ACTIVE LATTICES DETERMINE AW*-ALGEBRAS

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ABSTRACT. We prove that AW*-algebras are determined by their projections, their symmetries, and the action of the latter on the former. We introduce *active lattices*, which are formed from these three ingredients. More generally, we prove that the category of AW*-algebras is equivalent to a full subcategory of active lattices. Crucial ingredients are an equivalence between the category of piecewise AW*-algebras and that of piecewise complete Boolean algebras, and a refinement of the piecewise algebra structure of an AW*-algebra that enables recovering its total structure.

1. INTRODUCTION

Operator algebras play a major role in modern functional analysis and mathematical physics, and in particular algebras with an ample supply of projections. Such algebras display a rich interplay between their algebraic structure, the order-theoretic structure of their projections, the group-theoretic structure of their unitaries, and their various topological structures. It is therefore natural to wonder to what extent one of these aspects determines the others. We will consider AW*-algebras, where the topological angle plays a minor role, and focus on the other facets. Such algebras are not completely determined by the group-theoretic structure of their unitaries: for example, $U(A) \cong U(A^{\mathrm{op}})$, but $A \not\cong A^{\mathrm{op}}$ in general [9]. Adding the order-theoretic structure of their projections does not suffice to reconstruct the algebra either: again, $\mathrm{Proj}(A) \cong \mathrm{Proj}(A^{\mathrm{op}})$. Closely related to projections is the structure of the normal part $N(A)$ of A as a *piecewise*[1] *algebra* (see [4]). Roughly, these are algebras where one can only add or multiply commuting elements. But adding this structure is still not enough to determine the algebra, since $N(A)$ and $N(A^{\mathrm{op}})$ are isomorphic as piecewise algebras. It follows from our main result that taking into account one final ingredient does suffice to determine the algebra structure, namely the action by conjugation of the unitaries on the projections. Thus we answer the following preserver problem.

Corollary. *Let A and B be AW*-algebras. If $f \colon N(A) \to N(B)$ is an isomorphism of piecewise algebras, that restricts to isomorphisms $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$ and $U(A) \cong U(B)$, and satisfies $f(upu^*) = f(u)f(p)f(u)^*$, then $A \cong B$.*

There is considerable overkill in the previous corollary. For one thing, we could have stated the assumption on piecewise algebras in terms that a priori contain less information, such as, for example, the partial orders of commutative subalgebras of

[1]We prefer the terminology 'piecewise algebra' over the traditional 'partial algebra', because of the unfortunate conjunction 'partial complete Boolean algebra'.

A and B (see [12]), or various notions built on those. We will prove that any iso-morphism $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$ extends uniquely to an isomorphism $N(A) \to N(B)$ of piecewise algebras, that could therefore have been left out from the assumptions altogether. This puts the following two driving questions on an equal footing.

- What extra data make projections a complete invariant of AW*-algebras?
- What extra data on piecewise AW*-algebras enable extension to total ones?

Moreover, it suffices to consider the subgroup of the unitaries generated by so-called symmetries (see [1, Chapter 6]). Finally, the projection lattice injects into the symmetry group by $p \mapsto 1 - 2p$, and so the projection lattice acts on itself in a certain sense. We can package up the remaining data in an *active lattice*, which therefore completely determines the AW*-algebra structure. The precise definition can be found in Section 3, but let us emphasize here that it is expressed exclusively in terms of the projection lattice and the symmetry group that it generates.

In fact, we will be (quite) a bit more general, and work with arbitrary morphisms instead of just isomorphisms: we define a functor from the category of AW*-algebras to that of active lattices, and prove it to be full and faithful. Consequently, the category of AW*-algebras is equivalent to a full subcategory of the category of active lattices. This is summarized in the following commuting diagram of functors. Solid arrows represent functors that are faithful but not full, whereas the dashed functor we construct is both full and faithful.

$$
\begin{array}{ccc}
 & \textbf{AWstar} & \\
\swarrow^{\text{Proj}} \quad \downarrow \quad \searrow^{\text{Sym}} & & \\
\textbf{COrtho} \longleftarrow \textbf{Active} \longrightarrow \textbf{Group} &
\end{array}
$$

As W*-algebras (abstract von Neumann algebras) and normal $*$-homomorphisms form a full subcategory of AW*-algebras, this restricts to a full embedding of the category of W*-algebras into the category of active lattices.

Our results can be regarded as a particular answer to the Mackey–Gleason prob-lem, that has been studied in great detail for W*-algebras. This type of problem asks what properties of a function between projection lattices ensure that it extends to a linear function between operator algebras, or more generally, what properties of a function between operator algebras that is only piecewise linear make it linear [7].

Our main motivation is generalizing the duality of commutative C*-algebras and their Gelfand spectra to the noncommutative case. Many proposals for noncom-mutative spectra have been studied. One of them concerns *quantales* [25], that are based on projection lattices in the case of AW*-algebras. However, there are rig-orous obstructions to various categories being in duality with that of C*-algebras, including that of quantales [26, 3]. These obstructions leave open the possibility that a good notion of spectrum can be based on piecewise structures [14, 12]. Our active lattices come very close to quantales, but circumvent the obstruction afflict-ing them. Whereas a quantale is a monoid that is also a lattice, an active lattice can be regarded as a monoid that is generated by a lattice. Stone duality between Stonean spaces and complete Boolean algebras (see Section 2) allows one to con-sider $\mathrm{Proj}(A)$ as a substitute for the Gelfand spectrum in case A is a commutative AW*-algebra. So our results can also be regarded as a successful extension of this "substitute spectrum" to noncommutative AW*-algebras.

All in all, our results incorporate several technical contributions. First, we gen-eralize many constructions from W*-algebras to AW*-algebras, as the latter are the

natural home for our arguments. Second, our proofs require virtually no analytic methods; in particular, we will not rely on Gleason's theorem to extend piecewise linearity to linearity [8], but directly generalize results due to Dye [11] instead. These first two contributions should ease generalizing our results to C*-algebras. Third, we take morphisms seriously: instead of just focusing on isomorphisms, we deal with arbitrary morphisms with different domain and codomain. Fourth, our main results hold perfectly well for algebras with I_2 summands, which are exceptions to many classic theorems. These last two, nontrivial, contributions are a necessary preparation for a full noncommutative Gelfand duality.

The article proceeds as follows. Section 2 recalls AW*-algebras, complete Boolean algebras, and their piecewise versions. It then proves that the two resulting categories of piecewise structures are equivalent. Section 3 introduces active lattices after discussing the ingredients of projection lattices and symmetry groups. It also constructs the functor taking an AW*-algebra to its active lattice. Section 4 is devoted to proving that this functor is full.

2. (Piecewise) AW*-algebras and complete Boolean algebras

After reviewing commutative AW*-algebras and their equivalence to complete Boolean algebras, this section extends the equivalence to piecewise AW*-algebras and piecewise complete Boolean algebras, positively answering [4, Remark 3].

AW*-algebras and complete Boolean algebras. Kaplansky introduced AW*-algebras as an abstract generalization of von Neumann algebras [21, 2]. Their main characteristic is that they are algebraically determined by their projections, *i.e.* self-adjoint idempotents, to a great extent. We denote the set of projections of a $*$-ring A by $\mathrm{Proj}(A)$. This set is partially ordered by the relation $p \leq q \iff p = pq\,(= qp)$.

Definition 2.1. An *AW*-algebra* is a C*-algebra A that satisfies the following left-right symmetric and equivalent conditions:

(a) the right annihilator of any subset is generated as right ideal by a projection;
(b) the right annihilator of any element $a \in A$ is generated by a projection, and $\mathrm{Proj}(A)$ forms a complete lattice;
(c) the right annihilator of any element $a \in A$ is generated by a projection, and every orthogonal family in $\mathrm{Proj}(A)$ has a supremum;
(d) any maximal commutative subalgebra C is the closed linear span of $\mathrm{Proj}(C)$, and every orthogonal family in $\mathrm{Proj}(A)$ has a supremum.

A *morphism of AW*-algebras* is a $*$-homomorphism that preserves suprema of projections. We write **AWstar** for the category of AW*-algebras and their morphisms.

For a subset S of an AW*-algebra A, write $\mathrm{R}(S)$ for the (unique) projection of Definition 2.1(a): $\mathrm{R}(S)$ is the least projection annihilating every element of S, and is also the (unique) projection such that $xy = 0$ for all $x \in S$ if and only if $\mathrm{R}(S)y = y$. This is the *right annihilating projection* of S. With a slight abuse of notation, we write $\mathrm{R}(a)$ in place of $\mathrm{R}(\{a\})$ for a single element $a \in A$. The projection $\mathrm{RP}(a) = 1 - \mathrm{R}(a)$ is the *right supporting projection* of a. It is the least projection satisfying $a\,\mathrm{RP}(a) = a$.

Given the equivalent conditions defining AW*-algebras, there are several possible choices for morphisms of the category **AWstar**. Fortunately, the most obvious conditions one might impose on a $*$-homomorphism are also equivalent, as the

following lemma shows. Recall that a set of projections is called directed when every pair of its elements has an upper bound within the set.

Lemma 2.2. *For a* ∗*-homomorphism* $f\colon A \to B$ *between AW*-algebras, the following conditions are equivalent:*

(a) f preserves right annihilating projections of arbitrary subsets;
(b) f preserves suprema of arbitrary families of projections;
(c) f preserves suprema of orthogonal families of projections;
(d) f preserves suprema of directed families of projections.

If f satisfies these equivalent conditions, then the kernel of f is generated by a central projection and f preserves RP.

Proof. For a morphism f satisfying (c), the last sentence of the lemma follows from [2, Exercise 23.8]. That (b) implies (a) follows from the fact that such f preserves RP as well as the following equation for any $S \subseteq A$,

$$\mathrm{R}(S) = \bigvee_{x \in S} \mathrm{R}(x) = \bigvee_{x \in S} (1 - \mathrm{RP}(x))$$

(see also [2, Proposition 4.2]). Conversely, assume (a), and let $\{p_i\} \subseteq \mathrm{Proj}(A)$. We will prove that

$$\bigvee \{p_i\} = \mathrm{R}(\{1 - p_i\}),$$

from which (a) $\Rightarrow$ (b) will follow. Writing $p = \mathrm{R}(\{1 - p_i\})$, every $(1 - p_i) \perp p$, which gives $p_i \leq p$ for all i. And if all $p_i \leq q$ for any $q \in \mathrm{Proj}(A)$, then all $(1 - p_i) \perp q$, which means that $pq = q$ and thus $q \leq p$. Hence $p = \bigvee_i p_i$ as desired.

   Clearly (b) $\Rightarrow$ (d). To see (d) $\Rightarrow$ (c), let $\mathcal{P}$ be an orthogonal family of projections. Setting $q_S = \bigvee S$ for every finite subset $S \subseteq \mathcal{P}$ gives a directed family of projections with the same supremum as $\mathcal{P}$. Because each S is orthogonal and finite, we have $f(q_S) = f(\sum S) = \sum f(S) = \bigvee f(S)$. And because f is assumed to preserve directed suprema, $f(\bigvee \mathcal{P}) = f(\bigvee_S q_S) = \bigvee_S f(q_S) = \bigvee_S f(S) = \bigvee f(\mathcal{P})$.

   Finally, because any ∗-homomorphism $f\colon A \to B$ between AW*-algebras restricts to a lattice homomorphism $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$ (see [2, Proposition 5.7]), Lemma 3.2 below provides a direct proof of (c) $\Rightarrow$ (b). $\square$

   Observe that the proof of the previous lemma establishes more than was promised: it shows that direct sums provide finite products in the category **AWstar**. The initial object is the AW*-algebra $\mathbb{C}$, and the terminal object is the zero algebra. Observe also that the above lemma holds true if f is only assumed to be a ∗-ring homomorphism. This will be useful later in Section 4.

   Let **Wstar** denote the category of W*-algebras (*i.e.* abstract von Neumann algebras) and normal ∗-homomorphisms. Then **Wstar** is a full subcategory of **AWstar**. (The objects of **Wstar** are objects of **AWstar** by [2, Proposition 4.9], and the subcategory can be shown to be full, for instance, by composing a ∗-homomorphism $A \to B$ with all normal linear functionals on B and using [28, Corollary III.3.11]. See also [23, Lecture 11].) In particular, the lemma above provides equivalent conditions for a ∗-homomorphism between von Neumann algebras to be normal.

