

REACHABILITY IN TWO-DIMENSIONAL VECTOR ADDITION SYSTEMS WITH STATES IS PSPACE-COMPLETE

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ABSTRACT. Determining the complexity of the reachability problem for vector addition systems with states (VASS) is a long-standing open problem in computer science. Long known to be decidable, the problem to this day lacks any complexity upper bound whatsoever. In this paper, reachability for two-dimensional VASS is shown PSPACE-complete. This improves on a previously known doubly exponential time bound established by Howell, Rosier, Huynh and Yen in 1986. The coverability and boundedness problems are also noted to be PSPACE-complete. In addition, some complexity results are given for the reachability problem in two-dimensional VASS and in integer VASS when numbers are encoded in unary.

1. INTRODUCTION

Petri nets have a long history. Since their introduction [21] by Carl Adam Petri in 1962, thousands of papers on Petri nets have been published. Nowadays, Petri nets find a variety of applications, ranging, for instance, from modeling of biological, chemical and business processes to the formal verification of concurrent programs, see *e.g.* [9, 23, 28, 5, 1]. For the analysis of algorithmic properties of Petri nets, in the contemporary literature they are often equivalently viewed as *vector addition systems with states (VASS)*, and we will adopt this view in the remainder of this paper. A VASS comprises a finite-state controller with a finite number of counters ranging over the natural numbers. The number of counters is usually referred to as the dimension of the VASS, and we write *d*-VASS when we talk about VASS in dimension *d*. When taking a transition, a VASS can add or subtract an integer from a counter, provided that the resulting counter values are greater than or equal to zero; otherwise the transition is blocked. A configuration of a VASS is a tuple consisting of a control state and an assignment of the counters to natural numbers. The central decision problem for VASS is *reachability*: given two configurations, is there a path connecting them in the infinite graph induced by the VASS?

Even clarifying the decidability status of the reachability problem required tremendous efforts, and it actually took until 1981 for it to be shown decidable. This was achieved by Mayr [20], who built upon an earlier partial proof by Sacerdote and Tenney [25]. Mayr’s argument was then polished and simplified by Kosaraju [13] in 1982, and Kosaraju’s argument was in turn simplified ten years later by Lambert [14]. Only recently beginning in 2009, Leroux developed, in a series of papers, a fundamentally different approach to the decidability of the reachability problem [15, 16, 17]. But to this day, no explicit upper bound on the complexity of the general reachability problem for VASS is known. A primitive recursive upper bound claim made in 1998 [2] was dismissed in [12].

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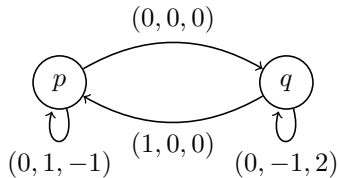


FIGURE 1. Example from [10] of a 3-VASS whose reachability set starting in configuration $p(0, 0, 1)$ is not semi-linear.

Milestones in the work on the computational complexity of the reachability problem for VASS include Lipton’s proof of EXPSPACE-hardness [19]. This lower bound is independent of the encoding of numbers, it does however require an unbounded number of counters. Deciding reachability of VASS in dimension one assuming unary encoding of numbers is easily seen to be NL-complete: the lower bound is inherited from graph reachability and the upper bound follows from a simple pumping argument. When numbers are encoded in binary, reachability in VASS in dimension one is known to be NP-complete [8]. A substantial contribution towards showing the decidability of the general reachability problem was made by Hopcroft and Pansiot in 1979, who showed that reachability in VASS in dimension two is decidable [10]. To this end, they developed an intricate algorithm that implicitly exploits the fact that the reachability set of a VASS in dimension two is semi-linear. Moreover, they could show that their method breaks down for VASS in any greater dimension, as the authors exhibited a VASS in dimension three with a reachability set that is not semi-linear. Yet, aspects of computational complexity were completely left unanswered in [10]. In 1986, Howell, Rosier, Huynh and Yen [11] analyzed Hopcroft and Pansiot’s algorithm and showed that it runs in nondeterministic doubly-exponential time, independently of whether numbers are presented in unary or binary. They could improve this nondeterministic doubly-exponential time upper bound to a deterministic doubly-exponential one and also identify a family of VASS in dimension two on which Hopcroft and Pansiot’s algorithm requires doubly-exponential time. In summary, since 1986 it has been state-of-the-art that reachability in VASS in dimension two is in 2-EXPTIME, and NL-hard and NP-hard, depending on whether numbers are encoded in unary or binary. Apart from EXPSPACE-hardness and decidability, no complexity-theoretic upper bound is known for the complexity of reachability in VASS in any dimension greater than two.

The main contribution of this paper is to show that reachability in VASS in dimension two is PSPACE-complete when numbers are encoded in binary. The PSPACE lower bound follows as an easy consequence of a recent result by Fearnley and Jurdziński who showed PSPACE-completeness of reachability in bounded one-counter automata [4]. Our PSPACE upper bound is obtained from showing that the length of a run witnessing reachability can be exponentially bounded in the size of the input, and consequently the existence of such a run can be decided by a PSPACE-algorithm. The difficult and main part of this paper is, of course, to establish the exponential upper bound on the length of witnessing runs. Our starting point is a careful analysis of an argument developed by Leroux and Sutre in [18] for the purpose of showing that reachability relations of VASS in dimension two can be captured by bounded regular languages, i.e., speaking in the terminology of [18], 2-VASS can be *flattened*. More precisely, this means that for any 2-VASS there is a finite set S of regular languages over the set of transitions, viewed as an alphabet, each of the form $u_0 v_1^* u_1 \cdots v_k^* u_k$ such that for any two reachable configurations there exists a witnessing run in the language defined by S . The paper of Leroux and Sutre reports that from any VASS in dimension two it is possible to construct such a bounded language. This immediately implies that the reachability relation of 2-VASS is semi-linear. In dimension three, the reachability relation is no longer semi-linear and hence such bounded languages cannot exist; the classical example by Hopcroft and Pansiot of a 3-VASS that does not possess a semi-linear reachability set is depicted in Fig. 1. The paper [18] has not appeared as a fully refereed publication and some proof details are omitted in it. Thus, while we follow closely the proof strategy presented in [18], we provide a complete proof that 2-VASS can be flattened, and in doing so develop new arguments in order to

allow for a tight analysis of our constructions with the overall goal of establishing the PSPACE upper bound. In summary, we make the following contributions:

- we show PSPACE-completeness of reachability in VASS in dimension two,
- for showing the former we provide a complete and rigorous proof that VASS in dimension two can be flattened by bounded languages that have small presentations, and
- we remark that reachability in VASS in dimension two with numbers encoded in unary is NL-hard and in NP.

The structure of this paper is as follows. In Section 2, we introduce our notation, give relevant definitions and formally define vector addition systems with states. Section 3 gives an overview of our main results. In Section 4, we prove our main technical result, namely that for any 2-VASS the global reachability relation can be characterized by small bounded languages; the latter are also known as linear path schemes in the literature. Section 5 is devoted to proving our main theorem, namely that reachability is PSPACE-complete. We also discuss some further corollaries and implications of our results there. Finally, we conclude in Section 6, where we discuss open problems and directions for future work.

2. PRELIMINARIES

In this section, we provide definitions relevant to this paper and introduce vector addition systems.

General notation. By $\mathbb{N} = \{0, 1, 2, \dots\}$, $-\mathbb{N} = \{0, -1, -2, \dots\}$ and \mathbb{Z} we denote the sets of non-negative integers, non-positive integers and integers, respectively. By \mathbb{Q} and $\mathbb{Q}_{\geq 0}$ we denote the set of rationals and non-negative rationals, respectively. We define $[i, j] \stackrel{\text{def}}{=} \{i, i+1, \dots, j\}$ for any $i, j \in \mathbb{Z}$. For each $k \in \mathbb{Z}$ we write $[k, \infty)$ to denote $\{z \in \mathbb{Z} : z \geq k\}$. A *quadrant* is one of the four sets \mathbb{N}^2 , $-\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times -\mathbb{N}$ and $-\mathbb{N} \times -\mathbb{N}$. Given two vectors $\mathbf{u} = (u_1, \dots, u_d)$, $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$, we denote by $\mathbf{u} + \mathbf{v} \stackrel{\text{def}}{=} (u_1 + v_1, \dots, u_d + v_d)$ the sum of their components. Given two sets $U, V \subseteq \mathbb{Z}^d$, we let $U + V \stackrel{\text{def}}{=} \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$. The *norm* of a vector $\mathbf{u} = (u_1, \dots, u_d)$ is defined as $\|\mathbf{u}\| \stackrel{\text{def}}{=} \max\{|u_i| : i \in [1, d]\}$. The *norm* of a matrix $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$ is defined as $\|A\| \stackrel{\text{def}}{=} n \cdot \max\{|a_{ij}| : i \in [1, m], j \in [1, n]\}$. For any word $w = a_1 \cdots a_n \in \Sigma^n$ over some alphabet Σ , $w[i, j]$ denotes $a_i a_{i+1} \cdots a_j$ for all $i, j \in [1, n]$.

Graphs, Parikh Images and Linear Path Schemes. For each set Σ , a Σ -labeled directed graph is a pair $G = (U, E)$, where U is a set of *vertices* and $E \subseteq U \times \Sigma \times U$ is a set of *edges*. We say G is *finite* if U and E are finite. Let $\pi = (u_1, a_1, u'_1) \cdots (u_k, a_k, u'_k) \in T^k$. The *Parikh image* Parikh_π of π is the mapping from Σ to \mathbb{N} such that $\text{Parikh}_\pi(a) = |\{i \in [1, k] : a_i = a\}|$ for each $a \in \Sigma$. If $X \subseteq E^*$, then Parikh_X denotes the set of Parikh images of X , i.e. $\text{Parikh}_X = \{\text{Parikh}_\pi : \pi \in X\}$. We say π is a *path* (from u_1 to u'_k) if $u'_i = u_{i+1}$ for all $i \in [1, k-1]$. A path π is a *cycle* if $k \geq 1$ and $u_1 = u'_k$, and *cycle-free* if no infix of π is a cycle. A cycle π is called *simple* if π is the only infix of π that is a cycle. A *linear path scheme* (from $u \in U$ to $u' \in U$) is a regular expression (whose language will be referred to implicitly) of the form

$$\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k,$$

where $\alpha_0 \beta_1 \alpha_1 \cdots \beta_k \alpha_k$ is a path (from u to u') and each β_i is a cycle. We define its *length* as $|\rho| \stackrel{\text{def}}{=} |\alpha_0 \beta_1 \alpha_1 \cdots \beta_k \alpha_k|$. We call β_1, \dots, β_k the *cycles of* ρ . Note that every path is a linear path scheme. The general structure of a linear path scheme is illustrated in Figure 2.

Vector Addition Systems with States. A *vector addition system with states* (VASS) in dimension d (d -VASS for short) is a finite \mathbb{Z}^d -labeled directed graph $V = (Q, T)$, where Q will be referred to as the *states* of V , and where T will be referred to as *transitions* of V . The *size of* V is defined as $|V| \stackrel{\text{def}}{=} |Q| + |T| \cdot d \cdot \lceil \log_2 \|T\| \rceil$, where $\|T\|$ denotes the absolute value of the largest number that appears in T , i.e. $\|T\| \stackrel{\text{def}}{=} \max\{\|\mathbf{z}\| : (p, \mathbf{z}, q) \in T\}$. We say that V is encoded in *binary* when we use this definition of $|V|$, which we will use as standard encoding in this paper. Alternatively, when we set $|V| \stackrel{\text{def}}{=} |Q| + |T| \cdot d \cdot \|T\|$ we say that V is encoded in *unary*.

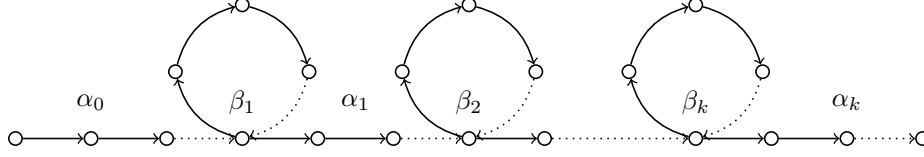


FIGURE 2. Illustration of the structure of a linear path scheme $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \sigma_k$.

Subsequently, $Q \times \mathbb{Z}^d$ denotes the set of *configurations* of V . Note that in the literature, the set of configurations is usually $Q \times \mathbb{N}^d$, however in this paper we will often deal with VASS whose counters can take integer values. For the sake of readability, we write configurations $(q, (z_1, \dots, z_d))$ and (q, \mathbf{z}) as $q(z_1, \dots, z_d)$ and $q(\mathbf{z})$, respectively.

For every subset $\mathbb{A} \subseteq \mathbb{Z}^d$, $p(\mathbf{u}), q(\mathbf{v}) \in Q \times \mathbb{A}$ and every transition $t = (p, \mathbf{z}, q)$, we write $p(\mathbf{u}) \xrightarrow{t}_{\mathbb{A}} q(\mathbf{v})$ whenever $\mathbf{v} = \mathbf{u} + \mathbf{z}$. We extend $\xrightarrow{t}_{\mathbb{A}}$ to sequences of transitions $\pi \in T^*$ as follows: $\xrightarrow{\pi}_{\mathbb{A}}$ is the smallest relation satisfying the following conditions for all configurations $p(\mathbf{u}), q(\mathbf{v}), r(\mathbf{w}) \in Q \times \mathbb{A}$ and all $t \in T$,

- $p(\mathbf{u}) \xrightarrow{\varepsilon}_{\mathbb{A}} p(\mathbf{u})$ and
- if $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{A}} q(\mathbf{v})$ and $q(\mathbf{v}) \xrightarrow{t}_{\mathbb{A}} r(\mathbf{w})$, then $p(\mathbf{u}) \xrightarrow{\pi \cdot t}_{\mathbb{A}} r(\mathbf{w})$.

We extend $\xrightarrow{t}_{\mathbb{A}}$ to languages $L \subseteq T^*$ in the natural way, $\xrightarrow{L}_{\mathbb{A}} \stackrel{\text{def}}{=} \bigcup \{ \xrightarrow{\pi}_{\mathbb{A}} : \pi \in L \}$. We write $\xrightarrow{*}_{\mathbb{A}}$ to denote $\xrightarrow{T^*}_{\mathbb{A}}$. An \mathbb{A} -run from $q_0(\mathbf{v}_0) \in Q \times \mathbb{A}$ to $q_k(\mathbf{v}_k) \in Q \times \mathbb{A}$ that is induced by a path $\pi = t_1 \cdots t_k$ is a sequence of configurations $q_0(\mathbf{v}_0) \xrightarrow{t_1}_{\mathbb{A}} q_1(\mathbf{v}_1) \cdots \xrightarrow{t_k}_{\mathbb{A}} q_k(\mathbf{v}_k)$ that we sometimes just abbreviate by $q_0(\mathbf{v}_0) \xrightarrow{\pi}_{\mathbb{A}} q_k(\mathbf{v}_k)$. When $\mathbb{A} = \mathbb{N}^d$ we also refer to an \mathbb{A} -run as a *run*.

In the remainder of this paper, we call $\xrightarrow{*}_{\mathbb{N}^d}$ the *reachability relation*, and $\xrightarrow{*}_{\mathbb{Z}^d}$ the *\mathbb{Z} -reachability relation*. Let $\pi = (p_1, \mathbf{z}_1, p_1) \cdots (p_k, \mathbf{z}_k, p_k) \in T^k$ for some $k \geq 0$. The *displacement* of π is $\delta(\pi) \stackrel{\text{def}}{=} \sum_{i=1}^k \mathbf{z}_i$, and the definition naturally extends to languages $L \subseteq T^*$ as $\delta(L) \stackrel{\text{def}}{=} \bigcup \{ \delta(\pi) : \pi \in L \}$. We say that a linear path scheme ρ over V *captures* a linear path scheme ρ' if $\delta(\rho') \subseteq \delta(\rho)$. Note in particular that if $\text{Parikh}_{\rho'} \subseteq \text{Parikh}_{\rho}$, then $\delta(\rho') \subseteq \delta(\rho)$. Similarly as in [18], we say that a linear path scheme $\alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ is *zigzag-free* if $\{ \delta(\beta_1), \dots, \delta(\beta_k) \} \subseteq Z$ for some quadrant Z .

3. MAIN RESULTS

In this paper, our main interest is in the reachability problem for 2-VASS, formally defined as follows:

2-VASS REACHABILITY

INPUT: A 2-VASS $V = (Q, T)$ and configurations $p(\mathbf{u})$ and $q(\mathbf{v})$ from $Q \times \mathbb{N}^2$.

QUESTION: Is there a run from $p(\mathbf{u})$ to $q(\mathbf{v})$, i.e. does $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ hold?

