A strong version of Cobham's theorem

and other thoughts about decidability in expansions of Presburger arithmetic

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ICALP, Trends in Arithmetic Theories, July 2022

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A strong version of Cobham's theorem.

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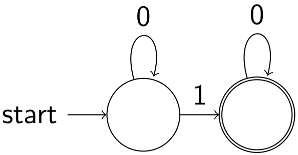

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- $k^{\mathbb{N}}$ is k-recognizable,
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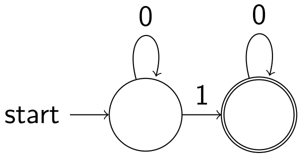

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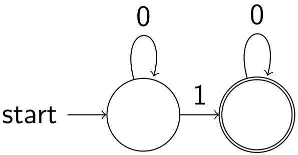

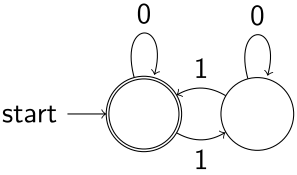

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Cobham-Semenov restated. Let $k, \ell \in \mathbb{N}_{\geq 2}$ be multiplicatively independent. A set $X \subseteq \mathbb{N}^n$ is definable in both $(\mathbb{N}, +, V_k)$ and $(\mathbb{N}, +, V_\ell)$ if and only if it is definable in $(\mathbb{N}, +)$.

H.-Schulz (2022). Let $k, \ell \in \mathbb{N}_{\geq 2}$ be multiplicatively independent, and let $X \subseteq \mathbb{N}^m$ and $Y \subseteq \mathbb{N}^n$ be such that

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Proof of Cobham-Semenov.

Suppose $X \subseteq \mathbb{N}^n$ is definable in both $(\mathbb{N}, +, V_k)$ and $(\mathbb{N}, +, V_\ell)$, but not in $(\mathbb{N}, +)$. Then the theory of $(\mathbb{N}, +, X, X)$ is undecidable. However, then the theory of $(\mathbb{N}, +, X)$ is undecidable.

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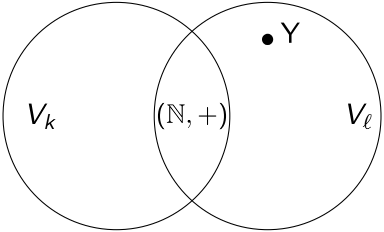

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This question is an old question. Bruyère, Cherlin and van den Dries asked this question as early as 1985, and it has been restated in the literature many times.

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In $(\mathbb{N}, +, 2^{\mathbb{N}}, 3^{\mathbb{N}})$:

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Open question. What fragments of the theory of $(\mathbb{N}, +, 2^{\mathbb{N}}, 3^{\mathbb{N}})$ are decidable?

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For $y \in 2^{\mathbb{N}}$, define $S(y)$ to be the set of all $x \in 3^{\mathbb{N}}$ such that $\lambda(x - \lambda(x)) = y$.

In words: $S(y)$ is the set of all powers of 3 for which y is the second largest power of 2 that appears in the binary representation of x.

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Fact. For all $y \in 2^{\mathbb{N}}$, $S(y)$ is finite. However, for all $m, n \in \mathbb{N}$ there is $y \in 2^{\mathbb{N}}$ such that $y > m$ and $|S(y)| > n$.

Main Lemma. Let $m, n \in \mathbb{N}$, let $Z_1, \ldots, Z_m$ be a partition of $\{1, \ldots, n\}$. Then there are $s \in \mathbb{N}_{>0}$ and $t_1 < \cdots < t_n$ such that for $i = 1, \ldots, m$

$$S(2^{s+i}) \cap [3^{t_1}, 3^{t_n}] = \{3^{t_j} \; : j \in Z_i\}.$$

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- This allows us to code/interpret arbitrary large finite subsets of $\mathbb{N}^2$.

