PML ENHANCED WITH A SELF-ADAPTIVE GOAL-ORIENTED *HP* FINITE-ELEMENT METHOD: SIMULATION OF THROUGH-CASING BOREHOLE RESISTIVITY MEASUREMENTS

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Abstract. We describe the application of a Perfectly Matched Layer (PML) combined with a self-adaptive goal-oriented hp Finite Element (FE) method to the simulation of resistivity measurements. The adaptive refinements and fast convergence of the self-adaptive hp FE method enhance the performance of the PML and, thus, enable the accurate and efficient truncation of the computational domain in open domain problems. We apply this methodology to the simulation of axisymmetric through-casing resistivity measurements in a borehole environment that are typically used for the assessment of rock formation properties. Our numerical results confirm the accuracy and efficiency of our method and provide evidence of highly accurate and reliable simulations of borehole logging measurements in the presence of a conductive steel casing and material contrast of fourteen orders of magnitude in conductivity. Moreover, the combination of adaptivity and PML enables us to significantly reduce the size of the computational domain.

Key words. hp-Finite Elements, Perfectly Matched Layer (PML), exponential convergence, goal-oriented adaptivity, through-casing resistivity measurements.

1. INTRODUCTION. In 1994, Berenger [3] introduced the concept of a Perfectly Matched Layer (PML) for exterior electromagnetic problems to reduce reflections from the boundary of a truncated computational domain. During the same year, the concept of PML was recognized as complex-coordinate stretching of Maxwell's equations by Chew *et al.* [5], which essentially means that the PML constitutes an analytic continuation of the governing equations into the complex plane; see also [22]. More recent contributions summarizing the mathematical developments and insight into PMLs can be found in [21, 25].

Let us briefly review the main idea of the PML and the difficulties pertaining to its implementation. Within the PML, both propagating and evanescent waves are transformed into evanescent waves with fast exponential decay. Thus, on the outer boundary of the PML, waves are highly attenuated in magnitude such that reflections due to an *ad-hoc* boundary condition (BC) (for example, a homogeneous Dirichlet boundary condition) become negligible. Since the exponential decay of the waves within the PML can be made arbitrarily large, reflections from the PML can be made arbitrarily small. However, the rapid decay of the solution in the PML produces a "boundary layer". Note that the resolution of such PML-induced boundary layers is crucial for the accuracy of the solution. Failing to ensure a sufficiently fine discretization in the PML typically results in spurious reflections that contaminate the solution in the entire computational domain.

Conventional discretization methods face a trade-off between using a highly attenuating PML that minimizes reflections and a PML with low attenuation that is typically easier to resolve. The more a wave is attenuated within the PML, the smaller are the reflections from the truncated domain boundary provided that the discretization can accurately resolve the rapidly changing solution in the PML. On the other hand, the more the solution is attenuated in the PML, the stronger are the resulting gradients and, hence, the more difficult it is for conventional discretization methods to provide adequate resolution.

We improve the performance of the PML by combining it with a numerical method that is capable of accurately resolving strong boundary layers; see [10] for some fundamental work on this subject in the context of acoustics, elasticity and electromagnetics. This numerical approach is based on a self-adaptive goal-oriented hp Finite Element (FE) method that automatically, i.e. without any user interaction, constructs an accurate approximation of boundary layers with relatively few unknowns. The use of this methodology renders the design of sophisticated PMLs unnecessary. We simply select *any* PML that provides sufficiently high attenuation to eliminate reflections from the boundary; then, the self-adaptive algorithm automatically produces a grid that accurately resolves the PML-induced boundary layer. Thus, no tuning of the PML is required.

Here, we apply the PML technique combined with the self-adaptive goal-oriented hp-FEM to problems arising in electromagnetic logging; see [15, 16] for details and validation of the adaptive method. To this end, we investigate different PMLs (including discontinuous PMLs) to simulate challenging axisymmetric through-casing resistivity measurements in a borehole environment. In this problem, currents propagate long distances through steel casing. To avoid the use of large computational domains, we employ the Maxwellian anisotropic PML formulation in cylindrical coordinates described in [23]. Thus, we significantly reduce the size of the computational domain, thereby eliminating unnecessarily elongated elements. It is important to note that the discretization in the PML needs to be highly accurate to avoid reflections from materials with a conductivity contrast of up to fourteen orders of magnitude.

The main challenges in the simulation of through-casing resistivity measurements pertain to high contrasts in the material properties, strong singularities, and large dynamic ranges (of up to twelve orders of magnitude for the case considered). Our main objective in these simulations is to determine the first vertical difference of electric current at two closely placed receiving coils, as we move the logging instrument in the vertical direction along the axis of the borehole. This objective function, also referred to as quantity of interest, can be used to determine the conductivity of the rock formation behind casing and, thus, characterize the rock formation; see [9, 14, 18, 24, 26] for details on through-casing resistivity measurements.