In order to determine the complexity of this problem, we show that the reachability relation of any 2-VASS can be defined by a finite union of linear path schemes. In particular, we are able to show strong bounds on their lengths and their number of cycles. For example, consider the 2-VASS V depicted in Fig. 3. Since V contains nested loops, e.g. $(t_1 t_3^* t_2)^*$, we cannot directly read

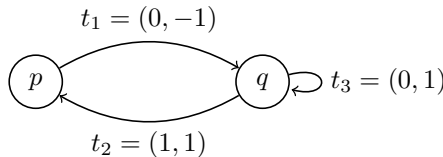


FIGURE 3. Example of a 2-VASS.

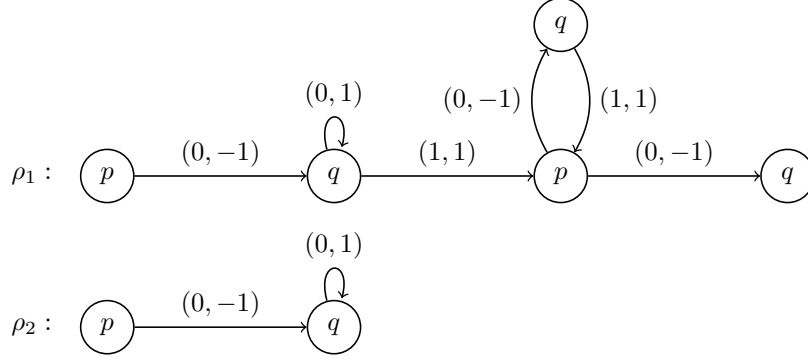


FIGURE 4. Illustration of a set $S = \{\rho_1, \rho_2\}$ of linear path schemes defining the reachability relation from p to q of the 2-VASS V depicted in Fig. 3. Here, $\rho_1 = t_1 t_3^* t_2 (t_1 t_2)^* t_1$, $\rho_2 = t_1 t_3^*$, and $p(u_1, u_2) \xrightarrow{*}_{\mathbb{N}^2} q(v_1, v_2)$ if, and only if, $p(u_1, u_2) \xrightarrow{S}_{\mathbb{N}^2} q(v_1, v_2)$.

off a characterization of its reachability set by a finite union of linear path schemes. However, by carefully unraveling loops we obtain the reachability set from the union of the subsequent linear path schemes, and in particular this means that V can be flattened:

$$\begin{aligned}
 p(u_1, u_2) \xrightarrow{*}_{\mathbb{N}^2} q(v_1, v_2) &\iff p(u_1, u_2) \xrightarrow{t_1 t_3^* \cup t_1 t_3^* t_2 (t_1 t_2)^* t_1}_{\mathbb{N}^2} q(v_1, v_2) && \text{(cf. Fig. 4)} \\
 p(u_1, u_2) \xrightarrow{*}_{\mathbb{N}^2} p(v_1, v_2) &\iff p(u_1, u_2) \xrightarrow{t_1 t_3^* t_2 (t_1 t_2)^* \cup \varepsilon}_{\mathbb{N}^2} p(v_1, v_2) \\
 q(u_1, u_2) \xrightarrow{*}_{\mathbb{N}^2} p(v_1, v_2) &\iff q(u_1, u_2) \xrightarrow{(t_2 t_1)^* t_3^* t_2}_{\mathbb{N}^2} p(v_1, v_2) \\
 q(u_1, u_2) \xrightarrow{*}_{\mathbb{N}^2} q(v_1, v_2) &\iff q(u_1, u_2) \xrightarrow{(t_2 t_1)^* t_3^*}_{\mathbb{N}^2} q(v_1, v_2) \quad .
 \end{aligned}$$

We will show that such a flattening exists for any 2-VASS. More precisely, our main technical result states that the global reachability relation of any 2-VASS $V = (Q, T)$ can be defined via a union of linear path schemes whose lengths can be polynomially bounded in $|Q| + \|T\|$, and *a fortiori* are at most exponential in $|V|$, and whose number of cycles is quadratic in $|Q|$:

Theorem 1. Let $V = (Q, T)$ be a 2-VASS. There is a finite set S of linear path schemes such that¹

- $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S}_{\mathbb{N}^2} q(\mathbf{v})$,
- $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ for every $\rho \in S$, and
- each $\rho \in S$ has at most $O(|Q|^2)$ cycles.

Having established Theorem 1, we can show that proving the existence of a path between two reachable configurations in a 2-VASS reduces to checking the existence of a solution for suitably constructed systems of linear Diophantine inequalities that depend on S and the properties listed in Theorem 1. The absence of nested cycles in linear path schemes in S is crucial to this reduction. By application of standard bounds from integer linear programming, this in turn enables us to bound the length of paths witnessing reachability, and to prove the main theorem of this paper in Section 5:

Theorem 2. 2-VASS REACHABILITY is PSPACE-complete.

¹The expanded technical meaning of this statement is that there are constants c_1 and c_2 such that for every 2-VASS $V = (Q, T)$ there exists a finite set S of linear path schemes with the properties that $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S}_{\mathbb{N}^2} q(\mathbf{v})$ and that each ρ in S has length at most $(|Q| + \|T\|)^{c_1}$ and has at most $c_2 |Q|^2$ cycles. The more familiar statements of this theorem and of lemmas of a similar nature in the rest of the paper were chosen to avoid clutter and to downplay the role of the precise constants.

4. PROOF OF THEOREM 1

In this section, we prove Theorem 1 and show that runs of a 2-VASS $V = (Q, T)$ are captured by a finite union of linear path schemes each of which have length at most $(|Q| + \|T\|)^{O(1)}$ and at most $O(|Q|^2)$ cycles. In order to construct this finite set of linear path schemes, we consider the following three types of runs $p(u_1, u_2) \xrightarrow{\pi}_{\mathbb{N}^2} q(v_1, v_2)$, depicted in Fig. 5:

- (1) Both counter values of $p(u_1, u_2)$ and of $q(v_1, v_2)$ are sufficiently large and $p = q$, but intermediate configurations on the run $p(u_1, u_2) \xrightarrow{\pi}_{\mathbb{N}^2} q(v_1, v_2)$ may have arbitrarily small counter values.
- (2) For all configurations of the run $p(u_1, u_2) \xrightarrow{\pi}_{\mathbb{N}^2} q(v_1, v_2)$ both counter values are sufficiently large.
- (3) For all configurations of the run $p(u_1, u_2) \xrightarrow{\pi}_{\mathbb{N}^2} q(v_1, v_2)$ at least one counter value is not too large.

In Sections 4.1, 4.2 and 4.3, we will show how to construct linear path schemes for these three types of runs. Then, in Section 4.4, we prove Theorem 1 by showing that any run can be decomposed as finitely many runs of these types.

In some more detail, the first step is to show in Section 4.1 that Parikh images of finite labeled graphs can be captured by linear path schemes of polynomial size. This will allow us to prove that \mathbb{Z} -reachability, i.e. runs in which counter values may drop below zero, can be captured by linear path schemes of polynomial size. We then give in Section 4.2 an effective decomposition of certain linear sets in dimension two into semi-linear sets with special properties, and use this decomposition in order to derive together with the results in Section 4.1 linear path schemes of size $(|Q| + \|T\|)^{O(1)}$ with a constant number of cycles for runs of type (1). Linear path schemes for runs of type (2) will then be seen to follow from the type (1) case.

For runs of type (3), in Section 4.3 we construct linear path schemes for 1-VASS and show that runs of a 2-VASS that stay within an “L-shaped band” are, essentially, runs of a 1-VASS. Our analysis of such runs of type (3) is a simple consequence of certain normal forms of shortest runs in one-counter automata, which 1-VASS are a subclass of, by Valiant and Paterson [27].

Similarities and differences in comparison with [18]. Our proof strategy of considering the three kinds of runs described above shares some similarities with [18], but in particular requires to explicate all implicit assumptions made in the conference paper [18]. There, the bounds on what is referred to as “large” and what is referred to as “not large” or “small” in the runs of type (1), (2) and (3) are not explicitly calculated. Our proofs for obtaining rather tight bounds require new insights. We capture runs of type (1) by linear path schemes of size $(|Q| + \|T\|)^{O(1)}$, whereas in [18] the linear path schemes were of size at least exponential in $|Q|$. To prove the former, we establish a new upper bound on the presentation size of Parikh images of finite automata in Lemma 4 below, which is a result of independent interest. The difference between our runs of type (2) and the ones analyzed in [18] is that our runs have to stay in the “outside region” entirely, whereas in [18] the set of displacements of paths from q to q' is analyzed. Runs of type (3) are treated as special cases of runs of type (2) in [18], whereas we invoke a result by Valiant and Paterson on normal forms of minimal runs in one-counter automata. Our final proof of Theorem 1 shows that each run can be factorized into segments of runs of types (1), (2) and (3) and requires a more careful treatment than in [18]. At every step, we have to ensure that the number of cycles of the linear path schemes we construct stays polynomial in the number of control states Q . This aspect is neglected in [18] as it is of no interest for the goal of [18], however, for us it is by far the technically most challenging part and one of the cornerstones of our PSPACE upper bound.

4.1. Parikh images of finite directed graphs and \mathbb{Z} -reachability of d -VASS. The main result of this section is the following proposition.

Proposition 3. Let $V = (Q, T)$ be a d -VASS. There exists a finite set S of linear path schemes such that

- (i) $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{Z}^d} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S}_{\mathbb{Z}^d} q(\mathbf{v})$,
- (ii) $|\rho| \leq 2 \cdot |Q| \cdot |T|$ for each $\rho \in S$, and

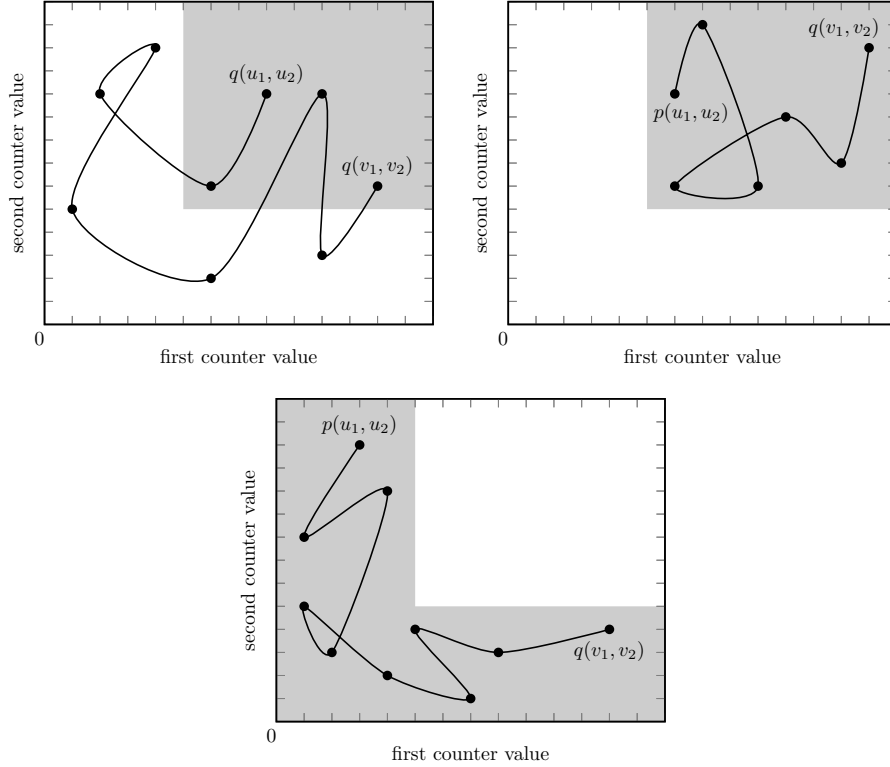


FIGURE 5. Example of the three types of runs. The region depicted in each case is the positive quadrant in the Cartesian plane. (1) top-left: run from q to q starting and ending sufficiently high; (2) top right: run staying sufficiently high; (3) bottom: run within an L-shaped band, i.e., running high on at most one component at a time.

(iii) each $\rho \in S$ has at most $|T|$ cycles.

In order to prove Proposition 3, we will prove suitable bounds on the representation size of the Parikh images of paths of a Σ -labeled finite graphs (or equivalently, nondeterministic finite automata) in terms of linear path schemes.

Lemma 4. Let $G = (U, E)$ be a finite Σ -labeled graph. There exists a finite set S of linear path schemes such that

- (i) $\{Parikh_\pi : \pi \text{ is a path}\} = \bigcup\{Parikh_\rho : \rho \in S\}$,
- (ii) $|\rho| \leq 2 \cdot |U| \cdot |E|$ for each $\rho \in S$, and
- (iii) each $\rho \in S$ has at most $|E|$ cycles.

Proof. We first provide some additional definitions. Let $\sigma, \sigma' : E \rightarrow \mathbb{N}$ be mappings and let X be a set of such mappings. We define $\sigma + \sigma' \in \mathbb{N}^E$ as $(\sigma + \sigma')(e) \stackrel{\text{def}}{=} \sigma(e) + \sigma'(e)$ for each $e \in E$ and $X + \sigma \stackrel{\text{def}}{=} \{\tau + \sigma : \tau \in X\}$. For each $u \in U$, let $\text{in}(u) \stackrel{\text{def}}{=} \{(u', a, u'') \in E : u'' = u\}$ and $\text{out}(u) \stackrel{\text{def}}{=} \{(u', a, u'') \in E \mid u' = u\}$ denote the set of incoming and outgoing edges of u , respectively. We say that σ is *flow-preserving* if for every $u \in U$ we have

$$\sum_{e \in \text{in}(u)} \sigma(e) = \sum_{e \in \text{out}(u)} \sigma(e) .$$

We will show the following claim:

Claim. Let $\pi \in E^*$ be a path. There exists some $h \geq 1$, a sequence of linear path schemes $\rho_1, \dots, \rho_h \subseteq E^*$, and a sequence $\sigma_1, \dots, \sigma_h \in \mathbb{N}^E$ such that

- (a) ρ_1 is a path of length at most $|U| \cdot |E|$ that visits each state of π at least once,
- (b) σ_1 is flow-preserving, and
- (c) $Parikh_\pi = Parikh_{\rho_1} + \sigma_1$,

and for every $1 < i \leq h$,

- (1) ρ_i is a linear path scheme that can be obtained from ρ_1 by inserting $i - 1$ simple cycles (of the form β^*),
- (2) σ_i is flow-preserving,
- (3) $Parikh_{\rho_{i-1}} + \sigma_{i-1} \subseteq Parikh_{\rho_i} + \sigma_i$,
- (4) $\sigma_{i-1}(e) \geq \sigma_i(e)$ for all $e \in E$ and there exists some $e \in E$ s.t. $\sigma_{i-1}(e) > \sigma_i(e) = 0$, and
- (5) $\sigma_h(e) = 0$ for all $e \in E$.

First observe that due to (4) we have $h \leq |E|$, and due to (1) we have $|\rho_i| \leq |\rho_1| + |U| \cdot (i - 1)$. Therefore $|\rho_h| \leq |\rho_1| + |U| \cdot |E| \leq 2 \cdot |U| \cdot |E|$, where the last inequality is due to (a). Moreover ρ_h has at most $|E|$ cycles due to (1) and $h \leq |E|$.

Before proving the claim, let us first see how it proves the lemma. We define

$$S \stackrel{\text{def}}{=} \{ \rho : \rho \text{ is a linear path scheme, } |\rho| \leq 2 \cdot |U| \cdot |E| \text{ and } \rho \text{ has at most } |E| \text{ cycles} \} .$$

Trivially, (ii) and (iii) are satisfied. To establish (i) let us fix an arbitrary path π . We apply the above Claim and obtain a linear path scheme $\rho_h \in S$ for π . It suffices to show $Parikh_\pi \in Parikh_{\rho_h}$ which holds due to

$$Parikh_\pi \stackrel{(c)}{=} Parikh(\rho_1) + \sigma_1 \stackrel{(3)}{\subseteq} \dots \stackrel{(3)}{\subseteq} Parikh_{\rho_h} + \sigma_h \stackrel{(5)}{=} Parikh_{\rho_h} .$$

We now prove the claim. Let π be a path and let us first define ρ_1 and σ_1 such that (a),(b) and (c) are satisfied. The path π can be decomposed as $\pi = e_1 \pi_1 \dots e_k \pi_k$ where $k \leq |U|$ and each $e_j = (u, a, u')$ is the first transition such that u or u' appears in π . We define ρ_1 and σ_1 as the result of the following iterative process: We initially set ρ_1 to π and set $\sigma_1(e) = 0$ for all $e \in E$; then we successively remove a simple cycle β from some π_j , and add $Parikh(\beta)$ to σ_1 . We repeat this process until no longer possible. The resulting ρ_1 is a path of length at most $|U| \cdot |E|$. Moreover, σ_1 is flow-preserving since we successively removed cycles only, and clearly $Parikh(\pi) = Parikh(\rho_1) + \sigma_1$, by construction. Thus (a), (b) and (c) hold.

