Complementarity in categorical quantum mechanics

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Abstract

We relate notions of complementarity in three layers of quantum mechanics: (i) von Neumann algebras, (ii) Hilbert spaces, and (iii) orthomodular lattices. Taking a more general categorical perspective of which the above are instances, we consider dagger monoidal kernel categories for (ii), so that (i) become (sub)endohomsets and (iii) become subobject lattices. By developing a 'point-free' definition of copyability we link (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean subalgebras.

1 Introduction

Complementarity is a supporting pillar of the Copenhagen interpretation of quantum mechanics. Unfortunately, Bohr's own formulation of the principle remained imprecise and flexible [23], and to date there is no concensus on a clear mathematical definition. Here, we understand it, roughly, to mean that complete knowledge of a quantum system can only be attained through examining all of its possible classical subsystems [10]. Notice that, perhaps unlike Bohr's own, this interpretation concerns *all* classical contexts, leading to a weaker notion of binary complementarity than usual. To avoid clashes with the various existing terminologies and their connotations, and to emphasize the distinction between talking about *two* (totally) incompatible classical contexts (as Bohr typically did), and mentioning all of them, we will speak of *partially complementary* classical contexts only when considering *two* of them. Only taken all together, (pairwise partially complementary) classical contexts give complete information, and we call them *completely complementary.* This paper considers instances of this interpretation of complementarity with regard to three aspects of quantum mechanics.

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(i) The observables of a quantum system form a von Neumann algebra. In this setting, complete complementarity is customarily taken to mean that one has to look at all commutative von Neumann subalgebras [16, 25, 4].

(ii) The states of a quantum system are unit vectors in a Hilbert space, which can be coordinatized by choosing any orthonormal basis. Here, complete complementarity may be interpreted as saying that it takes measurements in all possible orthonormal bases (of many identical copies of a system) to determine its state perfectly [26], as in state tomography [17, 18].

(iii) The measurable properties of a quantum system form an orthomodular lattice. Complete complementarity translates to this view as stating that the lattice structure is determined by all Boolean sublattices [21, 13].

In fact, we will take a more general perspective, as all three layers, separately, have recently been studied categorically.

(i) The set of commutative von Neumann subalgebras of a von Neumann algebra gives rise to a topos of set-valued functors, whose intuitionistic internal logic sheds light on the the original noncommutative algebra in so far as complete complementarity is concerned [12, 9].

(ii) The category of Hilbert spaces can be abstracted to a dagger monoidal category, in which much of quantum mechanics can still be formulated [1]. In this framework, orthonormal bases are characterized as so-called classical structures [7, 8, 2].

(iii) Orthomodular lattices can be obtained as kernel subobjects in a so-called dagger kernel category [11]. This paper considers Boolean sublattices systematically, in the tradition of *e.g.* [14].

We will take the view that of these three layers, (ii) is the primitive one, which the others derive from. Indeed, our main results are in categories that are simultaneously dagger monoidal categories and dagger kernel categories. We give definitions of partial and complete complementarity for (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean sublattices of the orthomodular lattice of kernels. By developing a point-free notion of copyability, we obtain a bijective correspondence between partially complementary classical structures and partially complementary Boolean sublattices. It is worth mentioning that this seems to be the first positive use of tensor products in the study of orthomodular lattices—the relation between tensor products and orthomodular lattices has resisted attempts at structural understanding so far, and only negative, restrictive, results are known. Our second main contribution is to characterize categorically what partially complementary commutative von Neumann subalgebras correspond to in terms of classical structures in the category of Hilbert spaces, conceptually improving upon previous work in this setting [20, 18]. The plan of the paper is as follows: Sections 2, 3 and 4 study layers (ii), (iii) and (i) respectively. Conclusions are then drawn in Section 5. The author is grateful to Samson Abramsky, Ross Duncan, Klaas Landsman, and Jamie Vicary for useful pointers and discussions.

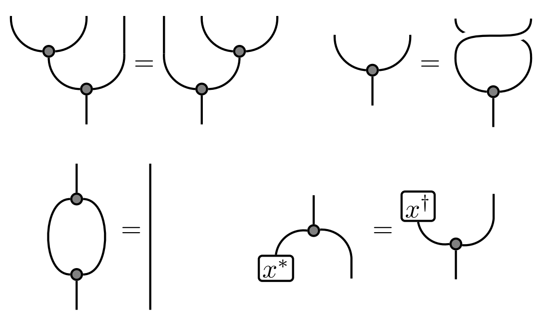
2 Classical structures

A *dagger* on a category $\mathbf{D}$ is a functor $\dagger\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{D}$ that acts on objects as $X^\dagger = X$ and satisfies $f^{\dagger\dagger} = f$ on morphisms. We will be interested in dagger categories that also have tensor products and kernels. By way of introduction we first recall these two extra structures separately, and then consider how they cooperate.

A *dagger symmetric monoidal category* is a dagger category that is simultaneously symmetric monoidal, satisfies $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and whose coherence isomorphisms such as $\lambda\colon X \otimes I \to X$ satisfy $\lambda^{-1} = \lambda^\dagger$. For more information about dagger monoidal categories and their uses in physics, we refer to [6, 1]. A morphism f is called *dagger monic* when $f^\dagger \circ f = \mathrm{id}$.

Definition 1 A *classical structure* in a dagger symmetric monoidal category $\mathbf{D}$ is a commutative semigroup $\delta\colon X \to X \otimes X$ that satisfies $\delta^\dagger \circ \delta = \mathrm{id}$ and the following so-called H*-axiom: there is an involution $*\colon \mathbf{D}(I,X)^{\mathrm{op}} \to \mathbf{D}(I,X)$ such that $\delta^\dagger \circ (x^* \otimes \mathrm{id}) = (x^\dagger \otimes \mathrm{id}) \circ \delta$.

Spelling out this terse definition in the graphical calculus [24], its conditions look as follows, where δ is depicted as .



We will not explicitly use much of a classical structure except its type and the fact that it is dagger monic; for more information we refer to the forthcoming article [2], and the aforementioned [6].