   If an AW*-algebra is commutative, its projections form a *complete Boolean algebra*: a distributive lattice in which every subset has a least upper bound, and in which every element has a complement. In fact, we now detail an equivalence between the categories of commutative AW*-algebras and complete Boolean algebras.

First recall Stone duality [17, Corollary II.4.4], which gives a dual equivalence between Boolean algebras and Stone spaces, *i.e.* totally disconnected compact Hausdorff spaces. If the Boolean algebra is complete, the corresponding Stone space is in fact a Stonean space, *i.e.* extremally disconnected, meaning that the closure of every open set is (cl)open. We write **CBool** for the category of complete Boolean algebras and homomorphisms of Boolean algebras that preserve arbitrary suprema. On the topological side, we write **Stonean** for the category of Stonean spaces and open continuous functions. With this choice of morphisms, Stone duality restricts to a dual equivalence between **CBool** and **Stonean**. See [5, Section 6].

Similarly, recall that Gelfand duality gives a dual equivalence between commutative C*-algebras and compact Hausdorff spaces. If the C*-algebra is an AW*-algebra, then the compact Hausdorff space is in fact a Stonean space [2, Theorem 7.1]. If we write **cAWstar** for the full subcategory of **AWstar** consisting of commutative AW*-algebras, then Gelfand duality restricts to a dual equivalence between **cAWstar** and **Stonean**. Hence we have the following equivalences.

$$(2.3) \qquad \textbf{cAWstar} \underset{\text{Cont}}{\overset{\text{Spec}}{\underset{\simeq}{\rightleftarrows}}} \textbf{Stonean}^{\text{op}} \underset{\text{Stone}}{\overset{\text{Clopen}}{\underset{\simeq}{\rightleftarrows}}} \textbf{CBool}$$

Explicitly, Spec is the functor taking characters and furnishing them with the Gelfand topology, and Clopen takes clopen subsets, so the composite Clopen ∘ Spec is naturally isomorphic to the functor Proj. We write Func for the composite Cont ∘ Stone. Explicitly, Stone = **CBool**$(-, 2)$ and Cont = $C(-, \mathbb{C})$. Thus Proj and Func form an equivalence between commutative AW*-algebras and complete Boolean algebras.

Piecewise structures. Piecewise algebras are sets of which only certain pieces carry algebraic structure, but in a coherent way. Before we can extend the equivalence above to a piecewise setting, we spell out the appropriate definitions. Definition 2.1(c) leads to a specialization of the definition of a piecewise C*-algebra, that we recall first [4].

Definition 2.4. A *piecewise C*-algebra* consists of a set A with:

- a reflexive and symmetric binary (*commeasurability*) relation $\odot \subseteq A \times A$;
- elements $0, 1 \in A$;
- a (total) involution $*: A \to A$;
- a (total) function $\cdot: \mathbb{C} \times A \to A$;
- a (total) function $\|-\|: A \to \mathbb{R}$;
- (partial) binary operations $+, \cdot: \odot \to A$;

such that every set $S \subseteq A$ of pairwise commeasurable elements is contained in a set $T \subseteq A$ of pairwise commeasurable elements that forms a commutative C*-algebra under the above operations.

A *piecewise AW*-algebra* is a piecewise C*-algebra A with

- a (total) function $\mathrm{RP}: A \to \mathrm{Proj}(A)$;
- a (partial) operation $\bigvee: \{X \subseteq \mathrm{Proj}(A) \mid X \times X \subseteq \odot\} \to \mathrm{Proj}(A)$;

such that every set $S \subseteq A$ of pairwise commeasurable elements is contained in a set $T \subseteq A$ of pairwise commeasurable elements that forms a commutative AW*-algebra under the above operations.

A *morphism of piecewise AW*-algebras* is a (total) function $f: A \to B$ such that:

- $f(a) \odot f(b)$ for commeasurable $a, b \in A$;
- $f(ab) = f(a)f(b)$ for commeasurable $a, b \in A$;
- $f(a + b) = f(a) + f(b)$ for commeasurable $a, b \in A$;
- $f(za) = zf(a)$ for $z \in \mathbb{C}$ and $a \in A$;
- $f(a)^* = f(a^*)$ for $a \in A$;
- $f(\bigvee_i p_i) = \bigvee_i f(p_i)$ for pairwise commeasurable projections $\{p_i\}$.

By Lemma 2.2, such a morphism automatically satisfies $f(\mathrm{RP}(a)) = \mathrm{RP}(f(a))$. Also, it follows from the last condition that $f(1) = 1$. Piecewise AW*-algebras and their morphisms organize themselves into a category denoted by **PAWstar**.

The prime example of a piecewise AW*-algebra is the set $N(A)$ of normal elements of an AW*-algebra A, where commeasurability is given by commutativity. Hence one can regard piecewise AW*-algebras as AW*-algebras of which the algebraic structure between noncommuting elements is forgotten.

Lemma 2.5. *The assignment sending an AW*-algebra A to its set of normal elements $N(A)$ defines a functor $N\colon$ **AWstar** $\to$ **pAWstar**.*

Proof. Let A be an AW*-algebra. The natural piecewise algebra structure on $N(A)$ is a piecewise C^*-algebra by [4, Proposition 3]. It is a piecewise AW*-algebra under the inherited RP and supremum operations, because every pairwise commuting subset of $N(A)$ is contained in a maximal commutative subalgebra of A, that is an AW*-subalgebra by Definition 2.1(d), and must itself necessarily be contained in $N(A)$. Functoriality of N is easy to check. $\qquad \square$

The next lemma observes that the structures $\bigvee$ and RP in Definition 2.4 are in fact properties. (Nonetheless morphisms in **pAWstar** have to preserve $\bigvee$.) Thus we may say that a certain piecewise C*-algebra "is a piecewise AW*-algebra" without ambiguity of the AW*-operations. We call a projection p of a piecewise C*-algebra a *least upper commeasurable bound* of a commeasurable set S of projections when $p \odot a$ for any a that makes $S \cup \{a\}$ commeasurable, and whenever a projection q is commeasurable with $S \le q$, then q is commeasurable with p as well and $p \le q$.

Lemma 2.6. *Let A be a piecewise C*-algebra. There is at most one choice of operations $\bigvee$ and RP as in Definition 2.4 making A a piecewise AW*-algebra.*

Proof. First, we claim that in any piecewise AW*-algebra, $\bigvee$ is characterized as giving the least upper commeasurable bound. Since these rely only the underlying piecewise C*-algebra structure, $\bigvee$ is then unique. The claim derives from Definition 2.4 as follows. Since $\bigvee$ makes A into a piecewise AW*-algebra, there exists a commutative AW*-algebra $T \subseteq A$ whose suprema are given by $\bigvee$, containing S. Hence T contains $\bigvee S$, making $S \cup \{\bigvee S\}$ commeasurable, and $\bigvee S$ majorizes S. If q is commeasurable with S and majorizes it, there exists an AW*-algebra T containing $S \cup \{q\}$. In particular, it is closed under suprema of projections, which are given by $\bigvee$. Thus it contains $\bigvee S$, which is therefore commeasurable with q and $\bigvee S \le q$. Finally, $\mathrm{RP}(a) = \bigwedge\{p \in \mathrm{Proj}(A) \mid ap = a\}$ equals $\bigvee\{q \in \mathrm{Proj}(A) \mid \forall p \in \mathrm{Proj}(A)\colon ap = a \Rightarrow p \le q\}$. $\qquad \square$

The next two results give convenient ways to recognize piecewise AW*-algebras among piecewise C*-algebras. The first shows that a piecewise AW*-algebra is a piecewise C*-algebra that is "covered" by sufficiently many AW*-algebras; recall that an AW*-algebra A is an *AW*-subalgebra* of an AW*-algebra B when the

inclusion $A \hookrightarrow B$ is a morphism in **AWstar**. The second is a characterization analogous to Kaplansky's original definition of AW*-algebras as C*-algebras with extra properties, Definition 2.1(d).

Lemma 2.7. *A piecewise C*-algebra A is a piecewise AW*-algebra when:*
- *any commeasurable subset S is contained in a commeasurable subset $T(S)$ that is an AW*-algebra, such that:*
- *if $S \subseteq S'$ are commeasurable subsets, then $T(S)$ is an AW*-subalgebra of $T(S')$.*

Proof. Define functions RP and $\bigvee$ by calculating $\mathrm{RP}(a)$ as in $T(\{a\})$, and calculating $\bigvee X$ as in $T(X)$. By [2, Proposition 3.8], then $\mathrm{RP}(a)$ is the same when calculated in any $T(S)$ with $a \in S$, because $T(\{a\})$ is an AW*-subalgebra of $T(S)$. Similarly, $\bigvee X$ is the same in any $T(S)$ with $X \subseteq S$ [2, Proposition 4.8]. Therefore RP and $\bigvee$ make A into a piecewise AW*-algebra. □

Proposition 2.8. *A piecewise C*-algebra A is a piecewise AW*-algebra when:*
- *commeasurable sets of projection have least upper commeasurable bounds; and*
- *any maximal commeasurable subalgebra is the closed linear span of projections.*

Proof. Assumption (a) defines a function $\bigvee$. If S is a commeasurable subset, Zorn's lemma provides a maximal commeasurable set $M \supseteq S$. By definition of piecewise C*-algebra, M is contained in a commeasurable C*-algebra. Hence maximality guarantees that M is a commutative C*-algebra under the operations of A. But now assumption (b) together with $\bigvee$ make M into an AW*-algebra [2, Exercise 7.1]. Taking $S = \{a\}$, we can define $\mathrm{RP}(a)$ as the unique right supporting projection in M. The functions $\bigvee$ and RP (uniquely) make A into a piecewise AW*-algebra. □

There is a similar definition of piecewise complete Boolean algebras that specializes the definition of piecewise Boolean algebras [4].

Definition 2.9. A *piecewise complete Boolean algebra* consists of a set B with
- a reflexive and symmetric binary (*commeasurability*) relation $\odot \subseteq B \times B$;
- a (total) unary operation $\neg \colon B \to B$;
- a (partial) operation $\bigvee \colon \{X \subseteq B \mid X \times X \subseteq \odot\} \to B$;

such that every set $S \subseteq B$ of pairwise commeasurable elements is contained in a pairwise commeasurable set $T \subseteq B$ that forms a complete Boolean algebra under the above operations. (Notice that these data uniquely determine elements $0 = \bigvee \emptyset$ and $1 = \neg 0$, and (partial) operations $x \vee y = \bigvee\{x, y\}$ and $x \wedge y = \neg(\neg x \vee \neg y)$.)

A *morphism of piecewise complete Boolean algebras* is a (total) function that preserves commeasurability and all the algebraic structure, whenever defined. We write **pCBool** for the resulting category.

A piecewise equivalence. The functor $\mathrm{Proj} \colon$ **AWstar** $\to$ **CBool** extends to a functor **pAWstar** $\to$ **pCBool** [4, Lemma 3]. We aim to prove that the latter functor is also (part of) an equivalence. By [4, Theorem 3], any piecewise complete Boolean algebra B can be seen as (a colimit of) a functor $\mathcal{C}(B) \to$ **CBool**, where $\mathcal{C}(B)$ is the diagram of (commeasurable) complete Boolean subalgebras of B and inclusions. Similarly, by the AW*-variation of [4, Theorem 7], any piecewise AW*-algebra A can be seen as a functor $\mathcal{C}(A) \to$ **cAWstar**, where $\mathcal{C}(A)$ is the diagram of (commeasurable) commutative AW*-subalgebras of A and inclusions. Hence postcomposition with Func should turn a piecewise complete Boolean algebra into a piecewise AW*-algebra. Below we explicitly compute the ensuing colimit

to get a functor $F\colon \mathbf{pCBool} \to \mathbf{pAWstar}$. Even though it is unclear how general coequalizers are computed in either category, the fact that $\mathcal{C}(B)$ is a diagram of monomorphisms makes the constructions manageable.