Let us prove (1) to (5) by induction on $1 < i \leq h$. We only prove the induction step, the base case can be proven analogously. Let $E' \stackrel{\text{def}}{=} \{e \in E : \sigma_{i-1}(e) > 0\}$. If $E' = \emptyset$, then (5) holds and we are done. Thus, we assume that $E' \neq \emptyset$. Let us fix a choice function $\chi : E' \rightarrow E'$ satisfying

$$\chi(u_1, a, u_2) = (u'_1, a, u'_2) \quad \implies \quad u_2 = u'_1 .$$

Note that χ exists since σ_{i-1} is flow-preserving by induction hypothesis. By the pigeonhole principle there exist some $e \in E'$ and some $\ell \geq 0$ such that $\beta \stackrel{\text{def}}{=}} e\chi(e)\chi^2(e) \dots \chi^\ell(e)$ is a simple cycle and $c \stackrel{\text{def}}{=} \sigma_{i-1}(e) \leq \sigma_{i-1}(\chi^j(e))$ for all $j \in [1, \ell]$. We define $\sigma_i \stackrel{\text{def}}{=} \sigma_{i-1} - Parikh(\beta^c)$ and observe that σ_i is flow-preserving due to minimality of c ; thus (2) and (4) are shown. Let β be a cycle from u to u . By (1) of induction hypothesis the linear path scheme ρ_{i-1} can be obtained from ρ_1 by inserting $(i - 2)$ simple cycles and can hence be factorized as $\rho_{i-1} = \alpha\gamma$, where α is a linear path scheme from some state to u . We set $\rho_i \stackrel{\text{def}}{=} \alpha\beta^*\gamma$ and hence (1) holds. Furthermore, (3) holds due to $Parikh_{\rho_{i-1}} + \sigma_{i-1} = Parikh_{\rho_{i-1}} + Parikh_{\beta^c} + \sigma_i \subseteq Parikh_{\rho_i} + \sigma_i$. \square

We are now prepared to prove Proposition 3.

Proof of Proposition 3. We have $T \subseteq Q \times \Sigma \times Q$ for some finite subset $\Sigma \subseteq \mathbb{Z}^d$. Let S be the finite union of linear path schemes from Lemma 4, then (2) and (3) are clear. For (1) we have the

following equivalences:

$$\begin{aligned}
p(\mathbf{u}) \xrightarrow{*}_{\mathbb{Z}^d} q(\mathbf{v}) &\iff \exists \text{ path } \pi \text{ from } p \text{ to } q \text{ in } V \text{ s.t. } \mathbf{v} - \mathbf{u} = \sum_{\mathbf{z} \in \Sigma} \text{Parikh}_\pi(\mathbf{z}) \cdot \mathbf{z} \\
&\stackrel{\text{Lemma 4 (i)}}{\iff} \exists \rho \in S \text{ from } p \text{ to } q, \exists f \in \text{Parikh}_\rho \text{ s.t. } \mathbf{v} - \mathbf{u} \in \sum_{\mathbf{z} \in \Sigma} f(\mathbf{z}) \cdot \mathbf{z} \\
&\iff \exists \rho \in S \text{ from } p \text{ to } q \text{ s.t. } \mathbf{v} - \mathbf{u} \in \delta(\rho) \\
&\iff p(\mathbf{u}) \xrightarrow{S}_{\mathbb{Z}^d} q(\mathbf{v})
\end{aligned}$$

□

4.2. Starting and ending in “sufficiently large” configurations. The goal of this section is to prove that, given a 2-VASS V , there exists a sufficiently small bound D such that the reachability relation between any two configurations $q(u_1, v_1)$ and $q(u_2, v_2)$ for which $u_1, u_1, u_2, v_2 \geq D$ can be captured by a finite set of small linear path schemes (in the sense of Theorem 1). In [18], this property is referred to as *ultimately flat*. As a consequence of this result, we can show that the reachability relation between arbitrary configurations for which there exists a run on which both counter values on all configurations stay above D can be captured by a finite union of small linear path schemes as well.

Proposition 5. Let $V = (Q, T)$ be a 2-VASS. There exist $D \leq (|Q| + \|T\|)^{O(1)}$ and sets of linear path schemes R, X such that for $\mathbb{O} \stackrel{\text{def}}{=} [D, \infty)^2$ and $\mathbf{u}, \mathbf{v} \in \mathbb{O}$,

- (a)
 - $q(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ if, and only if, $q(\mathbf{u}) \xrightarrow{R}_{\mathbb{N}^2} q(\mathbf{v})$, and
 - $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ and ρ has at most two cycles for every $\rho \in R$.
- (b)
 - $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{O}} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{X}_{\mathbb{N}^2} q(\mathbf{v})$, and
 - $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ and ρ has at most $2 \cdot |Q|$ cycles for every $\rho \in X$.

The proof of this proposition requires two intermediate steps. First, in Lemma 6 below we prove an effective decomposition of certain linear sets in dimension two into semi-linear sets with nice properties. Similar decompositions have been the cornerstone of the results by Hopcroft and Pansiot [10] and Leroux and Sutre [18]. The contribution of Lemma 6 is to establish a new proof from which we can obtain sufficiently small bounds on this decomposition. Next, in Lemma 7 we show how this decomposition can be applied in order to capture reachability instances by linear path schemes with two cycles whose displacements all point into the same quadrant. This in turn enables us to prove Part (a) of Proposition 5, from which we can then prove Part (b).

Let us recall some definitions concerning semi-linear sets. Let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_n\} \subseteq \mathbb{Z}^m$ and $\mathbb{D} \subseteq \mathbb{Q}$. The \mathbb{D} -cone generated by P is defined as

$$\text{cone}_{\mathbb{D}}(P) \stackrel{\text{def}}{=} \left\{ \sum_{i \in [1, n]} \lambda_i \cdot \mathbf{p}_i : \lambda_i \in \mathbb{D}, \lambda_i \geq 0 \right\}.$$

A linear set $L(\mathbf{b}; P)$ is given by a base vector $\mathbf{b} \in \mathbb{Z}^d$ and a finite set of period vectors $P \subseteq \mathbb{Z}^d$, where $L(\mathbf{b}; P) \stackrel{\text{def}}{=} \mathbf{b} + \text{cone}_{\mathbb{N}}(P)$. A semi-linear set is a finite union of linear sets. The norm $\|P\|$ of a finite set $P \subseteq \mathbb{Z}^d$ is defined as $\|P\| \stackrel{\text{def}}{=} \max\{\|\mathbf{p}\| : \mathbf{p} \in P\}$. Recall that $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ are *linearly dependent* if $\mathbf{0} = \lambda_1 \cdot \mathbf{u} + \lambda_2 \cdot \mathbf{v}$ for some $\lambda_1, \lambda_2 \in \mathbb{Q} \setminus \{0\}$, and *linearly independent* otherwise.

We now show the following statement: the intersection of a linear set $L(\mathbf{b}; P) \subseteq \mathbb{Z}^2$, such that $\mathbf{b} \in P$, with some quadrant Z is equal to a semi-linear set $\bigcup_{i \in I} L(\mathbf{c}_i; P_i)$ such that $\mathbf{c}_i \in L(\mathbf{b}; P)$ and each P_i contains only two “small” vectors from $(P \cup L(\mathbf{b}; P)) \cap Z$.

Lemma 6. Let $\mathbf{b} \in \mathbb{Z}^2$, let $P \subseteq \mathbb{Z}^2$ be finite with $\mathbf{b} \in P$ and let Z be a quadrant. Then $L(\mathbf{b}; P) \cap Z = \bigcup_{i \in I} L(\mathbf{c}_i; P_i)$ such that for each $i \in I$ we have

- $|P_i| \leq 2$,
- $P_i \subseteq (P \cup L(\mathbf{b}; P)) \cap Z$, and

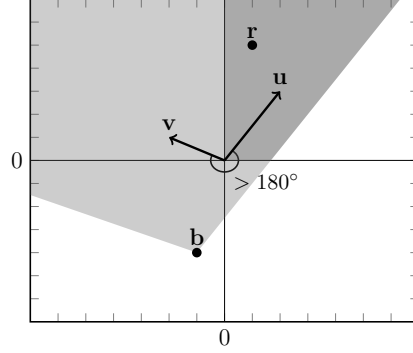


FIGURE 6. Example of Case 1 with the angle exceeding 180° . The filled area corresponds to $\mathbf{b} + \text{cone}_{\mathbb{Q}_{\geq 0}}(\{\mathbf{u}, \mathbf{v}\})$ and the darker filled area corresponds to $\mathbf{b} + \text{cone}_{\mathbb{Q}}(\{\mathbf{u}, \mathbf{v}\}) \cap (\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0})$.

- there exists $e \leq \|P\|^{O(1)}$ such that $\{\mathbf{c}_i\} \cup (P_i \cap L(\mathbf{b}; P)) \subseteq \mathbf{b} + \text{cone}_{[0, e]}(P)$.

Proof. Subsequently, we assume that $Z = \mathbb{N}^2$ and that P only contains pairwise linearly independent vectors, the general case can be obtained as an adaption of our argument. Let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ and $\mathbf{r} \in L(\mathbf{b}; P) \cap \mathbb{N}^2$, by definition $\mathbf{r} = \mathbf{b} + \lambda_1 \mathbf{p}_1 + \dots + \lambda_n \mathbf{p}_n$ for some $\lambda_i \in \mathbb{N}$. Denote $B \stackrel{\text{def}}{=} \|P\|$ and suppose there are more than two λ_i greater than B^2 , say $\lambda_1, \lambda_2, \lambda_3 > B^2$. An easy calculation shows that there exist $\gamma_1 \in [1, B^2]$ and $\gamma_2, \gamma_3 \in [-B^2, B^2]$ such that $\gamma_1 \mathbf{p}_1 = \gamma_2 \mathbf{p}_2 + \gamma_3 \mathbf{p}_3$. We can thus always decrease all but two λ_i below B^2 . Hence we can write $L(\mathbf{b}; P)$ as the following semi-linear set whose base vectors are sufficiently small and whose period vectors have cardinality at most two, where $W \stackrel{\text{def}}{=} \{\sum_{i=1}^n \lambda_i \mathbf{p}_i : \lambda_i \in [0, B^2]\}$:

$$L(\mathbf{b}; P) = \bigcup_{P' \subseteq P, |P'| \leq 2} \bigcup_{\mathbf{w} \in W} L(\mathbf{b} + \mathbf{w}; P').$$

Consequently, $\mathbf{r} = \mathbf{z} + \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$ where $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{b} + \mathbf{w} \in L(\mathbf{b}; P)$ for some $\mathbf{w} \in W$. Since $\|\mathbf{w}\| \leq \|P\|^{O(1)}$ for every $\mathbf{w} \in W$, we have $\|\mathbf{z}\| \leq \|P\|^{O(1)}$.

Our goal is to show that \mathbf{r} lies in a linear set fulfilling the properties required in the lemma. If $\{\mathbf{u}, \mathbf{v}\} \subseteq \mathbb{N}^2$, then we are done since then we have $\mathbf{u}, \mathbf{v} \in P \cap \mathbb{N}^2$. The cases when $\lambda = 0$ or $\zeta = 0$ are trivial, hence we subsequently assume $\lambda, \zeta > 0$. We thus consider the remaining cases separately up to symmetry. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Note that due to $\mathbf{r} = \mathbf{z} + \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \in \mathbb{N}^2$ we must have $\lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \in [-\|\mathbf{z}\|, \infty) \times [-\|\mathbf{z}\|, \infty)$.

Case 1: $\mathbf{u} \in \mathbb{N}^2$ and $\mathbf{v} \notin \mathbb{N}^2$. We only treat the case when the clockwise angle between \mathbf{u} and \mathbf{v} exceeds 180° , illustrated in Figure 6, i.e. when $\mathbf{v} \in -\mathbb{N}_{>0} \times \mathbb{Z}$ and $u_2/u_1 < v_2/v_1$. The case when the clockwise angle is below 180° can be treated symmetrically, i.e. when $\mathbf{v} \in \mathbb{Z} \times -\mathbb{N}_{>0}$ and $u_2/u_1 > v_2/v_1$. It cannot be exactly 180° since \mathbf{u} and \mathbf{v} are linearly independent by assumption.

Our first step is to show the existence of some $\alpha \in \mathbb{N}$ such that $(0, \alpha) = \beta \cdot \mathbf{u} + \gamma \cdot \mathbf{v} \in L(\mathbf{b}; P)$ for some $\beta, \gamma \in [1, B^4]$. Due to the linear independence of \mathbf{u} and \mathbf{v} , there exist $\eta \in [1, B^2]$ and $\chi, \theta \in [-B^2, B^2]$ such that $\eta \cdot \mathbf{b} = \chi \cdot \mathbf{u} + \theta \cdot \mathbf{v}$. Since the clockwise angle between \mathbf{u} and \mathbf{v} exceeds 180° there are positive $\alpha', \beta', \gamma' \in [1, B^2]$ such that $(0, \alpha') = \beta' \cdot \mathbf{u} + \gamma' \cdot \mathbf{v}$. Thus, we can choose α, β and γ as follows:

$$\begin{aligned} (1) \quad (0, \underbrace{\alpha'}_{\alpha} \cdot B^2) &= \underbrace{B^2 \cdot \beta'}_{\beta} \cdot \mathbf{u} + \underbrace{B^2 \cdot \gamma'}_{\gamma} \cdot \mathbf{v} \\ &= \beta \cdot \mathbf{u} + \gamma \cdot \mathbf{v} - \eta \cdot \mathbf{b} + \eta \cdot \mathbf{b} \\ &= \underbrace{(\beta - \chi)}_{\geq 0} \cdot \mathbf{u} + \underbrace{(\gamma - \theta)}_{\geq 0} \cdot \mathbf{v} + \eta \cdot \mathbf{b} \stackrel{\mathbf{b} \in P, \eta \geq 1}{\in} L(\mathbf{b}; P). \end{aligned}$$

As an intermediate step, we show that

$$\gamma \cdot \lambda - \beta \cdot \zeta > -\|P\|^{O(1)}.$$

To this end we rewrite $\lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$ as

$$\begin{aligned} \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v} &= \left(\left\lfloor \frac{\lambda}{\beta} \right\rfloor \cdot \beta + (\lambda \bmod \beta) \right) \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \\ &\stackrel{(1)}{=} \left\lfloor \frac{\lambda}{\beta} \right\rfloor \left((0, \alpha) - \gamma \cdot \mathbf{v} \right) + (\lambda \bmod \beta) \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \\ (2) \quad &= \underbrace{\left(\zeta - \left\lfloor \frac{\lambda}{\beta} \right\rfloor \cdot \gamma \right)}_{\kappa} \cdot \mathbf{v} + \underbrace{(\lambda \bmod \beta) \cdot \mathbf{u}}_{\text{has norm at most } B^5} + \left\lfloor \frac{\lambda}{\beta} \right\rfloor \cdot (0, \alpha). \end{aligned}$$

Recall that $v_1 < 0$. Since $\lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \in [-\|\mathbf{z}\|, \infty) \times \mathbb{Z}$, applying (2) we derive

$$\kappa \cdot v_1 + (\lambda \bmod \beta) \cdot \|\mathbf{u}\| \geq -\|\mathbf{z}\| \implies \kappa \leq H \text{ for some } H \leq \|P\|^{O(1)}.$$

We now obtain

$$\begin{aligned} \zeta - \frac{\lambda + \beta}{\beta} \cdot \gamma < \kappa \leq H &\implies \zeta < H + \frac{\lambda + \beta}{\beta} \cdot \gamma \\ (3) \quad &\implies \lambda > \frac{(\zeta - H) \cdot \beta}{\gamma} - \beta. \end{aligned}$$

In order to obtain $\mathbf{r} \in L(\mathbf{c}; \{\mathbf{x}, \mathbf{y}\})$ for suitable $\mathbf{c}, \mathbf{x}, \mathbf{y}$, we make a case distinction. Let $H' \stackrel{\text{def}}{=} H + 2 \cdot \gamma$.

- $\zeta \leq H'$: We choose $\mathbf{c} \stackrel{\text{def}}{=} \mathbf{z} + \zeta \cdot \mathbf{v}$, $\mathbf{x} = \mathbf{y} \stackrel{\text{def}}{=} \mathbf{u}$ and observe that $\|\mathbf{c}\| \leq \|P\|^{O(1)}$.
- $\zeta > H'$: Since $\mathbf{r} = \mathbf{z} + \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$, it suffices to show that $\lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$ can be written as $\varrho \cdot \mathbf{u} + \psi \cdot (0, \alpha) + \omega \cdot \mathbf{v}$ with $\omega \leq \|P\|^{O(1)}$. To this end, we first rewrite $\lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$ as

$$\begin{aligned} \lambda \cdot \mathbf{u} + (\zeta - H' + H') \cdot \mathbf{v} &= \lambda \cdot \mathbf{u} + \left\lfloor \frac{\zeta - H'}{\gamma} \right\rfloor \cdot \gamma \cdot \mathbf{v} + \underbrace{\left(((\zeta - H') \bmod \gamma) + H' \right)}_{\omega} \cdot \mathbf{v} \\ &\stackrel{(1)}{=} \underbrace{\left(\lambda - \left\lfloor \frac{\zeta - H'}{\gamma} \right\rfloor \cdot \beta \right)}_{\varrho} \cdot \mathbf{u} + \underbrace{\left(\left\lfloor \frac{\zeta - H'}{\gamma} \right\rfloor \right)}_{\psi} \cdot (0, \alpha) + \omega \cdot \mathbf{v}. \end{aligned}$$

Since $\psi \geq 0$ and $\omega \leq (B^4 + H') \leq \|P\|^{O(1)}$, it remains to prove that $\varrho \geq 0$:

$$\begin{aligned} \varrho &= \lambda - \left\lfloor \frac{\zeta - H'}{\gamma} \right\rfloor \cdot \beta = \lambda - \left\lfloor \frac{\zeta - H - 2 \cdot \gamma}{\gamma} \right\rfloor \cdot \beta \\ &> \lambda - \frac{\zeta - H - \gamma}{\gamma} \cdot \beta \\ &\stackrel{(3)}{>} \frac{(\zeta - H) \cdot \beta}{\gamma} - \beta - \frac{\zeta - H - \gamma}{\gamma} \cdot \beta \\ &= 0. \end{aligned}$$

Consequently, we set $\mathbf{c} = \mathbf{z} + \omega \cdot \mathbf{v}$, $\mathbf{x} = \mathbf{u}$ and $\mathbf{y} = (0, \alpha)$.

