Computing Science Group

Theory and Practice of Fusion

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Abstract. There are a number of approaches for eliminating intermediate data structures in functional programs—this elimination is commonly known as *fusion*. Existing fusion strategies are built upon various, but related, recursion schemes, such as folds and unfolds. We use the concept of *recursive coalgebras* as a unifying theoretical and notational framework to explore the foundations of these fusion techniques. We first introduce the calculational properties of recursive coalgebras and demonstrate their use with proofs and derivations in a calculational style, then provide an overview of fusion techniques by bringing them together in this setting.

1 Introduction

Functional programmers love modular programs. It is easy for them to create clear, concise, and reusable code by composing functions. Consider the following Haskell program as an example:

$$
\begin{aligned}
&f \ : \ (\mathit{Integer}, \mathit{Integer}) \to \mathit{Integer} \\
&f = \mathit{sum} \cdot \mathit{map}\ \mathit{sq} \cdot \mathit{filter}\ \mathit{odd} \cdot \mathit{between} \ .
\end{aligned}
$$

The program takes a pair of integers representing an interval and returns the sum of the squared odd integers in this interval. We have expressed this a composition of four functions: *between* generates an enumeration between two natural numbers as a list, *filter odd* removes any even numbers, *map sq* squares the remaining (odd) numbers, and *sum* adds them together. Unfortunately, the clarity of this program comes at a cost. The constituent functions of this program communicate with each other using intermediate data structures, the production and immediate consumption of which carries an obvious performance penalty. Yet, because these definitions are recursive, eliminating the need for these transient structures is beyond the reach of a typical compiler.

Nonetheless, such a transformation is possible. We can manually construct a program that is equivalent to f, but without the intermediate data structures

$$
\begin{aligned}
&f'\ (m, n) = \mathit{go}\ m \\
&\quad \textbf{where}\ \mathit{go}\ m \mid m > n \qquad = 0 \\
&\qquad\qquad\qquad \mid \mathit{otherwise} = \mathit{go}\ (m+1) + \textbf{if}\ \mathit{odd}\ m\ \textbf{then}\ \mathit{sq}\ m\ \textbf{else}\ 0 \ .
\end{aligned}
$$

This new program has lost the desirable qualities of the original—our concise, modular and declarative code has been hammered into a long, opaque and specialised function. In doing so, however, we have accomplished our goal of removing the intermediate data structures by transforming the numerous recursive traversals into a single one. This process is called *fusion*.

Fusing programs by hand quickly becomes infeasible for those of non-trivial length. Furthermore, it can be difficult to manually pinpoint all the opportunities for fusion. Instead, such a transformation should be performed automatically.

Difficulties arise, however, in automatically fusing functions defined using general recursion. Specifically, such transformations often have proof obligations that cannot be discharged by the compiler.

One remedy is to standardise the way data structures are produced or consumed by encapsulating the recursion scheme in a higher-order function. The arguments to these functions are the non-recursive ‘steps’. Simple syntactic transformations can then fuse many recursive traversals into a single one, and then non-recursive steps can be optimised using conventional methods. This approach is known as *shortcut fusion*. Different incarnations of this technique utilise different recursion schemes, e.g. *folds* for consumers or *unfolds* for producers. The steps of such a scheme are known as algebras or coalgebras, respectively.

The implementation of these fusion techniques is usually described syntactically, by giving a definition of the production and consumption combinators and accompanying rewrite rules. This alone does not really explain the underlying fusion mechanism. Furthermore, it is difficult to construct correctness proofs, or relate various fusion approaches to one another, despite the fact that such close relations exist. In this paper, we move fusion to a clearer setting, where the syntactic details of fusion fall away.

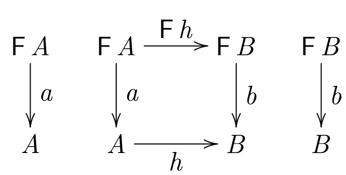
Category theory provides the tools we need to tackle the semantics of the recursion schemes. While some fusion techniques have been individually given this treatment before, our focus is to bring them all under one roof. In this paper, we propose using *recursive coalgebras* as that roof. We will show how recursive coalgebras enable us to explain the fusion rules underlying the various fusion techniques and give short, simple proofs of correctness.

The rest of the paper is structured as follows. In Section 2 we review initial algebras and folds, final coalgebras and unfolds, and their associated laws. In Section 3, we introduce hylomorphisms based on recursive coalgebras as a more general recursion scheme. We also relate initial algebras to final recursive coalgebras. In Section 4, we generalise the calculational properties of fold and unfold. We present some basic examples in Section 5.1 and Section 5.2 to warm up for the remainder of the section, where we apply our framework to *foldr*/*build* fusion (Section 5.3), *destroy*/*unfoldr* fusion (Section 5.4) and stream fusion (Section 5.6). Finally, we summarise the related work and conclude. We have placed a categorical refresher in Appendix A.

2 Background: Algebras and Coalgebras

Concepts from category theory have become an important tool in the theoretical understanding of functional programming languages. Of particular note is the concept of an *initial algebra*, which is key to giving a semantics to recursive datatypes [16]. In this section we will refresh the salient details of the theory: algebras, their initiality, and in true category-style, their dual situation. Appendix A serves as a further refresher of the categorical concepts that we assume knowledge of: functors, natural transformations and subcategories. For the more functional-programming-minded reader, we will parallel these developments with examples in Haskell, where possible.

Let $\mathsf{F} : \mathbb{C} \to \mathbb{C}$ be a functor. An F-*algebra* is a pair $\langle a, A \rangle$ consisting of an object $A : \mathbb{C}$ and an arrow $a : \mathsf{F}\,A \to A : \mathbb{C}$. An F-*algebra homomorphism* between algebras $\langle a, A \rangle$ and $\langle b, B \rangle$ is an arrow $h : A \to B : \mathbb{C}$ such that $h \cdot a = b \cdot \mathsf{F}\,h$.



A defining characteristic of functors is that they preserve identity and composition; this entails that identity is an algebra homomorphism and homomorphisms compose. Consequently, F-algebras and their homomorphisms form a category, called $\mathbf{Alg}(\mathsf{F})$. We abbreviate a homomorphism, $h : \langle a, A \rangle \to \langle b, B \rangle : \mathbf{Alg}(\mathsf{F})$, by $h : a \to b : \mathbf{Alg}(\mathsf{F})$ if the objects are obvious from the context, or simply by $h : a \to b$ if the functor F is also obvious.

In Haskell, we model a functor as a datatype whose constructors describe its action on data (i.e. objects). Its action on arrows is defined by making that data type an instance of the *Functor* typeclass

```
class Functor f where
  fmap : (a → b) → (f a → f b) .
```

We can then simply treat the concept of an F-algebra as a function, where F is a datatype that is an instance of the *Functor* class. The F-algebra $\langle a, A\rangle$ is simply a function $a : \mathsf{F}\,A \to A$. An F-algebra homomorphism between a and $b : \mathsf{F}\,B \to B$ is a function $h : A \to B$ that satisfies the side condition $h \cdot a = b \cdot \mathit{fmap}\,h$. This property cannot, however, be deduced from the type and must be checked by the programmer.

Here are two elementary facts that come in handy later on. First, if $\langle a, A\rangle$ is an F-algebra, then $\langle \mathsf{F}\,a, \mathsf{F}\,A\rangle$ is an F-algebra, as well. (This means that the functor $\mathsf{F} : \mathbb{C} \to \mathbb{C}$ can be lifted to a functor of type $\mathsf{F}^* : \mathbf{Alg}(\mathsf{F}) \to \mathbf{Alg}(\mathsf{F})$; Its action on objects is $\mathsf{F}^*\,\langle a, A\rangle = \langle \mathsf{F}\,a, \mathsf{F}\,A\rangle$ and its action on arrows is $\mathsf{F}^*\,h = \mathsf{F}\,h$.) Furthermore, since $a \cdot \mathsf{F}\,a = a \cdot \mathsf{F}\,a$, the algebra a is simultaneously an F-homomorphism of type

$$a \;:\; \mathsf{F}\,a \to a \;. \tag{1}$$

Second, if $\langle a, A\rangle$ is an F-algebra, and $\alpha : \mathsf{G} \dot{\to} \mathsf{F}$ is a natural transformation, then $\langle a \cdot \alpha\,A, A\rangle$ is a G-algebra. Homomorphisms in $\mathbf{Alg}(\mathsf{F})$ are also homomorphisms in $\mathbf{Alg}(\mathsf{G})$:

$$h : a \cdot \alpha\,A \to b \cdot \alpha\,B : \mathbf{Alg}(\mathsf{G}) \quad\Longleftarrow\quad h : a \to b : \mathbf{Alg}(\mathsf{F}) \;. \tag{2}$$

$$\begin{aligned} & h \cdot a \cdot \alpha\,A \\ = \;& \{\text{ assumption: } h : a \to b : \mathbf{Alg}(\mathsf{F}) \;\} \\ & b \cdot \mathsf{F}\,h \cdot \alpha\,A \\ = \;& \{\text{ naturality of } \alpha \;\} \\ & b \cdot \alpha\,B \cdot \mathsf{G}\,h \end{aligned}$$

(For the daring reader, the import of (2) is that $\mathbf{Alg}(-)$ is a functor of type $(\mathbb{C}^{\mathbb{C}})^{\mathsf{op}} \to \mathbf{Cat}$: it sends the functor F to the category of F-algebras and the natural transformation $\alpha : \mathsf{G} \dot{\to} \mathsf{F}$ to the functor $\mathbf{Alg}(\alpha) : \mathbf{Alg}(\mathsf{F}) \to \mathbf{Alg}(\mathsf{G})$, defined $\mathbf{Alg}(\alpha)\,\langle a, A\rangle = \langle a \cdot \alpha\,A, A\rangle$ and $\mathbf{Alg}(\alpha)\,h = h$.)