Lemma 2.10. *The monomorphisms in* $\mathbf{AWstar}$, $\mathbf{cAWstar}$, *and* $\mathbf{CBool}$ *are precisely the injective morphisms.*

Proof. Let $f\colon A \rightarrowtail B$ be a monomorphism in $\mathbf{AWstar}$ or $\mathbf{cAWstar}$. We first show that $\mathrm{Proj}(f)\colon \mathrm{Proj}(A) \rightarrowtail \mathrm{Proj}(B)$ is injective. Suppose that $f(p) = f(q)$ for $p, q \in \mathrm{Proj}(A)$. Define $g, h\colon \mathbb{C}^2 \to A$ by $g(1, 0) = p$ and $h(1, 0) = q$. Then $(f \circ g)(x, y) = xf(p) + yf(p)^\perp = (f \circ h)(x, y)$, so $g = h$ and hence $p = q$. In particular, f cannot map a nonzero projection of A to 0 in B. Thus $\ker(f) = 0$ by Lemma 2.2, and f is injective. Conversely, injective morphisms are trivially monic.

Monomorphisms $f\colon P \rightarrowtail Q$ in $\mathbf{CBool}$ factor as

$$P \cong \mathrm{Proj}(\mathrm{Func}(P)) \rightarrowtail \mathrm{Proj}(\mathrm{Func}(Q)) \cong Q.$$

Now, isomorphisms in $\mathbf{CBool}$ are bijective, and by the above, the middle arrow $\mathrm{Proj}(\mathrm{Func}(f))$ is injective, making f itself injective. $\qquad\square$

We are ready to define the object part of a functor $F\colon \mathbf{pCBool} \to \mathbf{pAWstar}$.

Definition 2.11. Let B be a piecewise complete Boolean algebra. Define $F(B)$ to be the following collection of data.

- The carrier set A is $(\coprod_{C \in \mathcal{C}(B)} \mathrm{Func}(C))/\sim$, where $\sim$ is the smallest equivalence relation satisfying $f \sim g$ for $f \in \mathrm{Func}(C)$ and $g \in \mathrm{Func}(D)$ when $C \subseteq D$ and $g = \mathrm{Func}(C \hookrightarrow D)(f)$.
- Two equivalence classes ρ and σ in A are commeasurable if and only if there exist $C \in \mathcal{C}(B)$ and $f, g \in \mathrm{Func}(C)$ such that $f \in \rho$ and $g \in \sigma$.
- Notice that $z \cdot 1_C \sim z \cdot 1_D$ for $C \subseteq D$ in $\mathcal{C}(B)$, and any $z \in \mathbb{C}$. Also, $\{0, 1\}$ is the minimal element of $\mathcal{C}(B)$. Hence $z \cdot 1_C \sim z \cdot 1_D$ for any $C, D \in \mathcal{C}(B)$ by transitivity.

  In particular, $[0_{\mathrm{Func}(\{0,1\})}] = [0_C]$ defines an element $0 \in A$ independently of C, and $1 \in A$ is defined by $[1_{\mathrm{Func}(\{0,1\})}] = [1_C]$ for any $C \in \mathcal{C}(B)$. Likewise, $z \cdot [f] = [z \cdot f]$ is well-defined for $z \in \mathbb{C}$.
- Similarly, $[f]^* = [f^*]$ gives a well-defined operation $*\colon A \to A$.
- If ρ and σ are two commeasurable elements of A, then by definition there are $C \in \mathcal{C}(B)$ and $f, g \in \mathrm{Func}(C)$ with $f \in \rho$ and $g \in \sigma$. Setting $\rho + \sigma = [f + g]$ and $\rho \cdot \sigma = [f \cdot g]$ gives well-defined operations $+, \cdot\colon \odot \to A$.
- If $C \subseteq D$, then $\mathrm{Func}(C \hookrightarrow D)\colon \mathrm{Func}(C) \to \mathrm{Func}(D)$ is an injective $*$-homomorphism by Lemma 2.10, and hence preserves norm [19, Theorems 4.1.8, 4.1.9]. So $\|[f]\| = \|f\|$ gives a well-defined operation $A \to \mathbb{R}$.

Proposition 2.12. *The data* $F(B)$ *defined above form a piecewise AW*-algebra.*

Proof. For ρ in A, define $C_\rho = \bigcap\{C \in \mathcal{C}(B) \mid \rho \cap \mathrm{Func}(C) \neq \emptyset\}$. Because $\mathcal{C}(B)$ is closed under arbitrary intersections, $C_\rho \in \mathcal{C}(B)$. If ρ and σ in A are commeasurable, then by definition there are $C \in \mathcal{C}(B)$ and $f, g \in \mathrm{Func}(C)$, so $C_\rho \subseteq C \supseteq C_\sigma$. But that implies any element of C_ρ is commeasurable in B with any element of C_σ.

Let $S \subseteq A$ be pairwise commeasurable. Then $\hat{S} = \bigcup_{\rho \in S} C_\rho \subseteq B$ is pairwise commeasurable by the last paragraph. Hence there exists a set $\hat{T} \subseteq B$ that contains $\hat{S}$, is pairwise commeasurable, and forms a complete Boolean algebra under

the operations from B. Therefore $T = \{[f] \mid f \in \mathrm{Func}(\hat{T})\} \subseteq A$ contains S, is commeasurable, and forms a commutative AW*-algebra under the operations from A. Hence A is a piecewise C*-algebra. Moreover, if $S \subseteq S'$, then $\hat{S} \subseteq \hat{S}'$, and $\hat{T} \subseteq \hat{T}'$ are both complete Boolean subalgebras of B under the same operation $\bigvee$, namely that of B. Hence T is an AW*-subalgebra of T', so that A is in fact a piecewise AW*-algebra by Lemma 2.7. $\square$

Lemma 2.13. *If $B \in$ **pCBool**, then $F(B)$ is a colimit of the diagram $\mathrm{Func}(C)$ with C ranging over $\mathcal{C}(B)$. Therefore F is functorial **pCBool** $\to$ **pAWstar**.*

Proof. Clearly there exists a cocone of morphisms $\mathrm{Func}(C) \to A$ for each $C \in \mathcal{C}(B)$, given by $f \mapsto [f]$. If $k_C \colon \mathrm{Func}(C) \to A'$ is another cocone, the unique mediating map $m \colon A \to A'$ is given by $m([f]) = k_C(f)$ when $f \in \mathrm{Func}(C)$.

Let $g \colon B_1 \to B_2$ be a morphism of **pCBool**. Because $F(B_1)$ is a colimit of $\{\mathrm{Func}(C) \mid C \in \mathcal{C}(B_1)\}$, to define a morphism $F(g) \colon F(B_1) \to F(B_2)$, it suffices to specify morphisms $\mathrm{Func}(C) \to F(B_2)$ in **pAWstar** for each $C \in \mathcal{C}(B_1)$. But g preserves commeasurability, so its restriction to C is a morphism in **CBool** and we can just take $F(g)\big|_{\mathrm{Func}(C)} = \mathrm{Func}(g\big|_C)$. This assignment is automatically functorial. Moreover, it is well-defined, even though colimits are only unique up to isomorphism, because Definition 2.11 fixed one specific colimit. $\square$

Theorem 2.14. *The functors F and Proj form an equivalence between the categories **pAWstar** and **pCBool**.*

Proof. For a piecewise AW*-algebra A we have

$$F(\mathrm{Proj}(A)) \cong \mathrm{colim}_{C \in \mathcal{C}(\mathrm{Proj}(A))} \mathrm{Func}(C)$$
$$\cong \mathrm{colim}_{C \in \mathcal{C}(A)} \mathrm{Func}(\mathrm{Proj}(C))$$
$$\cong \mathrm{colim}_{C \in \mathcal{C}(A)} C \cong A.$$

by Lemma 2.13, [4, Proposition 6], and [4, Theorem 7]. Each of the above isomorphisms is readily seen to be natural in A.

Next we establish an isomorphism $\mathrm{Proj}(F(B)) \cong B$. Let $\rho \in \mathrm{Proj}(F(B)) \subseteq F(B)$. If $\rho = [f]$ for $f \in \mathrm{Func}(C)$ and $C \in \mathcal{C}(B)$, then $f \in \mathrm{Proj}(\mathrm{Func}(C))$. So $\eta_C(f) \in C \subseteq B$, where η is the unit of the equivalence formed by Proj and Func. In fact, by naturality of η, if $C \subseteq D$ for another $D \in \mathcal{C}(B)$ with the inclusion denoted by $i \colon C \hookrightarrow D$, the following diagram commutes.

$$
\begin{array}{ccc}
\mathrm{Proj}(\mathrm{Func}(C)) & \xrightarrow{\ \eta_C\ } & C \\
{\scriptstyle \mathrm{Proj}(\mathrm{Func}(i))}\big\downarrow & & \big\downarrow{\scriptstyle i} \\
\mathrm{Proj}(\mathrm{Func}(D)) & \xrightarrow[\ \eta_D\ ]{} & D
\end{array}
$$

So if $g \sim f$ because $g = \mathrm{Func}(C \hookrightarrow D)(f)$, then $\eta_C(f) = \eta_D(g)$. That is, $\eta_C(f)$ is independent of the chosen representative f of ρ. Thus we have a map $\mathrm{Proj}(F(B)) \to B$ that is a morphism of piecewise complete Boolean algebras, because η is a morphism of complete Boolean algebras.

Conversely, for $b \in B$, consider the commeasurable subalgebra $B\langle b \rangle$ of B generated by b. Then $\eta_{B\langle b \rangle}^{-1}(b)$ is an element of $\mathrm{Proj}(\mathrm{Func}(B\langle b \rangle))$. Thus $b \mapsto [\eta_{B\langle b \rangle}^{-1}(b)]$ is a function $B \to \mathrm{Proj}(F(B))$, that is easily seen to be inverse to the function

$\mathrm{Proj}(F(B)) \to B$ above. Thus we have an isomorphism $B \cong \mathrm{Proj}(F(B))$ of piecewise complete Boolean algebras. Unfolding definitions shows that this isomorphism is natural in B. $\qquad\square$

As a consequence, the functor Proj preserves general coequalizers.

3. The category of active lattices

This section equips the piecewise AW*-algebra structure of AW*-algebras A with enough extra data to recover their full algebra structure, which will be done in the next section. The required structure consists of three ingredients: a lattice structure on $\mathrm{Proj}(A)$, a group structure on the so-called symmetry subgroup of the unitaries $U(A)$, and an action of the latter on the former. We will discuss each in turn.

The projection lattice. We start with some axioms satisfied by lattices of projections of AW*-algebras.

Definition 3.1. An *orthocomplementation* on a lattice P is an order-reversing involution $p \mapsto p^\perp$ satisfying $p \vee p^\perp = 1$ and $p \wedge p^\perp = 0$ (*i.e.*, $p^\perp$ is a *complement* of p). We say p and q are *orthogonal* when $p \leq q^\perp$. An orthocomplemented lattice is said to be *orthomodular* when $p \vee (p^\perp \wedge q) = q$ for all $p \leq q$. Complete orthomodular lattices form a category **COrtho** whose morphisms are functions that preserve the orthocomplementation as well as arbitrary suprema.

The condition of being an object of **COrtho** can be tested on orthogonal subsets, and the same is nearly true for morphisms.

Lemma 3.2. *An orthomodular lattice P is complete if and only if every orthogonal subset of P has a least upper bound. If P and Q are complete orthomodular lattices, a function $f\colon P \to Q$ is a morphism of **COrtho** if and only if it preserves orthocomplements, binary joins, and suprema of orthogonal sets.*

Proof. The first statement is [16, Corollary 1]. Let $f\colon P \to Q$ be as in the second statement, and let $\{p_i\}$ be any subset of P. Because f preserves finite joins, it preserves order, and so $\bigvee f(p_i) \leq f(\bigvee p_i)$; we prove the reverse comparison. Let $\{e_\alpha\}$ be a maximal orthogonal set of nonzero elements of P with $f(e_\alpha) \leq \bigvee f(p_i)$, and set $e = \bigvee e_\alpha$. By hypothesis, $f(e) = \bigvee f(e_\alpha) \leq \bigvee f(p_i)$. Thus it suffices to show that each $p_i \leq e$, for then $\bigvee p_i \leq e$ and $f(\bigvee p_i) \leq f(e) \leq \bigvee f(p_i)$ as desired. Assume for contradiction that some $p_j \not\leq e$. Then $e' = (p_j \vee e) \wedge e^\perp$ is a nonzero element of P orthogonal to e and hence orthogonal to each e_α. Furthermore

$$f(e') \leq f(p_j \vee e) = f(p_j) \vee f(e) \leq \bigvee f(p_i),$$

since $e' \leq p_j \vee e$. But this contradicts the maximality of $\{e_\alpha\}$. $\qquad\square$

The axioms defining AW*-algebras and their morphisms are such that the operation of passing to projection lattices defines a functor $\mathrm{Proj}\colon \mathbf{AWstar} \to \mathbf{COrtho}$.