Case 2: $\mathbf{u}, \mathbf{v} \notin \mathbb{N}^2$. The case $\text{cone}_{\mathbb{Q}}(\{\mathbf{u}, \mathbf{v}\}) \cap \mathbb{N}^2 = \{\mathbf{0}\}$ is trivial. Hence we assume $\text{cone}_{\mathbb{Q}}(\{\mathbf{u}, \mathbf{v}\}) \cap \mathbb{N}^2 \neq \{\mathbf{0}\}$ and it is easily seen that this implies $\mathbb{N}^2 \subseteq \text{cone}_{\mathbb{Q}}(\{\mathbf{u}, \mathbf{v}\})$. Without loss of generality we assume that $u_1, v_2 < 0$ and $u_2, v_1 > 0$ and consequently have $u_1/u_2 > v_1/v_2$, Figure 7 illustrates this case. In particular, for all $\lambda', \zeta' \in \mathbb{Z}$ we have

$$(4) \quad \lambda' \cdot \mathbf{u} + \zeta' \cdot \mathbf{v} \in \mathbb{N}^2 \quad \text{implies} \quad \lambda', \zeta' \in \mathbb{N} \quad .$$

Analogously to Case 1 there exist $\sigma, \tau, \xi, \alpha, \beta, \gamma \in [1, B^4]$ with

$$(5) \quad (\sigma, 0) = \tau \cdot \mathbf{u} + \xi \cdot \mathbf{v} \in L(\mathbf{b}; P) \quad \text{and} \quad (0, \alpha) = \beta \cdot \mathbf{u} + \gamma \cdot \mathbf{v} \in L(\mathbf{b}; P) \quad .$$

Similar to Case 1, it is sufficient to rewrite $(\ell_1, \ell_2) \stackrel{\text{def}}{=} \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v}$ as $\varrho \cdot (\sigma, 0) + \psi \cdot (0, \alpha) + \mathbf{w}'$, where $\varrho, \psi \in \mathbb{N}$, $\mathbf{w}' \in \text{cone}_{\mathbb{N}}\{\mathbf{u}, \mathbf{v}\}$ and $\|\mathbf{w}'\| \leq \|P\|^{O(1)}$. We observe that $\min(\ell_1, \ell_2) \geq -\|\mathbf{z}\|$ and $\max(\ell_1, \ell_2) \geq 0$ and make a case distinction.

Case 2(a): $\ell_1, \ell_2 \geq 0$. We have $(\ell_1, \ell_2) = (h_1 + r_1, h_2 + r_2)$, where $h_1 = \lfloor \frac{\ell_1}{\sigma} \rfloor \cdot \sigma$, $h_2 = \lfloor \frac{\ell_2}{\alpha} \rfloor \cdot \alpha$, $r_1 = (\ell_1 \bmod \sigma)$ and $r_2 = (\ell_2 \bmod \alpha)$. Due to

$$(h_1, h_2) = \left\lfloor \frac{\ell_1}{\sigma} \right\rfloor \cdot (\sigma, 0) + \left\lfloor \frac{\ell_2}{\alpha} \right\rfloor \cdot (0, \alpha) \stackrel{(5)}{=} \underbrace{\left(\left\lfloor \frac{\ell_1}{\sigma} \right\rfloor \cdot \tau + \left\lfloor \frac{\ell_2}{\alpha} \right\rfloor \cdot \beta \right)}_{\theta} \cdot \mathbf{u} + \underbrace{\left(\left\lfloor \frac{\ell_1}{\sigma} \right\rfloor \cdot \xi + \left\lfloor \frac{\ell_2}{\alpha} \right\rfloor \cdot \gamma \right)}_{\mu} \cdot \mathbf{v},$$

we set $\varrho \stackrel{\text{def}}{=} \lfloor \frac{\ell_1}{\sigma} \rfloor$ and $\psi \stackrel{\text{def}}{=} \lfloor \frac{\ell_2}{\alpha} \rfloor$. We argue that we can take $\mathbf{w}' \stackrel{\text{def}}{=} (r_1, r_2) \in \mathbb{N}^2$. Since $\|\mathbf{w}'\| \leq B^4$ it remains to show that $\mathbf{w} \in \text{cone}_{\mathbb{N}}(\{\mathbf{u}, \mathbf{v}\})$. To see the latter, we have $(r_1, r_2) = (\ell_1, \ell_2) - (h_1, h_2) = (\lambda - \theta) \cdot \mathbf{u} + (\zeta - \mu) \cdot \mathbf{v}$, and $\lambda - \theta, \zeta - \mu \in \mathbb{Z}$. But then (4) yields $\lambda - \theta, \zeta - \mu \in \mathbb{N}$, as required.

Case 2(b): $\ell_1 < 0, \ell_2 \geq 0$. First, we show that in all linear combinations $\ell_1 = \lambda \cdot u_1 + \zeta \cdot v_1$ such that $-\|\mathbf{z}\| \leq \ell_1 < 0$, λ and ζ only differ by a linear factor. Indeed, we have

$$(6) \quad \ell_1 = \lambda \cdot u_1 + \zeta \cdot v_1 \iff \lambda = -\frac{v_1}{u_1} \cdot \zeta + \frac{\ell_1}{u_1} \iff \zeta = -\frac{u_1}{v_1} \cdot \lambda + \frac{\ell_1}{v_1}.$$

By subtracting and adding $\alpha \cdot k \cdot v_1 \cdot u_1$, we get

$$(7) \quad \forall k \in \mathbb{N}: \quad \ell_1 = (\lambda - \alpha \cdot k \cdot v_1) \cdot u_1 + (\zeta + \alpha \cdot k \cdot u_1) \cdot v_1.$$

On the other hand, for $k > 0$ we have

$$(8) \quad (\lambda - \alpha \cdot k \cdot v_1) \cdot u_2 + (\zeta + \alpha \cdot k \cdot u_1) \cdot v_2 = \lambda \cdot u_2 + \zeta \cdot v_2 + \alpha \cdot k \cdot (u_1 \cdot v_2 - v_1 \cdot u_2) < \lambda \cdot u_2 + \zeta \cdot v_2$$

where the latter inequality follows from $u_1/u_2 > v_1/v_2$. Let us define

$$k_0 \stackrel{\text{def}}{=} \min \{ \max\{k \in \mathbb{N} : \lambda - \alpha \cdot k \cdot v_1 \geq 0\}, \max\{k \in \mathbb{N} : \zeta + \alpha \cdot k \cdot u_1 \geq 0\} \}.$$

Moreover, let $\lambda' \stackrel{\text{def}}{=} \lambda - \alpha \cdot k_0 \cdot v_1 \geq 0$ and $\zeta' \stackrel{\text{def}}{=} \zeta + \alpha \cdot k_0 \cdot u_1 \geq 0$. Clearly $\min\{\lambda', \zeta'\} \leq \|P\|^{O(1)}$ by the choice of k_0 and hence $\lambda', \zeta' \leq \|P\|^{O(1)}$ by (6). Moreover, from (7) we have that $\ell_1 = \lambda' \cdot u_1 + \zeta' \cdot v_1$. Finally, as required, we have

$$\begin{aligned} \begin{vmatrix} \ell_1 \\ \ell_2 \end{vmatrix} &= \lambda \cdot \mathbf{u} + \zeta \cdot \mathbf{v} \\ &\stackrel{(8)}{=} \lambda' \cdot \mathbf{u} + \zeta' \cdot \mathbf{v} + \underbrace{(-k_0 \cdot (u_1 \cdot v_2 - v_1 \cdot u_2))}_{\geq 0} \cdot \begin{vmatrix} 0 \\ \alpha \end{vmatrix}. \end{aligned}$$

Case 2(c): $\ell_1 \geq 0, \ell_2 < 0$. This case is symmetric to Case 2(b) and therefore omitted. □

Let us give an intuitive idea of how we can prove Proposition 5 (a) by an application of Lemma 6. Suppose we are given a run starting in $q(u_1, u_2)$ and ending in $q(v_1, v_2)$ such that $u_1 \leq v_1$ and $u_2 \leq v_2$. From Proposition 3 we know that the \mathbb{Z} -reachability relation can be captured by a union of linear path schemes. Since we start and end in the *same* state, any such linear path scheme can equivalently be viewed as a linear set $L(\mathbf{b}; P)$ such that $\mathbf{b} \in P$. An application of Lemma 6 then allows us to decompose such a linear set into a semi-linear set whose period vectors all point into the same \mathbb{N}^2 direction. The crucial point is that any linear set in this semi-linear set can again be translated back into a linear path scheme with at most two cycles whose displacements point to \mathbb{N}^2 . Consequently, any path obtained from such a linear path scheme does not, informally speaking, drift away too much, and if u_1 and u_2 are sufficiently large then \mathbb{N} -reachability and \mathbb{Z} -reachability coincide.

In order to make our intuition formal, we introduce some further additional notation. Interpreting Lemma 6 in terms of linear path schemes allows us to establish the following lemma.

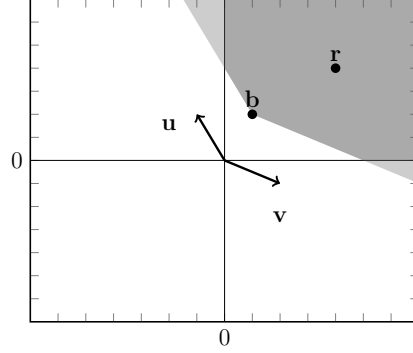


FIGURE 7. Example of Case 2. The filled area corresponds to $\mathbf{b} + \text{cone}_{\mathbb{Q}_{\geq 0}}(\{\mathbf{u}, \mathbf{v}\})$ and the darker filled area corresponds to $\mathbf{b} + \text{cone}_{\mathbb{Q}}(\{\mathbf{u}, \mathbf{v}\}) \cap (\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0})$.

Lemma 7. Let $q \in Q$. For every linear path scheme ρ from q to q , there exists a finite set R_ρ of zigzag-free linear path schemes such that

- (i) $\delta(\rho) \subseteq \delta(R_\rho)$,
- (ii) $|\sigma| \leq (|\rho| + \|T\|)^{O(1)}$ for each $\sigma \in R_\rho$, and
- (iii) each $\sigma \in R_\rho$ has at most two cycles.

Proof. Let $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ be a linear path scheme. Without loss of generality we assume that $\delta(\alpha_0 \cdots \alpha_k) \in \{\delta(\beta_i) : i \in [1, k]\}$, otherwise we apply the claim to the linear path scheme $\rho' \stackrel{\text{def}}{=} (\alpha_0 \cdots \alpha_k)^* \rho$ which satisfies this property and for which we have $\delta(\rho) \subseteq \delta(\rho')$. Moreover we assume that $\delta(\beta_i) \neq \delta(\beta_j)$ whenever $i \neq j$, since otherwise we can just remove β_j from ρ which results in a linear path scheme with the same displacements as ρ . We can write

$$\delta(\rho) = \bigcup_{Z \text{ is a quadrant}} \delta(\rho) \cap Z .$$

Hence for each quadrant Z it is sufficient to construct a set of appropriate zigzag-free linear path scheme $R_{\rho, Z}$ such that $\delta(\rho) \cap Z = \delta(R_{\rho, Z})$ since we can then just define our set of linear path schemes as $R_\rho \stackrel{\text{def}}{=} \bigcup \{R_{\rho, Z} : Z \text{ is a quadrant}\}$. Let $\mathbf{b} \stackrel{\text{def}}{=} \delta(\alpha_0 \cdots \alpha_k)$ and let $P \stackrel{\text{def}}{=} \{\delta(\beta_i) : i \in [1, k]\}$. By assumption we have $\mathbf{b} \in P$. Note that $\|P\| \leq |\rho| \cdot \|T\|$. By Lemma 6 there exists a semi-linear set $\bigcup_{i \in I} L(\mathbf{c}_i; P_i)$ with $\delta(\mathbf{b}; P) \cap Z = \bigcup_{i \in I} L(\mathbf{c}_i; P_i)$ satisfying for each $i \in I$,

- $|P_i| \leq 2$,
- $P_i \subseteq (P \cup L(\mathbf{b}; P)) \cap Z$, and
- there exists $e \leq \|P\|^{O(1)}$ such that $\{\mathbf{c}_i\} \cup (P_i \cap L(\mathbf{b}; P)) \subseteq \mathbf{b} + \text{cone}_{[0, e]}(P)$.

Let us fix an arbitrary $i \in I$. By the last item for each $\mathbf{u} \in \{\mathbf{c}_i\} \cup (P_i \cap L(\mathbf{b}; P))$ there exists a path $\pi_{\mathbf{u}}$ from q to q of the form $\alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$ for some $0 \leq e_1, \dots, e_k \leq \|P\|^{O(1)} \leq (|\rho| + \|T\|)^{O(1)}$ with $\mathbf{u} = \delta(\pi_{\mathbf{u}})$; thus $|\pi_{\mathbf{u}}| \leq (|\rho| + \|T\|)^{O(1)}$. Let $\pi_{\mathbf{c}_i} = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$ and define the linear path scheme σ_i to be obtained from $\pi_{\mathbf{c}_i}$ by inserting appropriate cycles β_j^* whenever $\delta(\beta_j) \in P_i \cap P$. Formally, we define

$$\sigma_i \stackrel{\text{def}}{=} \alpha_0 \beta_1^{e_1} \theta_1 \alpha_1 \cdots \beta_k^{e_k} \theta_k \alpha_k, \quad \text{where } \theta_j = \begin{cases} \beta_j^* & \text{if } \delta(\beta_j) \in P_i \cap P \\ \varepsilon & \text{otherwise} \end{cases} \quad \text{for every } j \in [1, k] .$$

Recalling that $P_i \subseteq (P \cup L(\mathbf{b}; P)) \cap Z$, it is now readily seen that

$$\rho_i \stackrel{\text{def}}{=} \sigma_i \cdot \prod_{\mathbf{u} \in P_i \setminus P} \pi_{\mathbf{u}}^*$$

is a zigzag-free linear path scheme with at most two cycles whose displacements point to Z , and satisfying $\delta(\rho_i) = L(\mathbf{c}_i; P_i)$ and $|\rho_i| \leq (|\rho| + \|T\|)^{O(1)}$. Finally, we define $R_{\rho, Z} \stackrel{\text{def}}{=} \bigcup_{i \in I} \rho_i$ due to

$$\delta(\rho) \cap Z = L(\mathbf{b}; P) \cap Z = \bigcup_{i \in I} L(\mathbf{c}_i; P_i) = \bigcup_{i \in I} \delta(\rho_i) = \delta(R_{\rho, Z}).$$

□

We are now fully prepared to give a proof of Proposition 5.

Proof of Proposition 5. Let us fix a 2-VASS $V = (Q, T)$.

Proof of (a): Let S be the finite set of linear path scheme from Proposition 3 such that

- $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{Z}^d} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S}_{\mathbb{Z}^d} q(\mathbf{v})$,
- $|\rho| \leq 2 \cdot |Q| \cdot |T|$ for each $\rho \in S$ and
- each $\rho \in S$ has at most $|T|$ cycles.