If the category $\mathbf{Alg}(\mathsf{F})$ has an initial object, then we call it the *intial* F-*algebra* $\langle \mathit{in}, \mu\mathsf{F}\rangle$. The presence of this initial object is conditional upon the underlying category $\mathbb{C}$ being ω-cocomplete and the functor F being ω-cocontinuous. Initiality means that there is a *unique* arrow from $\langle \mathit{in}, \mu\mathsf{F}\rangle$ to any other F-algebra $\langle a, A\rangle$. This arrow, called *fold*, is written $(\!|a|\!) : \mathit{in} \to a$. We construct elements of $\mu\mathsf{F}$ using in and deconstruct them using $(\!|a|\!)$. We can think of $(\!|a|\!) : \mathit{in} \to a$ as replacing constructors by functions, represented by the algebras in and a, respectively. Initiality is captured by the *uniqueness property* of folds:[1]

$$h = (\!|a|\!) \quad\Longleftrightarrow\quad h : \mathit{in} \to a \quad\Longleftrightarrow\quad h \cdot \mathit{in} = a \cdot \mathsf{F}\,h \;. \tag{3}$$

It is important to note that we have omitted the quantification of the names that appear in property (3); we have done so for presentational succinctness and we will continue in this manner. In this case we will spell out implicit quantification: the uniqueness property holds for all functors F, where the category $\mathbf{Alg}(\mathsf{F})$ has an initial object named $\langle \mathit{in}, \mu\mathsf{F}\rangle$, and for all F-algebras $\langle a, A\rangle$, and for all F-algebra homomorphisms $h : \mathit{in} \to a$.

This property provides us with a generic definition of $(\!|-|\!)$ in Haskell. First, we define the μ datatype, which takes a functor to its least fixed point:

$$\mathbf{data}\,\mu f = \mathit{in}\,\{\, \mathit{out} : f\,(\mu f)\,\}$$

The constructor *in* allows us to construct a structure of type μf out of something of type $f\,(\mu f)$ and *out* deconstructs it. The relationship between *in* and *out* in our setting is discussed further in Sections 3 and 4.

We can define $(\!|-|\!)$ as a higher-order function that takes an algebra $f\,a \to a$ and returns a function $\mu f \to a$, according to the uniqueness property:

$$\begin{aligned} & (\!|-|\!) : (\mathit{Functor}\,f) \Rightarrow (f\;a \to a) \to (\mu f \to a) \\ & (\!|a|\!) = a \cdot \mathit{fmap}\,(\!|a|\!) \cdot \mathit{out} \end{aligned}$$

[1] The formula $P \Longleftrightarrow Q \Longleftrightarrow R$ has to be read *conjunctively* as $P \Longleftrightarrow Q \wedge Q \Longleftrightarrow R$. Likewise, $P \Longleftarrow Q \Longleftrightarrow R$ is shorthand for $P \Longleftarrow Q \wedge Q \Longleftrightarrow R$.

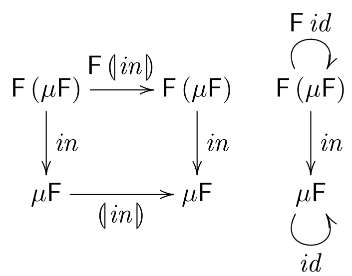
By allowing us to substitute $(\!|a|\!)$ for h, we see that the uniqueness property provides us with a definition of $(\!|-|\!)$ that recursively replaces occurrences of in by some algebra a. The placement of the recursive call for a given structure μf is determined by the definition $fmap$.

We will not employ the uniqueness property in an example proof just yet. In fact the uniqueness property is rarely used in its raw form; instead, there are a number of specific forms that we will introduce now.

Reflection law If we set h to the identity $h := id$, and a to the initial algebra $a := in$, we obtain the *reflection law*

$$(\!|in|\!) = id \quad . \tag{4}$$

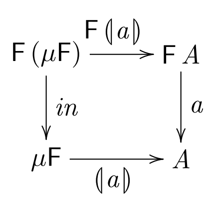
For the diagrammatically oriented, the following commutative diagram illustrates the above equation. Where possible, we will present diagrams along side equations, and we leave it to the reader to choose what suits them best.



Earlier we described $(\!|-|\!)$ as replacing constructors by functions. Here we are replacing the constructors with themselves, which clearly must be the identity.

Computation law If we substitute the left-hand side of the uniqueness property into the right-hand side, then we obtain the *computation law*: $(\!|a|\!) : in \to a$, or expressed in terms of the base category,

$$(\!|a|\!) \cdot in = a \cdot \mathsf{F}\,(\!|a|\!) \quad . \tag{5}$$

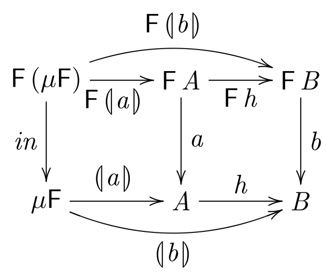


If we think of the functor F as describing the structure of a recursive datatype, then the computation law says that folding the substructures and then combining the result (with the algebra) is equal to folding over the whole structure.

Fusion law The most important consequence of the uniqueness property is the *fusion law*. It describes the fusion of an arrow with a fold to form a new fold.

$$h \cdot (\!|a|\!) = (\!|b|\!) \quad \Longleftarrow \quad h : a \to b \quad \Longleftrightarrow \quad h \cdot a = b \cdot \mathsf{F}\,h \tag{6}$$

Again, this equation can be illustrated as a commutative diagram, where fusion law is shown along the bottom, and the condition that h is a homomorphism is show in the right hand sub-diagram.

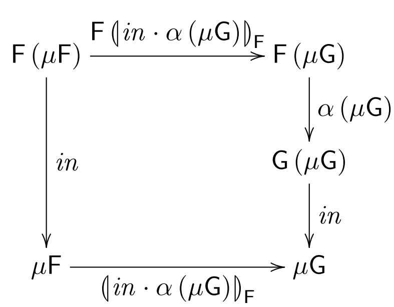


As its name would suggest, the fusion law is closely related to the program transformation techniques described in the introduction. It allows a fold to absorb a function on its left, thereby producing a single fold. The law also shows the difficulty of mechanising this process; in order to produce the fused program, we must invent a new algebra b that satisfies the precondition.

Functor fusion law Folds enjoy an additional fusion law. Whereas fusion allows us to absorb an additional function on the left, the *functor fusion law* allows us to absorb a function on the right. In order to formulate it, we have to turn μ into a higher-order functor of type $\mathbb{C}^{\mathbb{C}} \to \mathbb{C}$. The object part of this functor maps a functor to its initial algebra. (This is a bit loose as this is only well defined for functors that have an initial algebra. We noted before, we must assume that $\mathbb{C}$ is ω-cocomplete; $\mathbb{C}^{\mathbb{C}}$ is then the subcategory of ω-cocontinuous endofunctors.) The arrow part maps a natural transformation $\alpha : \mathsf{F} \dot{\to} \mathsf{G}$ to an arrow $\mu\alpha : \mu\mathsf{F} \to \mu\mathsf{G} : \mathbb{C}$. It is defined as

$$\mu\alpha \;=\; (\!|\, in \cdot \alpha\,(\mu\mathsf{G}) \,|\!) \;. \tag{7}$$

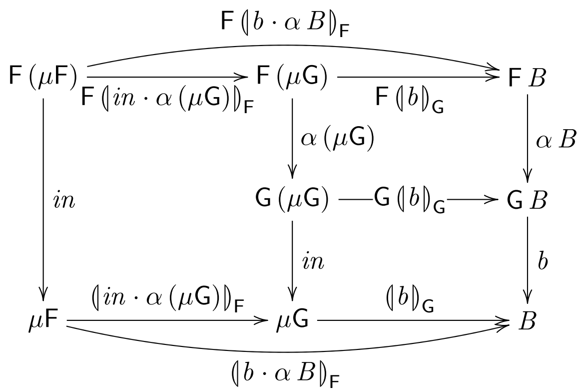
We can illustrate the definition of μ as a higher-order functor with the following diagram.



To reduce clutter, we will henceforth omit the argument of the natural transformation α. From these definitions we obtain the *functor fusion law* (we have annotated $(\!|-|\!)$ with the underlying functors):

$$(\!|b|\!)_{\mathsf{G}} \cdot \mu\alpha \;=\; (\!|b \cdot \alpha|\!)_{\mathsf{F}} \;. \tag{8}$$

The corresponding diagram appears to be complex, however, it is comprised of three simple sub-diagram: the left hand sub-diagram is the diagram we defined above for μ as a functor; the top right sub-diagram is the naturality of α; and the bottom right sub-diagram is the fold of the algebra b.



The functor fusion law states that a fold after a map can be fused into a single fold—the map $\mu\alpha$ can be seen as a 'base changer'.