Complete orthomodular lattices are tightly linked to piecewise complete Boolean algebras (rather than the more general orthocomplemented lattices). Indeed, any complete orthomodular lattice P canonically is a piecewise complete Boolean algebra, as follows. Define a commeasurability relation $\odot$ on P by the following equivalent conditions, for any $p, q \in P$:

  (i) there is a Boolean subalgebra of P that contains both p and q;
  (ii) there exist pairwise orthogonal $p', q', r \in P$ with $p = p' \vee r$ and $q = q' \vee r$;

(iii) $p \wedge (p \wedge q)^{\perp}$ is orthogonal to q;

(iv) $q \wedge (p \wedge q)^{\perp}$ is orthogonal to p;

(v) the *commutator* $(p \vee q) \wedge (p \vee q^{\perp}) \wedge (p^{\perp} \vee q) \wedge (p^{\perp} \vee q^{\perp})$ of p and q is zero.

For the equivalence of (i)–(iv) we refer to [30, Lemma 6.7]; for the equivalence of (i) and (v) see [24].

Lemma 3.3. *The assignment $P \mapsto (P, \odot)$ is a functor* **COrtho** $\to$ **pCBool**.

Proof. Given a complete orthomodular lattice P and the commeasurability relation $\odot$ above, it follows from [30, Lemma 6.10] that the supremum operation of P restricts to a partial operation $\bigvee \colon \{X \subseteq P \mid X \times X \subseteq \odot\} \to P$. $\square$

Composing this forgetful functor with the equivalence **pCBool** $\to$ **pAWstar** of Theorem 2.14 gives a canonical functor **COrtho** $\to$ **pAWstar**. Below, we will extend the structure of the piecewise complete Boolean algebra $\mathrm{Proj}(A)$ to that of a complete orthomodular lattice, where A is a piecewise AW*-algebra. As a converse to the above lemma, we now show that this is a property rather than structure.

For any piecewise Boolean algebra B, let $\leq$ be the union of the partial orders on each commeasurable subalgebra C of B. When this relation is transitive, it is a partial order, which we call the induced partial order. In that case we call B *transitive*. If every pair of (not necessarily commeasurable) elements of B have a least upper bound with respect to B, we say that B is *joined*. Similarly, we call a piecewise AW*-algebra A transitive or joined when $\mathrm{Proj}(A)$ is respectively transitive or joined.

Proposition 3.4. *The following categories are equivalent:*

(a) *the category* **COrtho** *of complete orthomodular lattices;*

(b) *the subcategory of* **pCBool** *whose objects are transitive and joined and whose morphisms preserve binary joins;*

(c) *the subcategory of* **pAWstar** *whose objects are transitive and joined and whose morphisms preserve binary joins of projections.*

Proof. The piecewise complete Boolean algebras that are in the image of the functor **COrtho** $\to$ **pCBool** from Lemma 3.3 are by definition transitive and joined. Next, we define a functor G in the opposite direction. Let B be a transitive, joined piecewise complete Boolean algebra and $\leq$ its induced partial order. By construction of $\leq$, it restricts to the given partial order on each commeasurable subalgebra of B. Furthermore, it is straightforward to verify that if $X \subseteq B$ is commeasurable then $\bigvee X$ is the least upper bound of X with respect to $\leq$. Kalmbach's bundle lemma [20, 1.4.22] now applies to show that $\leq$ and $\neg$ induce the structure of an orthomodular lattice on B. Because orthogonal subsets are commeasurable, and B has suprema of such subsets, it in fact has suprema of arbitrary subsets by Lemma 3.2. This makes B into a complete orthomodular lattice, and we can define $G(B) = (B, \leq)$. Setting $G(f) = f$ on for **pCBool** morphisms that preserve binary joins gives a well-defined functor, thanks to Lemma 3.2. It is straightforward to see that these two functors form an isomorphism of categories.

The equivalence of (b) and (c) follows from Theorem 2.14. $\square$

Remark 3.5. For an AW*-algebra A, recall that $\mathcal{C}(A)$ is the set of commutative AW*-subalgebras, ordered by inclusion. It carries the same information as the projection lattice $\mathrm{Proj}(A)$ [13, Theorem 2.5]. Therefore, everything that follows can equivalently be expressed in terms of $\mathcal{C}(A)$ instead of $\mathrm{Proj}(A)$.

The symmetry group. If A is a piecewise AW*- algebra, we let $U(A)$ denote the set of unitary elements of A, *i.e.* the set of all elements $u \in A$ such that $uu^* = 1$ (recall that $u \odot u^*$ for all $u \in A$). This set carries the structure of a *piecewise group*, *i.e.* one can multiply commeasurable elements, the multiplication has a unit (that is commeasurable with any element), and there is a total function giving inverses, such that every commeasurable subset generates a commutative subgroup. A *piecewise subgroup* is a subset that is a piecewise group in its own right under the inherited operations (and commeasurability relation). Every group is a piecewise group, and conversely, we will be extending the structure of the piecewise group $U(A)$ to that of a group. Piecewise groups form a category **pGroup** with the evident morphisms.

Definition 3.6. A *symmetry* in an AW*-algebra A is a self-adjoint unitary element; these are precisely the elements of the form $p^\perp - p = 1 - 2p$ for some $p \in \mathrm{Proj}(A)$. Let $U(A)$ denote the group of unitary elements of A, and define $\mathrm{Sym}(A)$ to be the subgroup of $U(A)$ generated by the symmetries of A. (Notice that if A is not commutative then $\mathrm{Sym}(A)$ contains elements that are not symmetries.)

Before moving on to actions of groups on lattices, we consider how large the symmetry $\mathrm{Sym}(A)$ group can become. We will see that this depends on the type: $\mathrm{Sym}(A)$ is (significantly) smaller than $U(A)$ for type I_n algebras, and just as large as $U(A)$ for other AW*-algebras.

If A is an AW*-algebra of type I_1, i.e. if A is commutative, then $\mathrm{Sym}(A)$ is as small as possible, namely in bijection with $\mathrm{Proj}(A)$, as the following example shows.

Example 3.7. If A is a commutative AW*-algebra, then the product of symmetries is again a symmetry, and so the sets $\mathrm{Sym}(A)$ and $\mathrm{Proj}(A)$ are bijective. In fact, $(1 - 2p)(1 - 2q) = 1 - 2((p + q - pq) - pq) = 1 - 2((p \vee q) - (p \wedge q)) = 1 - 2(p\Delta q)$, where Δ is the symmetric difference operation. Thus $\mathrm{Sym}(A)$ is the additive group of the Boolean ring structure associated to the Boolean algebra $\mathrm{Proj}(A)$.

For AW*-algebras of type I_n for $n \geq 2$, we will use the fact that traces and determinants are well-defined for matrices over commutative rings. Recall that any AW*-algebra of type I_n takes the form $\mathbb{M}_n(C)$ for a commutative AW*-algebra C [2, Proposition 18.2]. We will use roman letters $a, b, p, \dots$ for elements of a matrix algebra $\mathbb{M}_n(B)$ and greek letters $\alpha, \beta, \pi, \dots$ for elements of B when both are needed.

Lemma 3.8. *Let $A = \mathbb{M}_n(C)$ for $n \geq 2$ and a commutative AW*-algebra C.*

(a) *If $b, c \in C$ satisfy $0 \leq c = b^2 \leq 1$ and $b^* = b$, then there exists $u \in \mathrm{Sym}(C)$ with $b = uc_0$, where c_0 is the unique positive square root of c in C.*

(b) *If $u \in U(A)$ has $\det(u) = 1$, then $u = (1 - 2p)(1 - 2q)$ for some $p, q \in \mathrm{Proj}(A)$.*

(c) *If $u \in U(A)$ has $\det(u) = 1 - 2\pi$ for $\pi \in \mathrm{Proj}(C)$, then u can be written as $u = (1 - 2p)(1 - 2q)(1 - 2r)$ for some $p, q, r \in \mathrm{Proj}(A)$.*

(d) $\mathrm{Sym}(A)$ *is the normal subgroup $\{u \in U(A) \mid \det(u)^2 = 1\} = \det^{-1}(\mathrm{Sym}(C))$.*

Proof. For part (a), observe that the Gelfand spectrum X of C is extremally disconnected. So $\mathrm{int}(b^{-1}(-\infty, 0])$ is a clopen set, as is its complement $\mathrm{cl}(b^{-1}(0, \infty))$. So the function $u \colon X \to C$ defined by

$$u(x) = \begin{cases} -1 & \text{if } x \in \mathrm{int}(b^{-1}(-\infty, 0]), \\ 1 & \text{if } x \in \mathrm{cl}(b^{-1}(0, \infty)), \end{cases}$$

is continuous. It is clearly a self-adjoint unitary. If $x \in \mathrm{int}(b^{-1}(-\infty, 0])$, then $b(x) \leq 0$ and $u(x) = -1$, so $b(x) = u(x)c_0(x)$. If $x \in \mathrm{cl}(b^{-1}(0, \infty))$, then $b(x) \geq 0$ and $u(x) = 1$, so $b(x) = u(x)c_0(x)$. In either case $b = uc_0$.

For part (b) we generalize the argument of [11, page 87] from matrices with entries in $\mathbb{C}$ to entries in C. Let $u \in U(A)$ have determinant 1. Then u is unitarily equivalent to a diagonal matrix $\mathrm{diag}(\zeta_1, \dots, \zeta_n)$ with diagonal entries $\zeta_i \in U(C)$ satisfying $\prod \zeta_i = 1$ [10]. Such a matrix can be written as $\prod_{i=1}^{n-1} \mathrm{diag}(\zeta_{1,i}, \dots, \zeta_{n,i})$, where $\zeta_{i,i} = \prod_{k=1}^{i} \zeta_k$, $\zeta_{i+1,i} = \zeta_{i,i}^*$, and $\zeta_{k,i} = 1$ otherwise. Therefore, we may assume that $u = \mathrm{diag}(\zeta, \zeta^*, 1, \dots, 1)$ for fixed $\zeta \in U(C)$. Keeping the rest of the matrices involved equal to the identity matrix, we may in fact pretend that we are dealing with $n = 2$ and $u = \mathrm{diag}(\zeta, \zeta^*)$ for fixed $\zeta \in U(C)$. We may write $\zeta = \alpha + i\beta$ where $\alpha, \beta \in C$ are self-adjoint and satisfy $\alpha^2 + \beta^2 = 1$.

For each positive $\varphi \in C$, the element $1 + \varphi^2$ is invertible in C, so we can define

$$p_\varphi = \frac{1}{1 + \varphi^2} \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi^2 \end{pmatrix}.$$

Each p_φ is easily seen to be a projection in A, so $v_\varphi = (1 - 2p_\varphi)(1 - 2p_0)$ defines an element of $\mathrm{Sym}(A)$. Computing

$$v_\varphi = \frac{1}{1 + \varphi^2} \begin{pmatrix} 1 - \varphi^2 & -2\varphi \\ 2\varphi & 1 - \varphi^2 \end{pmatrix}$$

shows that $\det(v_\varphi) = 1$ and $\mathrm{tr}(v_\varphi) = 2 \cdot \frac{1 - \varphi^2}{1 + \varphi^2}$. Now, the function $\varphi \mapsto \frac{1 - \varphi^2}{1 + \varphi^2}$ is a composite of an order-automorphism $\varphi \mapsto \varphi^2$ of the positive cone of C with the Cayley transform $\varphi \mapsto \frac{1 - \varphi}{1 + \varphi}$, which maps the positive cone of C order-anti-isomorphically onto the interval $\{\gamma \in C \mid -1 < \gamma \leq 1\}$. Hence $\mathrm{tr}(v_\varphi)$ assumes all values in the interval $\{\gamma \in C \mid -2 < \gamma \leq 2\}$ as φ ranges over the positive cone of C, and actually achieves the value -2 by interpreting $p_\infty = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Diagonalizing v_φ to $\mathrm{diag}(\xi, \xi^*)$ with $\xi \in U(C)$, we can therefore make $\mathrm{tr}(v_\varphi) = \xi + \xi^* = 2\mathrm{Re}(\xi)$ assume all values in the positive cone of C by varying φ.