A *dagger kernel category* is a dagger category that has a zero object 0, and in which every morphism has a kernel that is dagger monic. We write $\ker(f)$ for the kernel of f, and $\mathrm{coker}(f) = \ker(f^\dagger)^\dagger$ for its cokernel. The definition $k^\perp = \ker(k^\dagger)$ for kernels k yields an orthocomplement on the partially ordered set $\mathrm{KSub}(X)$ of kernel subobjects of a fixed object X. The main result of [11], which we refer to for more information about dagger kernel categories, is that this poset $\mathrm{KSub}(X)$ is always an orthomodular lattice.

The goal of this section is to investigate when kernels $k\colon K \to X$ are 'compatible' with a given classical structure. To do so, we develop a notion of copyability that has to be 'point-free' because K is typically not the monoidal unit I.

2.1 Kernels and tensor products

Fix a category $\mathbf{D}$, and assume it to be a dagger symmetric monoidal category and a dagger kernel category simultaneously, which additionally satisfies

$$\ker(f) \otimes \ker(g) = \ker(f \otimes \mathrm{id}) \wedge \ker(\mathrm{id} \otimes g)$$

for all morphisms f and g.[1] The categories **Hilb** and **Rel** both satisfy the above relationship between tensor products and kernels. Some coherence properties follow easily from the assumptions:

$$\begin{aligned} \ker(f) \otimes 0 &= 0, & 0 \otimes \ker(g) &= 0, \\ \ker(f) \otimes \mathrm{id} &= \ker(f \otimes \mathrm{id}), & \mathrm{id} \otimes \ker(g) &= \ker(\mathrm{id} \otimes g), \\ \ker(f) \otimes \mathrm{id} = 0 &\Leftrightarrow \ker(f) = 0, & \mathrm{id} \otimes \ker(g) = 0 &\Leftrightarrow \ker(g) = 0. \end{aligned}$$

Notice that requiring $\ker(f \otimes g) = \ker(f) \otimes \ker(g)$ would have been too strong, for then $\ker(f) \otimes \mathrm{id} = \ker(f) \otimes \ker(0) = \ker(f \otimes 0) = \ker(0) = \mathrm{id}$ for any f. Nevertheless, the following lemma shows that this property does hold 'on the diagonal', *i.e.* when $f = g$.

Lemma 2 *We have* $\ker(f \otimes f) = \ker(f) \otimes \ker(f)$ *for any morphism* f.

Proof By definition of meet, $\ker(f \otimes f) = \ker(f \otimes \mathrm{id}) \wedge \ker(\mathrm{id} \otimes f)$ is the pullback of $\ker(f) \otimes \mathrm{id}$ and $\mathrm{id} \otimes \ker(f)$. Hence it suffices to prove that $\ker(f) \otimes \ker(f)$ is a pullback of $\ker(f) \otimes \mathrm{id}$ and $\mathrm{id} \otimes \ker(f)$, too. So suppose that $(\mathrm{id} \otimes \ker(f)) \circ p = (\ker(f) \otimes \mathrm{id}) \circ q$.

$$\begin{tikzcd}
\cdot \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & & \\
& K \otimes K \arrow[r, "\ker(f)\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\ker(f)"] & X \otimes K \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\ker(f)"] & Y \otimes K \arrow[d, "\mathrm{id}\otimes\ker(f)"] \\
& K \otimes X \arrow[r, "\ker(f)\otimes\mathrm{id}"'] & X \otimes X \arrow[r, "f\otimes\mathrm{id}"'] & Y \otimes X
\end{tikzcd}$$

Then

$$\begin{aligned} (\mathrm{id} \otimes \ker(f)) \circ (f \otimes \mathrm{id}) \circ p &= (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \ker(f)) \circ p \\ &= (f \otimes \mathrm{id}) \circ (\ker(f) \otimes \mathrm{id}) \circ q \\ &= (0 \otimes \mathrm{id}) \circ q \\ &= 0. \end{aligned}$$

Since $\ker(f)$ is dagger monic, we find $(f \otimes \mathrm{id}) \circ p = 0$. Therefore there exists φ such that $p = \ker(f \otimes \mathrm{id}) \circ \varphi$. A symmetric argument shows that $q = (\mathrm{id} \otimes \ker(f)) \circ \varphi$; as $\ker(f)$ is dagger monic, $\varphi = (\ker(f)^\dagger \otimes \mathrm{id}) \circ p = (\mathrm{id} \otimes \ker(f)^\dagger) \circ q$ is the unique such morphism. Hence $\ker(f) \otimes \ker(f)$ is indeed the pullback we were looking for. □

[1] One might consider additional coherence requirements such as $\ker(f \otimes g) = \ker(f \otimes \mathrm{id}) \vee \ker(\mathrm{id} \otimes g)$, but these are not necessary for our present purposes.

2.2 Copyability

Throughout this section we fix a classical structure $\delta\colon X \to X \otimes X$.

Definition 3 An endomorphism $p\colon X \to X$ is called *copyable* (along δ) when $\delta \circ p = (p \otimes p) \circ \delta$. A nonendomorphism $k\colon K \to X$ is called *copyable* (along δ) when $P(k) = k \circ k^\dagger$ is.

We start by relating the previous definition to copyability of vectors as used in [5].

Lemma 4 *The following are equivalent for a unit vector x in $H \in \mathbf{Hilb}$:*

(a) the morphism $\mathbb{C} \to H$ defined by $1 \mapsto x$ is copyable;

(b) there is a phase $z \in \mathbb{C}$ with $|z| = 1$ such that $\delta(x) = z \cdot (x \otimes x)$;

(c) there is a unit vector $x' \in H$ with $P(x) = P(x')$ and $\delta(x') = x' \otimes x'$.

Proof For (a)⇒(b):

$$\delta(x) = (\delta \circ P(x))(x) = (P(x) \otimes P(x)) \circ \delta(x) = \langle x \otimes x \,|\, \delta(x)\rangle \cdot (x \otimes x).$$

Taking $z = \langle x \otimes x \,|\, \delta(x)\rangle$ gives $|z| = \|\langle x \otimes x \,|\, \delta(x)\rangle \cdot (x \otimes x)\| = \|\delta(x)\| = \|x\| = 1$.