We apply Lemma 7 to each $\rho \in S$ and define $R \stackrel{\text{def}}{=} \bigcup_{\rho \in S} R_\rho$. Hence, for each $\sigma \in R$ we have $|\sigma| \leq (|T| \cdot |Q| + \|T\|)^{O(1)} = (|Q| + \|T\|)^{O(1)}$ by (ii) of Lemma 7. We set D required in Proposition 5 to $D \stackrel{\text{def}}{=} \max\{|\sigma| : \sigma \in R\} \cdot \|T\| \leq (|Q| + \|T\|)^{O(1)}$. The monotonicity of zigzag-free linear path schemes now provides the key ingredient for proving Proposition 5 (a). For the rest of the proof let us fix $\mathbf{u}, \mathbf{v} \in [D, \infty)^2$ and some zigzag-free linear path scheme $\sigma = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \alpha_2 \in R$. Suppose $q(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{Z}^2} q(\mathbf{v})$ for some $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \beta_2^{e_2} \alpha_2$, then by definition of D it is clear that

$$(9) \quad \forall i \in [0, |\pi|] : \quad \mathbf{0} \leq \mathbf{u} + \delta(\pi[1, i]) \leq \mathbf{v} + \begin{pmatrix} D \\ D \end{pmatrix}.$$

It remains to prove $q(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ if, and only if, $q(\mathbf{u}) \xrightarrow{\sigma}_{\mathbb{N}^2} q(\mathbf{v})$ for some $\sigma \in R$. The latter follows from the following circular sequence of implications and equivalences:

$$\begin{aligned} q(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v}) &\implies q(\mathbf{u}) \xrightarrow{*}_{\mathbb{Z}^2} q(\mathbf{v}) \\ &\stackrel{\text{Proposition 3}}{\iff} q(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{Z}^2} q(\mathbf{v}) \text{ for some } \rho \in S \\ &\stackrel{\text{Lemma 7(i)}}{\implies} q(\mathbf{u}) \xrightarrow{\sigma}_{\mathbb{Z}^2} q(\mathbf{v}) \text{ for some } \sigma \in R_\rho \text{ for some } \rho \in S \\ &\stackrel{(9)}{\implies} q(\mathbf{u}) \xrightarrow{\sigma}_{\mathbb{N}^2} q(\mathbf{v}) \text{ for some } \sigma \in R \\ &\implies q(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v}) \end{aligned}$$

Proof of (b): Suppose that $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{O}} q(\mathbf{v})$. Then π can be factorized as $\pi = \alpha_0 \beta_1 \alpha_1 \cdots \beta_k \alpha_k$ such that

$$p(\mathbf{u}) \xrightarrow{\alpha_0}_{\mathbb{O}} q_1(\mathbf{u}_1) \xrightarrow{\beta_1}_{\mathbb{O}} q_1(\mathbf{u}'_1) \xrightarrow{\alpha_1}_{\mathbb{O}} q_2(\mathbf{u}_2) \cdots q_k(\mathbf{u}_k) \xrightarrow{\beta_k}_{\mathbb{O}} q_k(\mathbf{u}'_k) \xrightarrow{\alpha_k}_{\mathbb{O}} q(\mathbf{v})$$

where $|\alpha_0|, |\alpha_1|, \dots, |\alpha_k| \leq |Q|$, each β_i is a cycle from q_i to q_i for some $q_i \in Q$, and $k \leq |Q|$. Since $\mathbf{u}_i, \mathbf{u}'_i \in \mathbb{O}$ for all $i \in [1, k]$, by (a) we have $q_i(\mathbf{u}_i) \xrightarrow{\rho_i}_{\mathbb{N}^2} q_i(\mathbf{u}'_i)$ for some linear path scheme $\rho_i \in R$. Consequently, we define X as

$$X \stackrel{\text{def}}{=} \{ \alpha_0 \rho_1 \alpha_1 \cdots \rho_k \alpha_k \text{ linear path scheme} : k \leq |Q|, \alpha_i \in T^*, |\alpha_i| \leq |Q|, \rho_i \in R \}$$

Let $\rho \in X$, then we have $|\rho| \leq |Q|^2 + |Q| \cdot (|Q| + \|T\|)^{O(1)} = (|Q| + \|T\|)^{O(1)}$, and ρ has at most $2 \cdot |Q|$ cycles.

□

4.3. Reachability in 2-VASS with One Bounded Component. The purpose of this section is to establish the following result on reachability between configurations for which there exists a run on which for all configurations *at most one* of the two counter values exceeds a certain bound. We refer to the bottom picture of Figure 5.

Proposition 8. Let $V = (Q, T)$ be a 2-VASS, $D \in \mathbb{N}$ and $\mathbb{L} = ([0, D] \times \mathbb{N}) \cup (\mathbb{N} \times [0, D])$. There exists a finite set $Y_{\mathbb{L}}$ of linear path schemes such that

- $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{L}} q(\mathbf{v})$ implies $p(\mathbf{u}) \xrightarrow{Y_{\mathbb{L}}}_{\mathbb{N}^2} q(\mathbf{v})$,
- $|\rho| \leq (|Q| + \|T\| + D)^{O(1)}$ for every $\rho \in Y_{\mathbb{L}}$; and
- each $\rho \in Y_{\mathbb{L}}$ has at most two cycles.

In its essence, restricting the set of admissible values of one of the two counters of a 2-VASS to $[0, D]$ as in Proposition 8 gives rise to a 1-VASS. This observation enables us to resort to techniques and results developed for 1-VASS (in fact even for one-counter automata), respectively. In particular, subsequently we make use of the following lemma established by Valiant and Paterson. It shows that reachability in a 1-VASS is captured by a finite union of linear path schemes each having at most one cycle.

Lemma 9 (Lemma 2 in [27]). Let $V = (Q, T)$ be a 1-VASS with unary updates (i.e. $T \subseteq Q \times \{-1, 0, 1\} \times Q$) and let $p(u) \xrightarrow{*}_{\mathbb{N}} q(v)$ for some configurations $p(u)$ and $q(u)$ such that $|u - v| \geq |Q| + |Q|^2$. There exist $\alpha, \beta, \gamma \in T^*$ and $\pi \in T^*$ such that $p(u) \xrightarrow{\pi}_{\mathbb{N}} p(v)$ and π has the the following properties,

- $\pi = \alpha\beta^i\gamma$ for some $i > 0$,
- $\alpha\beta^*\gamma$ is a linear path scheme with one cycle, and
- $|\alpha\gamma| < |Q|^2$ and β is a cycle with $|\beta| \leq |Q|$ and $|\delta(\beta)| \in [1, |Q|]$.

The following lemma states that in a 1-VASS with unary updates between any two reachable configurations with absolute counter difference D there is a run witnessing their reachability that has length at most $|Q|^{O(1)} + |Q| \cdot D$. It is obtained as an easy consequence of Lemma 9.

Lemma 10. Let $V = (Q, T)$ be a 1-VASS with unary updates, i.e. $T \subseteq Q \times \{-1, 0, 1\} \times Q$. Let $u, v \in \mathbb{N}$ and $D \stackrel{\text{def}}{=} |v - u|$. If $p(u) \xrightarrow{*}_{\mathbb{N}} q(v)$ then there is some run $p(u) \xrightarrow{\pi}_{\mathbb{N}} q(v)$ with $|\pi| \leq |Q|^{O(1)} + |Q| \cdot D$.

Proof. We first consider the case when $D \geq |Q| + |Q|^2$. By Lemma 9, we have $p(u) \xrightarrow{\alpha\beta^i\gamma}_{\mathbb{N}} q(v)$ for some $i \geq 0$, where $\alpha\beta^*\gamma$ is a linear path scheme, $|\alpha\gamma| < |Q|^2$ and β is a cycle with $|\beta| \leq |Q|$ and $|\delta(\beta)| \in [1, |Q|]$. Since $|\delta(\beta)| \in [1, |Q|]$ we have $i \leq D + |\alpha\gamma|$ and hence $|\alpha\beta^i\gamma| < |Q|^2 + |Q| \cdot i \leq |Q|^{O(1)} + |Q| \cdot D$.

We now turn to the case in which $D = |u - v| < |Q| + |Q|^2$. By the pigeonhole principle, for any run from $p(u)$ to $q(v)$ of minimal length either

- every configuration $r(w)$ on this minimal run satisfies $|w - v| < 2 \cdot (|Q| + |Q|^2)$, or
- there exists an intermediate configuration $r(w)$ on this minimal run with $|w - v| = 2 \cdot (|Q| + |Q|^2)$.

Clearly, any run of the form (i) is of length strictly less than $4 \cdot (|Q| + |Q|^2) \cdot |Q|$. Otherwise, for any minimal run π from $p(u)$ to $q(v)$ of the form (ii) there is some configuration $r(w)$ along this path with $|w - v| = 2 \cdot (|Q| + |Q|^2)$. Note that we have $|w - v| \leq |w - u| + |u - v|$ by the triangle inequality. This allows to conclude $|Q| + |Q|^2 \leq |w - u| \leq 3 \cdot (|Q| + |Q|^2)$ due to

$$|Q| + |Q|^2 \leq |w - v| - |u - v| \leq |w - u| \leq |w - v| + |u - v| \leq 3 \cdot (|Q| + |Q|^2).$$

Summarizing, we have $|Q| + |Q|^2 \leq |w - u| \leq 3 \cdot (|Q| + |Q|^2)$ and $|w - v| = 2 \cdot (|Q| + |Q|^2)$. Thus, $|\pi|$ is at most the length of two runs each of which has a counter difference of at least $|Q| + |Q|^2$, namely the length of a minimal run from $p(u)$ to $r(w)$ plus the length of a minimal run from $r(w)$ to $q(v)$:

$$|\pi| \leq (|Q|^2 + |Q| \cdot 3 \cdot (|Q| + |Q|^2)) + (|Q|^2 + |Q| \cdot 2 \cdot (|Q| + |Q|^2)) \leq |Q|^{O(1)}$$

□

We now combine the Lemmas 9 and 10 in order to show that the reachability relation of a 1-VASS $V = (Q, T)$ (with binary updates) can be captured by a union of linear path schemes that each have at most one cycle and length polynomially bounded in $|Q| + \|T\|$.

Lemma 11. Let $V = (Q, T)$ be a 1-VASS. There exists a finite set Y of linear path schemes such that

- (i) $p(u) \xrightarrow{*} q(v)$ if, and only if, $p(u) \xrightarrow{Y} q(v)$,
- (ii) $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ for each $\rho \in Y$, and
- (iii) each $\rho \in Y$ has at most one cycle.

Proof. The idea is to construct from $V = (Q, T)$ a unary 1-VASS $V' = (Q', T')$ with $T' \subseteq Q' \times \{-1, 0, +1\} \times Q'$ that mimics the behavior of V . We then apply Lemmas 9 and 10 to V' in order to obtain the set of linear path schemes Y for V . We mimic every transition $t = (q, z, q') \in T$ by a sequence of $|z| + 2$ transitions in V' of which $|z|$ either all increment or decrement the counter. Consequently, we define

$$Q' \stackrel{\text{def}}{=} Q \cup \{(t, i) : t = (p, z, q) \in T, i \in [0, |z|]\}$$

and

$$\begin{aligned} T' \stackrel{\text{def}}{=} & \{(p, 0, (t, 0)) : t = (p, z, q) \in T\} \cup \{((t, |z|), 0, q) : t = (p, z, q) \in T\} \\ & \cup \{((t, i), +1, (t, i+1)) : t = (p, z, q) \in T, z > 0, i \in [0, |z| - 1]\} \\ & \cup \{((t, i), -1, (t, i+1)) : t = (p, z, q) \in T, z < 0, i \in [0, |z| - 1]\} . \end{aligned}$$

Let us define the homomorphism $h : T \rightarrow T'^+$ such that

$$h(t) \stackrel{\text{def}}{=} \begin{cases} (q, 0, (t, 0)) \cdot \left(\prod_{i=1}^{|z|} ((t, i-1), +1, (t, i)) \right) \cdot ((t, |z|), 0, q') & \text{if } t = (q, z, q') \text{ and } z \geq 0 \\ (q, 0, (t, 0)) \cdot \left(\prod_{i=1}^{|z|} ((t, i-1), -1, (t, i)) \right) \cdot ((t, |z|), 0, q') & \text{if } t = (q, z, q') \text{ and } z < 0 \end{cases}$$

for every $t \in T$. The idea behind this definition is that for every run π in V we have that $h(\pi)$ is the run in V' that corresponds to π . The following conditions formalize this intuition and are easily verified:

- (i) $|h(t)| \leq \|T\| + 2$ for each $t \in T$,
- (ii) if $p(u) \xrightarrow{\pi} q(v)$ in V then $p(u) \xrightarrow{h(\pi)} q(v)$ in V' , and
- (iii) if $p, q \in Q$ and $p(u) \xrightarrow{\pi'} q(v)$ in V' then there is a unique $\pi \in T^*$ satisfying $\pi' = h(\pi)$ and $p(u) \xrightarrow{\pi} q(v)$ in V .

By (iii) for every $p(u) \xrightarrow{\pi'} q(v)$ with $p, q \in Q$ in V' we can write $h^{-1}(\pi')$ to denote the unique π such that $h(\pi) = \pi'$ and $p(u) \xrightarrow{\pi} q(v)$ in V . In this case, we have that π' is a cycle in V' if, and only if, $h^{-1}(\pi')$ is a cycle in V .

To show the existence of the finite set of linear path schemes satisfying the conditions required in the lemma, we show that whenever $p(u) \xrightarrow{*} q(v)$ in V then there exists a linear path scheme $\rho \subseteq T^*$ such that $p(u) \xrightarrow{\rho} q(v)$ in V and $|\rho| \leq (|Q| + \|T\|)^{O(1)}$. Let $D = |Q'| + |Q'|^2$ and assume $p(u) \xrightarrow{*} q(v)$ in V . Hence $p(u) \xrightarrow{*} q(v)$ in V' by (ii). We make a case distinction between $|u - v| \leq D$ and $|u - v| > D$.

Case 1: $|u - v| \leq D$. By Lemma 10 we have $p(u) \xrightarrow{\pi'} q(v)$ in V' for some path π' with $|\pi'| \leq |Q'|^{O(1)} + |Q'| \cdot D \leq (|Q| + \|T\| + D)^{O(1)}$. Thus, we set $\rho = h^{-1}(\pi')$ and note that $p(u) \xrightarrow{\rho} q(v)$ in V by (iii), and $|\rho| \leq |\pi'| \leq (|Q| + \|T\| + D)^{O(1)} \leq (|Q| + \|T\|)^{O(1)}$ as required.

Case 2: $|u - v| > D$. By Lemma 9, we have $p(u) \xrightarrow{\alpha'(\beta')^i \gamma'} q(v)$ in V' for some $i > 0$ and some linear path scheme $\rho' = \alpha' \beta'^* \gamma'$ from p to q satisfying $|\alpha' \gamma'| < |Q'|$ and $|\beta'| \leq |Q'|$.

Let $q' \in Q'$ be such that β' is a cycle from q' to q' . If $q' \in Q$ then α' is a path in V' from p to q' , β' is a cycle in V' from q' to q' and γ' is a path in V' from q' to q . Thus $\rho \stackrel{\text{def}}{=} \alpha'^{-1} (\beta'^{-1})^* \gamma'^{-1}$ is a linear path scheme in V for which we have $p(u) \xrightarrow{\rho} q(v)$ in V by (iii), and $|\rho| \leq |Q'| + |Q'|^2 \leq (|Q| + \|T\|)^{O(1)}$.

Otherwise, if $q' \in Q' \setminus Q$, we have $q' = (t, i)$ for some $t = (q_1, z, q_2) \in T$ and some $i \in [1, |z|]$. We only consider the case $z \geq 0$, the case $z < 0$ being symmetric. Since β' is a cycle from (t, i) to (t, i) , it follows from the definition of V' that $\alpha' = \alpha'_1 \alpha'_2$ and $\beta' = \beta'_1 \beta'_2$, where $\alpha'_2 = \beta'_2 = (q_1, 0, (t, 0)) \cdot \prod_{j=1}^i ((t, j-1), +1, (t, j))$. We have the following language equalities, where the last equality follows from $\alpha'_2 = \beta'_1$,

$$\alpha'(\beta')^* \gamma' = \alpha'_1 \alpha'_2 (\beta'_1 \beta'_2)^* \gamma' = \alpha'_1 (\beta'_2 \beta'_1)^* \beta'_2 \gamma'.$$

Moreover, in V' it holds that α'_1 is a path from $p \in Q$ to $q_1 \in Q$, $\beta'_2 \beta'_1$ is a cycle from $q_1 \in Q$ to q_1 , and $\beta'_2 \gamma'$ is a path from q_1 to $q \in Q$. Hence $\rho \stackrel{\text{def}}{=} h(\alpha'_1)^{-1} (h(\beta'_2 \beta'_1)^{-1})^* h(\beta'_2 \gamma')^{-1}$ is a linear path scheme in V with $|\rho| \leq |Q'| + |Q'|^2 \leq (|Q| + \|T\|)^{O(1)}$ for which we have $p(u) \xrightarrow{\rho} q(v)$ in V by (iii). \square

We are now in a position where we, informally speaking, can prove the first half of Proposition 8. The following lemma proves Proposition 8 when restricting the range of one counter.

Lemma 12. Let $V = (Q, T)$ be a 2-VASS, $D \in \mathbb{N}$ and $\mathbb{B} \in \{(\mathbb{N} \times [0, D]), ([0, D] \times \mathbb{N})\}$. Then there exists a finite set $Y_{\mathbb{B}}$ of linear path schemes such that

- $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{B}} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{Y_{\mathbb{B}}}_{\mathbb{B}} q(\mathbf{v})$;
- $|\rho| \leq (|Q| + \|T\| + D)^{O(1)}$ for each $\rho \in Y_{\mathbb{B}}$; and
- each $\rho \in Y_{\mathbb{B}}$ has at most one cycle.