In particular, for $\zeta = \alpha + i\beta$ as above, there exist positive $\varphi \in C$ and $\beta_0 = \sqrt{1 - \alpha^2}$ such that $\zeta_0 = \alpha + i\beta_0 \in U(C)$ and $\mathrm{diag}(\zeta_0, \zeta_0^*)$ is unitarily equivalent to v_φ. Part (a) gives $\sigma \in \mathrm{Sym}(C)$ with $\beta = \sigma\beta_0$. The $\mathbb{R}$-linear map θ fixing self-adjoint elements and sending i to $i\sigma$ defines a $*$-ring automorphism of C. Thus $\mathbb{M}_n(\theta)$ is a $*$-ring automorphism of A, and $\mathbb{M}_n(\theta)(v_\varphi)$ is unitarily equivalent to $\mathbb{M}_n(\theta)(\mathrm{diag}(\zeta_0, \zeta_0^*)) = \mathrm{diag}(\theta(\zeta_0), \theta(\zeta_0)^*) = \mathrm{diag}(\zeta, \zeta^*)$. Because v_φ is a product of two symmetries, the same is true for $\mathrm{diag}(\zeta, \zeta^*)$.

For part (c), suppose $\det(u) = 1 - 2\pi$. Set $r = \mathrm{diag}(\pi, 0) \in \mathrm{Proj}(A)$. Notice that $1 - 2r = \mathrm{diag}(1 - 2\pi, 1)$ has determinant $1 - 2\pi$. Then $u \cdot (1 - 2r)$ has determinant 1, so by part (b) there exist $p, q \in \mathrm{Proj}(A)$ such that $u(1 - 2r) = (1 - 2p)(1 - 2q)$. Multiplying on the right by $1 - 2r$, which is its own inverse, gives the desired representation of u.

Finally, part (d) follows from the observation $\mathrm{Sym}(C) = \{1 - 2\pi \mid \pi \in \mathrm{Proj}(C)\}$ and part (c), as follows. Because its generators $1 - 2p$ square to the identity, and the determinant is multiplicative, $\mathrm{Sym}(A) \subseteq \{u \in U(A) \mid \det(u)^2 = 1\}$. Next, if $\det(u)^2 = 1$, then $\det(u)$ is a symmetry in C, and hence of the form $1 - 2\pi$ for some $\pi \in \mathrm{Proj}(C)$, so that $\{u \in U(A) \mid \det(u)^2 = 1\} \subseteq \det^{-1}(\mathrm{Sym}(C))$. Finally, part (c) implies $\det^{-1}(\mathrm{Sym}(C)) \subseteq \mathrm{Sym}(A)$. $\qquad\square$

For AW*-algebras of infinite type, it is known that every unitary is a product of four symmetries [29], and therefore the symmetry group is the full unitary group.

That leaves AW*-algebras of type II_1. For W*-factors of this type, it is known that $\mathrm{Sym}(A) = U(A)$ [6]. If $\mathrm{Sym}(A)$ is closed in $U(A)$, it follows from from [18, Theorem 2], which holds for AW*-algebras, that $\mathrm{Sym}(A) = U(A)$. The general question of whether $\mathrm{Sym}(A) = U(A)$ for AW*-algebras A of type II_1 remains open.

Active lattices. The final piece of structure we will need to be able to recover the full algebra structure of an AW*-algebra is an action of the symmetry group.

Definition 3.9. An *action* of a group G on a piecewise AW*-algebra A is a group homomorphism from G to the group of isomorphisms $A \to A$ in **pAWstar**. Similarly, an action of a group G on a complete orthomodular lattice P is a group homomorphism from G to the group of isomorphisms $P \to P$ in **COrtho**. Explicitly, we can consider a function $G \times P \xrightarrow{\cdot} P$ satisfying:

- $1 \cdot p = p$ for all $p \in P$;
- $u \cdot (v \cdot p) = (uv) \cdot p$ for all $p \in P$ and $u, v \in G$;
- $u \cdot (-) \colon P \to P$ is a morphism of **COrtho** for each $u \in G$.

Alternatively, we can speak about a group homomorphism $\alpha \colon G \to \mathrm{Aut}(P)$. If the object being acted upon needs to be emphasized, we will speak of a *piecewise algebra action* or an *orthomodular action*, respectively.

If A is an AW*-algebra, then its unitary group $U(A)$ acts on its projection lattice $\mathrm{Proj}(A)$ by (left) conjugation: if p is a projection and u is a unitary, then upu^* is again a projection. Moreover, because conjugation with a unitary is an automorphism of AW*-algebras, $u(-)u^* \colon \mathrm{Proj}(A) \to \mathrm{Proj}(A)$ is a morphism of complete orthomodular lattices for each $u \in U(A)$. The group $\mathrm{Sym}(A)$ acts on $\mathrm{Proj}(A)$ by restricting the action of $U(A)$. This motivates the following definition.

Definition 3.10. The category **eAWstar** of *extended piecewise AW*-algebras* is defined as follows. Objects are 4-tuples $(A, P, G, \cdot)$ consisting of:

- a piecewise AW*-algebra A;
- an object P of **COrtho** that maps to $\mathrm{Proj}(A)$ under the forgetful functor **COrtho** $\to$ **pCBool**;
- a group G, that maps to a piecewise subgroup of $U(A)$ under the forgetful functor **Group** $\to$ **pGroup**, and that (contains and) is generated as a group by the elements $1 - 2p$ for all $p \in \mathrm{Proj}(A)$;
- an action of G on A, which restricts to (left) conjugation on $G \subseteq A$, that is, $g \cdot h = ghg^{-1}$ for $g \in G$ and $h \in G \subseteq A$.

A morphism $f \colon (A, P, G, \cdot) \to (A', P', G', \cdot')$ is a function $f \colon A \to A'$ such that:

- f is a morphism of piecewise AW*-algebras;
- f restricts to a morphism $P \to P'$ of complete orthomodular lattices;
- f restricts to a group homomorphism $G \to G'$;
- the equivariance condition $f(u \cdot a) = f(u) \cdot' f(a)$ holds for $u \in G$ and $a \in A$.

In fact, using the equivalence $F \colon$ **pCBool** $\to$ **pAWstar** of Theorem 2.14, we can whittle the data down further. In particular, if a group G has an orthomodular action on P, there is an induced piecewise algebra action on $F(P)$ as follows (applying Lemma 3.3 and Theorem 2.14):

$$G \to \mathrm{Aut}_{\textbf{COrtho}}(P) \subseteq \mathrm{Aut}_{\textbf{pCBool}}(P) \cong \mathrm{Aut}_{\textbf{pAWstar}}(F(P)).$$

Hence we can reformulate purely in terms of orthomodular lattices and groups.

Definition 3.11. An *active lattice* is a 3-tuple $(P, G, \cdot)$ consisting of:

- a complete orthomodular lattice P;
- a group G, that maps to a piecewise subgroup of $U(F(P))$ under the forgetful functor **Group** $\to$ **pGroup**, and that (contains and) is generated as a group by the elements $1 - 2p$ for all $p \in \mathrm{Proj}(F(P)) \cong P$;
- an orthomodular action of G on P such that the induced piecewise algebra action of G on $F(P)$ restricts to (left) conjugation on $G \subseteq F(P)$.

A *morphism of active lattices* $(P, G, \cdot) \to (P', G', \cdot')$ is a morphism $f \colon P \to P'$ of complete orthomodular lattices such that:

- Ff restricts to a group homomorphism $G \to G'$;
- equivariance $f(u \cdot p) = Ff(u) \cdot' f(p)$ holds for all $u \in G$ and $p \in P$.

Active lattices and their morphisms form a category **Active**.

Proposition 3.12. *The categories* **eAWstar** *and* **Active** *are equivalent.*

Proof. We use the unit $\eta_P \colon P \to \mathrm{Proj}(F(P))$ and counit $\varepsilon_A \colon F(\mathrm{Proj}(A)) \to A$ isomorphisms of the equivalence of Theorem 2.14 to define appropriate functors.

Define $G \colon$ **eAWstar** $\to$ **Active** by $G(A, P, G, \alpha) = (P, U(\varepsilon_A^{-1})(G), \alpha \circ U(\varepsilon_A))$ and $G(f) = f$. This is well-defined: if G is a piecewise subgroup of $U(A)$, then $U(\varepsilon_A^{-1})(G)$ is a piecewise subgroup of $U(F(P))$, and precomposing the action $\alpha \colon G \to \mathrm{Aut}(P)$ with $U(\varepsilon_A)$ turns it into an action of $U(\varepsilon_A^{-1}(G))$ on P. The equivariance condition on morphisms also follows directly.

In the reverse direction, define $H \colon$ **Active** $\to$ **eAWstar** on objects by setting $H(P, G, \alpha) = (F(P), \eta_P(P), G, \mathrm{Aut}(\eta_P^{-1}) \circ \alpha)$ and on morphisms by $H(f) = F(f)$. This is well-defined: the structure of P as a complete orthomodular lattice transfers via η_P to $\eta_P(P) = \mathrm{Proj}(F(P))$, and postcomposing the action $\alpha \colon G \to \mathrm{Aut}(P)$ with $\mathrm{Aut}(\eta_P^{-1})$ turns it into an action of G on $\mathrm{Proj}(F(P))$. The equivariance condition on morphisms also follows directly.

Now η_P implements a (natural) isomorphism $G \circ H(P, G, \cdot) \cong (P, G, \cdot)$, and ε_A implements a (natural) isomorphism $H \circ G(A, P, G, \cdot) \cong (A, P, G, \cdot)$. Hence G and H form an equivalence. $\square$

The functor. We can now define a functor from AW*-algebras to active lattices, and prove that it is faithful. In Section 4 we will prove that it is also full. The next proposition tacitly identifies a piecewise AW*-algebra A with $F(\mathrm{Proj}(A))$, as justified by Theorem 2.14.

Proposition 3.13. *There is a functor* $\mathrm{AProj} \colon$ **AWstar** $\to$ **Active** *acting as*

$$\mathrm{AProj}(A) = (\mathrm{Proj}(A), \mathrm{Sym}(A), \cdot),$$

on objects, where $u \cdot p = upu^*$. *It sends a morphism* $A \to B$ *to its restriction* $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$.

Proof. Follows directly from the definitions. $\square$

Via Proposition 3.12, we also write AProj for the functor **AWstar** $\to$ **eAWstar**.

Lemma 3.14. *The functor* AProj *is faithful.*

Proof. If $\mathrm{AProj}(f) = \mathrm{AProj}(f')$, the continuous linear functions $f, f' \colon A \to B$ coincide on $\mathrm{Proj}(A)$. But A is the closed linear span of $\mathrm{Proj}(A)$. $\square$

The reader might think that Definition 3.11 could be reduced further still by considering just complete orthomodular lattices acted upon by groups generated by them, and letting morphisms be equivariant pairs of group homomorphisms and morphisms of complete orthomodular lattices. The following example shows that one cannot ignore piecewise algebra structure this easily and hope to have a full and faithful functor out of **AWstar**.

Example 3.15. Consider $\mathrm{AProj}(\mathbb{M}_2(\mathbb{C})) = (\mathrm{Proj}(\mathbb{M}_2(\mathbb{C})), \mathrm{Sym}(\mathbb{M}_2(\mathbb{C})), \cdot)$. Define a morphism of complete orthomodular lattices $f \colon \mathrm{Proj}(\mathbb{M}_2(\mathbb{C})) \to \mathrm{Proj}(\mathbb{M}_2(\mathbb{C}))$ by $f(0) = 0$, $f(1) = 1$, and $f(p) = p^\perp$ for $p \neq 0, 1$. Recall from Lemma 3.8 that $\mathrm{Sym}(\mathbb{M}_2(\mathbb{C})) = \{u \in U_2(\mathbb{C}) \mid \det(u) = \pm 1\}$. Define a group homomorphism $g \colon \mathrm{Sym}(\mathbb{M}_2(\mathbb{C})) \to \mathrm{Sym}(\mathbb{M}_2(\mathbb{C}))$ by $g(u) = \det(u)u$. Write j for the injection $\mathrm{Proj}(\mathbb{M}_2(\mathbb{C})) \to \mathrm{Sym}(\mathbb{M}_2(\mathbb{C}))$ given by $j(p) = 1 - 2p$. For $p = 0, 1$ one easily checks that $j(f(p)) = g(j(p))$, and for $p \neq 0, 1$:

$$j(f(p)) = j(p^\perp) = p^\perp - p = \det(p - p^\perp) \cdot (p - p^\perp) = g(p - p^\perp) = g(j(p)).$$

Finally, for $u \in \mathrm{Sym}(\mathbb{M}_2(\mathbb{C}))$ and $p \neq 0, 1$:

$$g(u)f(p)g(u)^* = |\det(u)|^2 u p^\perp u^* = 1 - u p u^* = f(u p u^*),$$

and for $p = 0, 1$ this formula is also easily seen to hold. Hence f and g satisfy the equivariance condition.