The remaining sections of this paper are organized as follows. In Section 2, we describe the variational formulation for axisymmetric problems and we construct the PML. In Section 3, we present the self-adaptive goal-oriented hp-FE method. In Section 4, we describe the through-casing resistivity problem, and we present numerical results that demonstrate that the PML combined with the self-adaptive hp-FE method enables a considerable reduction of the size of the computational domain without compromising the accuracy of the solution. We also demonstrate that, in combination with the PML, adaptivity in both element size h and polynomial approximation order p is necessary to ensure accuracy and efficiency of the simulations, since h-adaptivity alone turns out to be insufficient to do so. Finally, in Section 5, we provide concluding remarks.

2. Maxwell's Equations and PML Formulation. Using cylindrical coordinates (ρ, ϕ, z) , the variational formulation of Maxwell's equations in terms of the azimuthal component of the magnetic field H_{ϕ} for axisymmetric problems is given by (see [12] for a detailed derivation):

(2.1)
$$\begin{cases} \text{Find } H_{\phi} \in H_{\phi,\Gamma} + \tilde{H}_{\Gamma}^{1}(\Omega) \text{ such that:} \\ \int_{\Omega} \left[(\bar{\bar{\sigma}}_{\rho,z} + j\omega\bar{\bar{\epsilon}}_{\rho,z})^{-1} \nabla \times \mathbf{H}_{\phi} \right] \cdot (\nabla \times \bar{\mathbf{F}}_{\phi}) \, dV + j\omega \int_{\Omega} (\bar{\bar{\mu}}_{\phi}\mathbf{H}_{\phi}) \cdot \bar{\mathbf{F}}_{\phi} \, dV \\ = -\int_{\Omega} M_{\phi}^{imp} \, \bar{F}_{\phi} \, dV \quad \forall F_{\phi} \in \tilde{H}_{\Gamma}^{1}(\Omega) , \end{cases}$$

where $\mathbf{H}_{\phi,\Gamma}$ is a lift (in our case, $\mathbf{H}_{\phi,\Gamma} = 0$) of the essential boundary condition data, $\tilde{H}_{\Gamma}^{1}(\Omega) = \{F_{\phi} : \mathbf{F}_{\phi} = (0, F_{\phi}, 0) \in H_{\Gamma}(\operatorname{curl}; \Omega)\} = \{F_{\phi} \in L^{2}(\Omega) : \frac{1}{\rho}F_{\phi} + \frac{\partial F_{\phi}}{\partial \rho} \in L^{2}(\Omega), \frac{\partial F_{\phi}}{\partial z} \in L^{2}(\Omega), F_{\phi}|_{\Gamma} = 0\}, F_{\phi} \in \tilde{H}_{\Gamma}^{1}(\Omega) \text{ is an arbitrary test function, } \bar{F}_{\phi} \text{ is the complex conjugate of } F_{\phi}, \Gamma = \partial\Omega \text{ is the boundary of domain } \Omega, \bar{\bar{\sigma}}_{\rho,z}, \bar{\bar{\epsilon}}_{\rho,z}, \text{ are the meridian components of the electrical conductivity and dielectric permittivity of the medium, respectively, <math>\bar{\mu}_{\phi}$ is the azimuthal component of the magnetic permeability of the medium, and M_{ϕ}^{imp} is the azimuthal component of a prescribed impressed magnetic current density.

We model source toroid antennas by prescribing an impressed volume magnetic current M_{ϕ}^{imp} on a toroidal coil equal to that induced by an electric excitation with a Vertical Electric Dipole (VED) — also known as Hertzian dipole — of current equal to $(\sigma + j\omega\epsilon)$ Amperes. Thus, the magnetic moment of the toroid is independent of its geometrical dimensions and, in addition, curves at different frequencies may be compared.

A variety of BC's can be imposed to truncate the computational domain. For example, it is possible to use an infinite element technique [4], a PML, a boundary element technique [8] or an absorbing BC. For a comparison of all these truncations methods, see [7]. In this paper, we impose a homogeneous Dirichlet boundary condition ($H_{\phi} = 0$) on the outer boundary of the computational domain, and we compare it against results obtained using a PML.

REMARK: The axis of symmetry is not a boundary of the original 3D problem, and therefore, no boundary condition on this axis is needed to solve this problem. Nevertheless, since the formulation of problem (2.1) requires the use of space $\tilde{H}_{\Gamma}^{1}(\Omega)$ and since this space involves the weight $\frac{1}{\rho}$ that becomes singular for $\rho \to 0$, a homogeneous Dirichlet condition at the axis of symmetry $(H_{\phi}|_{\rho=0}=0)$ must be specified for the discrete solution. That is, we utilize the artificial condition $H_{\phi} = 0$ at the axis of symmetry to ensure the proper integrability for variational formulation (2.1). Therefore, the condition imposed at the axis of symmetry should be called *integrability condition* rather than *boundary condition*. Note that different proper integrability conditions may be selected in this context, $H_{\phi}|_{\rho=0} = 0$ being the most natural one.