Conversely, to see (b)⇒(a):

$$\begin{aligned}
(\delta \circ P(x))(y) &= \delta \circ x \circ x^\dagger(y) \\
&= \langle x \,|\, y\rangle \cdot \delta(x) \\
&= \langle x \,|\, y\rangle \cdot z \cdot (x \otimes x) \\
&= \langle \delta(x) \,|\, \delta(y)\rangle \cdot z \cdot (x \otimes x) \\
&= |z|^2 \cdot \langle x \otimes x \,|\, \delta(y)\rangle \cdot (x \otimes x) \\
&= \langle x \otimes x \,|\, \delta(y)\rangle \cdot (x \otimes x) \\
&= (x \otimes x) \circ (x^\dagger \otimes x^\dagger) \circ \delta(y) \\
&= P(x \otimes x) \circ \delta(y).
\end{aligned}$$

The equivalence of (b) and (c) is established by the equality $x' = z \cdot x$. □

Example 5 In any dagger kernel category with tensor products satisfying the coherence set out in section 2.1, zero morphisms and identity morphisms are always copyable:

$$\begin{aligned}
\delta \circ P(0) = \delta \circ 0 = 0 \circ \delta = (0 \otimes 0) \circ \delta = P(0 \otimes 0) \circ \delta, \\
\delta \circ P(\mathrm{id}) = \delta \circ \mathrm{id} = \delta = (\mathrm{id} \otimes \mathrm{id}) \circ \delta = P(\mathrm{id} \otimes \mathrm{id}) \circ \delta.
\end{aligned}$$

These two kernels are called the *trivial* kernels.

Example 6 In the category **Hilb** of Hilbert spaces, a classical structure δ corresponds to the choice of an orthonormal basis (e_i) [2], whereas a kernel corresponds to a (closed) linear subspace [11]. A kernel is copyable if and only if it is the linear span of a subset of the orthonormal basis.

Example 7 In the category **Rel** of sets and relations, a classical structure δ on X corresponds to (a disjoint union of) Abelian group structure(s) on X [19], and a kernel is to a subset $K \subseteq X$ [11]. A kernel is copyable iff

$$\begin{aligned}\{(x \cdot y, (x, y)) \mid x, y \in X, x \cdot y \in K\} &= \delta \circ P(k)\\ &= P(k \otimes k) \circ \delta\\ &= \{(x \cdot y, (x, y)) \mid x \in K, y \in K\},\end{aligned}$$

i.e. if and only if $x \in K \wedge y \in K \Leftrightarrow x \cdot y \in K$. One direction of this equivalence implies that K is a subsemigroup. Fixing $k \in K$, we see that for any $x \in X$ there is $y = x^{-1} \cdot k$ such that $x \cdot y \in K$. Therefore, the other direction implies that $x \in K$. That is, $K = X$. We conclude that the only copyable kernels in **Rel** are the trivial ones.

This signifies that **Rel** does not 'have enough kernels'. There is an order isomorphism between $\{p \in \mathbf{Rel}(X, X) \mid p = p^\dagger = p \circ p \leq \mathrm{id}\}$ and $\mathrm{KSub}(X)$ for a certain order $\leq$ on endohomsets [11, Proposition 12]. The situation improves when we consider all $p \in \mathbf{Rel}(X, X)$ with $p = p^\dagger = p^2$ instead of just the ones below the identity: these are precisely *partial equivalence relations*, *i.e.* the symmetric and transitive relations. One finds that such an endomorphism $\sim$ is copyable if and only if it is a 'groupoid congruence' in the following sense:

$$x \cdot y \sim z \Longleftrightarrow \exists_{x', y'}[x \sim x', y \sim y', x' \cdot y' = z]$$

Lemma 8 *A dagger monic k is copyable if and only if there is a (unique) morphism δ_k making the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, two heads, "k^\dagger"] \arrow[d, tail, "\delta"'] & K \arrow[r, tail, "k"] \arrow[d, dashed, "\delta_k"] & X \arrow[d, tail, "\delta"] \\ X \otimes X \arrow[r, two heads, "k^\dagger \otimes k^\dagger"'] & K \otimes K \arrow[r, tail, "k \otimes k"'] & X \otimes X. \end{tikzcd}$$

Proof

$$\begin{aligned} & k \text{ is copyable}\\ \Longleftrightarrow\ & (P(k) \otimes P(k)) \circ \delta = \delta \circ P(k)\\ \Longleftrightarrow\ & (P(k) \otimes P(k)) \circ \delta = (P(k) \otimes P(k)) \circ \delta \circ P(k) = \delta \circ P(k)\\ \Longleftrightarrow\ & \exists_{\delta_k}.\delta \circ k = (k \otimes k) \circ \delta_k, \delta_k \circ k^\dagger = (k^\dagger \otimes k^\dagger) \circ \delta. \end{aligned}$$

□

We say that f is a *dagger retract* of g if there are dagger monic a and b making the following diagram commute:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] & \cdot \arrow[r, "a^\dagger"] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "b"'] & \cdot \arrow[r, "b^\dagger"'] & \cdot \end{tikzcd}$$

Notice that if f and f' are both dagger retracts of g (along the same a and b), then $f = b^\dagger \circ b \circ f = b^\dagger \circ g \circ a = f' \circ a^\dagger \circ a = f'$.

Proposition 9 *If k is a copyable dagger monic, δ_k is a classical structure.*

PROOF If k is a copyable dagger monic, then it follows from Lemma 8 that δ_k is a dagger retract of δ, and $\delta_k^\dagger$ is a dagger retract of $\delta^\dagger$. Therefore, δ_k is associative, commutative, and is dagger monic. For example, to verify commutativity, notice that $\gamma_k \colon K \otimes K \to K \otimes K$ is a dagger retract of $\gamma \colon X \otimes X \to X \otimes X$. Since dagger retracts compose, this means that $\gamma_k \circ \delta_k$ and δ_k are both dagger retracts (along the same morphisms) of $\gamma \circ \delta = \delta$. Hence $\gamma_k \circ \delta_k = \delta_k$. The other algebraic properties are verified similarly (including the Frobenius equation).