But if there is a linear map $h \colon \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})$ that restricts to f on $\mathrm{Proj}(\mathbb{M}_2(\mathbb{C}))$ and to g on $\mathrm{Sym}(\mathbb{M}_2(\mathbb{C}))$, then for $\zeta \in U(\mathbb{C})\backslash\{\pm 1\}$, $p \in \mathrm{Proj}(\mathbb{M}_2(\mathbb{C}))\backslash\{0, 1\}$, and $u = \zeta p + \zeta^* p^\perp \in \mathrm{Sym}(\mathbb{M}_2(\mathbb{C}))$, we would have

$$u = g(u) = g(\zeta p + \zeta^* p^\perp) = \zeta f(p) + \zeta^* f(p^\perp) = \zeta p^\perp + \zeta^* p = u^*,$$

contradicting $\zeta \neq \pm 1$. Therefore it cannot be the case that h restricts to f.

In the commutative case, the functor AProj has nice properties.

Example 3.16. There is a functor **CBool** $\to$ **Active**, that maps a complete Boolean algebra B to the active lattice $(B, B_{\mathrm{add}}, \cdot)$. Here, we identify B with $\mathrm{Proj}(F(B))$ using Theorem 2.14, and B_{add} is the additive group of B qua Boolean ring, which acts trivially on the Boolean algebra B itself. This functor is full and faithful. Moreover, it factors through the functor AProj. If we restrict to the full subcategory **cActive** of **Active** consisting of the objects $(P, G, \cdot)$ for which P is a complete Boolean algebra, then it follows from Example 3.7 that the functor AProj becomes an equivalence of categories. This makes the left triangle in the following diagram commute. The other faces obviously commute.
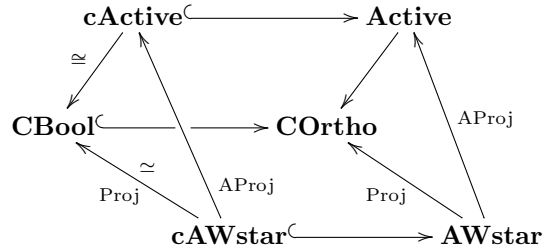
4. Recovering total algebras from piecewise algebras

This section proves that the functor AProj of Proposition 3.13 is full. The proof distinguishes two cases. First, we adapt a theorem of Dye to deal with algebras without type I_2 summands. Subsequently we deal with algebras of type I_2 directly.

Algebras without I_2 summand and a theorem of Dye. To facilitate the proof of Theorem 4.6 below, we give a sequence of preparatory lemmas. Several of these are adapted from Dye's results in [11, Section 3]. Let A be an AW*-algebra. Any matrix ring $\mathbb{M}_n(A)$ is an AW*-algebra; see [2, Section 62]. If x is a row vector in A^n one of whose entries is a projection, then there is a projection in $\mathbb{M}_n(A)$ whose range is the submodule $Ax \subseteq A^n$ according to [11, Lemma 2]. We shall refer to these projections in $\mathbb{M}_n(A)$ as *vector projections*.

In particular, given two distinct indices $1 \le i, j \le n$ and an element $\alpha \in A$, there is a projection as above where the vector x is taken to have 1 in the ith entry, α in the jth entry, and all other entries zero. We denote the corresponding projection in $\mathbb{M}_n(A)$ by $p_{ij}(\alpha)$. For instance, when $n = 2$, $i = 1$, and $j = 2$, we have

$$p_{12}(\alpha) = \begin{pmatrix} (1 + \alpha\alpha^*)^{-1} & (1 + \alpha\alpha^*)^{-1}\alpha \\ \alpha^*(1 + \alpha\alpha^*)^{-1} & \alpha^*(1 + \alpha\alpha^*)^{-1}\alpha \end{pmatrix}.$$

For larger n, we follow the convention to only write down the nonzero 2-by-2 parts of such n-by-n matrices. Notice that if $p_{ij}(\alpha) = p_{ij}(\beta)$ for some $\alpha, \beta \in A$, then $\alpha = \beta$.

Lemma 4.1. *Let A be an AW*-algebra.*

(a) Sublattices of $\mathrm{Proj}(\mathbb{M}_n(A))$ containing all $p_{ij}(\alpha)$ contain all vector projections.
(b) Any projection in $\mathbb{M}_n(A)$ is the supremum of (orthogonal) vector projections.

Hence the $p_{ij}(\alpha)$ generate $\mathrm{Proj}(\mathbb{M}_n(A))$ as a complete orthomodular lattice.

Proof. Part (a) is proven as in [11, Lemma 7]. For (b), first note that the proof of [11, Lemma 7] illustrates that every nonzero element of $\mathrm{Proj}(\mathbb{M}_n(A))$ contains a nonzero vector projection. Fix $p \in \mathrm{Proj}(\mathbb{M}_n(A))$. Zorn's lemma gives a maximal set S of orthogonal nonzero homogeneous projections below p. We claim that p equals $p_0 = \bigvee S$. Otherwise $p_0 < p$, so that there would be a nonzero vector projection $q \le p - p_0$. Because $p - p_0 \le p$, transitivity gives $q \le p$. Combined with $q \le p - p_0$, this implies q is orthogonal to p_0. It follows that q is orthogonal to $\mathrm{Proj}(S)$, so $S \sqcup \{q\}$ is an orthogonal set of projections below p, contradicting maximality. □

We denote by $e_{ij} \in \mathbb{M}_n(A)$ the matrix unit whose i, j-entry is 1 and every other entry is zero. Note that $e_{ii} = p_{ij}(0)$ for any $j \ne i$. For a projection p in an AW*-algebra, we denote by $s_p = 1 - 2p$ the corresponding symmetry.

Lemma 4.2. *Let A and B be AW*-algebras. If $f \colon \mathrm{Proj}(\mathbb{M}_n(A)) \to \mathrm{Proj}(\mathbb{M}_n(B))$ is a function satisfying $f(e_{ii}) = e_{ii}$ and $f(s_p q s_p) = s_{f(p)} f(q) s_{f(p)}$, then for each i, j and each $\zeta \in U(A)$ there is a unique $\xi \in U(B)$ with $f(p_{ij}(\zeta)) = p_{ij}(\xi)$.*

Proof. Notice that for $\zeta \in U(A)$, we have (in "2-by-2 shorthand")

$$p_{ij}(\zeta) = \frac{1}{2}\begin{pmatrix} 1 & \zeta \\ \zeta^* & 1 \end{pmatrix}.$$

It is easy to see that conjugation by $1 - 2p_{ij}(\zeta)$ swaps e_{ii} and e_{jj} while leaving the remaining diagonal matrix units fixed. Conversely, if $p \in \mathrm{Proj}(\mathbb{M}_n(A))$ is such that

conjugation by $1 - 2p$ leaves e_{kk} fixed for $k \neq i, j$, then it must equal the identity everywhere except in rows and columns i and j. Hence we can write $p = \left(\begin{smallmatrix} \alpha & \beta \\ \beta^* & \gamma \end{smallmatrix} \right)$ in "2-by-2 shorthand". If $e_{ii} = (1 - 2p)e_{jj}(1 - 2p)$, then $\alpha = \frac{1}{2}$ and $\beta^*\beta = \frac{1}{4}$, and it follows from $p = p^2$ that $\gamma = \frac{1}{2}$ and $\beta\beta^* = \frac{1}{4}$. Hence the projections of the form $p_{ij}(\zeta)$ with ζ unitary are precisely those projections p for which conjugation with $1 - 2p$ swaps e_{ii} and e_{jj} while leaving the other e_{kk} fixed.

Now, because of the assumptions that f sends diagonal matrix units to diagonal matrix units, and is equivariant, the same statement is true about $f(p_{ij}(\zeta))$. Hence there is some unitary $\xi \in U(B)$ such that $f(p_{ij}(\zeta)) = p_{ij}(\xi)$; uniqueness follows. $\square$

Recall that a $\mathbb{C}$-linear function $f \colon A \to B$ between C^*-algebras that preserves the involution is a *Jordan $*$-homomorphism* if it preserves the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$; this is readily seen to be equivalent to the property that f preserves the square of every element.

Lemma 4.3. *Given a $*$-ring homomorphism $A \to B$ between C^*-algebras, there is a unique Jordan $*$-homomorphism $A \to B$ that equals it on self-adjoint elements.*

Proof. Let $f \colon A \to B$ be a $*$-ring homomorphism. As it preserves 1 it is $\mathbb{Q}$-linear, and it follows from preserving positivity that it is in fact $\mathbb{R}$-linear. Define complementary projections $q_- = \frac{1}{2}(1 + if(i))$ and $q_+ = \frac{1}{2}(1 - if(i))$ in B. Setting

$$f_- \colon A \to q_- B q_- \qquad\qquad f_-(a) = \tfrac{1}{2}(f(a) + if(ia))$$
$$f_+ \colon A \to q_+ B q_+ \qquad\qquad f_+(a) = \tfrac{1}{2}(f(a) - if(ia))$$

gives $*$-ring homomorphisms, where f_- is $\mathbb{C}$-anti-linear and f_+ is $\mathbb{C}$-linear. Clearly $f = f_+ + f_-$. Now define $g \colon A \to B$ by

$$g(a) = f_+(a) + (f_-(a))^*.$$

This $\mathbb{C}$-linear function preserves the involution and agrees with f on self-adjoint elements. It is easy to verify that it preserves the operation of squaring because the images of f_+ and f_- are orthogonal in B. Uniqueness is straightforward. $\square$

The following lemma records some results of Dye [11] about properties of the "coordinate assignment" from Lemma 4.2. Basically, it expresses algebraic operations on the coordinates in lattice-theoretic terms. The subsequent lemma will use these properties to establish a $*$-ring homomorphism, following [11, Lemmas 6 and 8]. Recall that a *lattice polynomial* is an expression combining a finite number of variables using $\wedge$ and $\vee$; these are preserved by morphisms in **COrtho**.

Lemma 4.4. *There exist lattice polynomials P, Q, and R such that, for any elements α, β, γ of a C^*-algebra A with γ invertible, any integer $n \geq 3$, and any distinct indices $1 \leq i, j, k \leq n$, the following hold:*

(a) $p_{ij}(\alpha + \beta) = P\big(p_{ij}(\alpha), p_{ij}(\beta), p_{ik}(\gamma), e_{ii}, e_{jj}, e_{kk}\big)$;
(b) $p_{ij}(-\alpha\beta) = Q\big(p_{ik}(\alpha), p_{kj}(\beta), e_{ii}, e_{jj}\big)$;
(c) $p_{ij}(-\alpha^*) = R\big(p_{ji}(\alpha), e_{ii}, e_{jj}\big)$.

Proof. See [11, Lemma 5], [11, Lemma 4], and [11, Lemma 3(i)], respectively. $\square$

Lemma 4.5. *Let $f \colon \mathrm{Proj}(\mathbb{M}_n(A)) \to \mathrm{Proj}(\mathbb{M}_n(B))$ be a morphism of **COrtho** for AW*-algebras A, B, and $n \geq 3$. Suppose $f(e_{ii}) = e_{ii}$ for all i, and that for any distinct i, j and any $\zeta \in U(A)$ there is $\xi \in U(B)$ with $f(p_{ij}(\zeta)) = p_{ij}(\xi)$. Then there is a diagonal $W \in U(\mathbb{M}_n(B))$ such that:*

(a) there is a function $\varphi\colon U(A) \to U(B)$ satisfying the "coordinate condition"

$$f(p_{ij}(\alpha)) = W^* p_{ij}(\varphi(\alpha))W$$

for all $\alpha \in U(A)$ and distinct indices i, j;

(b) φ extends to a $*$-ring homomorphism $A \to B$ satisfying the coordinate condition for all $\alpha \in A$ and distinct i, j.