2.1. PML Formulation. Following [23], we construct an anisotropic Maxwellian PML by considering material properties within the PML of the form

(2.2)
$$\bar{\bar{\sigma}} = \bar{\bar{\Lambda}}\sigma \quad ; \quad \bar{\bar{\epsilon}} = \bar{\bar{\Lambda}}\epsilon \quad ; \quad \bar{\bar{\mu}} = \bar{\bar{\Lambda}}\mu \; ,$$

where σ , ϵ , and μ are the conductivity, dielectric permittivity, and magnetic permeability of the media (assuming isotropic materials), respectively,

(2.3)
$$\bar{\overline{\Lambda}} = \begin{bmatrix} \frac{\bar{\rho}}{\rho} \frac{s_z}{s_\rho} & 0 & 0\\ 0 & \frac{\rho}{\bar{\rho}} s_z s_\rho & 0\\ 0 & 0 & \frac{\bar{\rho}}{\rho} \frac{s_\rho}{s_z} \end{bmatrix},$$

where $\tilde{\rho} = \int_{0}^{\rho} s_{\rho}(\rho') d\rho'$, and, s_{ρ} , s_{ϕ} , and s_z are the so-called stretching coordinate functions. We shall define these functions as

(2.4)
$$s_{\rho} = s_{\phi} = s_z = 1 + \psi_i - j\psi_i$$
,

where $\psi_i = \psi_i(x, x_0, x_1)$ is given by

(2.5)
$$\psi_i(x, x_0, x_1) = \begin{cases} 0 & x < x_0 \text{ or } x > x_1 \\ g_i(x) & x \in (x_0, x_1) \end{cases},$$

and the interval (x_0, x_1) specifies the location of the PML. We consider three different PMLs by defining three different functions $g_i(x)$ according to

(2.6)
$$g_1(x) = \left[2\left(\frac{x-x_0}{x_1-x_0}\right)\right]^{17} \quad \text{PML 1,}$$
$$g_2(x) = 20000 \left(\frac{x-x_0}{x_1-x_0}\right) \quad \text{PML 2,}$$
$$g_3(x) = 10000 \qquad \qquad \text{PML 3.}$$

The main difference between the three PMLs described in equation (2.6) pertains to the desired degree of smoothness. While $\psi_1(x) \in C^{16}$ (sixteen continuous derivatives) is a smooth function, $\psi_2(x) \in C^0$ (continuous function) and $\psi_3(x)$ (discontinuous function) are non-smooth functions. The use of smooth functions $\psi_i(x)$ is commonly advocated; see, for instance, [21]. In Section 5, we show that this smoothness property does not provide any additional advantages when the PML is enhanced with the selfadaptive goal-oriented hp-FE Method. In general, the solution within the PML decays (asymptotically) as an exponential function of the integral of $g_i(x)$. The higher the value of $\int g_i(x)$ the more pronounced is the decay of the solution. We shall assume a PML thickness of $x_1 - x_0 = 0.5m$ throughout this paper.

We note that in order to work in a large range of frequencies, it may be more adequate to consider a frequency dependent PML such that the corresponding solution has an asymptotically frequency independent decay. For simplicity and clarity of the results, in here we avoid dealing with a frequency dependent PML.

REMARK: It is well-known that the use of the PML typically results in a high condition number of the associated stiffness matrix, and thus, iterative solvers may face convergence difficulties; see e.g. [20] and references therein. However, such convergence problems can be overcome by designing adequate preconditioners that effectively improve the condition number of the preconditioned stiffness matrix and thereby accelerate the rate of convergence of the iterative solver. Design of such preconditioners constitutes an active research area.

3. The Self-Adaptive Goal-Oriented *hp*-FE Method. Our numerical method is designed to have the following capabilities:

- to resolve PML-induced boundary layers sufficiently accurately to avoid spurious reflections,
- to resolve the strong variation in conductivity that occurs between the casing and the rock formation (of up to fourteen orders of magnitude),
- to accurately compute the quantity of interest $L(\mathbf{H})$ the first vertical difference of the azimuthal component of the magnetic field —, which is expected to be several orders of magnitude smaller than the magnetic field itself. The total dynamic range (the ratio between the largest value of the solution and $L(\mathbf{H})$) is expected to be of the order of $10^8 10^{13}$.

In the sequel, we present the self-adaptive goal-oriented hp FEM that possesses all of the above capabilities.

We utilize a numerical technique that is based on an hp FE discretization (see [6] for details), where h denotes the element size, and p is the polynomial element order (degree) of approximation. Both h and p can vary *locally* throughout the grid. The main advantage of the hp-FEM over conventional discretization methods is that it provides exponential convergence rates of the solution with respect to the number of unknowns (as well as the CPU time), independent of the number, intensity, and/or distribution of singularities in the solution. For a proof of this result we refer to [1, 2, 19].