We are left to check the H*-axiom. Let $x \colon I \to K$. Since δ satisfies the H*-axiom, there is $(k \circ x)^* \colon I \to X$ such that

$$\delta^\dagger \circ ((k \circ x)^* \otimes \mathrm{id}) = ((k \circ x)^\dagger \otimes \mathrm{id}) \circ \delta.$$

Now put $x^* = k^\dagger \circ (k \circ x)^* \colon I \to K$. Then:

$$\begin{aligned}
\delta_k^\dagger \circ (x^* \otimes \mathrm{id}) &= \delta_k^\dagger \circ (k^\dagger \otimes \mathrm{id}) \circ ((k \circ x)^* \otimes \mathrm{id}) \\
&= \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ ((k \circ x)^* \otimes \mathrm{id}) \circ k \\
&= k^\dagger \circ \delta^\dagger \circ ((k \circ x)^* \otimes \mathrm{id}) \circ k \\
&= k^\dagger \circ ((k \circ x)^\dagger \otimes \mathrm{id}) \circ \delta \circ k \\
&= (x^\dagger \otimes \mathrm{id}) \circ (k^\dagger \otimes k^\dagger) \circ \delta \circ k \\
&= (x^\dagger \otimes \mathrm{id}) \circ \delta_k \circ k^\dagger \circ k \\
&= (x^\dagger \otimes \mathrm{id}) \circ \delta_k.
\end{aligned}$$

Hence δ_k satisfies the H*-axiom, too. □

Observe that it follows from the previous proposition that a dagger monic k is copyable if and only if its domain carries a classical structure δ_k and k is simultaneously a homomorphism of nonunital monoids and of nonunital comonoids. It stands to reason to define categories of classical structures to have such morphisms. This is a natural generalization of Definition 3 and also matches [15, 2.4.4].

Definition 10 Let $\mathbf{CS}[\mathbf{D}]$ denote the category whose objects are classical structures in $\mathbf{D}$. A morphism $(X, \delta_X) \to (Y, \delta_Y)$ is a morphism $f \colon X \to Y$ in $\mathbf{D}$ satisfying $\delta_Y \circ f = (f \otimes f) \circ \delta_X$ and $\delta_Y^\dagger \circ (f \otimes f) = f \circ \delta_X^\dagger$.

Proposition 11 *If $\mathbf{D}$ is a dagger kernel category, so is $\mathbf{CS}[\mathbf{D}]$.*

PROOF It is straightforward to see that $\mathbf{CS}[\mathbf{D}]$ inherits daggers and zero objects from $\mathbf{D}$, so it suffices to prove that it also inherits kernels. Let $f \colon (X, \delta_X) \to (Y, \delta_Y)$ be a morphism in $\mathbf{CS}[\mathbf{D}]$. We will use Lemma 8 to establish that $\ker(f)$ is copyable. Since $\ker(f) \otimes \ker(f) = \ker(f \otimes f)$ by Lemma 2, and

$$(f \otimes f) \circ \delta_X \circ \ker(f) = \delta_Y \circ f \circ \ker(f) = \delta_Y \circ 0 = 0,$$

there is a $\varphi\colon K \to K\otimes K$ satisfying $\delta_X \circ \ker(f) = (\ker(f)\otimes\ker(f))\circ\varphi$. Similarly, since

$$f \circ \delta_X^\dagger \circ (\ker(f) \otimes \ker(f)) = \delta_Y^\dagger \circ (f \otimes f) \circ (\ker(f) \otimes \ker(f)) = 0,$$

there is a $\psi\colon K \otimes K \to K$ with $\delta_X^\dagger \circ (\ker(f) \otimes \ker(f)) = \ker(f) \circ \psi$. As all morphisms involved are dagger monic, one finds $\varphi^\dagger = \psi$. Thus $\ker(f)$ is a well-defined morphism in $\mathbf{CS}[\mathbf{D}]$ and in fact a kernel. $\square$

The previous proposition enables the following satisfactory rephrasing of Proposition 9.

Corollary 12 *Let δ be a classical structure on X in $\mathbf{D}$. The kernels that are copyable along δ are precisely the kernel subobjects of δ in $\mathbf{CS}[\mathbf{D}]$.* $\square$

2.3 Complementarity and mutual unbiasedness

Definition 13 Two classical structures are *partially complementary* if no nontrivial kernel is simultaneously copyable along both.

The above definition clashes with complementarity of classical structures as defined in [5]. Let us spend some time developing a notion that does correspond to complementarity in the sense of [5] directly.

Definition 14 A morphism $x\colon U \to X$ is called *unbiased* (relative to δ) if and only if $P(x^\dagger \circ k) = P(x^\dagger \circ l)$ for all copyable kernels k and l.

Recall that if the ambient category is simply well-pointed, then $\mathrm{KSub}(X)$ is atomic, and its atoms are precisely the kernels with domain I [11]. If such 'points' are unbiased in the above sense, then they are unbiased vectors in the sense of [5].

Another advantage of the previous definition in this point-based setting is that it does not need to specify what the scalars $\langle k \,|\, x\rangle$ are. In the traditional point-based setting this scalar involves the dimension of the carrier Hilbert space, and is therefore limited to finite-dimensional spaces. The above definition can be interpreted regardless of dimensional aspects.

Lemma 15 *If a nonzero kernel is copyable (along δ) then it is not unbiased (relative to δ).*

Proof Let x be a nonzero copyable kernel. Then $x^\perp$ is copyable, too, by Lemma 19. The trivial kernels are always copyable by Example 5. Hence $k = \mathrm{id}$ and $l = x^\perp$ are both copyable kernels. But

$$P(x^\dagger \circ k) = P(x^\dagger) = x^\dagger \circ x \neq 0 = P(0) = P(x^\dagger \circ \ker(x^\dagger)) = P(x^\dagger \circ l),$$

and therefore x cannot be unbiased. $\square$

Definition 16 Two classical structures are *mutually unbiased* if a nontrivial kernel is unbiased relative to one whenever it is copyable along the other.

Proposition 17 *Mutually unbiased classical structures are partially complementary.*

Proof Suppose that δ and δ' are mutually unbiased, and let k be a kernel in the intersection. That is, k is copyable along both δ and δ'. By Lemma 15, k cannot be unbiased relative to δ'. This contradicts mutual unbiasedness unless k were a trivial kernel. $\square$

There is no converse to the previous proposition. For example, consider the object $\mathbb{C}^2$ in the category **Hilb**. A classical structure corresponds to an orthonormal basis of $\mathbb{C}^2$, and hence corresponds (up to sign) to a single ray. The collections of copyables of classical structures induced by two different rays always have trivial intersection. But certainly not every pair of different orthonormal bases is mutually unbiased. We conclude that Definition 16 is too strong for our purposes.