To establish functor fusion we reason

$$
\begin{array}{ll}
 & \llparenthesis b \rrparenthesis_{\mathsf{G}} \cdot \mu\alpha = \llparenthesis b \cdot \alpha \rrparenthesis_{\mathsf{F}} \\
\iff & \{ \text{ definition of } \mu \ \} \\
 & \llparenthesis b \rrparenthesis_{\mathsf{G}} \cdot \llparenthesis in \cdot \alpha \rrparenthesis_{\mathsf{F}} = \llparenthesis b \cdot \alpha \rrparenthesis_{\mathsf{F}} \\
\Leftarrow & \{ \text{ fusion (6) } \} \\
 & \llparenthesis b \rrparenthesis_{\mathsf{G}} : in \cdot \alpha \to b \cdot \alpha : \mathbf{Alg}(\mathsf{F}) \\
\Leftarrow & \{ \ \alpha \text{ natural and (2) } \} \\
 & \llparenthesis b \rrparenthesis_{\mathsf{G}} : in \to b : \mathbf{Alg}(\mathsf{G}) \ .
\end{array}
$$

Given these prerequisites, it is straightforward to show that μ preserves identity

$$
\begin{array}{ll}
 & \mu id \\
= & \{ \text{ definition of } \mu \text{ (7) } \} \\
 & \llparenthesis in \cdot id \rrparenthesis \\
= & \{ \text{ identity and reflection } \} \\
 & id
\end{array}
$$

and composition

$$
\begin{array}{ll}
 & \mu\beta \cdot \mu\alpha \\
= & \{ \text{ definition of } \mu \text{ (7) } \} \\
 & \llparenthesis in \cdot \beta \rrparenthesis \cdot \mu\alpha \\
= & \{ \text{ functor fusion (8) } \} \\
 & \llparenthesis in \cdot \beta \cdot \alpha \rrparenthesis \\
= & \{ \text{ definition of } \mu \text{ (7) } \} \\
 & \mu(\beta \cdot \alpha) \ .
\end{array}
$$

We can also provide a Haskell definition of μ as a functor. The action on data is given by its datatype declaration. The action on functions is given by (7):

$$
\begin{array}{l}
\mu : (\mathit{Functor}\, f) \Rightarrow (\forall\, a \,.\, f\ a \to g\ a) \to (\mu f \to \mu g) \\
\mu\alpha = \llparenthesis in \cdot \alpha \rrparenthesis
\end{array}
$$

Note that we use a rank-2 polymorphic type to express the idea that μ maps natural transformation from f to g to a function between their fixpoints.

Finally, the initial algebra $\mu\mathsf{F}$ is the least fixed point of F—this is known as Lambek's Lemma [14]. One direction of the isomorphism $\mathsf{F}\,(\mu\mathsf{F}) \cong \mu\mathsf{F}$ is given by in, its inverse is $in^{\circ} = \llparenthesis \mathsf{F}\, in \rrparenthesis$. Lambek's Lemma is the key to giving a semantics to recursively defined datatypes. To illustrate, consider that the recursive definition of lists of natural numbers in Haskell

$$\textbf{data}\,\mathsf{List} = \mathit{Nil} \mid \mathit{Cons}\,(\mathbb{N}, \mathsf{List})$$

implicitly defines an underlying functor $\mathsf{L}\,X = 1 + \mathbb{N} \times X$, the so-called base functor of List. (This notation is a categorical rendering of sum-of-products algebraic datatypes, and defines a functor L with an argument X, where 1 denotes the terminal object of the underlying category). Since the initial object $\mu\mathsf{L}$ satisfies the equation $X \cong \mathsf{L}\,X$, we can use it to assign meaning to the recursive datatype definition.

As an aside, the fold of the List datatype is a specialisation of Haskell's library function *foldr*:

$$\begin{aligned} &\mathit{foldr} : (\mathbb{N} \to a \to a) \to a \to \mathsf{List} \to a \\ &\mathit{foldr}\; c\; n\; \mathit{Nil} = n \\ &\mathit{foldr}\; c\; n\; (\mathit{Cons}\,(a, as)) = c\; a\; (\mathit{foldr}\; c\; n\; as) \ . \end{aligned}$$

The function *foldr* replaces each of the List constructors, which constitute in, with c and n, which together constitute an algebra. (The type of algebras $\mathsf{L}\,A \to A$ is isomorphic to the type of the arguments $A \times (\mathbb{N} \times A \to A)$.)

To establish the isomorphism we must show that $in \cdot \llparenthesis \mathsf{F}\, in \rrparenthesis = id$,

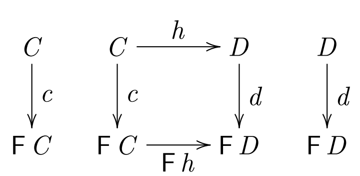
$$\begin{aligned} & in \cdot \llparenthesis \mathsf{F}\, in \rrparenthesis = id \\ \iff\ & \{ \text{reflection} \} \\ & in \cdot \llparenthesis \mathsf{F}\, in \rrparenthesis = \llparenthesis in \rrparenthesis \\ \Longleftarrow\ & \{ \text{fusion (6)} \} \\ & in : \mathsf{F}\, in \to in \ , \end{aligned}$$

and $\llparenthesis \mathsf{F}\, in \rrparenthesis \cdot in = id$,

$$\begin{aligned} & \llparenthesis \mathsf{F}\, in \rrparenthesis \cdot in \\ =\ & \{ \text{computation} \} \\ & \mathsf{F}\, in \cdot \mathsf{F}\, \llparenthesis \mathsf{F}\, in \rrparenthesis \\ =\ & \{ \mathsf{F}\ \text{functor} \} \\ & \mathsf{F}\,(in \cdot \llparenthesis \mathsf{F}\, in \rrparenthesis) \\ =\ & \{ \text{see above} \} \\ & \mathsf{F}\, id \\ =\ & \{ \mathsf{F}\ \text{functor} \} \\ & id \ . \end{aligned}$$

The initial F-algebra is the least solution of the equation $X \cong \mathsf{F}\,X$. If we dualise the development above, we obtain another canonical solution, namely the greatest one. In category theory, dualisation is denoted by the prefix 'co-'.

An F-*coalgebra* is a pair $\langle C, c\rangle$ consisting of an object $C : \mathbb{C}$ and an arrow $c : C \to \mathsf{F}\,C : \mathbb{C}$. An F-*coalgebra homomorphism* between coalgebras $\langle C, c\rangle$ and $\langle D, d\rangle$ is an arrow $h : C \to D : \mathbb{C}$ such that $\mathsf{F}\,h \cdot c = d \cdot h$.



Coalgebras and coalgebra homomorphisms also form a category, called $\mathbf{Coalg}(\mathsf{F})$. The dual of the initial algebra is the final coalgebra, whose carrier $\nu\mathsf{F}$ is the greatest fixed point of F. Finality means that, for any other coalgebra, there is a unique arrow from it to the final coalgebra. Whereas a fold consumes a data structure, an unfold produces some data structure from a given seed.

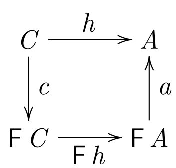
Unfortunately, least and greatest fixed points are different beasts in general. In the category **Set** of sets and total functions, $\mu\mathsf{L}$ is the set of finite lists, whereas $\nu\mathsf{L}$ also contains infinite lists. This means that folds and unfolds are incompatible, in general. In the following section we will focus on a restricted species of coalgebras, enabling us to work with folds and unfolds under the same roof.

3 Recursive Coalgebras

In this section we will introduce *recursive* coalgebras. We follow the work of Capretta et al. [3], who motivate the use of hylomorphisms based on recursive coalgebras as a structured recursion scheme. We shall continue to parallel our developments with examples in Haskell.

A coalgebra $\langle C, c\rangle$ is called *recursive* if for *every* algebra $\langle a, A\rangle$ the equation in the unknown $h : A \leftarrow C$,

$$h \;=\; a \cdot \mathsf{F}\,h \cdot c \;, \tag{9}$$



has a *unique* solution. The equation captures the *divide-and-conquer* pattern of computation: a problem is divided into sub-problems (c), the sub-problems are solved recursively ($\mathsf{F}\,h$), and finally the sub-solutions are combined into a single solution (a). The uniquely defined function h is called a *hylomorphism* or *hylo* for short and is written $\llparenthesis a \leftarrow c \rrparenthesis_{\mathsf{F}} : A \leftarrow C$. The notation is meant to suggest that h takes a coalgebra to an algebra, with the type $A \leftarrow C$ mirroring the hylo notation. We omit the subscripted functor name if it is obvious from the context. Uniqueness of h is captured by the following property.

$$h = \llparenthesis a \leftarrow c \rrparenthesis \quad\iff\quad h = a \cdot \mathsf{F}\,h \cdot c \tag{10}$$

In Haskell, $\llparenthesis - \leftarrow - \rrparenthesis$ becomes a function that takes an algebra and a recursive coalgebra as arguments and returns resulting hylo according to the definition in the universal property:

$$\llparenthesis - \leftarrow - \rrparenthesis : (\mathit{Functor}\, f) \Rightarrow (f\ a \to a) \to (c \to f\ c) \to (c \to a)$$
$$\llparenthesis a \leftarrow c \rrparenthesis = a \cdot \mathit{fmap}\,\llparenthesis a \leftarrow c \rrparenthesis \cdot c \ .$$

This function takes an algebra and a recursive coalgebra, yielding a hylo. Note that the type of this function does not guarantee that c is a *recursive* coalgebra and therefore does not guarantee that the resulting hylo has a unique solution; the programmer needs to discharge this obligation by some other means.

The category of recursive coalgebras and coalgebra homomorphisms forms a full subcategory of **Coalg**(F), called **Rec**(F). (See Appendix A for a the definition of a subcategory.) If the subcategory **Rec**(F) has a final object $\langle F, \mathit{out}\rangle$, then there is a unique arrow from any other *recursive* coalgebra $\langle C, c\rangle$ to $\langle F, \mathit{out}\rangle$. This arrow, called *unfold*, is written $\llbracket c \rrbracket : c \to \mathit{out}$. Finality is captured by the following uniqueness property.

$$h = \llbracket c \rrbracket \quad\iff\quad h : c \to \mathit{out} \quad\iff\quad \mathsf{F}\,h \cdot c = \mathit{out} \cdot h \tag{11}$$

This is the usual property of unfolds, except that we are working in the category **Rec**(F), *not* **Coalg**(F). As with folds, we can draw out a Haskell definition of unfolds from the uniqueness property:

$$\llbracket - \rrbracket : (\mathit{Functor}\, f) \Rightarrow (c \to f\ c) \to (c \to \mu f)$$
$$\llbracket c \rrbracket = \mathit{in} \cdot \mathit{fmap}\,\llbracket c \rrbracket \cdot c \ .$$

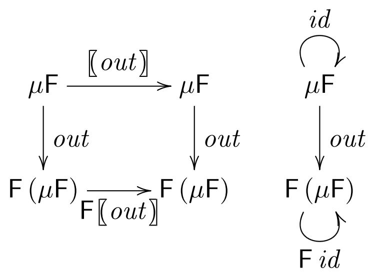
In contrast to folds, we are *creating* a structure of type μf from a seed value. The recursion, similarly, is determined by the form of the underlying functor f through the use of *fmap*.