In order to ensure an optimal distribution of element size h and polynomial order of approximation p, we utilize a self-adaptive goal-oriented hp-adaptive strategy that minimizes the error in a user-prescribed "quantity of interest" which, for the problem under consideration, is the first vertical difference of the azimuthal component of the magnetic field, $L(\mathbf{H})$. An adaptive algorithm based on minimizing a quantity of interest of the error rather than the energy-norm is referred to as a *qoal-oriented* adaptive algorithm [17]. Goal-oriented adaptivity utilizes the solutions of two related problems: the original "direct" problem, and an additional "dual" or adjoint problem that is composed of the stiffness matrix used for solving the direct problem and a righthand-side term given by the user-prescribed quantity of interest. The corresponding solution of this dual problem is called the *influence* function, which can be interpreted as a generalization of Green's function. This influence function is utilized to guide optimal refinements. The above mentioned convergence result also extends to goaloriented adaptivity, i.e. our algorithm delivers exponential convergence rates also for the user-prescribed quantity of interest. In particular, discretization within the PML is such that it provides optimal accuracy in terms of the user-prescribed quantity of interest. We note that even if the energy-norm within the PML is small, intensive refinements within the PML may be required in order to accurately reproduce the quantity of interest.

The automatic adaptivity is based on the following two-grid paradigm: A given (coarse) conforming hp mesh is first refined globally in both h and p to yield a corresponding *fine mesh*, i.e. each element is broken into four element sons in 2D (eight element sons in 3D), and the discretization order of approximation p is raised uniformly by one. Subsequently, the problem of interest is solved on the fine mesh. The next optimal coarse mesh is then determined as the one that maximizes the decrease of the projection-based interpolation error divided by the number of added unknowns. Since the mesh optimization process is based on the minimization of the interpolation error of the solution rather than the residual, the algorithm is, in principle, problem

independent. A detailed description of the hp self-adaptive goal-oriented algorithm can be found in [12, 13]. For a validation of this numerical methodology, see [14–16].

4. Numerical Applications.

4.1. Through-Casing Resistivity Applications. In this subsection, we describe the problem setup for the numerical experiments in subsection 4.2. We consider a through-casing resistivity problem that is commonly utilized to probe subsurface rock formations with electromagnetic waves.

Using cylindrical coordinates (ρ, ϕ, z) , we specify the following geometry, sources, receivers, and materials (see also the illustration in Fig. 4.1):

- One 9 cm radius toroidal coil with a 1 cm x 1 cm cross-section located on the axis of symmetry and moving along the z-axis, and two receiving coils of the same dimensions located 150 cm and 175 cm, respectively, above the source antenna.
- Borehole: a cylinder Ω_I of radius 10 cm surrounding the axis of symmetry $(\Omega_I = \{(\rho, \phi, z) : \rho \leq 10 \text{ cm}\})$, with resistivity R=0.1 Ω · m.
- Casing: a steel casing Ω_{II} of width 1.27 cm surrounding the borehole ($\Omega_{II} = \{(\rho, \phi, z) : 10 \text{ cm} \le \rho \le 11.27 \text{ cm}\}$), with resistivity R=0.000001 Ω · m = $10^{-6} \Omega$ · m.
- Formation material I: a subdomain Ω_{III} defined by $\Omega_{III} = \{(\rho, \phi, z) : \rho > 11.27 \text{cm}, 0 \text{ cm} \le z \le 100 \text{ cm}\}$, with resistivity R=10000 Ω · m.
- Formation material II: a subdomain Ω_{IV} defined by $\Omega_{IV} = \{(\rho, \phi, z) : \rho > 11.27 \text{ cm}, -50 \text{ cm} \le z < 0 \text{ cm}\}$, with resistivity R=0.01 Ω · m.
- Formation material III: a subdomain Ω_V defined by $\Omega_V = \{(\rho, \phi, z) : \rho > 11.27 \text{ cm}, z < -50 \text{ cm} \text{ or } z > 100 \text{ cm}\}$, with resistivity R=5 Ω · m.

The quantity of interest $L(\mathbf{H})$ for these simulations is the first difference of electric current I at the two receiving coils $(l_1 \text{ and } l_2)$ of radius a = 9 cm divided by the (vertical) distance Δz between the two receiving coils, i.e.,

(4.1)
$$L(\mathbf{H}) = \frac{I_1 - I_2}{\Delta z} = \frac{\oint_{\mathbf{I}_1} \mathbf{H}(l) \, dl - \oint_{\mathbf{I}_2} \mathbf{H}(l) \, dl}{\Delta z}.$$

In this paper, we consider an operating frequency of 1 Hz, and a variation of resistivity of the casing from $10^{-10}\Omega \cdot m$ to $10^{-6}\Omega \cdot m$.

4.2. Numerical Results. Below, we present the numerical simulations of the resistivity logging problem in a cased well described in subsection 4.1. We place our transmitter coil at z = -1.65m.