3 Boolean subalgebras of orthomodular lattices

This section concerns level (iii) of the Introduction. We will prove that kernels that are copyable along δ form a Boolean subalgebra of the orthomodular lattice of all kernel subobjects of X.

Lemma 18 *The copyable kernels form a sub-meetsemilattice of* $\mathrm{KSub}(X)$*.*

Proof The bottom element 0 is always copyable by Example 5. So we have to prove that if k and l are copyable kernels, then so is $k \wedge l$. Recall that $k \wedge l$ is defined as the pullback.

$$\begin{tikzcd}
K \wedge L \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "l"] \\
K \arrow[r, "k"'] & X
\end{tikzcd}$$

Together with the assumption that k and l are copyable, this means that the top, back, right and bottom face of the following cube commute:

$$\begin{tikzcd}
& K \arrow[rr, "k"] & & X \\
K \wedge L \arrow[ur, "q"] \arrow[rr, "p"] & & L \arrow[ur, "l"] & \\
& K \otimes K \arrow[uu, "\delta_k^\dagger"'] \arrow[rr, "k\otimes k"'] & & X \otimes X. \arrow[uu, "\delta^\dagger"'] \\
(K \wedge L)^{\otimes 2} \arrow[uu, dashed, "\varphi"] \arrow[ur, "q\otimes q"] \arrow[rr, "p\otimes p"'] & & L \otimes L \arrow[uu, "\delta_l^\dagger"'] \arrow[ur, "l\otimes l"'] &
\end{tikzcd}$$

Hence $l \circ \delta_l^\dagger \circ (p \otimes p) = k \circ \delta_k^\dagger \circ (q \otimes q)$. Therefore, by the universal property of pullbacks, there exists a dashed morphism φ making the left and front sides of the above cube commute. Using the fact that p and q are dagger monic, we deduce $\varphi = (k \wedge l)^\dagger \circ \delta^\dagger \circ ((k \wedge l) \otimes (k \wedge l))$. This means that the left square in the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "(k\wedge l)^\dagger"] \arrow[d, "\delta"'] & K \wedge L \arrow[r, "k\wedge l"] \arrow[d, "\varphi^\dagger"] & X \arrow[d, "\delta"] \\
X \otimes X \arrow[r, "(k\wedge l)^\dagger \otimes (k\wedge l)^\dagger"'] & (K \wedge L) \otimes (K \wedge L) \arrow[r, "(k\wedge l)\otimes(k\wedge l)"'] & X \otimes X.
\end{tikzcd}
$$

The right square is seen to commute analogously—take daggers of all the vertical morphisms in the cube. Therefore the whole rectangle commutes. In other words, $\delta \circ P(k \wedge l) = (P(k \wedge l) \otimes P(k \wedge l)) \circ \delta$, that is, $k \wedge l$ is copyable. $\square$

Lemma 19 *The copyable kernels form an orthocomplemented sublattice of the orthomodular lattice* $\mathrm{KSub}(X)$.

Proof We have to prove that if k is a copyable kernel, then so is $k^\perp = \ker(k^\dagger)$.

$$
\begin{tikzcd}
K \arrow[r, tail, "k"] & X \arrow[r, "(k^\perp)^\dagger"] & K^\perp \arrow[r, tail, "k^\perp"] & X \arrow[r, "k^\dagger"] & K \\
K \otimes K \arrow[u, "\delta_k^\dagger"] \arrow[r, tail, "k\otimes k"'] & X \otimes X \arrow[u, "\delta^\dagger"'] \arrow[r, two heads, "(k^\perp)^\dagger \otimes (k^\perp)^\dagger"'] & K^\perp \otimes K^\perp \arrow[u, dashed, shift left, "g"] \arrow[u, dashed, shift right, "f"'] \arrow[r, tail, "k^\perp \otimes k^\perp"'] & X \otimes X \arrow[u, "\delta^\dagger"] \arrow[r, two heads, "k^\dagger \otimes k^\dagger"'] & K \otimes K \arrow[u, "\delta_k^\dagger"']
\end{tikzcd}
$$

Since k is copyable, we have $k^\dagger \circ \delta^\dagger \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (0 \otimes 0) = 0$, so that the dashed arrow f in the above diagram exists, making the square to its right commute. Since $k^\perp$ is dagger monic, f must equal $\mathrm{coker}(k) \circ \delta^\dagger \circ (\ker(k^\dagger) \otimes \ker(k^\dagger))$.

Similarly, it follows from copyability of k that $(k^\perp)^\dagger \circ \delta_k^\dagger \circ (k \otimes k) = 0$, so that the dashed arrow g exists. Since g must be $\mathrm{coker}(k) \circ \delta^\dagger \circ (\mathrm{coker}(k)^\dagger \otimes \mathrm{coker}(k)^\dagger)$, we see that f and g coincide. Hence the rectangle composed of the middle two squares commutes. Taking its dagger yields the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r, "(k^\perp)^\dagger"] \arrow[d, tail, "\delta"'] \arrow[rr, bend left, "P(k^\perp)"] & K^\perp \arrow[r, tail, "k^\perp"] \arrow[d, "f^\dagger"'] & X \arrow[d, tail, "\delta"] \\
X \otimes X \arrow[r, two heads, "(k^\perp)^\dagger \otimes (k^\perp)^\dagger"'] \arrow[rr, bend right, "P(k^\perp) \otimes P(k^\perp)"'] & K^\perp \otimes K^\perp \arrow[r, tail, "k^\perp \otimes k^\perp"'] & X \otimes X.
\end{tikzcd}
$$

That is, $k^\perp$ is copyable. $\square$

Remark 20 Notice that if the classical structure had a unit ε, the previous result would have been impossible if we had additionally demanded $\varepsilon \circ P(k) = \varepsilon$ for k to be copyable, since then $\varepsilon = \varepsilon \circ P(k^{\perp}) = \varepsilon \circ P(k) \circ P(k^{\perp}) = \varepsilon \circ P(k \wedge k^{\perp}) = \varepsilon \circ 0 = 0$. Compare [2].