Just as for folds, the uniqueness property implies the reflection, computation and fusion laws.

Reflection law If we set h to the identity $h := id$, and c to the final recursive coalgebra $c := out$, we obtain the *reflection law*

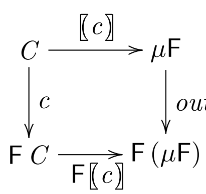
$$[\![out]\!] = id \ . \tag{12}$$

The duality of algebras and coalgebras is readily apparent in the diagrams: compare the following diagram to the one for the fold reflection law.



Computation law If we substitute the left-hand side of the uniqueness property into the right-hand side, then we obtain the *computation law*: $[\![c]\!] : c \to out$, or expressed in terms of the base category,

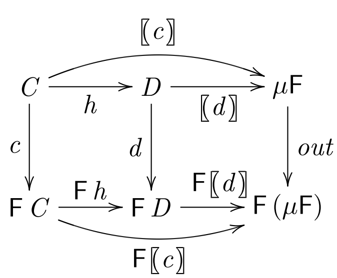
$$\mathsf{F}[\![c]\!] \cdot c = out \cdot [\![c]\!] \ . \tag{13}$$



Fusion law This time the fusion law describes the fusion of an arrow with an unfold to form a new unfold.

$$[\![c]\!] = [\![d]\!] \cdot h \quad \Longleftarrow \quad h : c \to d \quad \Longleftrightarrow \quad \mathsf{F}\, h \cdot c = d \cdot h \ . \tag{14}$$

Similarly to the fusion law for folds, in this case the law can be found along the top edge of the following diagram, with the left hand sub-diagram showing the precondition on h.



We have seen in Section 2 that *in* has an inverse—Lambek's Lemma. If $\langle F, out \rangle$ is the final recursive coalgebra, then *out* has an inverse, as well:

Lemma 1. *A recursive coalgebra is final if and only if it is invertible: (1) If $\langle F, out \rangle$ is the final recursive coalgebra, then out is invertible with $out^{\circ} = [\![\mathsf{F}\, out]\!]$. (2) If $\langle C, c \rangle$ is a recursive coalgebra and c is invertible, then $\langle C, c \rangle$ is final. Furthermore, $[\![d]\!] = (\!| c^{\circ} \leftarrow d |\!)$.*

For the proof of Lemma 1 we require:

Lemma 2. *Let* $\langle C, c\rangle$ *be a recursive* F*-coalgebra. Then* $\langle \mathsf{F}\,C, \mathsf{F}\,c\rangle$ *is also recursive.*

Proof. We have to show that $h = a \cdot \mathsf{F}\,h \cdot \mathsf{F}\,c$ has a solution, and furthermore, that it is unique. We shall accomplish this concurrently.

$$
\begin{array}{ll}
 & h = a \cdot \mathsf{F}\,h \cdot \mathsf{F}\,c \\
\Longleftrightarrow & \{\ \mathsf{F}\ \text{functor}\ \} \\
 & h = a \cdot \mathsf{F}\,(h \cdot c) \\
\Longleftrightarrow & \{\ \text{logic and}\ f = g \Longrightarrow f \cdot c = g \cdot c\ \} \\
 & h = a \cdot \mathsf{F}\,(h \cdot c) \quad \text{and} \quad h \cdot c = a \cdot \mathsf{F}\,(h \cdot c) \cdot c \\
\Longleftrightarrow & \{\ \text{uniqueness property and assumption: } c \text{ is recursive}\ \} \\
 & h = a \cdot \mathsf{F}\,(h \cdot c) \quad \text{and} \quad h \cdot c = \llparenthesis a \leftarrow c \rrparenthesis \\
\Longleftrightarrow & \{\ \text{Leibniz}\ \} \\
 & h = a \cdot \mathsf{F}\,\llparenthesis a \leftarrow c \rrparenthesis \quad \text{and} \quad h \cdot c = \llparenthesis a \leftarrow c \rrparenthesis \\
\Longleftrightarrow & \{\ \text{computation}\ \} \\
 & h = a \cdot \mathsf{F}\,\llparenthesis a \leftarrow c \rrparenthesis \quad \text{and} \quad h \cdot c = a \cdot \mathsf{F}\,\llparenthesis a \leftarrow c \rrparenthesis \cdot c \\
\Longleftrightarrow & \{\ \text{logic and}\ f = g \Longrightarrow f \cdot c = g \cdot c\ \} \\
 & h = a \cdot \mathsf{F}\,\llparenthesis a \leftarrow c \rrparenthesis \ \Box
\end{array}
$$

Proof (of Lemma 1).

1. First of all, $\llbracket \mathsf{F}\,out \rrbracket$ is well-defined, since $\mathsf{F}\,out$ is recursive, by Lemma 2. To establish the isomorphism, we must show that $id = \llbracket \mathsf{F}\,out \rrbracket \cdot out$,

$$
\begin{array}{ll}
 & id = \llbracket \mathsf{F}\,out \rrbracket \cdot out \\
\Longleftrightarrow & \{\ \text{unfold reflection}\ \} \\
 & \llbracket out \rrbracket = \llbracket \mathsf{F}\,out \rrbracket \cdot out \\
\Longleftarrow & \{\ \text{unfold fusion}\ \} \\
 & out : out \to \mathsf{F}\,out\ ,
\end{array}
$$

and $id = out \cdot \llbracket \mathsf{F}\,out \rrbracket$,

$$
\begin{array}{ll}
 & out \cdot \llbracket \mathsf{F}\,out \rrbracket \\
= & \{\ \text{unfold computation}\ \} \\
 & \mathsf{F} \llbracket \mathsf{F}\,out \rrbracket \cdot \mathsf{F}\,out \\
= & \{\ \mathsf{F}\ \text{functor}\ \} \\
 & \mathsf{F}\,(\llbracket \mathsf{F}\,out \rrbracket \cdot out) \\
= & \{\ \text{see above}\ \} \\
 & \mathsf{F}\,id \\
= & \{\ \mathsf{F}\ \text{functor}\ \} \\
 & id\ .
\end{array}
$$

2. We have to prove the uniqueness property of unfolds (11) with $out := c$ and $\llbracket d \rrbracket := \llparenthesis c^{\circ} \leftarrow d \rrparenthesis$.

$$
\begin{array}{ll}
 & h = \llparenthesis c^{\circ} \leftarrow d \rrparenthesis \Longleftrightarrow \mathsf{F}\,h \cdot d = c \cdot h \\
\Longleftrightarrow & \{\ \text{inverses}\ \} \\
 & h = \llparenthesis c^{\circ} \leftarrow d \rrparenthesis \Longleftrightarrow c^{\circ} \cdot \mathsf{F}\,h \cdot d = h
\end{array}
$$

The latter equivalence is an instance of the uniqueness property of hylos (10). □

The definition of a hylomorphism does not assume that the initial F-algebra exists. The powerset functor, for instance, admits no fixed points, yet we may want to divide a problem into a *set* of sub-problems. However, if the initial algebra exists, then it coincides with the final recursive coalgebra and, furthermore, folds and unfolds emerge as special cases of hylos. We can state this more formally:

Theorem 1. *Initial* F*-algebras and final recursive* F*-coalgebras coincide: (1) If* $\langle C, out\rangle$ *is the final recursive* F*-coalgebra, then* $\langle out^{\circ}, C\rangle$ *is the initial* F*-algebra. Furthermore,* $(\!| a |\!) = (\!| a \leftarrow out |\!)$*. (2) If* $\langle in, A\rangle$ *is the initial* F*-algebra, then* $\langle A, in^{\circ}\rangle$ *is the final recursive* F*-coalgebra. Furthermore,* $[\![c]\!] = (\!| in \leftarrow c |\!)$*.*

Proof (of Theorem 1).

1. By Lemma 1–(1) *out* has an inverse. We have to prove the uniqueness property of folds (3) with $in := out^{\circ}$ and $(\!| a |\!) := (\!| a \leftarrow out |\!)$.

$$
\begin{aligned}
& h = (\!| a \leftarrow out |\!) \iff h \cdot out^{\circ} = a \cdot \mathsf{F}\, h \\
\iff & \quad \{ \text{ inverses } \} \\
& h = (\!| a \leftarrow out |\!) \iff h = a \cdot \mathsf{F}\, h \cdot out
\end{aligned}
$$

   The latter equivalence is an instance of the uniqueness property of hylos (10).

2. We first have to show that $\langle A, in^{\circ}\rangle$ is recursive.

$$
\begin{aligned}
& h = a \cdot \mathsf{F}\, h \cdot in^{\circ} \\
\iff & \quad \{ \text{ inverses } \} \\
& h \cdot in = a \cdot \mathsf{F}\, h \\
\iff & \quad \{ \text{ uniqueness property of folds (3) } \} \\
& h = (\!| a |\!)
\end{aligned}
$$

   The statement then follows from Lemma 1–(2) with $c := in^{\circ}$. □

Theorem 1 allows us to treat folds and unfolds in the same setting—note that an unfold produces an element of an initial algebra! An alternative is to work in a setting where $\mu\mathsf{F}$ and $\nu\mathsf{F}$ coincide; an *algebraically compact* category is such a setting [10]. Haskell's ambient category $\mathbf{Cpo}_{\perp}$ serves as the standard example. This is the usual approach [9], however, the downside is that the hylo equation (9) only has a canonical, least solution, not a unique solution, so (10) does not hold.