Proof. We only consider the case $\mathbb{B} = \mathbb{N} \times [0, D]$, the other case follows by symmetry. Starting from V we construct a 1-VASS $\bar{V} = (\bar{Q}, \bar{T})$ such that the following holds:

- (1) $\bar{Q} = \{q_i : q \in Q, i \in [0, D]\}$, and
- (2) for each $p, q \in Q$ and each $(u_1, u_2), (v_1, v_2) \in \mathbb{B}$ we have $p(u_1, u_2) \xrightarrow{*}_{\mathbb{B}} q(v_1, v_2)$ in V if, and only if, $p_{u_2}(u_1) \xrightarrow{*}_{\mathbb{N}} q_{v_2}(v_1)$ in \bar{V} .

To achieve (2) note that we can simply define \bar{T} as follows,

$$\bar{T} = \{(p_n, i, q_{n+j}) : (p, (i, j), q) \in T \text{ and } n, n+j \in [0, D]\}.$$

This gives rise to a homomorphism $\phi : \bar{T}^* \rightarrow T^*$ with $\phi(p_n, i, q_{n+j}) \stackrel{\text{def}}{=} (p, (i, j), q)$ for each $(p_n, i, q_{n+j}) \in \bar{T}$. For each path $\bar{\pi}$ in \bar{V} we have

- (3) if $p_{u_2}(u_1) \xrightarrow{\bar{\pi}}_{\mathbb{N}} q_{v_2}(v_1)$ in \bar{V} then $p(u_1, u_2) \xrightarrow{\phi(\bar{\pi})}_{\mathbb{B}} q(v_1, v_2)$ in V .

It follows immediately from the definition of \bar{T} that any linear path scheme $\bar{\rho} = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ over the 1-VASS \bar{V} induces the linear path scheme $\phi(\bar{\rho}) = \phi(\alpha_0) \phi(\beta_1)^* \phi(\alpha_1) \cdots \phi(\beta_k)^* \phi(\alpha_k)$ over the 2-VASS V . Furthermore, ϕ is naturally extended to any set of linear path schemes \bar{S} : we put $\phi(\bar{S}) \stackrel{\text{def}}{=} \bigcup \{\phi(\bar{\rho}) : \bar{\rho} \in \bar{S}\}$. Applying Lemma 11 to \bar{V} yields a set \bar{Y} of linear path schemes such that

- (4) $p(u) \xrightarrow{*}_{\mathbb{N}} q(v)$ in \bar{V} if, and only if, $p(u) \xrightarrow{\bar{\rho}}_{\mathbb{N}} q(v)$ in \bar{V} for some $\bar{\rho} \in \bar{Y}$, where $\bar{\rho}$ has at most one cycle and $|\bar{\rho}| \leq (|\bar{Q}| + \|\bar{T}\|)^{O(1)} \leq (|Q| \cdot D + \|T\|)^{O(1)} = (|Q| + \|T\| + D)^{O(1)}$.

We define $Y_{\mathbb{B}}$ required in the lemma as $Y_{\mathbb{B}} \stackrel{\text{def}}{=} \phi(\bar{Y})$. By definition, $Y_{\mathbb{B}}$ already fulfills the second and third condition required in the lemma. The first condition now follows from the following circular sequence of implications. Let $p, q \in Q$ and $u_1, u_2, v_1, v_2 \in \mathbb{N}$, we have

$$\begin{aligned} p(u_1, u_2) \xrightarrow{*}_{\mathbb{B}} q(v_1, v_2) \text{ in } V & \stackrel{(2)}{\iff} p_{u_2}(u_1) \xrightarrow{*}_{\mathbb{N}} q_{v_2}(v_1) \text{ in } \bar{V} \\ & \stackrel{(4)}{\iff} p_{u_2}(u_1) \xrightarrow{\bar{\rho}}_{\mathbb{N}} q_{v_2}(v_1) \text{ in } \bar{V} \\ & \stackrel{(3)}{\implies} p(u_1, u_2) \xrightarrow{\phi(\bar{\rho})}_{\mathbb{B}} q(v_1, v_2) \text{ in } V \\ & \iff p(u_1, u_2) \xrightarrow{S}_{\mathbb{B}} q(v_1, v_2) \text{ in } V \\ & \implies p(u_1, u_2) \xrightarrow{*}_{\mathbb{B}} q(v_1, v_2) \text{ in } V. \end{aligned}$$

\square

In the remainder of this section, by application of Lemma 12 we prove Proposition 8 which, given some $D \in \mathbb{N}$, states that runs which stay inside the L -shaped band $\mathbb{L} = ([0, D] \times \mathbb{N}) \cup (\mathbb{N} \times [0, D])$ can be captured by a union of small linear path schemes with at most two cycles.

Proof of Proposition 8. Let us define $E \stackrel{\text{def}}{=} D + \|T\|$ and $\mathbb{L}' \stackrel{\text{def}}{=} ([0, E] \times \mathbb{N}) \cup (\mathbb{N} \times [0, E])$. Let $p, q \in Q$ and $\mathbf{u}, \mathbf{v} \in \mathbb{L}$, and let $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{L}} q(\mathbf{v})$ be such that $|\pi|$ is minimal. In order to prove Proposition 8, it suffices to provide some linear path scheme ρ such that $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{L}'} q(\mathbf{v})$, $|\rho| \leq (|Q| + \|T\| + D)^{O(1)}$ and ρ has at most two cycles. Let $\mathbb{B}_1 \stackrel{\text{def}}{=} [0, E] \times \mathbb{N}$, $\mathbb{B}_2 \stackrel{\text{def}}{=} \mathbb{N} \times [0, E]$ and let $H \stackrel{\text{def}}{=} \mathbb{B}_1 \cap \mathbb{B}_2 = [0, E] \times [0, E]$. Due to minimality of π and by choice of H we can factorize π as $\pi = \pi_1 \cdots \pi_k$, where

$$p_0(\mathbf{u}_0) \xrightarrow{\pi_1}_{\mathbb{L}} p_1(\mathbf{u}_1) \cdots \xrightarrow{\pi_k}_{\mathbb{L}} p_k(\mathbf{u}_k)$$

and

- (i) $p_0 = p, p_k = q, \mathbf{u}_0 = \mathbf{u}, \mathbf{u}_k = \mathbf{v}$;
- (ii) $p_{i-1}(\mathbf{u}_{i-1}) \xrightarrow{\pi_i}_{\mathbb{C}_i} p_i(\mathbf{u}_i)$, where $\mathbb{C}_i \in \{\mathbb{B}_1, \mathbb{B}_2\}$ for every $i \in [1, k]$;
- (iii) $\mathbf{u}_i \in H$ for each $i \in [1, k-1]$; and
- (iv) $k \leq |H| = (E+1)^2 \leq D^{O(1)}$.

By combining (ii) with Lemma 12 we have that for each run $p_{i-1}(\mathbf{u}_{i-1}) \xrightarrow{\pi_i}_{\mathbb{C}_i} p_i(\mathbf{u}_i)$ there exists a linear path scheme $\rho_i = \alpha_i(\beta_i)^* \gamma_i$ such that $p_{i-1}(\mathbf{u}_{i-1}) \xrightarrow{\rho_i}_{\mathbb{C}_i} p_i(\mathbf{u}_i)$ and $|\rho_i| \leq (|Q| + \|T\| + E)^{O(1)} = (|Q| + \|T\| + D)^{O(1)}$. For simplicity, here we only treat the case where each ρ_i has precisely one cycle, the cases when some ρ_i contains no cycle can be dealt with analogously. Note that whenever $i \in [2, k-1]$ we have $p_{i-1}(\mathbf{u}_{i-1}), p_i(\mathbf{u}_i) \in Q \times H$ by (iii). Since ρ_i has only one cycle, $|\rho_i| \leq (|Q| + \|T\| + D)^{O(1)}$ and $\mathbf{u}_{i-1}, \mathbf{u}_i \in H$ there exists some $e_i \leq (|Q| + \|T\| + E)^{O(1)}$ such that

$$p_{i-1}(\mathbf{u}_{i-1}) \xrightarrow{\alpha_i(\beta_i)^{e_i} \gamma_i}_{\mathbb{C}_i} p_i(\mathbf{u}_i), \quad \text{thus in particular} \quad p_{i-1}(\mathbf{u}_{i-1}) \xrightarrow{\alpha_i(\beta_i)^{e_i} \gamma_i}_{\mathbb{L}'} p_i(\mathbf{u}_i)$$

Consequently, we have

$$p_0(\mathbf{u}_0) \xrightarrow{\alpha_1 \beta_1^* \gamma_1}_{\mathbb{C}_1} p_1(\mathbf{u}_1) \xrightarrow{\prod_{i=2}^{k-1} \alpha_i \beta_i^{e_i} \gamma_i}_{\mathbb{L}'} p_{k-1}(\mathbf{u}_{k-1}) \xrightarrow{\alpha_k \beta_k^* \gamma_k}_{\mathbb{C}_k} p_k(\mathbf{u}_k).$$

Hence, we define

$$\rho \stackrel{\text{def}}{=} \alpha_1 \beta_1^* \gamma_1 \cdot \left(\prod_{i=2}^{k-1} \alpha_i \beta_i^{e_i} \gamma_i \right) \cdot \alpha_k \beta_k^* \gamma_k$$

which has at most two cycles and for which we have $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{L}'} q(\mathbf{v})$ and

$$\begin{aligned} |\rho| &\leq k \cdot \max\{e_i : i \in [2, k-1]\} \cdot \max\{|\rho_i| : i \in [1, k]\} \\ &\stackrel{\text{(iv)}}{\leq} D^{O(1)} \cdot (|Q| + \|T\| + E)^{O(1)} \cdot (|Q| + \|T\| + D)^{O(1)} \\ &= (|Q| + \|T\| + D)^{O(1)}. \end{aligned}$$

This concludes the proof of Proposition 8.

4.4. Factorizing arbitrary runs: Proof of Theorem 1. By application of the results established in Sections 4.2 and 4.3, we will now prove Theorem 1. In Section 4.2, we showed that the following two kinds of runs can be captured by small linear path schemes:

- Type (1): Runs between two configurations $q(\mathbf{u})$ and $q(\mathbf{v})$ where both components of \mathbf{u} and \mathbf{v} are sufficiently large, but intermediate configurations could have small counter values.
- Type (2): Runs on which for all configurations both counter values are sufficiently large.

Complementary, in Section 4.3 we showed that there are small linear path schemes with at most two cycles that capture the following runs:

- Type (3): Runs on which for all configurations at least one counter value is not too large.

The goal of this section is to show that any run can be factorized into few runs that are each of types (1), (2) or (3). To this end, let us fix a 2-VASS $V = (Q, T)$. Let $D \leq (|Q| + \|T\|)^{O(1)}$ be the constant from Proposition 5. Informally speaking, we have hereby defined that “sufficiently large” means to be greater or equal to D . Moreover we set $\mathbb{L} \stackrel{\text{def}}{=} ([0, D + \|T\|] \times \mathbb{N}) \cup (\mathbb{N} \times [0, D + \|T\|])$, $\mathbb{O} \stackrel{\text{def}}{=} [D, \infty)^2$, and $\mathbb{B} \stackrel{\text{def}}{=} \mathbb{L} \cap \mathbb{O} = ([D, D + \|T\|] \times \mathbb{N}) \cup (\mathbb{N} \times [D, D + \|T\|])$. Again, informally speaking, we have hereby defined that “not too large” means to be smaller or equal to $D + \|T\|$.

Let us summarize what we have proven in Sections 4.2 and 4.3:

- Runs of type (1) can be captured by a set of linear path schemes R , where each $\rho \in R$ has at most two cycles and length at most $(|Q| + \|T\|)^{O(1)}$ by Proposition 5(a).
- Runs of type (2) can be captured by a set of linear path schemes X , where each $\rho \in X$ has at most $2 \cdot |Q|$ cycles and length at most $(|Q| + \|T\|)^{O(1)}$ by Proposition 5(b).
- Runs of type (3) can be captured by a set of linear path schemes $Y_{\mathbb{L}}$, where each $\rho \in Y_{\mathbb{L}}$ has at most two cycles and length at most $(|Q| + \|T\| + D)^{O(1)} = (|Q| + \|T\|)^{O(1)}$ by Proposition 8.

Given $p(\mathbf{u})$ and $q(\mathbf{v})$, let us fix an arbitrary run $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$, where $\pi = t_1 \cdots t_k \in T^k$ and

$$p(\mathbf{u}) = q_0(\mathbf{u}_0) \xrightarrow{t_1}_{\mathbb{N}^2} q_1(\mathbf{u}_1) \cdots \xrightarrow{t_k}_{\mathbb{N}^2} q_k(\mathbf{u}_k) = q(\mathbf{v}) \quad .$$

We will be interested in the indices of configurations whose counter values lie in \mathbb{B} and define

$$I \stackrel{\text{def}}{=} \{i \in [0, k] : \mathbf{u}_i \in \mathbb{B}\} \quad .$$

Let us define the function $x : I \rightarrow I$ that maps each index $i \in I$ to the smallest element in I larger than i (and i if $i = \max I$), i.e.

$$x(i) \stackrel{\text{def}}{=} \begin{cases} \min\{j \in I : j > i\} & \text{if } i < \max I, \\ i & \text{otherwise, i.e. } i = \max I \end{cases} \quad .$$

We also define the function $\ell : \{q_i \in Q : i \in I\} \rightarrow I$ that maps each state q that appears in a configuration in $Q \times \mathbb{B}$ to the largest index in I where it appears, i.e.

$$\ell(q) \stackrel{\text{def}}{=} \max\{i \in I : q = q_i\} \quad .$$

We are now interested in factorizing the run $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ into runs between configurations that start and end in $\mathbb{B} = \mathbb{L} \cap \mathbb{O}$. More precisely, by the choice of \mathbb{O} , \mathbb{L} and \mathbb{B} and by the pigeonhole principle there exist indices $i_1, \dots, i_h \in I$ such that the run $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ can be factorized as (cf. Figure 8):

$$\begin{aligned} q_0(\mathbf{u}_0) &\xrightarrow{\pi_{0,1}}_{\mathbb{D}_{0,1}} q_{i_1}(\mathbf{u}_{i_1}) \xrightarrow{\pi_1}_{\mathbb{N}^2} q_{\ell(q_{i_1})}(\mathbf{u}_{\ell(q_{i_1})}) \xrightarrow{\pi_{1,2}}_{\mathbb{D}_{1,2}} q_{i_2}(\mathbf{u}_{i_2}) \xrightarrow{\pi_2}_{\mathbb{N}^2} q_{\ell(q_{i_2})}(\mathbf{u}_{\ell(q_{i_2})}) \quad \cdots \\ &\cdots \xrightarrow{\pi_{h-1,h}}_{\mathbb{D}_{h,h-1}} q_{i_h}(\mathbf{u}_{i_h}) \xrightarrow{\pi_h}_{\mathbb{N}^2} q_{\ell(q_{i_h})}(\mathbf{u}_{\ell(q_{i_h})}) \xrightarrow{\pi_{h,h+1}}_{\mathbb{D}_{h,h+1}} q_k(\mathbf{u}_k) \quad , \end{aligned}$$

where

- (i) $h \leq |Q|$,
- (ii) $i_t \in I$ and thus we have $\mathbf{u}_{i_t} \in \mathbb{B}$ and $q_{i_t} = q_{\ell(q_{i_t})}$ for each $t \in [1, h]$,
- (iii) $\mathbb{D}_{t,t+1} \in \{\mathbb{O}, \mathbb{L}\}$ for each $t \in [1, h]$, and
- (iv) $i_{t+1} = x(\ell(q_{i_t}))$ for each $t \in [1, h-1]$.

By (ii) each run of the form $q_{i_t}(\mathbf{u}_{i_t}) \xrightarrow{\pi_t}_{\mathbb{N}^2} q_{\ell(q_{i_t})}(\mathbf{u}_{\ell(q_{i_t})})$ is a run of type (1) and can hence be replaced by some linear path scheme from R (recall that $\mathbb{B} \subseteq \mathbb{O}$). By (iii) and (iv), each run of the form $\xrightarrow{\pi_{t,t+1}}_{\mathbb{D}_{t,t+1}}$ is a run of type (2) or of type (3) and can hence be replaced by some linear path scheme from $X \cup Y_{\mathbb{L}}$. In summary, the run $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ can be replaced by a linear path scheme that has at most $(h+1) \cdot 2 \cdot |Q| \leq O(|Q|^2)$ cycles and size at most $(h+1) \cdot (|Q| + \|T\|)^{O(1)} = (|Q| + \|T\|)^{O(1)}$. This concludes the proof of Theorem 1.