First, we study the importance of the size of the computational domain when we consider *ad hoc* boundary conditions — homogeneous Neumann boundary conditions at the top and bottom of the domain, and homogeneous Dirichlet boundary conditions at the side — without a PML. In this context, we shall refer to the error due to the truncation of the computational domain as modeling error and use the solution obtained with PML 1 (10 m x 5 m domain) as reference solution. This choice of reference solution is justified, since the selected PML strongly attenuates the solution, thereby minimizing reflections. Thus, the modeling error of the reference solution, which is due to the replacement of the almost-zero solution on the outer-part of the PML by a homogeneous Dirichlet BC ($H_{\phi} = 0$) is negligible. Our numerical results further support this choice of reference solution. Figures 4.2 and 4.3 display the relative error of the real and imaginary parts of the quantity of interest given by Eq. (4.1) as a function of the vertical length of the computational domain. The horizontal



FIG. 4.1. 2D cross-section of the geometry of a through-casing resistivity problem. Measurement instruments consist of one transmitter and two receiver coils. The model includes a conductive borehole, a metallic casing, and four layers in the formation material with varying resistivities.

length of the domain is taken to be one fourth of the vertical length, since casing is present only in the vertical direction.

From Figures 4.2 and 4.3, we can deduce that in order to guarantee a modeling error below 1% we need to consider large computational domains with several kilometers in size. In particular, for a casing resistivity equal to $10^{-6}\Omega \cdot m$, a domain with vertical length equal to 10000 m still delivers relative errors in the imaginary part greater than 1%. This error can be attributed to the truncation of the computational domain¹. Moreover, Figures 4.2 and 4.3 indicate a non-monotonic behavior of the error as a function of the size of the computational domain. Although we do not have a rigorous explanation for this observation, such non-monotonic behavior prompts a word of caution in that it renders the selection of an optimal domain size difficult. In particular, assuming a monotonic behavior based on an extrapolation of a small set of numerical results may be misleading, since it does not reflect the actual dependence of the error on the domain length and, thus, may lead to the selection of an inappropriate size of the computational domain.

Table 4.1 complements the results shown in Figures 4.2 and 4.3 by displaying the quantity of interest given by Eq. (4.1) corresponding to the three different PMLs considered in this paper. We observe that with any of the three different PMLs (and

¹The discretization error is several orders of magnitude smaller than the modeling error. We verified this claim by using an error estimator based on the solution of the problem on a globally refined (in both h and p) grid. See [11] for additional details on the error estimator.



FIG. 4.2. Through-casing resistivity problem. Relative error of the real part of the quantity of interest according to Eq. (4.1) as a function of the vertical length of the computational domain. Different curves indicate different resistivities of the casing, ranging from $10^{-10}\Omega \cdot m$ to $10^{-6}\Omega \cdot m$. Results obtained by goal-oriented hp-adaptivity.

a computational domain of $5m \times 2.5m$) we obtain more accurate results than those obtained by considering a computational domain of $12800m \times 3200m$ without a PML. This confirms the high accuracy obtained by the use of PMLs enhanced with the self-adaptive goal-oriented hp-FE method.

Next, we analyze the computational cost for PMLs combined with our adaptive algorithm. For each computational domain we study the behavior of the discretization error (disregarding the modeling error due to the truncation of the computational domain) as a function of the problem size (number of unknowns). In Table 4.2 we display the discretization error and the corresponding number of unknowns for different computational domains. Note that these results depend on the initial grid, which is different for each computational domain (although all of them are based on geometrically graded grids). Nevertheless, the results of Table 4.2 provide an indication of the performance associated with each computational domain and PML. These results show a considerable reduction (about 50%) in terms of the number of unknowns when introducing PML 2 as opposed to all other cases displayed in Table 4.2. It is noteworthy that even with the poorly designed discontinuous PML 3 we obtain results that are competitive with those of the other cases, since the corresponding solution is also piecewise smooth.

Figures 4.4 and 4.5 display the logs² corresponding to a casing resistivity equal to $10^{-6}\Omega \cdot m$ and $10^{-10}\Omega \cdot m$, respectively. We have computed these logs by utilizing

 $^{^{2}}$ A log is a plot displaying the response of the logging instrument as we move it in the vertical direction along the axis of the borehole.



FIG. 4.3. Through-casing resistivity problem. Relative error of the imaginary part of the quantity of interest according to Eq. (4.1) as a function of the vertical length of the computational domain. Different curves indicate different resistivities of the casing, ranging from $10^{-10}\Omega \cdot m$ to $10^{-6}\Omega \cdot m$. Results obtained by goal-oriented hp-adaptivity.

two different computational domains, viz. a $10m \times 5m$ domain with PML 2, and a $3200m \times 800m$ domain without a PML. Fig. 4.4 shows that the computed amplitude is similar for both computational domains, whereas the phase significantly differs. This illustrates that a large computational domain of $3200m \times 800m$ without a PML can induce considerable phase error. In Fig. 4.5, we obtain similar results when considering either of the two computational domains described above.