4 Calculational Properties

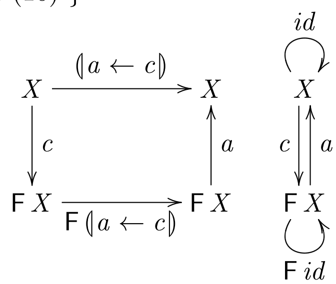
In this section we will cover the calculational properties of our hylomorphisms. In a similar fashion to folds and unfolds, hylomorphisms have an identity law and a computation law, and they follow similarly from the uniqueness property (10).

Identity law If we set h in (10) to the identity $h := id$, we obtain the *identity law*

$$
(\!| a \leftarrow c |\!) = id \quad \iff \quad a \cdot c = id \ . \tag{15}
$$

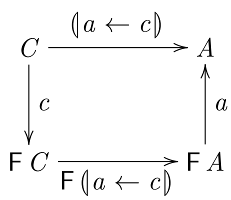
Proof.

$$
\begin{aligned}
& (\!|a \leftarrow c|\!) = id \\
\iff & \quad \{\ \text{uniqueness property of hylos (10)}\ \} \\
& id = a \cdot \mathsf{F}\, id \cdot c \\
\iff & \quad \{\ \mathsf{F}\ \text{functor}\ \} \\
& id = a \cdot c \quad \square
\end{aligned}
$$



Computation law If we substitute the left-hand side of the uniqueness property into the right-hand side, then we obtain the *computation law*:

$$(\!|a \leftarrow c|\!) \quad = \quad a \cdot \mathsf{F}\,(\!|a \leftarrow c|\!) \cdot c \ . \tag{16}$$

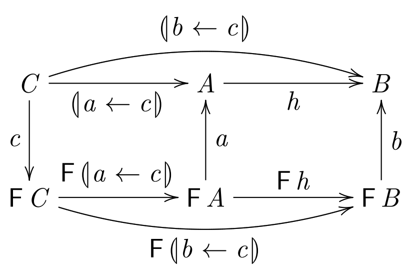


For hylomorphisms, we have *three* fusion laws: algebra fusion, coalgebra fusion, and composition.

Algebra fusion An algebra homomorphism after a hylo can be fused to form a single hylo.[2]

$$h \cdot (\!|a \leftarrow c|\!) = (\!|b \leftarrow c|\!) \quad \Longleftarrow \quad h : a \to b \quad \iff \quad h \cdot a = b \cdot \mathsf{F}\, h \tag{17}$$

The corresponding diagram is very similar to that of the fold fusion law.



For the proof we appeal to the uniqueness property.

$$
\begin{aligned}
& h \cdot (\!|a \leftarrow c|\!) = (\!|b \leftarrow c|\!) \\
\iff & \quad \{\ \text{uniqueness property of hylos (10)}\ \} \\
& h \cdot (\!|a \leftarrow c|\!) = b \cdot \mathsf{F}\,(h \cdot (\!|a \leftarrow c|\!)) \cdot c
\end{aligned}
$$

[2] Note that h appears as both an algebra homomorphism in $\mathbf{Alg}(\mathsf{F})$ and as the underlying arrow in the underlying category.

The obligation is discharged as follows:

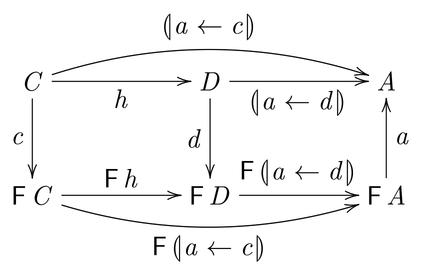
$$
\begin{aligned}
& h \cdot (\!| a \leftarrow c |\!) \\
= \; & \{ \text{ hylo computation (16) } \} \\
& h \cdot a \cdot \mathsf{F}\,(\!| a \leftarrow c |\!) \cdot c \\
= \; & \{ \text{ assumption: } h : a \to b \; \} \\
& b \cdot \mathsf{F}\,h \cdot \mathsf{F}\,(\!| a \leftarrow c |\!) \cdot c \\
= \; & \{ \; \mathsf{F} \text{ functor } \} \\
& b \cdot \mathsf{F}\,(h \cdot (\!| a \leftarrow c |\!)) \cdot c \;\; .
\end{aligned}
$$

As an aside, in the calculation the coalgebra c is totally passive.

Coalgebra fusion Dually, we can fuse a coalgebra homomorphism before a hylo to form a single hylo.

$$(\!| a \leftarrow c |\!) = (\!| a \leftarrow d |\!) \cdot h \quad \Longleftarrow \quad h : c \to d \quad \Longleftrightarrow \quad \mathsf{F}\,h \cdot c = d \cdot h \tag{18}$$

Again, the diagram is very similar to that of the unfold fusion law.

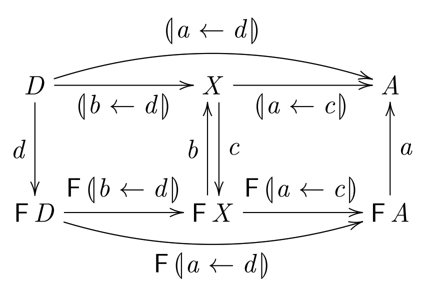


Like the law, the proof is the dual of that for algebra fusion.

Composition law A composition of hylos can be merged into a single one if the coalgebra of the hylo on the left inverts the algebra of the right hylo.

$$(\!| a \leftarrow c |\!) \cdot (\!| b \leftarrow d |\!) = (\!| a \leftarrow d |\!) \quad \Longleftarrow \quad c \cdot b = id \tag{19}$$

The following diagram illustrates the composition visually, as it is the composition of two hylo diagrams.



Composition is, in fact, a simple consequence of algebra fusion as the hylomorphism $(\!| a \leftarrow c |\!) : b \to a$ is simultaneously an F-algebra homomorphism.

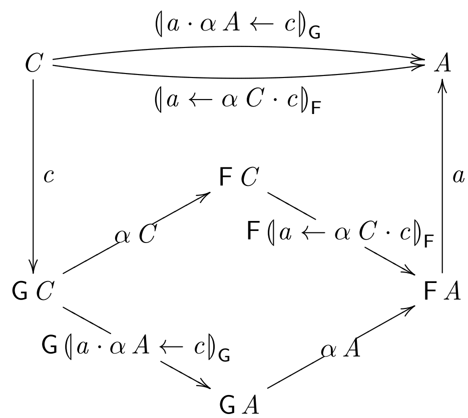
$$
\begin{aligned}
& (\!| a \leftarrow c |\!) \cdot b \\
= \; & \{ \text{ hylo computation (16) } \} \\
& a \cdot \mathsf{F}\,(\!| a \leftarrow c |\!) \cdot c \cdot b \\
= \; & \{ \text{ assumption: } c \cdot b = id \; \} \\
& a \cdot \mathsf{F}\,(\!| a \leftarrow c |\!)
\end{aligned}
$$

Alternatively, we can derive the composition law from coalgebra fusion by showing that $\llparenthesis b \leftarrow d \rrparenthesis : d \to c$ is an F-coalgebra homomorphism. The composition law, together with the next law, generalises the functor fusion law of folds.

Hylo shift law or base change law If we have a natural transformation $\alpha : \mathsf{G} \dot{\to} \mathsf{F}$, then

$$\llparenthesis a \cdot \alpha\, A \leftarrow c \rrparenthesis_{\mathsf{G}} \;=\; \llparenthesis a \leftarrow \alpha\, C \cdot c \rrparenthesis_{\mathsf{F}} \;. \tag{20}$$

Like the diagram for the composition law, the following diagram is also comprised of two hylo diagrams, however, in this case they are intertwined with the natural transformation α. (The naturality square has been turned into a naturality diamond.)



In fact, the statement can be strengthened: if c is recursive, then $\alpha\, C \cdot c$ is recursive, as well.

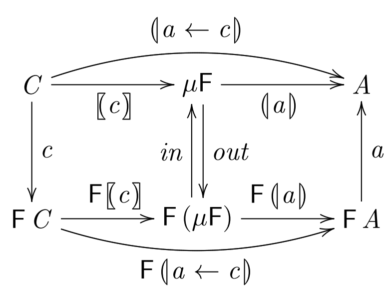
$$\begin{aligned}
& h = a \cdot \mathsf{F}\, h \cdot \alpha\, C \cdot c \\
\iff\; & \{\ \alpha \text{ natural}\ \} \\
& h = a \cdot \alpha\, A \cdot \mathsf{G}\, h \cdot c \\
\iff\; & \{\ \text{uniqueness property of hylos (10)}\ \} \\
& h = \llparenthesis a \cdot \alpha\, A \leftarrow c \rrparenthesis_{\mathsf{G}}
\end{aligned}$$

It is worth pointing out that the laws stated thus far are independent of the existence of initial algebras. Only the following law makes this assumption.

Fold/unfold law A fold after an unfold is a hylo.

$$\llparenthesis a \rrparenthesis \cdot [\![c]\!] \;=\; \llparenthesis a \leftarrow c \rrparenthesis \tag{21}$$

In the following diagrams, a fold and an unfold diagram are juxtaposed to form a hylo diagram.



From left to right we are performing fusion and thus deforesting an intermediate data structure. From right to left we are turning a control structure into a data structure. The *fold/unfold law* is a direct consequence of Theorem 1 and any of the fusion laws.

5 Fusion

In the previous sections, we have presented fusion laws and demonstrated their use in proofs. In this section, we will show how these laws can help us formalise a collection of fusion techniques that we collectively brand *shortcut fusion*. These techniques share the common characteristic of standardising the way data structures are recursively consumed and produced. Program transformations rewrite syntactically explicit instances of data structure consumption immediately followed by production. This approach has the benefit of allowing us to target fusible functions without affecting other parts of the program. Where shortcut fusion techniques differ is in their choice of recursion scheme. By using recursive coalgebras, we can clearly lay out and compare these approaches within the *same* framework. (Previously these recursion schemes were only compatible for analysis by restricting the working category to one that is algebraically compact, such as $\mathbf{Cpo}_{\bot}$.) This allows us to examine the relationships among these fusion approaches, which are not readily apparent when examining their individual implementations.