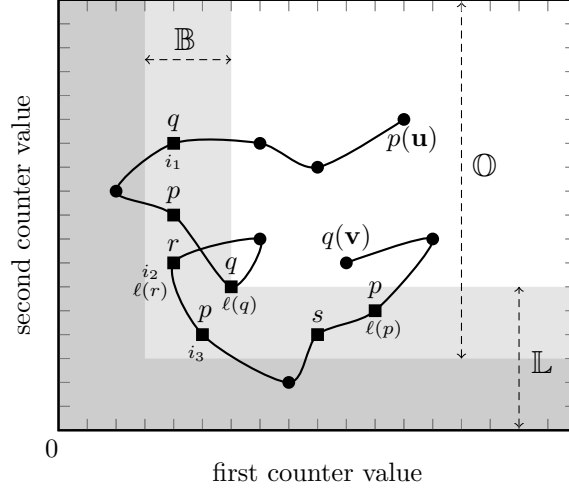


FIGURE 8. Example of the decomposition of a path in the proof of Theorem 1. The region depicted is the positive quadrant in the Cartesian plane. Here, $I = \{3, 5, 6, 8, 9, 11, 12\}$ is marked with squares, and $i_1 = 3$, $\ell(q) = 6$, $i_2 = \ell(r) = 8$, $i_3 = 9$ and $\ell(p) = 12$.

5. COMPLEXITY RESULTS

Having established Theorem 1, it is now not difficult to show that reachability in 2-VASS is in PSPACE by application of bounds from integer linear programming. A complementary lower bound follows via a reduction from reachability in bounded one-counter automata, which is known to be PSPACE-complete [4]. This is the subject of Section 5.1 below which proves Theorem 2. The PSPACE lower bound does, however, crucially depend on binary encoding of numbers. In fact, we show in Section 5.2 that reachability in unary 2-VASS is in NP and NL-hard. The precise complexity of this problem remains an open problem of this paper. Finally, for the sake of completeness, in Section 5.3 we briefly state some corollaries of our results on the complexity of reachability in \mathbb{Z} -VASS, and on coverability and boundedness in 2-VASS.

Before we begin, let us recall some definitions and results from integer linear programming. Let A be a $d \times k$ integer matrix and $\mathbf{c} \in \mathbb{Z}^d$. A *system of linear Diophantine inequalities* (resp. a *system of linear Diophantine equations*) is given as $\mathcal{I} : A \cdot \mathbf{x} \geq \mathbf{c}$ (resp. as $\mathcal{E} : A\mathbf{x} = \mathbf{c}$) and we say that \mathcal{I} (resp. \mathcal{E}) is *feasible* if there exists some $\mathbf{e} \in \mathbb{N}^k$ such that $A \cdot \mathbf{e} \geq \mathbf{c}$ (resp. $A \cdot \mathbf{e} = \mathbf{c}$), i.e., every inequality (resp. equality) holds in every row of \mathcal{I} (resp. \mathcal{E}). Subsequently, we refer to \mathbf{e} as a *solution* of \mathcal{I} or \mathcal{E} , respectively. By $[[\mathcal{I}]] \subseteq \mathbb{N}^k$ we denote the set of all solutions of \mathcal{I} , the set of solutions $[[\mathcal{E}]] \subseteq \mathbb{N}^k$ is defined analogously.

Let us now recall two bounds on solutions of systems of linear Diophantine inequalities and equations that we subsequently rely upon. The first bound we use in this paper concerns systems of linear Diophantine inequalities.

Proposition 13 ([26], p. 239). Let $\mathcal{I} : A \cdot \mathbf{x} \geq \mathbf{c}$ be a feasible system of linear Diophantine inequalities, where A is a $d \times k$ matrix. Then there exists a solution $\mathbf{e} \in \mathbb{N}^k$ of \mathcal{I} such that

$$\|\mathbf{e}\| \leq 2^{k^{O(1)}} \cdot O(\|A\| + \|\mathbf{c}\|) \quad .$$

Next, we consider a bound for feasible homogeneous systems of linear Diophantine equations.

Proposition 14 ([22], Theorem 1). Let $\mathcal{E} : A \cdot \mathbf{x} = \mathbf{0}$ be a system of linear Diophantine equations, where A is a $d \times k$ integer matrix. Then there exists $P \subseteq \mathbb{N}^k$ such that $\|P\| \leq (\|A\| + 1)^d$ and

$$[[\mathcal{E}]] = \text{cone}_{\mathbb{N}}(P) \quad .$$

From this proposition it is now easy to generalize to the non-homogeneous case.

Corollary 15. Let $\mathcal{E} : A \cdot \mathbf{x} = \mathbf{c}$ be a feasible system of linear Diophantine equations such that A is a $d \times k$ matrix. Then there exists a solution $\mathbf{e} \in \mathbb{N}^k$ of \mathcal{E} such that

$$\|\mathbf{e}\| \leq (\|A\| + \|\mathbf{c}\|)^{O(d)} .$$

Proof. Define

$$\mathcal{E}' : \begin{bmatrix} A & -\mathbf{c} \end{bmatrix} \begin{vmatrix} \mathbf{x} \\ y \end{vmatrix} = \mathbf{0} ,$$

where \mathbf{x} ranges over \mathbb{N}^k and y is a fresh variable ranging over \mathbb{N} . From Proposition 14 we have that $\llbracket \mathcal{E}' \rrbracket = \text{cone}_{\mathbb{N}}(P)$ for some $P \subseteq \mathbb{N}^{k+1}$ such that $\|P\| \leq (\|A\| + \|\mathbf{c}\| + 1)^d$. Now \mathcal{E} is feasible if, and only if, there is some $\mathbf{p} \in \text{cone}_{\mathbb{N}}(P)$ whose $(k+1)$ -st component is equivalent to 1. From such a \mathbf{p} we obtain a solution of \mathcal{E} with the desired bounds. \square

5.1. Reachability in 2-VASS is PSPACE-complete. In this section, we prove Theorem 2 and show that reachability in 2-VASS is PSPACE-complete. Given an instance $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ of reachability, by Theorem 1 we have that $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{N}^2} q(\mathbf{v})$ for some linear path scheme ρ such that $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ and ρ has $O(|Q|^2)$ cycles. Writing $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$, we have

$$(10) \quad p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{N}^2} q(\mathbf{v}) \iff \text{there exist } e_1, \dots, e_k \in \mathbb{N} \text{ such that } p(\mathbf{u}) \xrightarrow{\alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k}_{\mathbb{N}^2} q(\mathbf{v}).$$

Consequently, obtaining a PSPACE upper bound for reachability reduces to bounding the binary representation of the e_i polynomially in the sizes of V , \mathbf{u} and \mathbf{v} . Without loss of generality, in the following we may assume that $e_i \geq 1$ for all $i \in [1, k]$.

Our approach is straightforward: we rephrase the existential question from (10) in terms of finding solutions to a system of linear Diophantine inequalities and then apply standard bounds from integer linear programming in order to bound the e_i . For our reduction, let us first discuss the particular case when we wish to decide whether the repetition of a cycle corresponds to a run. In this case, it is sufficient to only check whether its initial and final segments lead to counter values greater or equal to zero, formalized by the following lemma.

Lemma 16. Let $V = (Q, T)$ be d -VASS, $\mathbf{u} \in \mathbb{N}^d$ and let $\beta \in T^m$ be a cycle. Then there exists a system of linear Diophantine inequalities $\mathcal{I} : \mathbf{a} \cdot x \geq \mathbf{c}$ such that

- $e \in \llbracket \mathcal{I} \rrbracket$ if, and only if, $q(\mathbf{u}) \xrightarrow{\beta^e}_{\mathbb{N}^d} q(\mathbf{u} + e \cdot \delta(\beta))$ and $e \geq 1$ for every $e \in \mathbb{N}$,
- $\mathbf{a}, \mathbf{c} \in \mathbb{Z}^{d+1}$, and
- $\|\mathbf{a}\| \leq |\beta| \cdot \|T\|$ and $\|\mathbf{c}\| \leq 2 \cdot |\beta| \cdot \|T\| + \|\mathbf{u}\|$.

Proof. Consider the following linear Diophantine inequalities containing two rows for every $1 \leq j \leq m$:

$$(11) \quad \mathbf{u} + \delta(\beta[1, j]) \geq \mathbf{0}$$

$$(12) \quad \mathbf{u} + (x - 1) \cdot \delta(\beta) + \delta(\beta[1, j]) \geq \mathbf{0}$$

The first row expresses that on the first traversal of β we do not drop below zero. This row is independent from x , and if the constraints are infeasible we can chose \mathcal{I} to be any infeasible system of linear Diophantine inequalities.

Next, in (12) we assert that the last time we traverse β no counter drops below zero. In particular, we have

$$\mathbf{u} + (x - 1) \cdot \delta(\beta) + \delta(\beta[1, j]) \geq \mathbf{0} \iff \delta(\beta) \cdot x \geq \underbrace{\delta(\beta) - \delta(\beta[1, j]) - \mathbf{u}}_{\stackrel{\text{def}}{=} \mathbf{c}_j} .$$

Consequently, we define \mathbf{a} required in the lemma as $\mathbf{a} \stackrel{\text{def}}{=} (1, \delta(\beta))$. For every j , let $\mathbf{c}_j = (c_{1,j}, \dots, c_{d,j})$, we set \mathbf{c} to

$$\mathbf{c} \stackrel{\text{def}}{=} (1, \max\{c_{1,1}, \dots, c_{1,m}\}, \dots, \max\{c_{d,1}, \dots, c_{d,m}\}) .$$

The first row of $\mathcal{I} : \mathbf{a} \cdot x \geq \mathbf{c}$ asserts that any solution $e \in \llbracket \mathcal{I} \rrbracket$ is greater-equal to one, and the subsequent rows that *all* constraints of type (12) are fulfilled by our particular choice of \mathbf{c} . In

particular $q(\mathbf{u}) \xrightarrow{\beta^e}_{\mathbb{N}^d} q(\mathbf{u} + e \cdot \delta(\beta))$. It is easily checked that the norms of \mathbf{a} and \mathbf{c} fulfill the requirements of the lemma. \square

The restriction to non-zero solutions in Lemma 16 is due the fact that the inequality constraints on prefixes of β could wrongly exclude zero from a solution. Therefore we have to consider the cases when cycles are taken at least once or not at all separately. In doing so, we generalize the previous lemma to arbitrary linear path schemes. The function $\text{sign} : \mathbb{N} \rightarrow \{0, 1\}$ of naturals is defined as expected, $\text{sign}(n) = 1$ if $n \geq 1$ and $\text{sign}(n) = 0$ if $n = 0$.

Lemma 17. Let $V = (Q, T)$ be a d -VASS, $\mathbf{u} \in \mathbb{N}$ and $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ be a linear path scheme from p to q and let $\chi : [1, k] \rightarrow \{0, 1\}$. Then there exists a system of linear Diophantine inequalities $\mathcal{I} = \mathcal{I}(\mathbf{u}, \rho, \chi)$ of the form $\mathcal{I} : A \cdot \mathbf{x} \geq \mathbf{c}$ such that

- $\mathbf{e} \in \llbracket \mathcal{I} \rrbracket$ if, and only if, $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$ and $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^d} q(\mathbf{u} + \delta(\pi))$ and $\chi(i) = \text{sign}(e_i)$ for every $\mathbf{e} = (e_1, \dots, e_k) \in \mathbb{N}^k$,
- A is a $((d+1) \cdot k) \times k$ -matrix,
- $\|A\| \leq k \cdot |\rho| \cdot \|T\|$, and
- $\|\mathbf{c}\| \leq O(\|\mathbf{u}\| + |\rho| \cdot \|T\|)$.

Proof. We only prove the lemma for the concrete function $\chi : [1, k] \rightarrow \{0, 1\}$, where $\chi(i) = 1$ for all $i \in [1, k]$. In the following, we write $\mathbf{x} = (x_1, \dots, x_k)$. First, we assert that the solutions e_i are greater or equal to 1, i.e.,

$$(13) \quad I_k \cdot \mathbf{x} \geq \mathbf{1},$$

where I_k is the k -th unit matrix and $\mathbf{1} = (1, \dots, 1)$. Next, informally speaking, we have to construct \mathcal{I} in a way such that we assert that the counter value does not drop below zero on any infix of ρ in any dimension. For segments of ρ between cycles, this can be ensured by the following constraints for every $j \in [0, k]$ and $\ell \in [1, |\alpha_j|]$, which simply enforce the accumulated counter value to be non-negative:

$$(14) \quad \begin{aligned} \mathbf{u} + \sum_{0 \leq i < j} (\delta(\alpha_i) + \delta(\beta_{i+1}) \cdot x_{i+1}) + \delta(\alpha_j[1, \ell]) &\geq \mathbf{0} \\ \iff \sum_{1 \leq i \leq j} \delta(\beta_i) \cdot x_i &\geq -\mathbf{u} - \sum_{0 \leq i < j} \delta(\alpha_i) - \delta(\alpha_j[1, \ell]) \end{aligned}$$

For counter values which, informally speaking, occur along cycles β_j of ρ , we follow the construction from Lemma 16 and assert the following constraints for every $j \in [1, k]$ and $\ell \in [1, |\beta_j|]$:

$$(15) \quad \begin{aligned} \mathbf{u} + \delta(\alpha_0) + \sum_{1 \leq i < j} (\delta(\beta_i) \cdot x_i + \delta(\alpha_i)) + \delta(\beta_j[1, \ell]) &\geq \mathbf{0} \\ \mathbf{u} + \delta(\alpha_0) + \sum_{1 \leq i < j} (\delta(\beta_i) \cdot x_i + \delta(\alpha_i)) + \delta(\beta_j) \cdot (x_j - 1) + \delta(\beta_j[1, \ell]) &\geq \mathbf{0} \\ \iff \sum_{1 \leq i \leq j-1} \delta(\beta_i) \cdot x_i &\geq -\mathbf{u} - \sum_{0 \leq i < j} \delta(\alpha_i) - \delta(\beta_j[1, \ell]) \end{aligned}$$

$$(16) \quad \sum_{1 \leq i \leq j} \delta(\beta_i) \cdot x_i \geq -\mathbf{u} - \sum_{0 \leq i < j} \delta(\alpha_i) + \delta(\beta_j) - \delta(\beta_j[1, \ell])$$

By our construction, it is easily verified that for every $\mathbf{e} = (e_1, \dots, e_k) \in \mathbb{N}^k$ we have $\chi(i) = 1$ for all $i \in [1, k]$ and $p(\mathbf{u}) \xrightarrow{\alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k}_{\mathbb{N}^d} q(\mathbf{u} + \delta(\pi))$ if, and only if, \mathbf{e} fulfills *all* constraints defined in (13), (14), (15) and (16). It thus remains to, informally speaking, extract the required system \mathcal{I} of linear Diophantine inequalities from those constraints.

For every fixed $j \in [1, k]$, by combining the constraints from (14), (15) and (16), we obtain systems of linear Diophantine inequalities $\mathcal{I}'_j : B_j \cdot \mathbf{x} \geq \mathbf{d}_j$ such that B_j consists of at most d *different* rows, since every x_i is multiplied by the *same* $\delta(\beta_i)$. Let A_j be the following $(d \times k)$ -matrix: $A_j \stackrel{\text{def}}{=} [\delta(\beta_1) \cdots \delta(\beta_j) \mathbf{0} \cdots \mathbf{0}]$. For the i -th row of A_i , let $c_{j,i} \in \mathbb{Z}$ be the maximum value in \mathbf{d}_j of the rows with the same coefficients in \mathcal{I}'_j , similar as in the construction of \mathbf{c}_j in

Lemma 16. We define $\mathbf{c}_j \stackrel{\text{def}}{=} (c_{j,1}, \dots, c_{j,d})$ and set $\mathcal{I}_j : A_j \cdot \mathbf{x} \geq \mathbf{c}_j$. By construction, we now have that $\mathbf{e} \in \mathbb{N}^k$ is a solution of \mathcal{I}_j if, and only if, \mathbf{e} is a solution to \mathcal{I}'_j and in particular fulfills all relevant constraints in (14), (15) and (16).

In order to obtain the matrix A and \mathbf{c} required in the lemma, we define

$$A \stackrel{\text{def}}{=} \begin{bmatrix} I_k \\ A_1 \\ \vdots \\ A_k \end{bmatrix} \quad \text{and} \quad \mathbf{c} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{1} \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_k \end{bmatrix} .$$

The dimension of A and \mathbf{c} is as required. It thus remains to estimate the norm of A and \mathbf{c} . We have

$$\|A\| \leq \sum_{1 \leq i \leq k} \|\delta(\beta_i)\| \leq k \cdot |\rho| \cdot \|T\| .$$

For \mathbf{c} , the following inequality bounds the norm of the right-hand sides of (14), (15) and (16):

$$\|\mathbf{c}\| \leq \|\mathbf{u}\| + 2 \cdot |\rho| \cdot \|T\|$$

□

By application of Proposition 13, this lemma now enables us to give bounds on the length of a run witnessing reachability for two given configurations.

Lemma 18. Let $V = (Q, T)$ be a d -VASS, let $p(\mathbf{u})$ and $q(\mathbf{v})$ be configurations of V , and let $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ be a linear path scheme from p to q . Then $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{N}^d} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^d} q(\mathbf{v})$ for some $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$ such that $e_i \leq 2^{k^{O(1)}} \cdot O(\|\mathbf{u}\| + \|\mathbf{v}\| + |\rho| \cdot \|T\|)$ for each $i \in [1, k]$.