Main Lemma. Let $m, n \in \mathbb{N}$, let $Z_1, \dots, Z_m$ be a partition of $\{1, \dots, n\}$. Then there are $s \in \mathbb{N}_{>0}$ and $t_1 < \dots < t_n$ such that for $i = 1, \dots, m$

$$S(2^{s+i}) \cap [3^{t_1}, 3^{t_n}] = \{3^{t_j} \ : j \in Z_i\}.$$

$Z_1 = 1, 3, 5,$

$Z_2 = 2, 6, 7,$

$Z_3 = 4$

	3^{t_1}	3^{t_2}	3^{t_3}	3^{t_4}	3^{t_5}	3^{t_6}	3^{t_7}
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- Proof of Main Lemma just uses density of $2^{-\mathbb{N}}3^{\mathbb{N}}$ in $\mathbb{R}_{>0}$.
- This allows us to code/interpret arbitrary large finite subsets of $\mathbb{N}^2$.
- Such theories are known to be undecidable, as the halting problem or the tiling problem can be encoded in such theories.

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Folklore. The set of square-free integers is not piecewise syndetic and does not have zero natural density.

Combinatorics on words.

We observed that $\{n \in \mathbb{N} \ : \ s_2(n) \text{ is even } \}$ is 2-recognizable, where $s_2(n)$ is the binary digit sum. Thus the function $f : \mathbb{N} \to \{0, 1\}$ given by

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Example. To check that the Thue-Morse sequence in not eventually periodic, we have to decide

$$(\mathbb{N}, +, V_2) \models \forall p \; (p > 0) \to \Big(\forall i \; \exists j \; j > i \;\wedge\; f(j) \neq f(j+p)\Big)$$

Different words.

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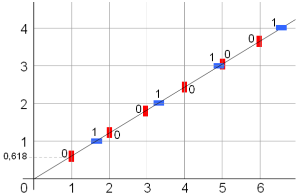

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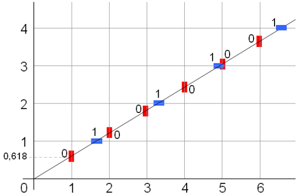

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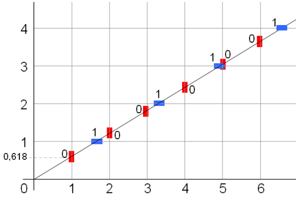

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$$S_0 := 0, S_1 := 01, S_2 := 010, S_3 := 01001, S_4 := 01001010, S_5 := 0100101001001\dots$$

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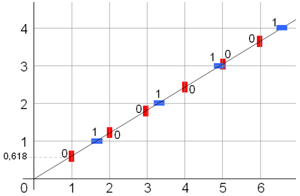

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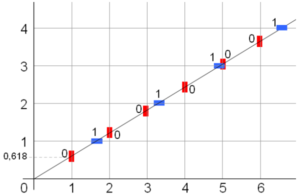

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Solution. Replace k-ary representations by different non-standard representations.

A **continued fraction expansion** $[a_0; a_1, \ldots, a_k, \ldots]$ is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

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Zeckendorf representation (1972). Let $N \in \mathbb{N}$ and F_k be the k-th Fibonacci number. Then N can be written uniquely as

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Ostrowski (1918). Every natural number N can be written uniquely as

$$N = \sum_{k=0}^{n} b_{k+1} q_k,$$

where $b_k \in \mathbb{N}$ such that $b_1 < a_1$, $b_k \leq a_k$ and, if $b_k = a_k$, $b_{k-1} = 0$.

Let $V_a : \mathbb{N} \to \mathbb{N}$ be the function that maps $x \geq 1$ with Ostrowski representation $b_n \dots b_1$ to the least q_k with $b_{k+1} \neq 0$, and 0 to 1.

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Proof Strategy. Show that these structures are uniformly ω-automatic. Uses general adder in Ostrowski numeration systems due to Baranwal, Schaeffer and Shallit.