Proof. Abbreviate the coordinate condition as $(*)$. By hypothesis, for all indices $j > 1$ there exist $\beta_j \in U(B)$ such that $f(p_{1j}(1)) = p_{1j}(\beta_j)$. Define $W = \mathrm{diag}(1, \beta_2, \ldots, \beta_n) \in U(B)$. Then $p_{1j}(\beta_j) = W^* p_{1j}(1)W$ for all j. Notice that conjugation by a diagonal unitary fixes all e_{ii}, and leaves the set $\{p_{ij}(\alpha)\}$ invariant as α ranges over $U(A)$ (respectively, over A). Thus, replacing f with the morphism $p \mapsto W f(p)W^*$, we may assume that $f(p_{1j}(1)) = p_{1j}(1)$ for all $j > 1$, and prove that $(*)$ holds in both (a) and (b) with $W = 1$.

Towards (a), define $\varphi\colon U(A) \to U(B)$ by the condition $f(p_{12}(\alpha)) = p_{12}(\varphi(\alpha))$. In case $f(p_{1j}(\alpha)) = p_{1j}(\varphi(\alpha))$, for some $\alpha \in U(A)$ and distinct $i, j > 1$, it follows by applying Lemma 4.4 that

$$
\begin{aligned}
f(p_{ij}(\alpha)) &= f\big(Q\big(p_{i1}(-1), p_{1j}(\alpha), e_{ii}, e_{jj}\big)\big) \\
&= f\big(Q\big(R\big(p_{1i}(1), e_{ii}, e_{11}\big), p_{1j}(\alpha), e_{ii}, e_{jj}\big) \\
&= Q\big(R\big(p_{1i}(1), e_{ii}, e_{11}\big), p_{1j}(\varphi(\alpha)), e_{ii}, e_{jj}\big) = p_{ij}(\varphi(\alpha)).
\end{aligned}
$$

In particular, because $(*)$ is known to hold in case $i = 1$ and $\alpha = 1$, this shows that $(*)$ in fact holds for $\alpha = 1$ and any distinct $i, j > 1$ (and, of course, when $i = 1$ and $j = 2$). Now since $(*)$ for the case $\alpha = 1$ and $j = 2$, and it holds by assumption for $i = 1$, $j = 2$ and all $\alpha \in U(A)$, then for $j > 2$ we find:

$$
\begin{aligned}
f(p_{1j}(\alpha)) &= f\big(Q\big(p_{12}(\alpha), R\big(p_{j2}(1), e_{22}, e_{jj}\big), e_{11}, e_{jj}\big)\big) \\
&= Q\big(p_{12}(\varphi(\alpha)), R\big(p_{j2}(1), e_{22}, e_{jj}\big), e_{11}, e_{jj}\big)\big) = p_{1j}(\varphi(\alpha)).
\end{aligned}
$$

Thus the above shows that $(*)$ holds for all $\alpha \in U(A)$ and any $j \geq 2$. For the remaining case where $i > 1$ and $j = 1$, simply note that

$$
\begin{aligned}
f(p_{i1}(\alpha)) &= f(R\big(p_{1i}(-\alpha^*), e_{ii}, e_{11}\big)) \\
&= R\big(p_{1i}(-\alpha^*), e_{ii}, e_{11}\big) = p_{i1}(\alpha).
\end{aligned}
$$

To prove part (b), we start by defining a function $\psi\colon A \to B$ on self-adjoint elements α of A. Set $\zeta = \frac{\alpha}{2n} + i\sqrt{1 - (\frac{\alpha}{2n})^2}$, where n is an integer satisfying $\|\alpha\| \leq 2n$. Then $\zeta \in U(A)$ satisfies $\zeta + \zeta^* = \frac{\alpha}{n}$. Now, an application of Lemma 4.4(a) with $c = 1$ shows $f(p_{ij}(\alpha_1 + \alpha_2)) = p_{ij}(\beta_1 + \beta_2)$ if $f(p_{ij}(\alpha_k)) = p_{ij}(\beta_k)$. Therefore, in particular, $f(p_{ij}(\frac{\alpha}{n})) = f(p_{ij}(\zeta + \zeta^*)) = p_{ij}(\varphi(\zeta) + \varphi(\zeta^*))$, and there is a $\beta \in B$ with $f(p_{ij}(\alpha)) = p_{ij}(\beta)$. Setting $\psi(\alpha) = \beta$ satisfies $f(p_{ij}(\alpha)) = p_{ij}(\psi(\alpha))$ for self-adjoint $\alpha \in A$. Because arbitrary elements $\alpha \in A$ can be written as $\alpha_1 + i\alpha_2$ for self-adjoint α_k, we may define $\psi(\alpha) = \psi(\alpha_1) + i\psi(\alpha_2)$, still satisfying $f(p_{ij}(\alpha)) = p_{ij}(\psi(\alpha))$. It follows that $p_{ij}(\psi(\alpha)) = p_{ij}(\varphi(\alpha))$ for unitary α, whence ψ extends φ.

Next we prove that ψ is a $*$-ring homomorphism. First apply Lemma 4.4(a) with $c = 1$ and use part (a) to deduce

$$
\begin{aligned}
p_{ij}\big(\psi(\alpha) + \psi(\beta)\big) &= P\big(p_{ij}(\psi(\alpha)), p_{ij}(\psi(\beta)), p_{ik}(\psi(1)), e_{ii}, e_{jj}, e_{kk}\big) \\
&= P\big(f(p_{ij}(\alpha)), f(p_{ij}(\beta)), f(p_{ik}(1)), f(e_{ii}), f(e_{jj}), f(e_{kk})\big) \\
&= f\big(P\big(p_{ij}(\alpha), p_{ij}(\beta), p_{ik}(1), e_{ii}, e_{jj}, e_{kk}\big)\big) \\
&= f(p_{ij}(\alpha + \beta)) = p_{ij}\big(\psi(\alpha + \beta)\big),
\end{aligned}
$$

and conclude that ψ is additive. Hence also $\psi(0) = \psi(0+0) - \psi(0) = 0$. It similarly follows from Lemma 4.4(b) that ψ is multiplicative. Finally, Lemma 4.4(c) shows that ψ preserves the involution. □

The assumption that each $\zeta \in U(A)$ allows $\xi \in U(A)$ such that $f(p_{ij}(\zeta)) = p_{ij}(\xi)$ is slightly stronger than necessary and is only used to shorten the proof above. With more work, one may simply assume that this is the case when $i = 1$ and $j = 2$.

We are now ready to prove an AW*-analogue of Dye's theorem [11, Theorem 1].

Theorem 4.6. *Let A be an AW*-algebra without type I_2 summands, and B any AW*-algebra. An **COrtho**-morphism $f\colon \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ extends to a Jordan $*$-homomorphism $A \to B$ if and only if $f(s_p q s_p) = s_{f(p)} f(q) s_{f(p)}$.*

Proof. The "only if" direction follows because the expression to be preserved can be written in terms of Jordan operations:

$$
\begin{aligned}
s_p q s_p = (1 - 2p)q(1 - 2p) &= q - 2(pq + qp) + 4pqp \\
&= q - 2(p \circ q) + 4(pqp + p^{\perp}qp^{\perp}) \circ p \\
&= q - 2(p \circ q) + 4(p + q - 1)^2 \circ p.
\end{aligned}
$$

For the converse we first reduce the problem to the case $A = \mathbb{M}_n(D)$ for $n \geq 3$ and AW*-algebra D. Indeed, [2, Theorem 15.3] and [2, Theorem 18.4] provide unique orthogonal central projections $p_1, p_2, \ldots, p_\infty$ with sum 1 such that $p_n A$ is of type I_n for $n < \infty$ and $p_\infty A$ has no finite type I summands. Then A is the C*-sum of $p_i A$ [2, Proposition 10.2], and it suffices to consider one summand at a time because Jordan $*$-homomorphisms are closed under direct sums. By [2, Exercise 19.2], $p_\infty A \cong \mathbb{M}_3(D)$ for some AW*-algebra D. For each finite n, by [2, Proposition 18.2], $p_n A \cong \mathbb{M}_n(C)$ for some commutative AW*-algebra C. By assumption $p_2 = 0$, leaving us with commutative AW*-algebras $p_1 A$. But this case is taken care of by the duality (2.3), since morphisms in **cAWstar** are definitely Jordan $*$-homomorphisms. Thus we may replace A with $\mathbb{M}_n(A)$ for $n \geq 3$.

Next, we make another reduction (replicating the proof of [11, Theorem 1]) to show that we may also replace B with $\mathbb{M}_n(B)$. Any two distinct diagonal matrix units e_{ii} in $\mathbb{M}_n(A)$ have a common complement, so the same is true for their images under f. By [21, Theorem 6.6] this means that their images $f(e_{ii})$ are equivalent projections. These n orthogonal equivalent projections sum to 1, so by [2, Proposition 16.1] they form the diagonal projections of a set of n-by-n matrix units in B. Thus we may replace B by $\mathbb{M}_n(B)$ and assume that $f(e_{ii}) = e_{ii}$.

So we are assuming that A and B are AW*-algebras with an **COrtho**-morphism $f\colon \mathrm{Proj}(\mathbb{M}_n(A)) \to \mathrm{Proj}(\mathbb{M}_n(B))$ for $n \geq 3$. Combining Lemmas 4.2 and 4.5 produces a $*$-ring homomorphism $\varphi\colon A \to B$ and a diagonal $W \in U(M_n(B))$ such that $f(p_{ij}(\alpha)) = W^* p_{ij}(\varphi(\alpha))W$ for all $\alpha \in A$ and all distinct i, j. It follows from the definition of p_{ij} that $f(p_{ij}(\alpha)) = W^*\big(\mathbb{M}_n \varphi(p_{ij}(\alpha))\big)W$ for all i, j and $\alpha \in A$.

Next we show that φ preserves suprema of projections, using an auxiliary function $\pi \mapsto p_{12}(\pi) \wedge e_{22}$. It is a well-defined morphism $j_A \colon \operatorname{Proj}(A) \to \operatorname{Proj}(M_n(A))$ of complete orthomodular lattices that is injective. Hence

$$j_B(\bigvee_i \varphi(\pi_i)) = \bigvee_i p_{12}(\varphi(\pi_i)) \wedge e_{22} = \bigvee_i Wf(j_A(\pi_i))W^*$$

$$= Wf(j_A(\bigvee_i \pi_i))W^* = p_{12}(\varphi(\bigvee_i \pi_i)) \wedge e_{22} = j_B(\varphi(\bigvee_i \pi_i)),$$

and so $\bigvee_i \varphi(\pi_i) = \varphi(\bigvee_i \pi_i)$ by injectivity of j_B.

Consequently, the $*$-ring homomorphism $\mathbb{M}_n(\varphi) \colon \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ also preserves suprema of projections by [15, Theorem 8.2 and Remark 8.3]. Hence so does its conjugation with W. Now Lemma 4.1 guarantees that $W^*\mathbb{M}_n(\varphi)W$ equals f on all of $\operatorname{Proj}(\mathbb{M}_n(A))$. The proof is concluded by an application of Lemma 4.3. $\square$

It remains an open question whether every morphism of complete orthomodular lattices $\operatorname{Proj}(A) \to \operatorname{Proj}(B)$ extends to a Jordan $*$-homomorphism $A \to B$ when A and B are AW*-algebras and A has no type I$_2$ summands. This is known to be the case when A and B are W*-algebras [8, Corollary 1]. Our proof suffices to answer this question for AW*-algebras if Lemma 4.2 holds more generally without the equivariance assumption.