We observe a large frequency shift of approximately 160 degrees in Fig. 4.5 due to the presence of a highly conductive casing in a highly resistive formation, as can be physically expected. In addition, a "horn" in the amplitude appears when compared to Figure 4.4.

An hp-grid produced by the self-adaptive goal-oriented hp-FE method with PML 2 and a casing resistivity equal to $10^{-6}\Omega \cdot m$ is displayed in Figure 4.6 for exemplification. The grid contains 7421 unknowns (of which 80 % are used in the PML), and it delivers a discretization error in the quantity of interest below 0.2%. Large elements of highorder approximation are effective in approximating the smooth part of the solution, while small elements of low order are more suitable for approximating abrupt spatial variations due to singularities. On the outer part of the casing, we observe more refinements than on the inner part of the casing, since it is physically known that the quantity of interest exhibits little sensitivity to the conductivity inside the borehole.

To assess the importance of refinements in both h and p, let us consider below an adaptive goal-oriented FE method with fixed polynomial order of approximation pbut variable mesh size h. To guide optimal refinements for a given coarse h-grid, we utilize the globally h-refined grid, that is, the h/2-grid, where, in two dimensions, all

Table 4.1

Through-casing resistivity problem. Real and Imaginary parts of the quantity of interest — given by Eq. (4.1) — as a function of the size of the computational domain and presence of a PML. Results obtained by goal-oriented hp-adaptivity.

Resistivity	Domain Size	Real Part	Imag Part
Casing	(m)	(A/m)	(A/m)
$10^{-6} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	1.2320E-6	-8.5928E-9
$10^{-6} \Omega \cdot m$	PML 2 (5 x 2.5)	1.2320E-6	-8.5960E-9
$10^{-6} \Omega \cdot m$	PML 3 (5 x 2.5)	1.2320E-6	-8.6016E-9
$10^{-6} \Omega \cdot m$	400 x 100	1.3382E-6	-2.0264E-8
$10^{-6} \Omega \cdot m$	1600 x 400	1.2169E-6	-1.2622E-8
$10^{-6} \Omega \cdot m$	6400 x 1600	1.2295E-6	-8.1076E-9
$10^{-6} \Omega \cdot m$	12800 x 3200	1.2320E-6	-8.3840E-9
$10^{-6} \Omega \cdot m$	$25600 \ge 6400$	1.2320E-6	-8.5968E-9
$10^{-8} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	1.7188E-10	-2.0959E-11
$10^{-8} \Omega \cdot m$	PML 2 (5 x 2.5)	1.7188E-10	-2.0959E-11
$10^{-8} \Omega \cdot m$	PML 3 (5 x 2.5)	1.7188E-10	-2.0957E-11
$10^{-8} \Omega \cdot m$	400 x 100	1.8772E-10	-1.4463E-11
$10^{-8} \Omega \cdot m$	1600 x 400	1.7077E-10	-2.3905E-11
$10^{-8} \Omega \cdot m$	6400 x 1600	1.7124E-10	-1.9548E-11
$10^{-8} \Omega \cdot m$	12800 x 3200	1.7120E-10	-2.0900E-11
$10^{-8} \Omega \cdot m$	25600 x 6400	1.7188E-10	-2.0960E-11
$10^{-10} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	1.4255E-13	2.2710E-15
$10^{-10} \Omega \cdot m$	PML 2 (5 x 2.5)	1.4255E-13	2.2710E-15
$10^{-10} \Omega \cdot m$	PML 3 (5 x 2.5)	1.4255E-13	2.2709E-15
$10^{-10} \Omega \cdot m$	400 x 100	1.4198E-13	2.1780E-15
$10^{-10} \Omega \cdot m$	1600 x 400	1.4256E-13	2.3695E-15
$10^{-10} \Omega \cdot m$	6400 x 1600	1.4256E-13	2.2188E-15
$10^{-10} \Omega \cdot m$	12800 x 3200	1.4255E-13	2.2703E-15
$10^{-10} \Omega \cdot m$	25600 x 6400	1.4255E-13	2.2710E-15

elements are divided into four element sons. We denote the globally h-refined grid as the fine grid. Then, we employ an adaptive strategy that is analogous to the one for hp-adaptivity with the only provision that the polynomial order of approximation p remains unchanged.

The convergence behavior of the *h*-adaptive method restricted to polynomialapproximation order p = 2 (quadratic basis functions) and combined with a PML is displayed in Figure 4.7, where the discretization error is measured with reference to the solution that is provided by the *hp*-adaptive method. The results displayed in Figure 4.7 and their comparison to the results given in Table 4.2 illustrate that the *h*adaptive method combined with a PML is less accurate for solving the through-casing resistivity problem under consideration. This observation indicates that in the present case, a restriction of the approximation order to p = 2 significantly limits the efficiency of the method when compared to the *hp*-adaptive method. In particular, we observe that the sequence of coarse *h*-grids delivers for 50,000 unknowns an error that is still larger than 10%. Furthermore, for any coarse grid with fewer than 15,000 unknowns, the fine grid delivers large errors (above 30 %), and therefore, is unable to guide optimal refinements. To further support this observation, we display in Figure 4.8 two

TABLE 4.2

Through-casing resistivity problem. Number of unknowns employed by the self-adaptive goaloriented hp-FE method as a function of the size of the computational domain and presence of a PML.