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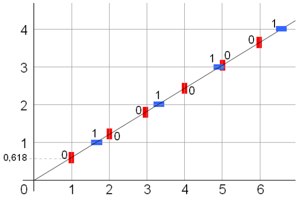

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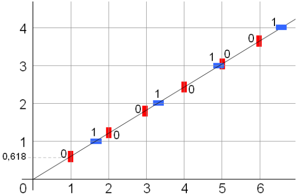

Fact. Let $n \in \mathbb{N}_{\geq 1}$. Then the following are equivalent:

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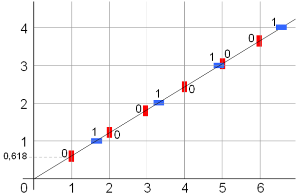

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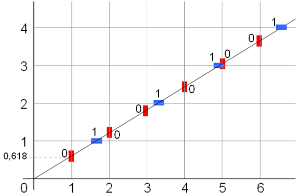

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$$T_{\text{Sturmian}} \models \varphi \text{ if and only if all Sturmian words are not evenututally periodic.}$$

The decision procedure for T_{Sturmian} allows us to check that no Sturmian word is eventually periodic.

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Thus

$$T_{\text{Sturmian}} \models \forall n > 0 \exists i \ \psi(i,n)$$

if and only if

every characteristic Sturmian word contains palindromes of every length.

For $w \in \{0,1\}^*$, let $\overline{w}$ be the $\{0,1\}$-word obtained by replacing each 1 in w by 0 and each 0 in w by 1. We say v is an **antisquare** if $v = w\overline{w}$.

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$$\mathcal{N}_a \models \chi(i,n) \text{ if and only if } c_a(i) \dots c_a(i+2n) \text{ is an antisquare.}$$

Thus

$$T_{\text{Sturmian}} \models \exists m \forall i \forall n \, (\chi(i,n) \to n \leq m)$$

if and only if

every characteristic Sturmian word contains finitely many antisquares.

An implementation: **Pecan**

- Try Pecan at `http://reedoei.com/pecan`
- Git: `https://github.com/ReedOei/Pecan`

Pecan improves on **Walnut** by Mousavi, another automated theorem prover for deciding combinatorial properties of automatic words, by using Büchi automata instead of finite automata.

This difference enables Pecan to handle uncountable families of sequences, allowing us quantify over all Sturmian words.

We used Pecan to re-prove theorems from papers about Sturmian words (papers as recent as 2020) and have established first new results. Currently working to prove a conjecture of Jason Bell.

				Max		Final	
Name	Quant.	At.	Runtime	States	Edges	States	Edges
Mirror invariant	$\exists$	1	8.1	1440	16840	1129	9666
Unbordered	$\exists^3$	2	0.5	275	1156	92	410
Cube	$\exists$	4	0.7	936	5956	126	561
Least period	$\forall$	4	2605.2	352577	6098198	577	4161
Max unb. subf.	$\forall$	4	26.4	25200	196575	585	4345
Palindrome	$\exists^2$	4	5.1	1934	12337	922	6274
Period	$\exists^2$	5	64.1	5853	103886	1660	17570
Recurrent	$\forall\exists$	5	272.6	61713	960207	34	212
Special factor	$\exists^3\forall$	8	1361.8	17738	103274	4594	25349
Factor Lt (idx)	$\exists\forall^2$	11	702.7	1057221	22348882	2204	25026
Ev. periodic	$\exists^2\forall\exists^2$	12	216.6	78338	1001075	1	0
Reverse factor	$\exists\forall^2$	12	842.0	1408050	22780414	1440	16840
Antipalindrome	$\exists^2\forall^3$	13	242.2	78396	1668960	200	834
Antisquare	$\forall^3$	13	1844.3	2542937	31570114	136	539
Square	$\forall^3$	13	2138.0	1908657	23683717	155	747
$(01)^*\|(10)^*$	$\forall$	16	77.9	5409	72739	103	456

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