Lemma 21 *[11, Theorem 1] An orthocomplemented sublattice L of* $\mathrm{KSub}(X)$ *is Boolean if and only if $k \wedge l = 0$ implies $l^{\dagger} \circ k = 0$ for all $k, l \in L$.* □

Theorem 22 *The copyable kernels form a Boolean subalgebra of the orthomodular lattice* $\mathrm{KSub}(X)$*.*

Proof By the previous lemmas, it suffices to prove that if $k \wedge l = 0$ for copyable kernels k and l, then $l^{\dagger} \circ k = 0$. So let k and l be copyable kernels and suppose $k \wedge l = 0$. Say $k = \ker(f)$ and $l = \ker(g)$. Then

$$(f \otimes \mathrm{id}) \circ (k \otimes l) = (f \circ k) \otimes l = 0 \otimes l = 0,$$

so that $k \otimes l \leq \ker(f \otimes \mathrm{id}) = k \otimes \mathrm{id} \leq (k \otimes \mathrm{id}) \wedge (\mathrm{id} \otimes k) = k \otimes k$. Similarly, $k \otimes l \leq l \otimes l$. Therefore the bottom, top, back and right faces of the following cube commute:

$$\begin{tikzcd}
& K \arrow[rr, "k"] & & X \\
0 \arrow[ur, "0"] \arrow[rr, "0"] & & L \arrow[ur, "l"] & \\
& K \otimes K \arrow[uu, "\delta_k^{\dagger}"] \arrow[rr, "k \otimes k"] & & X \otimes X. \arrow[uu, "\delta^{\dagger}"'] \\
K \otimes L \arrow[uu, dashed, "\varphi"] \arrow[ur, "\mathrm{id} \otimes (k^{\dagger} \circ l)"] \arrow[urrr, "k \otimes l"] \arrow[rr, "(l^{\dagger} \circ k) \otimes \mathrm{id}"'] & & L \otimes L \arrow[uu, "\delta_l^{\dagger}"'] \arrow[ur, "l \otimes l"'] &
\end{tikzcd}$$

The universal property of the pullback formed by the top face yields the dashed morphism φ making the left and front faces commute. Hence $\delta_l^{\dagger} \circ ((l^{\dagger} \circ k) \otimes \mathrm{id}) = 0$. But then, as k and l are copyable:

$$\begin{aligned}
l^{\dagger} \circ k &= l^{\dagger} \circ k \circ \delta_k^{\dagger} \circ \delta_k \\
&= l^{\dagger} \circ \delta^{\dagger} \circ (k \otimes k) \circ \delta_k \\
&= \delta_l^{\dagger} \circ (l^{\dagger} \otimes l^{\dagger}) \circ (k \otimes k) \circ \delta_k \\
&= \delta_l^{\dagger} \circ ((l^{\dagger} \circ k) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (l^{\dagger} \circ k)) \circ \delta_k \\
&= 0 \circ (\mathrm{id} \otimes (l^{\dagger} \circ k)) \circ \delta_k \\
&= 0.
\end{aligned}$$

This finishes the proof. □

It follows immediately that the category **CS**[**D**] is a Boolean dagger kernel category (in the sense of [11]).

In the category **Hilb**, Example 6 shows that the copyable kernels in fact form a maximal Boolean subalgebra. Example 7 shows that this does not hold generally. For in **Rel**, one has $\mathrm{KSub}(X) = \mathcal{P}(X)$, but the copyable kernels are always $\{0, 1\}$, which are only maximal if X has cardinality 1. Similarly, not every maximal Boolean subalgebra B of $\mathrm{KSub}(X)$ induces a classical structure on X of which B are the copyables, as does happen to be the case in **Hilb**.

The following definition expresses the standard view in (order-theoretic) quantum logic that Boolean subalgebras of orthomodular lattices are regarded as embodying complete complementarity. It is precisely what is needed to make Theorem 24 true.

Definition 23 Two Boolean subalgebras of an orthomodular lattice are called *partially complementary* when they have trivial intersection.

Theorem 24 *Two classical structures are partially complementary if and only if their collections of copyable kernels are partially complementary.* □

Hence we have linked, fully abstractly, partial complementarity in the order-theoretic sense to partial complementarity in the sense of classical structures. Various order-theoretic questions about the lattices of copyable kernels suggest themselves for further investigation. For example, one could imagine that the width or height of the Boolean sublattice of copyable kernels is independent of the classical structure, enabling a general notion of dimension. One could also study how copyability interacts with closed or compact structure in the category.

4 Von Neumann algebras

Finally, this section advances to level (i) of the Introduction. We instantiate the dagger monoidal kernel category $\mathbf{D}$ to be **Hilb**. For any object $H \in \mathbf{Hilb}$, the endohomset $A = \mathbf{Hilb}(H, H)$ is then a type I von Neumann algebra (and every type I von Neumann algebra is of this form). At this level, the notion of complete complementarity is formalized by considering all commutative von Neumann subalgebras C of A. We denote the collection of all such subalgebras of A by $\mathcal{C}(A)$. Let us recall some facts about this situation.

(a) The set $\mathrm{Proj}(A) = \{p \in A \mid p^\dagger = p = p^2\}$ of projections is a complete, atomic, atomistic orthomodular lattice [22, p85].

(b) There is an order isomorphism $\mathrm{Proj}(A) \cong \mathrm{KSub}(H)$ [11, Proposition 12].

(c) Any von Neumann algebra is generated by its projections [22, 6.3], so in particular $C = \mathrm{Proj}(C)''$.

(d) Since C is a subalgebra of A, also $\mathrm{Proj}(C)$ is a sublattice of $\mathrm{Proj}(A)$.

(e) Because C is commutative, $\mathrm{Proj}(C)$ is a Boolean algebra [22, 4.16].

The following lemma draws a conclusion of interest from these facts.

Lemma 25 *Commutative von Neumann subalgebras C of $A = \mathbf{Hilb}(H, H)$ are in bijective correspondence with Boolean subalgebras of* $\mathrm{KSub}(H)$.