5.1 Warm-up: Type Functors

We have seen in Section 2 that μ is a functor, whose action on arrows is defined $\mu\alpha = (\!|in \cdot \alpha|\!)$. Using Theorem 1 and the hylo shift law (20) we can actually express $\mu\alpha$ as a fold, an unfold or a hylo.

$$\mu\alpha \;=\; (\!|in \cdot \alpha|\!) \;=\; (\!|in \cdot \alpha \leftarrow out|\!) \;=\; (\!|in \leftarrow \alpha \cdot out|\!) \;=\; [\![\alpha \cdot out]\!] \;.$$

As a warm-up let us show that μ preserves identity

$$\begin{aligned}
&\mu id \\
= &\quad \{ \text{ definition of } \mu \text{ } \} \\
&(\!|in \leftarrow out|\!) \\
= &\quad \{ \text{ identity (15) and } in \cdot out = id \text{ } \} \\
&id \;,
\end{aligned}$$

and composition

$$\begin{aligned}
&\mu(\beta \cdot \alpha) \\
= &\quad \{ \text{ definition of } \mu \text{ } \} \\
&(\!|in \cdot \beta \cdot \alpha \leftarrow out|\!) \\
= &\quad \{ \text{ shift law (20) } \} \\
&(\!|in \cdot \beta \leftarrow \alpha \cdot out|\!) \\
= &\quad \{ \text{ composition (19) and } out \cdot in = id \text{ } \} \\
&(\!|in \cdot \beta \leftarrow out|\!) \cdot (\!|in \leftarrow \alpha \cdot out|\!) \\
= &\quad \{ \text{ definition of } \mu \text{ } \} \\
&\mu\beta \cdot \mu\alpha \;.
\end{aligned}$$

In Section 5.6 we shall see a key use of μ for stream fusion. For now, let us show a use of μ with the base functor of parameterized List

```
data L a b = Nil | Cons (a, b) .
```

This is a higher-order functor of type $\mathsf{L} : \mathbb{C} \to \mathbb{C}^{\mathbb{C}}$ that takes objects to functors and arrows to natural transformations. In Haskell, we can make this datatype an instance of the *Functor* class:

```
instance Functor (L a) where
  fmap f Nil           = Nil
  fmap f (Cons (a, b)) = Cons (a, f b) .
```

We define this instance for the functor obtained by applying L to some type a. Haskell allows us to define this polymorphically for all a. The list datatype defined in terms of its base functor is $\mathsf{List}\,A = \mu(\mathsf{L}\,A)$. The parametric type List is itself a functor, a so-called type functor, whose action on arrows is Haskell's *map* function, defined in this setting by $\mathsf{List}\,f = \mu(\mathsf{L}\,f)$. Note that μ expects a natural transformation and that L delivers one.

5.2 Warm-up: Simple Fusion

Using algebras and recursive coalgebras, we can readily express simple program transformations. For example, the program $sum \cdot between$ creates a list of integers and then sums them together. We can express this program as a fold after an unfold: $(\!|\mathfrak{s}|\!) \cdot [\![\mathfrak{b}]\!]$. The algebra $\mathfrak{s}$ and the *recursive* (we omit a proof of this fact) coalgebra $\mathfrak{b}$ are given by

$$\begin{aligned}
&\mathfrak{s} : \mathsf{L}\,\mathbb{N}\,\mathbb{N} \to \mathbb{N} \\
&\mathfrak{s}\,Nil = 0 \\
&\mathfrak{s}\,(Cons\,(x, y)) = x + y
\end{aligned}
\qquad
\begin{aligned}
&\mathfrak{b} : (\mathbb{N}, \mathbb{N}) \to \mathsf{L}\,\mathbb{N}\,(\mathbb{N}, \mathbb{N}) \\
&\mathfrak{b}\,(m, n) = \mathbf{if}\ m > n\ \mathbf{then}\ Nil \\
&\qquad\qquad\quad \mathbf{else}\ Cons\,(m, (m+1, n)) \ .
\end{aligned}$$

Expressed thus, an intermediate list is used to build the sequence of numbers and then consumed to calculate the sum. It would instead be more efficient to write a program that sums the numbers as they are produced. The existence of such a transformation is given by the *fold/unfold law* (21) which we can apply to obtain $(\!|\mathfrak{s}|\!) \cdot [\![\mathfrak{b}]\!] = (\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)$. We can derive the definition of the function that corresponds to this hylo:

$$\begin{aligned}
& (\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\,(m, n) \\
=\ & \{\ \text{unfold } (\!| - \leftarrow - |\!)\ \} \\
& \mathfrak{s}\,(fmap\,(\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\,(\mathfrak{b}\,(m, n))) \\
=\ & \{\ \text{unfold } \mathfrak{b}\ \} \\
& \mathbf{if}\ m > n\ \mathbf{then}\ \mathfrak{s}\,(fmap\,(\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\ Nil) \\
& \mathbf{else}\ \mathfrak{s}\,(fmap\,(\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\,(Cons\,(m, (m+1, n)))) \\
=\ & \{\ \text{unfold } fmap\ \} \\
& \mathbf{if}\ m > n\ \mathbf{then}\ \mathfrak{s}\ Nil \\
& \mathbf{else}\ \mathfrak{s}\,(Cons\,(m, (\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\,(m+1, n))) \\
=\ & \{\ \text{unfold } \mathfrak{s}\ \} \\
& \mathbf{if}\ m > n\ \mathbf{then}\ 0\ \mathbf{else}\ m + (\!|\mathfrak{s} \leftarrow \mathfrak{b}|\!)\,(m+1, n).
\end{aligned}$$

Therefore, the hylo corresponds to the function:

$$\begin{aligned}
& gau\text{ß} : (\mathbb{N}, \mathbb{N}) \to \mathbb{N} \\
& gau\text{ß}\,(m, n) = \mathbf{if}\ m > n\ \mathbf{then}\ 0\ \mathbf{else}\ m + gau\text{ß}\,(m+1, n) \ .
\end{aligned}$$

This example is one of many cases where the laws of algebras and recursive coalgebras correspond to program transformations in functional programming. We move on to consider more interesting examples in the following sections.

Termination Let us take a minor digression and explore how one might show that a coalgebra is recursive and discuss how this relates to termination.

We defined *between* as an unfold and gave $\mathfrak{b}$ as the coalgebra. In our setting, an unfold is only well-defined if the coalgebra is recursive, but we left this claim unproven. The chosen base category of the coalgebra has an impact on how one might go about proving its recursiveness. If we choose to work in **Set** and our base functors are polynomial, then unique solutions of the hylo equation (9) are the terminating solutions [1]. Therefore, to show that a coalgebra is recursive in this setting, it is sufficient to show that it defines a terminating computation; we can do so with the various tools available, such as well-founded relations, or measure functions. In the case of *between*, the measure function $\lambda(n, m) \to n + 1 - m$ is sufficient to show termination.

If we step outside the restriction of polynomial functors, then recursiveness no longer guarantees termination (a unique solution is not necessarily a terminating one) as the following example demonstrates (not valid Haskell). Define the **Set** functor

$$\mathit{Square}\,X = \mathit{Nothing} \mid \mathit{Just}\,\{\,(x_1, x_2) \mid x_1 \in X, x_2 \in X, x_1 \neq x_2\,\}$$

which gives the square of X with the diagonal removed. The type $\mathit{Square}\,\mathit{Bool}$, for instance, has three elements: $\mathit{Nothing}$, $\mathit{Just}\,(\mathit{False}, \mathit{True})$ and $\mathit{Just}\,(\mathit{True}, \mathit{False})$. The action on arrows is defined

$$\begin{aligned}
&\mathit{Square}\,f\,\mathit{Nothing} = \mathit{Nothing}\\
&\mathit{Square}\,f\,(\mathit{Just}\,(x_1, x_2))\\
&\quad \mid f\,x_1 \neq f\,x_2 \quad = \mathit{Just}\,(f\,x_1, f\,x_2)\\
&\quad \mid \mathit{otherwise} \quad = \mathit{Nothing}\ .
\end{aligned}$$

Note that Square preserves the invariant, possibly changing Just to $\mathit{Nothing}$. Now, define the coalgebra $c : \mathit{Bool} \to \mathit{Square}\,\mathit{Bool}$ by $c\,b = \mathit{Just}\,(\mathit{False}, \mathit{True})$. The coalgebra c is recursive as the equation

$$x = a \cdot \mathit{Square}\,x \cdot c$$

has the unique solution $x\,b = a\,\mathit{Nothing}$. (Since c is constant, x has to be constant, as well. For a constant x, the call $\mathit{Square}\,x$ yields $\mathit{Nothing}$.) If executed, however, $\mathit{Square}\,x$ will issue two recursive calls, $x\,\mathit{False}$ and $x\,\mathit{True}$.

5.3 Generalised *foldr*/*build* Fusion

We now move on to the main target of our new setting: shortcut fusion. The original shortcut fusion technique is a fold-centric approach called *foldr/build* fusion [11]. As its name would suggest, its original intention was to provide fusion for list functions written in terms of *foldr* and an additional combinator *build*. In this section, we will explore the foundations of this technique.

