}} \to \mathrm{ClSub}(\mathbf{H})$ by extending its action on morphisms as follows. Let (f, g) be a morphism $m \to n$, that is, let $f : X \to Y$ and $g : M \to N$ satisfy $fm = ng$. We are to define a morphism $(f, g)^\perp : n^\perp \to m^\perp$, that is, a pair $f^\perp : Y \to X$ and $g^\perp : N^\perp \to M^\perp$ satisfying $f^\perp \circ n^\perp = m^\perp \circ g^\perp$. Put $f^\perp = f^\dagger$. Then

$$m^\dagger \circ f^\dagger \circ n^\perp = g^\dagger \circ n^\dagger \circ \ker(n^\dagger) = g^\dagger \circ 0 = 0,$$

so there is a $g^\perp$ such that $f^\dagger \circ n^\perp = \ker(m^\dagger) \circ g^\perp = m^\perp \circ g^\perp$. It must be

$$g^\perp = (m^\perp)^\dagger \circ m^\perp \circ g^\perp = \ker(m^\dagger)^\dagger \circ f^\dagger \circ n^\perp = \mathrm{coker}(m) \circ f^\dagger \circ \ker(n^\dagger).$$

This explicitly defines the functor $(-)^\perp : \mathrm{ClSub}(\mathbf{H})^{\mathrm{op}} \to \mathrm{ClSub}(\mathbf{H})$. It makes the square (2) commute. Now, a morphism $(f, g) : m \to n$ of $\mathrm{ClSub}(\mathbf{H})$ is Cartesian (over f) iff $f = ngm^\dagger = nn^\dagger fmm^\dagger$. Consequently, the morphism $(f, g)^\perp : n^\perp \to m^\perp$ in $\mathrm{ClSub}(\mathbf{H})^{\mathrm{op}}$ is Cartesian iff

$$f^\dagger = \ker(m^\dagger) \circ \ker(m^\dagger)^\dagger \circ f^\dagger \circ \ker(n^\dagger) \circ \ker(n^\dagger)^\dagger.$$

Thus, $(-)^\perp$ is a fibred functor iff $f^\dagger = \ker(m^\dagger)\ker(m^\dagger)^\dagger f^\dagger \ker(n^\dagger)\ker(n^\dagger)^\dagger$ whenever $f = nn^\dagger fmm^\dagger$ for any morphism f and †-mono's m and n.

Finally, we come to our counterexample. Take $m = \kappa_M$ for $M \neq 0$, $f = mm^\dagger$ and $n = \mathrm{id}_{M \oplus M}$. Then $f = mm^\dagger = mm^\dagger mm^\dagger = nn^\dagger fmm^\dagger$. But $\ker(m^\dagger) = \kappa'$ so $\ker(m^\dagger)^\dagger = \pi'$ and $\ker(n^\dagger) = 0$, so

$$\ker(m^\dagger) \circ \ker(m^\dagger)^\dagger \circ f^\dagger \circ \ker(n^\dagger) \circ \ker(n^\dagger)^\dagger. = \kappa' \circ \pi' \circ f^\dagger \circ 0 = 0 \neq f^\dagger.$$

Hence $(-)^\perp$ is not a fibred functor. □

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