PROOF As in (b) above, we identify $\mathrm{Proj}(A)$ with $\mathrm{KSub}(H)$. A commutative subalgebra C corresponds to the Boolean subalgebra $\mathrm{Proj}(C)$. Conversely, a Boolean subalgebra B corresponds to the commutative subalgebra B'' generated by it. These mappings are inverses because $\mathrm{Proj}(C)'' = C$ and $\mathrm{Proj}(B'') = B$.□

We now set out to establish the relation between commutative subalgebras of A and classical structures on H.

Lemma 26 *An orthocomplemented sublattice L of* $\mathrm{KSub}(H)$ *is Boolean if and only if the following equivalent conditions hold:*

- *there exists a classical structure on the greatest element of L along which every element of L is copyable;*
- *there exists a classical structure on H along which every element of L is copyable.*

PROOF Necessity is established by Theorem 22. For sufficiency, let L be a Boolean sublattice of $\mathrm{KSub}(H)$. Since $\mathrm{KSub}(H)$ is complete by (a) above, $\bigvee L$ exists. By atomicity (a), $\bigvee L$ is completely determined by the set of atoms a_i below it. By definition of atoms, $a_i \wedge a_j = 0$ when $i \neq j$. Because L is Boolean, it follows from Lemma 21 that a_i and a_j are orthogonal. Also, because **Hilb** is simply well-pointed, the kernels a_i correspond to one-dimensional subspaces [11, Lemma 11]. That is, the a_i give an orthonormal basis for (the domain of) the greatest element of L (which can be extended to an orthonormal basis of H). This, in turn, induces a classical structure δ on the greatest element of L (or H) [2]. Finally, Example 6 shows that the kernels a_i, and hence all $l \in L$, are copyable along δ. □

Theorem 27 *For the von Neumann algebra $A = \mathbf{Hilb}(H, H)$:*

$$\mathcal{C}(A) \cong \{L \subseteq \mathrm{KSub}(H) \mid L \textit{ orthocomplemented sublattice}, \\ \exists_{\delta \colon 1_L \to 1_L \otimes 1_L} \forall_{l \in L} \,.\, l \textit{ copyable along } \delta\}.$$

PROOF This is just a combination of Lemma 25 and Lemma 26. □

The previous theorem implies that for any classical structure δ on H, there is an induced commutative von Neumann subalgebra $C \in \mathcal{C}(A)$ corresponding to the lattice L of all copyable kernels. The following definition and corollary finish the connections of partial complementarity across the three levels discussed in the Introduction.

Definition 28 Von Neumann subalgebras of $\mathbf{Hilb}(H, H)$ are *partially complementary* when their intersection is the trivial subalgebra $\{z \cdot \mathrm{id} \mid z \in \mathbb{C}\}$.

Notice that this definition does not need the subalgebras to be commutative. It has no need for finite dimension, either, in contrast to works that rely on the Hilbert-Schmidt inner product to make $\mathbf{Hilb}(H,H)$ into a Hilbert space [20]. Compare also [18, Definition 1.1].

Corollary 29 *Two classical structures on an object H in* **Hilb** *are partially complementary if and only if they induce partially complementary commutative von Neumann subalgebras of* $\mathbf{Hilb}(H,H)$. □

Unlike $\mathrm{Proj}(A)$, the sublattice $\mathrm{Proj}(C)$ is not atomic for general $C \in \mathcal{C}(A)$; for a counterexample, take $H = L^2([0,1])$ and $C = L^\infty([0,1])$. If this does happen to be the case, for example if we restrict the ambient category $\mathbf{D}$ to that of finite-dimensional Hilbert spaces, we can strengthen the characterization of $\mathcal{C}(A)$ in Theorem 27.

Proposition 30 *For a finite-dimensional Hilbert space H and the von Neumann algebra $A = \mathbf{fdHilb}(H,H)$:*

$$\mathcal{C}(A) \cong \{(k_i)_{i\in I} \mid k_i \in \mathrm{KSub}(H)\backslash\{0\},\ k_i \wedge k_j = 0 \textit{ for } i \neq j, \\ \exists_{\delta\colon H\to H\otimes H} \,.\, k_i \textit{ copyable along } \delta\}.$$

Proof Since every $C \in \mathcal{C}(A)$ is finite-dimensional and hence $\mathrm{Proj}(C)$ is atomic, an orthocomplemented sublattice L as in Theorem 27 is completely determined by its atoms (k_i). □

Every finite-dimensional C*-algebra is a von Neumann algebra, and hence in case H is finite-dimensional and $A = \mathbf{Hilb}(H,H)$, we find that $\mathcal{C}(A)$ is the collection of all commutative C*-subalgebras of A. Notice that the characterization of Proposition 30 above has no need for the cumbersome combinatorial symmetry considerations of [12, 1.4.5]. Corollary 12 gives another characterization of $\mathcal{C}(A)$ in the finite-dimensional case, purely in terms of classical structures and their morphisms.

Proposition 31 *For a finite-dimensional Hilbert space H and the von Neumann algebra $A = \mathbf{fdHilb}(H,H)$:*

$$\mathcal{C}(A) \cong \{(\delta_i)_{i\in I} \mid \delta_i \in \mathbf{CS}[\mathbf{fdHilb}], \\ \delta_i, \delta_j \textit{ partially complementary for } i \neq j, \\ \exists_{\delta\in\mathbf{CS}[\mathbf{fdHilb}]}\forall_i \,.\, \mathbf{CS}[\mathbf{fdHilb}](\delta_i,\delta) \neq \emptyset\}.$$

Hence $\mathcal{C}(A)$ is isomorphic to the collection of cocones in the category of classical substructures on H that are pairwise partially complementary. □

Another, very concrete, characterization in terms of traces and determinants is known in the finite-dimensional case [18, Proposition 1.3]. It is easy to compute, but the above characterization seems conceptually more informative and lends itself more readily to generalization.

5 Concluding remarks

Observing the similarities across the three levels of quantum mechanics considered, we can now propose the following precise formulation of complete complementarity.

> A collection of classical structures is *completely complementary* when its members are pairwise partially complementary and jointly epic.