Proof. The set of those $e_1, \dots, e_k \in \mathbb{N}$ that achieve $\mathbf{u} + \delta(\pi) = \mathbf{v}$ can be obtained from the set of solutions of the system $\mathcal{E} : B \cdot \mathbf{x} = \mathbf{d}$ of linear Diophantine equations with unknowns $\mathbf{x} = (x_1, \dots, x_k)$, where

$$A \stackrel{\text{def}}{=} [\delta(\beta_1) \cdots \delta(\beta_k)] \quad \text{and} \quad \mathbf{d} \stackrel{\text{def}}{=} \mathbf{v} - \mathbf{u} - \sum_{0 \leq i \leq k} \delta(\alpha_i) .$$

The constraint matrix of \mathcal{E} is of dimension $d \times k$ and has norm bounded by $|\rho| \cdot \|T\|$. The norm of the right-hand side of \mathcal{E} is bounded by $\|\mathbf{u}\| + \|\mathbf{v}\| + |\rho| \cdot \|T\|$. Let us fix an arbitrary $\chi : [1, k] \rightarrow \{0, 1\}$. Lemma 17 yields a system of linear Diophantine inequalities $\mathcal{I} = \mathcal{I}(\mathbf{u}, \rho, \chi)$ of the form $\mathcal{I} : A \cdot \mathbf{x} \geq \mathbf{c}$ whose set of solutions $\mathbf{e} = (e_1, \dots, e_k) \in \mathbb{N}$ corresponds to all runs $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{u} + \delta(\pi))$, where $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$ and $\chi(i) = \text{sign}(e_i)$ for all $i \in [1, k]$. Consequently, for any $(e_1, \dots, e_k) \in \llbracket \mathcal{I} \rrbracket \cap \llbracket \mathcal{E} \rrbracket$ and $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$, we have $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^d} q(\mathbf{v})$ and $\chi(i) = \text{sign}(e_i)$ for all $i \in [1, k]$. Now we obtain $\mathcal{I} \cap \mathcal{E}$ as

$$\mathcal{I} \cap \mathcal{E} : \begin{bmatrix} A \\ B \\ -B \end{bmatrix} \cdot \mathbf{x} \geq \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \\ -\mathbf{d} \end{bmatrix} .$$

From Lemma 17 and our observations above we conclude that the norm of the constraint matrix of $\mathcal{I} \cap \mathcal{E}$ is bounded by $k \cdot |\rho| \cdot \|T\|$. Moreover, the norm on right-hand side is bounded by $O(\|\mathbf{u}\| + \|\mathbf{v}\| + |\rho| \cdot \|T\|)$. By application of Proposition 13, the bounds on the solutions of $\mathcal{I} \cap \mathcal{E}$ follow. □

An immediate corollary of Lemma 18 and Theorem 1 is that reachability in 2-VASS is in PSPACE.

Corollary 19. Reachability in 2-VASS is in PSPACE.

Proof. Let $V = (Q, T)$ be a 2-VASS and $p(\mathbf{u}), q(\mathbf{v})$ be configurations of V . By Theorem 1, there exists a set S of linear path schemes such that

- $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S}_{\mathbb{N}^2} q(\mathbf{v})$,

- $|\rho| \leq (|Q| + \|T\|)^{O(1)}$ for every $\rho \in S$, and
- each $\rho \in S$ has at most $O(|Q|^2)$ cycles.

By Lemma 18, if $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{N}^2} q(\mathbf{v})$ for some $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k \in S$ then $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ for some $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k \in \rho$ such that $e_1, \dots, e_k \in [0, e]$, where e can be bounded as

$$\begin{aligned} e &\leq 2^{|Q|^{O(1)}} \cdot O\left(\|\mathbf{u}\| + \|\mathbf{v}\| + (|Q| + \|T\|)^{O(1)} \cdot \|T\|\right) \\ &\leq 2^{(|V| + \log\|\mathbf{u}\| + \log\|\mathbf{v}\|)^{O(1)}}. \end{aligned}$$

Since $|\pi| \leq |\rho| \cdot e$, the run $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ can be guessed nondeterministically in polynomial space by storing only the intermediate configurations in an on-the-fly manner. Consequently, reachability in 2-VASS is in PSPACE. \square

In order to complete the proof of Theorem 2, it remains to show hardness for PSPACE. We reduce from reachability in bounded one-counter automata, which is known to be PSPACE-complete [4]. A bounded one-counter automaton is given by a tuple $V = (Q, T, b)$, where (Q, T) is a 1-VASS and $b \in \mathbb{N}$ is a bound encoded in binary. Let $\mathbb{B} = [0, b]$, given configurations $p(u), q(u)$ of V such that $u, v \in \mathbb{B}$, reachability is to decide whether $p(u) \xrightarrow{*}_{\mathbb{B}} q(v)$.

Lemma 20. Reachability in 2-VASS is PSPACE-hard.

Proof. Let $V = (Q, T, b)$ be a bounded one-counter automaton, and let $V' \stackrel{\text{def}}{=} (Q, T')$ be the 2-VASS obtained from V by setting $T' \stackrel{\text{def}}{=} \{h(t) : t \in T\}$, where $h(p, z, q) \stackrel{\text{def}}{=} (p, (z, -z), q)$. We define an injection φ from configurations of V to configurations of V' as follows:

$$\varphi(q(z)) \stackrel{\text{def}}{=} q(z, b - z)$$

For any path π , it is now easily checked by induction on $|\pi|$ that

$$p(u) \xrightarrow{\pi}_{\mathbb{B}} q(v) \text{ in } V \iff \varphi(p(u)) \xrightarrow{h(\pi)}_{\mathbb{N}^2} \varphi(q(v)) \text{ in } V'.$$

\square

This concludes the proof of Theorem 2 and shows that reachability in 2-VASS is PSPACE-complete.

5.2. Reachability in 2-VASS with Unary Updates. For unary 2-VASS we can show that reachability is in NP and NL-hard.

Given a unary 2-VASS V , by Theorem 1 whenever $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^2} q(\mathbf{v})$ then there exists a linear path scheme $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k$ whose length is *polynomial* in $|V|$ such that $p(\mathbf{u}) \xrightarrow{\rho}_{\mathbb{N}^2} q(\mathbf{v})$. Moreover, the proof of Corollary 19 shows that there exist $e_1, \dots, e_k \leq 2^{(|V| + \log\|\mathbf{u}\| + \log\|\mathbf{v}\|)^{O(1)}}$ such that for $\pi = \alpha_0 \beta_1^{e_1} \alpha_1 \cdots \beta_k^{e_k} \alpha_k$, we have $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$. In particular, every e_i can be represented using a polynomial number of bits. Hence, (ρ, e_1, \dots, e_k) may serve as a certificate that can be guessed in polynomial time. It remains to show that this certificate can be verified in polynomial time. Checking that ρ is a linear path scheme is easily verified in polynomial time. In order to check if $p(\mathbf{u}) \xrightarrow{\pi}_{\mathbb{N}^2} q(\mathbf{v})$ in polynomial time we can construct the system of linear Diophantine equations from Lemma 17 and verify that $\mathbf{e} = (e_1, \dots, e_k)$ is a solution to this system. This shows that reachability in unary 2-VASS is in NP.

NL-hardness of reachability trivially follows from NL-hardness of reachability in directed graphs. Here, we wish to slightly strengthen this result and remark that reachability is NL-hard already for unary 2-VASS, whose underlying graph corresponds structurally to a linear path scheme (formally, every state lies on at most one cycle and the deletion of all cycles yields a union of isolated vertices and a cycle-free path, cf. Figure 2 at the beginning of this document). Let $G = (U, E)$ be a directed graph such that $U = \{u_0, \dots, u_{m-1}\}$ and $E = \{e_0, \dots, e_{n-1}\} \subseteq U \times U$. We define an injection

$h : U \rightarrow [0, m-1]^2$ as $h(u_i) = (i, m-1-i)$ that relates vertices of G with vectors from bounded intervals. Let $\ell \stackrel{\text{def}}{=} m \cdot n - 1$, the flat unary 2-VASS $V = (Q, T)$ can now be defined as follows:

$$\begin{aligned} Q &\stackrel{\text{def}}{=} \{q_0, q'_0, \dots, q_\ell, q'_\ell\} \\ T &\stackrel{\text{def}}{=} \{(q_j, \mathbf{0}, q_{j+1}) : j \in [0, \ell-1]\} \\ &\cup \{(q_j, -h(u_i), q'_j), (q'_j, h(u_k), q_j) : e_j = (u_i, u_k), i = j \bmod n, i \in [0, \ell]\} \quad . \end{aligned}$$

Suppose we wish to decide whether u_{m-1} is reachable from u_0 , we claim that this is the case if, and only if, $q_0(h(u_0)) \xrightarrow{*} q_\ell(h(u_{m-1}))$. Informally speaking, the vertex currently visited along a path is encoded in the counter values of V . Every loop between q_j and q'_j allows for simulating the edge $e_{(j \bmod n)} = (u_i, u_k)$ of G . The transition from q_j to q'_j can only be traversed if the vertex encoded into the current counter values corresponds to u_i . If we are able to reach q'_j , the transition back to q_j then updates the currently visited vertex to u_k . Since a path from u_0 to u_{m-1} of minimal length in G traverses at most m vertices, $\ell + 1 = m \cdot n$ states q_j suffice.

Theorem 21. Reachability in unary 2-VASS is in NP and NL-hard.

5.3. Derived Results. In this section, we explicitly state and remark some results that can additionally be derived from the technical results established in this paper.

5.3.1. \mathbb{Z} -Reachability in Unary d -VASS is NL-complete for each fixed d . The decomposition established in Proposition 3 enables us to obtain a new result on \mathbb{Z} -reachability of d -VASS when d is fixed. The complexity of this problem depends on the encoding of numbers as well as the dimension d . When numbers are encoded in binary, reachability is NP-complete even when $d = 1$ [7, 8], and reachability is also NP-complete when numbers are encoded in unary and d is part of the input to the problem [7]. By application of Proposition 3 and Corollary 15, we can solve the case of reachability under unary encoding of numbers for each fixed dimension d .

Theorem 22. For every fixed $d \geq 1$, \mathbb{Z} -reachability in unary d -VASS is NL-complete.

Proof. NL-hardness trivially follows from NL-hardness of reachability in directed graphs. Let $d \geq 1$ be fixed. Let $V = (Q, T)$ be a d -VASS and $p(\mathbf{u}), q(\mathbf{v}) \in Q \times \mathbb{Z}^d$ be two configurations as input to the \mathbb{Z} -reachability problem. By Proposition 3, there exists a finite set S of linear path schemes such that

- $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$ if, and only if, $p(\mathbf{u}) \xrightarrow{S} q(\mathbf{v})$,
- $|\rho| \leq 2 \cdot |Q| \cdot |T|$ for each $\rho \in S$, and
- each $\rho \in S$ has at most $|T|$ cycles.

Suppose $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$, then $p(\mathbf{u}) \xrightarrow{\rho} q(\mathbf{v})$ for some $\rho = \alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k \in S$ with $k \leq |T|$. Let $\mathcal{E} : A \cdot \mathbf{x} = \mathbf{c}$ be the system of linear Diophantine equations, where

$$A \stackrel{\text{def}}{=} [\delta(\beta_1) \quad \cdots \quad \delta(\beta_k)] \in \mathbb{Z}^{d \times k} \quad \text{and} \quad \mathbf{c} \stackrel{\text{def}}{=} \mathbf{v} - (\mathbf{u} + \delta(\alpha_0 \alpha_1 \cdots \alpha_k)) \in \mathbb{Z}^d \quad .$$

Then, we have

$$\begin{aligned} p(\mathbf{u}) \xrightarrow{\rho} q(\mathbf{v}) &\iff p(\mathbf{u}) \xrightarrow{\alpha_0 \beta_1^* \alpha_1 \cdots \beta_k^* \alpha_k} q(\mathbf{v}) \text{ for some } \mathbf{e} = (e_1, \dots, e_k) \in \mathbb{N}^k \\ &\iff \mathbf{e} \in \llbracket \mathcal{E} \rrbracket \text{ for some } \mathbf{e} \in \mathbb{N}^k \quad . \end{aligned}$$

By Corollary 15, if $\llbracket \mathcal{E} \rrbracket \neq \emptyset$ then \mathcal{E} has a solution \mathbf{e} such that $\|\mathbf{e}\| \leq (\|A\| + \|\mathbf{c}\|)^{O(d)}$. Hence, by definition of A and \mathbf{c} , the norm of solutions can be bounded by some b , where

$$b \leq (|T| \cdot |\rho| \cdot \|T\| + |\rho| \cdot \|T\| + \|\mathbf{u}\| + \|\mathbf{v}\|)^{O(d)} \leq ((|T| + \|T\|)^{O(1)} + \|\mathbf{u}\| + \|\mathbf{v}\|)^{O(d)} \quad .$$

Since $\|T\|$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are encoded in unary (i.e. $|V| = |Q| + |T| \cdot d \cdot \|T\|$) and d is fixed, we obtain $b \leq |V|^{O(1)}$.

Thus, $p(\mathbf{u}) \xrightarrow{\rho} q(\mathbf{v})$ implies that $p(\mathbf{u}) \xrightarrow{\pi} q(\mathbf{v})$ for some $\pi \in T^*$, where $|\pi| \leq b \cdot |\rho| \leq |V|^{O(1)}$. Therefore, in order to decide reachability it suffices to guess on-the-fly the intermediate configurations of a path of polynomial length from $p(\mathbf{u})$ to $q(\mathbf{v})$, which can be done nondeterministically in logarithmic space. \square

5.3.2. *Boundedness and Coverability in d -VASS.* For the sake of completeness, here we wish to discuss some consequences of PSPACE-hardness of reachability in 2-VASS to the complexity of coverability and boundedness in d -VASS that were left open in the literature.

The boundedness problem can be stated as follows.

d -VASS BOUNDEDNESS

INPUT: A d -VASS $V = (Q, T)$ and a configuration $p(\mathbf{u})$.

QUESTION: Is $\{q(\mathbf{v}) : p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^d} q(\mathbf{v})\}$ an infinite set?

The coverability problem can be stated as follows.

d -VASS COVERABILITY

INPUT: A d -VASS $V = (Q, T)$ and configurations $p(\mathbf{u})$ and $q(\mathbf{v})$.

QUESTION: Does there exist $\mathbf{w} \geq \mathbf{v}$ such that $p(\mathbf{u}) \xrightarrow{*}_{\mathbb{N}^d} q(\mathbf{w})$?

The complexity of boundedness and coverability for d -VASS in a fixed dimension d has been studied by Rosier & Yen in [24]. They show that both problems are PSPACE-complete for any fixed $d \geq 4$. Chan [3] later noted that boundedness is already PSPACE-complete for $d = 3$, leaving the case $d = 2$ as an open problem.

Theorem 23 ([24, 3]). Boundedness and coverability in d -VASS are PSPACE-complete for any fixed $d \geq 3$.

It is moreover known that for $d = 1$ those problems are NP-complete [6]. From the results in [4] and Lemma 20, it is now easy to improve the lower bounds from [24, 3] and show that reachability and coverability are PSPACE-complete for every fixed $d \geq 2$. An instance of reachability between $p(u)$ and $q(v)$ in a bounded one-counter automaton with bound b can be reduced to boundedness and coverability in 2-VASS by using the construction provided in Lemma 20 as a gadget and adding an extra transition from q to a fresh control state r . This transition simply checks whether the current counter values are equal to $(v, b - v)$ by subtracting this value from the counter, and r has a single self-loop which increments both counters by one, say. Together with the upper bounds established in [24], the above-mentioned proof sketch yields the following theorem as a corollary.

Corollary 24. Boundedness and coverability in d -VASS are PSPACE-complete for any fixed $d \geq 2$.

6. CONCLUSION AND FUTURE WORK

In this paper, we have located the complexity, i.e., PSPACE-completeness, of the reachability problem for 2-VASS. We have also noted that the coverability and boundedness problems for 2-VASS are PSPACE-complete. When numbers are encoded in unary we showed that \mathbb{Z} -reachability in d -VASS is NL-complete for any fixed d . Reachability for unary 2-VASS was shown to be NL-hard and in NP. Our approach does not immediately lead to a better upper bound than NP mainly due to the following reason. Our proof showed that the reachability relation can be captured by a set of linear path schemes whose number of cycles is quadratically bounded. The matrix of the resulting system of linear Diophantine inequalities thus has quadratically many columns and its smallest solutions can thus become exponentially large. The latter correspond to the exponents of the cycles of the linear path scheme and hence of the length of the run.

It could be interesting to study the reachability problem in d -VASS with a single control state, known as d -VAS, for $2 \leq d \leq 5$, since 5-VAS (resp. 2-VAS) are slightly more (resp. less) general than 2-VASS and have semi-linear reachability sets [10]. A more challenging problem seems to be to obtain a first complexity upper bound for reachability in 3-VASS.

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