The mother of all fusion rules is algebra fusion (17). It allows us to fuse a hylo followed by an algebra homomorphism into a single hylo. It is similar to fold fusion in the sense that to use this law, we must construct a new algebra that satisfies a pre-condition. To illustrate, the pipeline $\mathit{sum} \cdot \mathit{filter}\,\mathit{odd}$ can be expressed as a composition of two folds: $(\!|\mathfrak{s}|\!) \cdot (\!|\mathfrak{f}|\!)$. The algebra $\mathfrak{s}$ is the one of Section 5.2 and the algebra $\mathfrak{f}$ is given by

$$\begin{aligned}
&\mathfrak{f} : \mathsf{L}\,\mathbb{N}\,(\mu(\mathsf{L}\,\mathbb{N})) \to \mu(\mathsf{L}\,\mathbb{N})\\
&\mathfrak{f}\,\mathit{Nil} \qquad\qquad\quad = \mathit{in}\,\mathit{Nil}\\
&\mathfrak{f}\,(\mathit{Cons}\,(x, y)) = \mathbf{if}\ \mathit{odd}\,x\ \mathbf{then}\ \mathit{in}\,(\mathit{Cons}\,(x, y))\ \mathbf{else}\ y\ .
\end{aligned}$$

Our intention is to fuse $(\!|\mathfrak{s}|\!) \cdot (\!|\mathfrak{f}|\!)$ using algebra fusion. Let us first remind ourselves of the algebra fusion law (17).

$$h \cdot (\!|a \leftarrow c|\!) = (\!|b \leftarrow c|\!) \quad \Longleftarrow \quad h : a \to b \quad \Longleftrightarrow \quad h \cdot a = b \cdot \mathsf{F}\,h$$

We are instantiating left-hand side of the fusion law with $h := (\!|\mathfrak{s}|\!)$ and $(\!|a \leftarrow c|\!) := (\!|\mathfrak{f}|\!)$, where (following Theorem 1) $a := \mathfrak{f}$ and $c := \mathit{out}$. To be able to apply algebra fusion , we have to show that $(\!|\mathfrak{s}|\!)$ is an algebra homomorphism from $\mathfrak{f}$ to some unknown algebra $\mathfrak{sf}$—the instantiation of $h : a \to b$, where $b := \mathfrak{sf}$. By hand, it is not hard to derive $\mathfrak{sf}$ so that $(\!|\mathfrak{s}|\!) \cdot \mathfrak{f} = \mathfrak{sf} \cdot \mathsf{F}\,(\!|\mathfrak{s}|\!)$.

$$\begin{aligned}
&\mathfrak{sf} : \mathsf{L}\,\mathbb{N}\,\mathbb{N} \to \mathbb{N}\\
&\mathfrak{sf}\,\mathit{Nil} \qquad\qquad\quad = \mathfrak{s}\,\mathit{Nil}\\
&\mathfrak{sf}\,(\mathit{Cons}\,(x, y)) = \mathbf{if}\ \mathit{odd}\,x\ \mathbf{then}\ \mathfrak{s}\,(\mathit{Cons}\,(x, y))\ \mathbf{else}\ y
\end{aligned}$$

Since $(\!|\mathfrak{s}|\!)$ replaces in by $\mathfrak{s}$, we simply have to replace the occurrences of in in $\mathfrak{f}$ by $\mathfrak{s}$. While this is an easy task to perform by hand, it is potentially difficult to mechanise as it requires analysis of the body of $\mathfrak{f}$; within it, the constructor in could easily have any name and conversely any function could be named in. Also, $\mathfrak{f}$ could contain unrelated occurrences of in. This transformation is therefore not purely syntactic, but also involves some further analysis of the source program; this is not an approach we wish to pursue.

The central idea of *foldr/build* fusion is to expose *in* so that replacing it by the algebra a is simple to implement. Consider fold fusion (6) again.

$$h \cdot (\!|a|\!) = (\!|b|\!) \quad \Longleftarrow \quad h : a \to b$$

A fold $(\!|-|\!)$ is a transformation that turns an algebra into a homomorphism. Assume that we have another such transformation, say, β that satisfies

$$h \cdot \beta\, a = \beta\, b \quad \Longleftarrow \quad h : a \to b\ . \tag{22}$$

The generalisation of *foldr/build* from lists to arbitrary datatypes, the so-called *acid rain rule* [20], is then

$$(\!|a|\!) \cdot \beta\, in \;=\; \beta\, a\ . \tag{23}$$

Using β we expose *in* so that we can replace *in* by a simply by replacing β's argument. Instead of building a structure and then folding over it, we eliminate the *in* and pass a directly to β. The proof of correctness is painless.

$$\begin{aligned} & (\!|a|\!) \cdot \beta\, in = \beta\, a \\ \Longleftarrow\ & \{ \text{ assumption (22) } \} \\ & (\!|a|\!) : in \to a \end{aligned}$$

But, have we made any progress? After all, before we can apply (23), we have to prove (22). Fold satisfies this property, but this instance of (23) is trivial: $(\!|a|\!) \cdot (\!|in|\!) = (\!|a|\!)$. Now, it turns out that in a *relationally parametric* programming language [17], the proof obligation (22) amounts to the *free theorem* [22] of the polymorphic type

$$\beta : \forall A\ .\ (\mathsf{F}\, A \to A) \to (B \to A)\ , \tag{24}$$

where B is some fixed type. In other words, in such a language the proof obligation can be discharged by the type checker.

Returning to our example, we redefine *filter odd* as $(\lambda\, a\ .\ (\!|\phi\, a|\!))\ in$ where

$$\begin{aligned} & \phi : (\mathsf{L}\,\mathbb{N}\, b \to b) \to (\mathsf{L}\,\mathbb{N}\, b \to b) \\ & \phi\, a\, \mathit{Nil} \qquad\qquad\quad = a\, \mathit{Nil} \\ & \phi\, a\, (\mathit{Cons}\, (x, y)) = \textbf{if}\ \mathit{odd}\, x\ \textbf{then}\ a\, (\mathit{Cons}\, (x, y))\ \textbf{else}\ y\ . \end{aligned}$$

We derived ϕ from the algebra $\mathfrak{f}$ by abstracting away from *in*. The reader should convince herself that $\lambda\, a\ .\ (\!|\phi\, a|\!)$ has indeed the desired polymorphic type (24). We can then invoke the acid rain rule (23) to obtain

$$(\!|\mathfrak{s}|\!) \cdot (\lambda\, a\ .\ (\!|\phi\, a|\!))\ in = (\lambda\, a\ .\ (\!|\phi\, a|\!))\ \mathfrak{s} = (\!|\phi\, \mathfrak{s}|\!)\ .$$

The example also shows that the *acid rain* rule is somewhat unstructured in that a hylo is hidden inside the abstraction $\lambda\, a$. Without performing an additional beta-reduction, we can apply the rule only once. We obtain a more structured rule if we shift the abstraction to the algebra and achieve *cata-hylo fusion:* If τ is a transformation that takes F-algebras to G-algebras satisfying

$$h : \tau\, a \to \tau\, b : \mathbf{Alg}(\mathsf{G}) \quad \Longleftarrow \quad h : a \to b : \mathbf{Alg}(\mathsf{F})\ , \tag{25}$$

then

$$(\!|a|\!)_{\mathsf{F}} \cdot (\!|\tau\, in \leftarrow c|\!)_{\mathsf{G}} \;=\; (\!|\tau\, a \leftarrow c|\!)_{\mathsf{G}}\ . \tag{26}$$

If τ is $\lambda\, a\ .\ a$, then this is just the fold/unfold law (21): $(\!|a|\!) \cdot (\!|in \leftarrow c|\!) = (\!|a|\!) \cdot [\![c]\!] = (\!|a \leftarrow c|\!)$. For $\tau\, a = a \cdot \alpha$, this is essentially functor fusion (8)—note that (2) is an example of (25). The proof of correctness is straightforward.

$$
\begin{aligned}
& (\!|a|\!)_{\mathsf{F}} \cdot (\!|\tau\, in \leftarrow c|\!)_{\mathsf{G}} = (\!|\tau\, a \leftarrow c|\!)_{\mathsf{G}} \\
= \; & \{ \text{ algebra fusion (17) } \} \\
& (\!|a|\!)_{\mathsf{F}} : \tau\, in \to \tau\, a : \mathbf{Alg}(\mathsf{G}) \\
= \; & \{ \text{ assumption (25) } \} \\
& (\!|a|\!)_{\mathsf{F}} : in \to a : \mathbf{Alg}(\mathsf{F})
\end{aligned}
$$

The proof obligation (25) once again amounts to a theorem for free, this time of the polymorphic type [3]

$$\tau : \forall A\, .\, (\mathsf{F}\, A \to A) \to (\mathsf{G}\, A \to A) \ .$$

Using cata-hylo fusion, the running example simplifies to

$$(\!|\mathfrak{s}|\!) \cdot (\!|\phi\, in|\!) = (\!|\phi\, \mathfrak{s}|\!) \ .$$

We can now also fuse a composition of folds:

$$(\!|a|\!) \cdot (\!|\tau_1\, in|\!) \cdot \ldots \cdot (\!|\tau_n\, in|\!) \cdot [\![c]\!] = (\!|(\tau_n \cdot \ldots \cdot \tau_1)\, a \leftarrow c|\!) \ .$$

This demonstrates how the rewrite rule is able to achieve fusion over an entire pipeline of functions.