Notice that this formulation is almost information-theoretic. Compare also [3]; in combination with Corollary 12, this suggests that a dagger kernel category could be seen as a colimit (or amalgamation of some other kind) of its Boolean subcategories.

The view on $\mathcal{C}(A)$ provided by Section 4 holds several promises for the study of functors on $\mathcal{C}(A)$ that we intend to explore further in future work:

- One can consider variations in the study of **Set**-valued functors on $\mathcal{C}(A)$ by choosing different morphisms on $\mathcal{C}(A)$: *e.g.* inclusions [12], or reverse inclusions [9]. In the above perspective, the natural direction that suggests itself is that of morphisms between classical structures, *i.e.* inclusions. Moreover, a more interesting choice of morphisms based on classical structures (see *e.g.* [7]) could make $\mathcal{C}(A)$ into a category that is not just a partially ordered set.

- The topos of functors on $\mathcal{C}(A)$ can be abstracted away from **Hilb** to any dagger monoidal kernel category that satisfies a suitable 'spectral assumption' linking commutative submonoids of endohomsets to classical structures. For example, one could lift Theorem 27 or even Proposition 31 to a definition, and study **Set**-valued functors on these characterizations of $\mathcal{C}(A)$ in any dagger monoidal kernel category.

  In fact, in this generalized setting, there is no need for the base category to be **Set**. After all, the basic objects of study of *e.g.* [9] are really partial orders of subobjects in a functor category. This just happens to be a Heyting algebra because the functors take values in the topos **Set**, but in principle less structured partial orders of subobjects are just as interesting, and perhaps are also justifiable physically.

- One of the weak points of the study of functors on $\mathcal{C}(A)$ to date is that there is no obvious way to study compound systems. That is, there is no obvious (satisfactory) relation between $\mathcal{C}(A \otimes B)$ and $\mathcal{C}(A)$ and $\mathcal{C}(B)$. Considering A as (a submonoid of) an endohomset opens the broader context of a fibred setting in which studying entanglement is possible.

All in all, the above considerations strongly suggest studying fibrations of all classical structures over all objects of a dagger (kernel) monoidal category, *i.e.* studying the forgetful functor $\mathbf{CS}[\mathbf{D}] \to \mathbf{D}$. As a first step in this direction, observe that it follows from Proposition 11 that copyable kernels are stable under pullback (along morphisms of classical structures).

Finally, we remark that we have not used the H*-axiom (or the Frobenius equation) at all in this paper. Apparently, the combination of (copyable) kernels with the dagger monic type $X \to X \otimes X$ of classical structures suffices for these purposes.

References

[1] Samson Abramsky and Bob Coecke. Categorical quantum mechanics. In *Handbook of quantum logic and quantum structures: quantum logic*, pages 261–324. Elsevier, 2009.

[2] Samson Abramsky and Chris Heunen. Classical structures in infinite-dimensional categorical quantum mechanics. *preprint*, 2010.

[3] Benno van den Berg and Chris Heunen. Noncommutativity as a colimit. *arXiv:1003.3618, under consideration for Applied Categorical Structures*, 2010.

[4] Jeremy Butterfield and Christopher J. Isham. A topos perspective on the Kochen-Specker theorem: I. Quantum states as generalized valuations. *International Journal of Theoretical Physics*, 37(11):2669–2733, 1998.

[5] Bob Coecke and Ross Duncan. Interacting quantum observables: Categorical algebra and diagrammatics. In *Automata, Languages and Programming, ICALP 2008*, number 5126 in Lecture Notes in Computer Science, pages 298–310. Springer, 2008.

[6] Bob Coecke, Éric O. Paquette, and Duško Pavlović. *Semantic techniques in quantum computation*, chapter Classical and quantum structuralism, pages 29–70. Cambridge University Press, 2010.

[7] Bob Coecke and Duško Pavlović. Quantum measurements without sums. In *Mathematics of Quantum Computing and Technology*. Taylor and Francis, 2007.

[8] Bob Coecke, Duško Pavlović, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 2009.

[9] Andreas Döring and Christopher J. Isham. 'What is a thing?': Topos theory in the foundations of physics. In *New Structures for Physics*, Lecture Notes in Physics. Springer, 2009.

[10] Carsten Held. The meaning of complementarity. *Studies in History and Philosophy of Science Part A*, 25:871–893, 1994.

[11] Chris Heunen and Bart Jacobs. Quantum logic in dagger kernel categories. *Order*, 2010.

[12] Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. Bohrification. In *Deep Beauty*. Cambridge University Press, 2009.

[13] Gudrun Kalmbach. *Orthomodular Lattices*. Academic Press, 1983.

[14] Simon Kochen and Ernst Specker. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17:59–87, 1967.

[15] Joachim Kock. *Frobenius algebras and 2-D Topological Quantum Field Theories*. Number 59 in London Mathematical Society Student Texts. Cambridge University Press, 2003.

[16] Nicolaas P. Landsman. *Handbook of the philosophy of science vol 2: Philosophy of Physics*, chapter Between classical and quantum, pages 417–554. North-Holland, 2007.

[17] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[18] Kalyanapuram R. Parthasarathy. On estimating the state of a finite level quantum system. *Infinite Dimensional Analytical Quantum Probabilities and Related Topics*, 7:607–617, 2006.

[19] Duško Pavlović. Quantum and classical structures in nondeterministic computation. In P. Bruza et al., editor, *Third International symposium on Quantum Interaction*, volume 5494 of *Lecture Notes in Artificial Intelligence*, pages 143–157. Springer, 2009.

[20] Dénes Petz. Complementarity in quantum systems. *Reports on Mathematical Physics*, 59(2):209–224, 2007.

[21] Constantin Piron. *Foundations of quantum physics*. Number 19 in Mathematical Physics Monographs. W.A. Benjamin, 1976.

[22] Miklós Rédei. *Quantum Logic in Algebraic Approach*. Kluwer, 1998.

[23] Erhard Scheibe. *The logical analysis of quantum mechanics*. Pergamon, 1973.

[24] Peter Selinger. A survey of graphical languages for monoidal categories. In *New Structures for Physics*, Lecture Notes in Physics. Springer, 2010.

[25] Franco Strocchi. *Elements of quantum mechanics of infinite systems*. World scientific, 1985.

[26] John von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer, 1932.