Lemma 4.7. *Let A and B be AW*-algebras. If $f \colon \operatorname{AProj}(A) \to \operatorname{AProj}(B)$ is a morphism in* **eAWstar** *such that f extends to a continuous $\mathbb{C}$-linear function $g \colon A \to B$, then g is a morphism of AW*-algebras satisfying $\operatorname{AProj}(g) = f$.*

Proof. We first show that $g(a)g(b) = g(ab)$ for all $a, b \in A$. Because the functions $A \times A \to A$ on each side of the equation above are continuous and bilinear, and because A is the closed linear span of its projections, it suffices to consider the case where a and b are projections. Now, for $p, q \in \operatorname{Proj}(A)$,

$$1 - 2g(p) - 2g(q) + 4g(pq) = g(1 - 2p - 2q + 4pq)$$
$$= f((1 - 2p)(1 - 2q))$$
$$= f(1 - 2p)f(1 - 2q)$$
$$= (1 - 2f(p))(1 - 2f(q))$$
$$= 1 - 2g(p) - 2g(q) + 4g(p)g(q),$$

and therefore $g(pq) = g(p)g(q)$ as desired. The above equations used, respectively: linearity of g; g extends f; f is a group homomorphism on $\operatorname{Sym}(A)$; f is a piecewise algebra morphism; g extends f.

So g is an algebra homomorphism, and it is readily seen to be a $*$-homomorphism using linearity and the fact that it equals f on normal elements. Because f preserves suprema of projections and g extends it, we see that g is a morphism in **AWstar**, which obviously satisfies $\operatorname{AProj}(g) = f$. $\square$

Corollary 4.8. *Let A and B be AW*-algebras, and $f \colon \operatorname{AProj}(A) \to \operatorname{AProj}(B)$ a morphism of* **eAWstar**. *If A has no type I$_2$ summand, f is in the image of AProj.*

Proof. Theorem 4.6 extends $f \colon \operatorname{Proj}(A) \to \operatorname{Proj}(B)$ to a Jordan $*$-homomorphism $g \colon A \to B$, which is continuous [27, Page 439]. Because A is the closed linear span of $\operatorname{Proj}(A)$, in fact f and g coincide as functions $N(A) \to N(B)$. Hence the result follows from Lemma 4.7. $\square$

Type I_2 algebras. Next we focus on algebras of type I_2. As in Lemma 3.8, we will use the fact that determinants and traces are well-defined for matrices with entries in a commutative ring.

Proposition 4.9. *Let A and B be AW*-algebras, and $f\colon \mathrm{AProj}(A) \to \mathrm{AProj}(B)$ a morphism of* **eAWstar**. *If A is type I_2, then f is in the image of* AProj.

Proof. Let C be a commutative AW*-algebra with $A = \mathbb{M}_2(C)$; this exists by [2, Proposition 18.2]. Fix $p = e_{11} \in \mathrm{Proj}(A)$ and $u = e_{12} + e_{21} \in \mathrm{Sym}(A)$. Since $upu = p^\perp$, we deduce

$$f(u)f(p)f(u) = f(p)^\perp,$$
$$f(u)f(p) = f(p)^\perp f(u),$$
$$f(p)f(u) = f(u)f(p)^\perp.$$

It follows that $e'_{11} = f(p)$, $e'_{12} = f(p)f(u)$, $e'_{21} = f(u)f(p)$, and $e'_{22} = f(p)^\perp$ form a self-adjoint set of 2-by-2 matrix units in B (see [19, Page 429]). The image $D = f(C) \subseteq B$ is a commutative $*$-subalgebra centralizing all e'_{ij}. Letting $V \subseteq B$ be the D-span of the e'_{ij}, it follows that V is a $*$-subalgebra of B isomorphic to $\mathbb{M}_2(D)$. Define a C-linear function $g\colon A \to V \subseteq B$ by $g(e_{ij}) = e'_{ij}$ and $g(\gamma) = f(\gamma)$ for $\gamma \in C$; it is a $*$-homomorphism.

Next we will prove that g equals f on all $q \in \mathrm{Proj}(A)$. Notice that $\det(q)$ is a projection in C. Using properties of the adjugate matrix [22, XIII.4.16] we find $\det(q)1_A = \mathrm{adj}(q)q = \mathrm{adj}(q)q^2 = \det(q)q$, and so $\det(q)1_A \le q$ in $\mathrm{Proj}(A)$. Because $\mathrm{adj}\colon \mathbb{M}_2(C) \to \mathbb{M}_2(C)$ is C-linear, the projection $q' = q - \det(q)1_A$ has determinant

$$\det(q') = \mathrm{adj}(q')q' = (\mathrm{adj}(q) - \det(q)1_A)(q - \det(q)1_A) = 0.$$

So without loss of generality we may assume $\det(q) = 0$. In this case one can compute that $\tau = \mathrm{tr}(q)$ is a projection of C. As any projection in A with trace τ and determinant zero, q can be written (in standard matrix units e_{ij}) in the form

$$q = \frac{1}{2}\begin{pmatrix} \tau + \alpha & \zeta\beta \\ \zeta^*\beta & \tau - \alpha \end{pmatrix}$$

where $\alpha, \beta \in C$ are self-adjoint, satisfy $\alpha^2 + \beta^2 = \tau$, and $\zeta \in C$ is a partial isometry with $\zeta\zeta^*\beta = \beta$. Replacing ζ with $\zeta + (1 - \zeta\zeta^*)$ if necessary, we may in fact assume $\zeta \in U(C)$. Because the algebra C has square roots [10, Corollary 2.3], there exists $\xi \in U(C)$ such that $i\xi^2 = \zeta$. From $\alpha^2 + \beta^2 = \tau$ one deduces that τ supports α and β, so $\tau^\perp$ annihilates α and β. Then

$$(*) \quad 1 - 2q = \begin{pmatrix} \tau^\perp - \alpha & -\zeta\beta \\ -\zeta^*\beta & \tau^\perp + \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \tau - \tau^\perp & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\tau^\perp - \alpha & -i\xi^2\beta \\ i(\xi^*)^2\beta & \tau^\perp + \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \tau - \tau^\perp & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\xi & 0 \\ 0 & \xi^* \end{pmatrix}\begin{pmatrix} \tau^\perp + \alpha & i\beta \\ i\beta & \tau^\perp + \alpha \end{pmatrix}\begin{pmatrix} \xi^* & 0 \\ 0 & \xi \end{pmatrix}$$
$$= \big((\tau - \tau^\perp)p + p^\perp\big)\big(-\xi p + \xi^* p^\perp\big)\big((\tau^\perp + \alpha)1 + i\beta u\big)\big(\xi^* p + \xi p^\perp\big).$$

The four factors in the right hand side are elements of $\mathrm{Sym}(A)$ by Lemma 3.8(d). Because f is piecewise linear and is multiplicative when restricted to $\mathrm{Sym}(\mathbb{M}_2(C))$,

applying f to $(*)$ and invoking piecewise linearity gives

$$
\begin{aligned}
1 - 2f(q) &= f\big((\tau - \tau^\perp)p + p^\perp\big)f\big(-\xi p + \xi^* p^\perp\big)f\big((\tau^\perp + \alpha)1 + i\beta u\big)f\big(\xi^* p + \xi p^\perp\big)\\
&= (\tau^\perp - \alpha)f(p) - \zeta\beta f(p)f(u) - \zeta^*\beta f(u)f(p) + (\tau^\perp + \alpha)f(p)^\perp\\
&= (\tau^\perp - \alpha)g(E_{11}) - \zeta\beta g(E_{12}) - \zeta^*\beta g(E_{21}) + (\tau^\perp + \alpha)g(E_{22})\\
&= g(1 - 2q) = 1 - 2g(q),
\end{aligned}
$$

whence $f(q) = g(q)$. Finally, because $*$-homomorphisms are continuous, an application of Lemma 4.7 finishes the proof. $\qquad\square$

Fullness of the functor and some open problems. We summarize by showing that $\mathrm{AProj}\colon \mathbf{AWstar} \to \mathbf{Active}$ is a full functor.

Theorem 4.10. *If A and B are AW*-algebras, and $f\colon \mathrm{AProj}(A) \to \mathrm{AProj}(B)$ is a morphism in* $\mathbf{Active}$*, then $f = \mathrm{AProj}(g)$ for some $g\colon A \to B$ in* $\mathbf{AWstar}$*.*

Proof. Proposition 3.12 allows us to replace $\mathbf{Active}$ by $\mathbf{eAWstar}$. As any AW*-algebra, A is a direct sum $A = p_1 A \oplus p_2 A$ for central projections p_1 and $p_2 = 1 - p_1$, where $p_1 A$ is a type I_2 AW*-algebra and $p_2 A$ is an AW*-algebra without type I_2 summands [2, Section 15]. Because the p_i are central and sum to 1 in A, they are central in $\mathrm{Sym}(A)$. As f is a morphism in $\mathbf{Active}$, the same statements are true for the projections $q_i = f(p_i)$ in B. Thus f restricts to two morphisms $f_i\colon \mathrm{AProj}(p_i A) \to \mathrm{AProj}(q_i B)$ of $\mathbf{eAWstar}$. Corollary 4.8 provides a morphism $g_1\colon p_1 A \to q_1 B$ of $\mathbf{AWstar}$ with $\mathrm{AProj}(g_1) = f_1$, and Proposition 4.9 provides a morphism $g_2\colon p_2 A \to q_2 B$ of $\mathbf{AWstar}$ with $\mathrm{AProj}(g_2) = f_2$. Their sum $g\colon A \to B$, defined by $g(a) = g_1(p_1 a) + g_2(p_2 a)$, is a morphism of $\mathbf{AWstar}$ satisfying $\mathrm{AProj}(g) = f$. $\qquad\square$

Corollary 4.11. $\mathbf{AWstar}$ *is equivalent to a full subcategory of* $\mathbf{Active}$*.*

Proof. Follows directly from Lemma 3.14 and Theorem 4.10. $\qquad\square$

This corollary immediately presents the problem of characterizing those active lattices in the essential image of AProj. That is, for which active lattices $(P, G, \cdot)$ does there exist an AW*-algebra A such that $(P, G, \cdot) \cong \mathrm{AProj}(A)$ as active lattices? This is a *coordinatization problem,* reminiscent of von Neumann's coordinatization of continuous geometries by continuous regular rings [31]. The authors are currently unaware of any active lattices that are not in the essential image of AProj. A solution to this problem should provide deeper insight into how exactly the active lattice $\mathrm{AProj}(A)$ "encodes" the ring structure of an AW*-algebra A.

In conclusion, our results also suggest the following natural question for general C*-algebras: can one reconstruct a C*-algebra A from the piecewise C*-algebra $N(A)$, the unitary group $U(A)$, and the action by conjugation of the latter on the former? The following lemma shows that this comes down to a Mackey–Gleason type problem once again.

Proposition 4.12. *Let A, B be C*-algebras, and $f\colon N(A) \to N(B)$ a morphism of piecewise C*-algebras that restricts to a group homomorphism $U(A) \to U(B)$. Then f extends to a $*$-homomorphism $A \to B$ if and only if it is additive on self-adjoints.*

Proof. One direction is obvious. For the other, assume that f is additive on self-adjoints. Since any $a \in A$ can be written as $a = a_1 + a_2$ for self-adjoint $a_1 = \frac{1}{2}(a + a^*)$ and $a_2 = \frac{1}{2i}(a - a^*)$, if f extends to a linear function $f\colon A \to B$, then it does so

uniquely, by $f(a) = f(a_1) + if(a_2)$. First notice that this is well-defined and coincides with the given values for $a \in N(A)$, since in that case $a_1 \odot a_2$.

As $(a+b)_1 = a_1 + b_1$ and $(a+b)_2 = a_2 + b_2$, the assumption makes the extension $f: A \to B$ additive. Next, for $z \in \mathbb{C}$, say $z = x + iy$ for real x and y, compute

$$f(za) = f(xa_1 - ya_2) + if(xa_2 + ya_1),$$
$$zf(a) = f(xa_1) - f(ya_2) + if(xa_2) + if(ya_1).$$

So the assumption in fact makes the extension $f: A \to B$ a $\mathbb{C}$-linear function. It also clearly satisfies $f(a^*) = f(a)^*$ and $f(1) = 1$.

Finally, any self-adjoint $a \in A$ can be written as $a = \frac{1}{2}a_+ + \frac{1}{2}a_-$ for unitaries $a_\pm = a \pm i\sqrt{1 - a^*a}$ that commute with each other and a. Therefore any element of A is a linear combination of four unitaries. Because f restricts to a homomorphism $U(A) \to U(B)$, it therefore preserves multiplication on all of A. $\square$

One could take into account the action by conjugation of $U(A)$ on $N(A)$, but it is not clear at all how additionally assuming that $f(uau^*) = f(u)f(a)f(u)^*$ should guarantee that f is additive on self-adjoints.

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