Resistivity	Domain Size (m)	Nr. Unknowns	Nr. Unknowns
Casing		$(\approx 1\% \text{ error})$	$(\approx 0.01\% \text{ error})$
$10^{-6} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	$19541 \ (0.083\%)$	24886~(0.037%)
$10^{-6} \Omega \cdot m$	PML 2 $(5 \ge 2.5)$	7095 (0.29%)	13345~(0.006%)
$10^{-6} \Omega \cdot m$	PML 3 (5 x 2.5)	8679~(1.04%)	$19640 \ (0.009\%)$
$10^{-6} \Omega \cdot m$	6400 x 1600	12327~(0.43%)	$18850 \ (0.014\%)$
$10^{-6} \Omega \cdot m$	12800 x 3200	12327~(0.43%)	$18850 \ (0.014\%)$
$10^{-6} \Omega \cdot m$	25600 x 6400	12099~(1.22%)	19828~(0.037%)
$10^{-8} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	$12702 \ (0.214\%)$	$27291 \ (0.002\%)$
$10^{-8} \Omega \cdot m$	PML 2 $(5 \ge 2.5)$	7787~(0.832%)	18115~(0.003%)
$10^{-8} \Omega \cdot m$	PML 3 (5 x 2.5)	10857~(0.750%)	28954~(0.083%)
$10^{-8} \Omega \cdot m$	6400 x 1600	11433~(0.262%)	20109~(0.013%)
$10^{-8} \Omega \cdot m$	12800 x 3200	14305~(1.262%)	28834 (0.012%)
$10^{-8} \Omega \cdot m$	25600 x 6400	21400 (0.384%)	32178 (0.011%)
$10^{-10} \Omega \cdot m$	PML 1 $(5 \ge 2.5)$	5957 (0.811%)	14497 (0.004%)
$10^{-10} \Omega \cdot m$	PML 2 $(5 \ge 2.5)$	4714 (1.056%)	12378~(0.004%)
$10^{-10} \Omega \cdot m$	PML 3 $(5 \ge 2.5)$	5942~(0.988%)	11812 (0.011%)
$10^{-10} \Omega \cdot m$	6400 x 1600	8805(1.530%)	$13786 \ (0.004\%)$
$10^{-10} \Omega \cdot m$	12800 x 3200	8545 (0.872%)	$17521 \ (0.019\%)$
$10^{-10} \Omega \cdot m$	25600 x 6400	6597~(1.103%)	18245~(0.039%)

optimal coarse *h*-grids. The first one (left panel) contains an intermediate (optimal) *h*-grid with 14,105 unknowns. The estimated relative error (computed as the difference between the solutions obtained on the *h*- and h/2-grids, respectively) is below 1%. However, the actual (exact) error is above 30%, since the fine grid fails to provide an accurate reference solution. Indeed, we observe in Figure 4.8 (left panel) that the PML in the upper part of the domain is under-resolved.

Let us briefly summarize our findings for h-adaptivity when restricting ourselves to linear elements (p = 1) that are commonly used in engineering practice. Our numerical experiments indicate that for linear elements an h-adapted grid with over 100,000 unknowns is still not sufficiently fine to reduce the relative error in the quantity of interest below 50% for the problem under consideration. The final h-grid, displayed in Figure 4.9, contains over 100,000 unknowns. However, it still exhibits a relative error in the quantity of interest over 50%. As above, we observe in the final h-grid that the upper-part of the domain, including the PML, is under-resolved.

Finally, we address another challenging problem. We reconsider our original through-casing resistivity problem, and we replace the upper layer of the formation (with resistivity equal to $5 \Omega \cdot m$) by an anisotropic material with horizontal resistivity equal to $1 \Omega \cdot m$ and vertical resistivity equal to $5 \Omega \cdot m$. We consider a casing resistivity equal to $10^{-6}\Omega \cdot m$. Utilizing the solution obtained with PML 1 as reference solution, we plot in Fig. 4.10 the error due to truncation of the computational domain versus the frequency. By employing either PML 2 or a large computational domain of size equal to $25000m \times 6400m$, we obtain a total error below 0.01% which is the discretization error tolerance that we selected for this problem. When considering



FIG. 4.4. Through-casing resistivity problem with a resistivity of casing equal to $10^{-6}\Omega \cdot m$. Amplitude (left) and phase (right) of the final log. The two curves were obtained on two different computational domains: A $10m \times 5m$ domain with PML 3, and a $3200m \times 800m$ domain without a PML. Results obtained by goal-oriented hp-adaptivity.



FIG. 4.5. Through-casing resistivity problem with a resistivity of casing equal to $10^{-10}\Omega \cdot m$. Amplitude (left) and phase (right) of the final log. The two curves were obtained on two different computational domains: A $10m \times 5m$ domain with PML 3, and a $3200m \times 800m$ domain without a PML. Results obtained by goal-oriented hp-adaptivity.

smaller domains, we observe the effect of the domain truncation error which may be-



FIG. 4.6. Through-casing resistivity problem. hp-grid with 7421 unknowns delivered by the self-adaptive goal-oriented hp-FE method. Different colors indicate different polynomials orders of approximation, from 1 (light) up to 9 (dark). Size of computational domain: $5m \times 2.5m$, including a 0.5m thick PML.



FIG. 4.7. Through-casing resistivity problem with resistivity of casing equal to $10^{-6}\Omega \cdot m$. Convergence history (number of degrees of freedom — unknowns — of the coarse grid vs. relative error of the quantity of interest in percentage) delivered by the self-adaptive goal-oriented h-FE method, with p = 2, combined with the PML 2 (5m × 2.5m).



FIG. 4.8. Through-casing resistivity problem. Two h-grids (with p = 2) delivered by the selfadaptive goal-oriented h-FE method. Left panel: Intermediate h-grid with 14105 unknowns, delivering an error of over 30%. Right panel: Final h-grid with 76643 unknowns, delivering an error of 2%. Size of computational domain: $5m \times 2.5m$, including a 0.5m thick PML 2.

come as large as 5% for a domain of size $1600m \times 400m$. As a general trend, Fig. 4.10 indicates a monotonic decrease of the error with increasing frequency (as physically expected). Note, however, that there are exceptions to this monotonic behavior as evidenced by Fig. 4.10.



FIG. 4.9. Through casing resistivity problem. Final h-grid with p = 1, containing 104834 unknowns, and delivering an error over 50%. Size of computational domain: $5m \times 2.5m$, including a 0.5m thick PML 2.

5. CONCLUSIONS. We have shown that numerical reflections from domain truncation for a layered medium with high-material constrasts are minimized with a self-adaptive goal-oriented *hp*-FE method in combination with a PML. Such an adaptive method is capable of delivering optimal grid refinements and, thus, it can substantially improve the performance of PMLs.

For problems with material coefficients varying by up to ten orders of magnitude within the PML, the self-adaptive goal-oriented *hp*-FE method automatically constructs a grid that exhibits minimum reflections (below 0.001% relative error in the quantity of interest for a moderate number of unknowns), even when considering discontinuous PMLs. Thus, for through-casing resistivity measurements, we can reduce the size of the computational domain from, for instance, 25,000m to 5m by using a PML in combination with our adaptive algorithm. We have demonstrated that this neither compromises the accuracy of the magnitude nor of the phase of the computed logging response. Moreover, the number of unknowns that are needed to solve the problem to a prescribed accuracy can be significantly reduced by using a PML enhanced with adaptivity instead of a large domain without PML. Thus, we have shown that PMLs can be successfully employed not only for pure wave propagation problems with quasi-uniform materials, but also for engineering applications with high material contrast within the PML region.

Furthermore, we have shown that a PML combined with a goal-oriented FE method that is self-adaptive only in the element size h may be considerably less accurate than the corresponding hp-method and require orders of magnitude more unknowns to achieve the same level of accuracy. This indicates that adaptivity in



FIG. 4.10. Anisotropic through-casing resistivity problem with resistivity of casing equal to $10^{-6}\Omega \cdot m$. Relative error of the absolute value of the quantity of interest as a function of the frequency. Different curves indicate different domain sizes: $400m \times 100m$, $1600m \times 400m$, $6400m \times 1600m$, $25600m \times 6400m$, and $5m \times 2.5m$ (with PML 2). Results obtained by goal-oriented hp-adaptivity.

both element size h and polynomial approximation order p is essential for the accuracy of the solution.

Finally, we have demonstrated that PMLs combined with the self-adaptive goaloriented hp-FE method maintain good performance in the presence of anisotropic materials, as well as at different frequencies.

ACKNOWLEDGMENTS. This work was financially supported by Baker Atlas and The University of Texas at Austin's *Joint Industry Research Consortium on Formation Evaluation* sponsored by Aramco, Baker Atlas, BP, British Gas, Chevron, ConocoPhillips, ENI E&P, ExxonMobil, Halliburton, Marathon, Mexican Institute for Petroleum, Hydro, Occidental Petroleum, Petrobras, Schlumberger, Shell E&P, Statoil, TOTAL, and Weatherford International Ltd. Moreover, the fourth author was supported by the Netherlands Organization for Scientific Research (NWO) and the ICES Postdoctoral Fellowship Program of the Institute for Computational Engineering and Sciences (ICES) at The University of Texas at Austin. This support is gratefully acknowledged.

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