5.4 Generalised *destroy*/*unfoldr* Fusion

The *foldr*/*build* brand of shortcut fusion, and its generalisation to algebraic datatypes, is *fold-centric*. This limits the kind of functions that we can fuse, simply because some functions such as *zip* or *take* are not folds, or are not naturally written as folds. We can dualise *foldr*/*build* fusion to achieve an *unfold-centric* approach, called *destroy*/*unfoldr* [19]. To illustrate, consider the simple pipeline $take\, 5 \cdot between$, where $take\, n$ takes n elements (if available) from a list. It can be written as an unfold after an initialisation step: $take\, n = [\![\mathfrak{t}]\!] \cdot start\, n$, where $start\, n = (\lambda\, l\, .\, (n, l))$, and where the coalgebra $\mathfrak{t}$ is given by

$$
\begin{aligned}
& \mathbf{type}\, \mathsf{State}\, a = (\mathbb{N}, a) \\
& \mathfrak{t} : \mathsf{State}\, (\mu(\mathsf{L}\, a)) \to \mathsf{L}\, a\, (\mathsf{State}\, (\mu(\mathsf{L}\, a))) \\
& \mathfrak{t}\, (0, x) \qquad = Nil \\
& \mathfrak{t}\, (n+1, x) = \mathbf{case}\, out\, x\, \mathbf{of}\, Nil \to Nil; Cons\, (a, y) \to Cons\, (a, (n, y)) \ .
\end{aligned}
$$

Here we make explicit the notion that an unfold models the steps of a stateful computation. The coalgebra takes a state as an argument and uses it to produce a value and a new state. In this example, the state type pairs the input list with a natural number, enabling us to track the overall number of values produced. The number of elements to take, paired with the list where the values are to be taken from, forms the initial state.

We can dualise the *acid rain rule* to fuse the pipeline. If β is a transformation that satisfies

$$\beta\, c = \beta\, d \cdot h \quad \Longleftarrow \quad h : c \to d \ , \tag{27}$$

then

$$\beta\, c \;=\; \beta\, out \cdot [\![c]\!] \ . \tag{28}$$

Previously we exposed in, now we expose out. To apply the dual of acid rain we redefine $take\, n$ as $(\lambda\, c\, .\, [\![\gamma\, c]\!] \cdot start\, n)\, out$, where

$$
\begin{aligned}
& \gamma : (c \to \mathsf{L}\, a\, c) \to (\mathsf{State}\, c \to \mathsf{L}\, a\, (\mathsf{State}\, c)) \\
& \gamma\, c\, (0, x) \ = Nil \\
& \gamma\, c\, (n, x) = \mathbf{case}\, c\, x\, \mathbf{of}\, Nil \to Nil; Cons\, (a, y) \to Cons\, (a, (n-1, y)) \ .
\end{aligned}
$$

[3] The original formulation of cata-hylo fusion, by Takano and Meijer [20], unnecessarily requires F and G to be the same.

The transformation γ is derived from $\mathfrak{t}$ by abstracting away from *out*. We can now tackle our example:

$$(\lambda\, c\,.\, [\![\gamma\, c]\!] \cdot \mathit{start}\, 5)\; \mathit{out} \cdot [\![\mathfrak{b}]\!] = (\lambda\, c\,.\, [\![\gamma\, c]\!] \cdot \mathit{start}\, 5)\; \mathfrak{b} = [\![\gamma\, \mathfrak{b}]\!] \cdot \mathit{start}\, 5\ \ .$$

The proof obligation (27) corresponds to the free theorem of

$$\beta : \forall\, C\,.\, (C \to \mathsf{F}\, C) \to (C \to D)\ \ , \tag{29}$$

where D is fixed. And, indeed, $\lambda\, c\,.\, [\![\gamma\, c]\!] \cdot \mathit{start}\, 5$ has the required type.

Similarly, we can dualise our more structured *cata-hylo fusion* to achieve *hylo-ana fusion:* If τ is a transformation that takes recursive F-coalgebras to recursive G-coalgebras satisfying

$$h : \tau\, c \to \tau\, d : \mathbf{Rec}(\mathsf{G}) \quad \Longleftarrow \quad h : c \to d : \mathbf{Rec}(\mathsf{F})\ \ , \tag{30}$$

then

$$(\!| a \leftarrow \tau\, c |\!)_{\mathsf{G}} \;=\; (\!| a \leftarrow \tau\, \mathit{out} |\!)_{\mathsf{G}} \cdot [\![c]\!]_{\mathsf{F}}\ \ . \tag{31}$$

The proof of correctness follows similarly to the proof of cata-hylo fusion.

$$\begin{aligned}
&\quad (\!| a \leftarrow \tau\, c |\!)_{\mathsf{G}} = (\!| a \leftarrow \tau\, \mathit{out} |\!)_{\mathsf{G}} \cdot [\![c]\!]_{\mathsf{F}} \\
&= \quad \{\ \text{coalgebra fusion (18)}\ \} \\
&\quad [\![c]\!]_{\mathsf{F}} : \tau\, c \to \tau\, \mathit{out} : \mathbf{Rec}(\mathsf{G}) \\
&= \quad \{\ \text{assumption (30)}\ \} \\
&\quad [\![c]\!]_{\mathsf{F}} : c \to \mathit{out} : \mathbf{Rec}(\mathsf{F})
\end{aligned}$$

This time the proof obligation (30) cannot be discharged by the type checker alone as τ has to transform a *recursive* coalgebra into a *recursive* coalgebra! Note that this more structured rule is unable to handle our *take* example, as this function is actually a state initialisation followed by an unfold even though this a common method for defining functions as unfolds. We leave this as future work.

Our example has focused on fusing the list parameter of *take*, yet if we admit to the fact that natural numbers are an inductive datatype, then *take* is really a function that consumes *two* data structures. The aforementioned *zip* is another function that consumes two data structures, and therefore has the potential to be fused with both of these inputs. Let us employ the expression $\mathit{zip} \cdot (\mathit{between} \times \mathit{between})$ as another example that can be written in terms of unfolds: $[\![\mathfrak{z}]\!] \cdot ([\![\mathfrak{b}]\!] \times [\![\mathfrak{b}]\!])$. The algebra $\mathfrak{z}$ is given by

$$\begin{aligned}
&\mathfrak{z} : (\mu(\mathsf{L}\, a_1), \mu(\mathsf{L}\, a_2)) \to \mathsf{L}\, (a_1, a_2)\, (\mu(\mathsf{L}\, a_1), \mu(\mathsf{L}\, a_2)) \\
&\mathfrak{z}\, (x_1, x_2) = \mathbf{case}\, (\mathit{out}\, x_1, \mathit{out}\, x_2)\, \mathbf{of} \\
&\qquad\qquad (\mathit{Cons}\, (a_1, b_1), \mathit{Cons}\, (a_2, b_2)) \to \mathit{Cons}\, ((a_1, a_2), (b_1, b_2)) \\
&\qquad\qquad \mathit{otherwise} \qquad\qquad\qquad\qquad\quad\ \to \mathit{Nil}\ \ .
\end{aligned}$$

Our rules (28) and (31) are not applicable as we have *two* producers to the right of *zip*. Now, to fuse such a function, we need to employ *parallel hylo-ana fusion:* If τ satisfies,

$$\begin{aligned}
&h_1 \times h_2 : \tau\, (c_1, c_2) \to \tau\, (d_1, d_2) : \mathbf{Rec}(\mathsf{G}) \\
&\qquad \Longleftarrow \quad h_1 : c_1 \to d_1 : \mathbf{Rec}(\mathsf{F}_1)\ \wedge\ h_2 : c_2 \to d_2 : \mathbf{Rec}(\mathsf{F}_2)\ \ ,
\end{aligned} \tag{32}$$

then

$$(\!| a \leftarrow \tau\, (c_1, c_2) |\!)_{\mathsf{G}} \;=\; (\!| a \leftarrow \tau\, (\mathit{out}, \mathit{out}) |\!)_{\mathsf{G}} \cdot ([\![c_1]\!]_{\mathsf{F}_1} \times [\![c_2]\!]_{\mathsf{F}_2})\ \ . \tag{33}$$

The proof of correctness follows the follows the same steps as for cata-hylo and hylo-ana.

$$\begin{aligned}
&(\!|a \leftarrow \tau\,(c_1, c_2)|\!)_{\mathsf{G}} = (\!|a \leftarrow \tau\,(out, out)|\!)_{\mathsf{G}} \cdot ([\![c_1]\!]_{\mathsf{F}_1} \times [\![c_2]\!]_{\mathsf{F}_1})\\
= &\quad \{ \text{ coalgebra fusion (18) } \}\\
&([\![c_1]\!]_{\mathsf{F}_1} \times [\![c_2]\!]_{\mathsf{F}_1}) : \tau\,(c_1, c_2) \to \tau\,(out, out) : \mathbf{Rec}(\mathsf{G})\\
= &\quad \{ \text{ assumption (32) } \}\\
&[\![c_1]\!]_{\mathsf{F}_1} : c_1 \to out : \mathbf{Rec}(\mathsf{F}_1) \ \wedge\ [\![c_2]\!]_{\mathsf{F}_2} : c_2 \to out : \mathbf{Rec}(\mathsf{F}_2)
\end{aligned}$$

Using the parallel hylo-ana rule, we are now able to fuse the *zip* example:

$$[\![\zeta\,(out, out)]\!] \cdot ([\![\mathfrak{b}]\!] \times [\![\mathfrak{b}]\!]) = [\![\zeta\,(\mathfrak{b}, \mathfrak{b})]\!] \ ,$$

where the transformation ζ is defined

$$\begin{aligned}
&\zeta : (b_1 \to \mathsf{L}\,a_1\,b_1, b_2 \to \mathsf{L}\,a_2\,b_2) \to (b_1, b_2) \to \mathsf{L}\,(a_1, a_2)\,(b_1, b_2)\\
&\zeta\,(c_1, c_2)\,(x_1, x_2) = \mathbf{case}\,(c_1\,x_1, c_2\,x_2)\,\mathbf{of}\\
&\qquad (Cons\,(a_1, b_1), Cons\,(a_2, b_2)) \to Cons\,((a_1, a_2), (b_1, b_2))\\
&\qquad otherwise \to Nil\ .
\end{aligned}$$

Note that $\mathsf{F}_1 := \mathsf{L}\,A_1$, $\mathsf{F}_2 := \mathsf{L}\,A_2$ and $\mathsf{G} := \mathsf{L}\,(A_1 \times A_2)$.

Let us review the structured fusion rules presented so far. The cata-hylo rule absorbs a fold on the left of a hylo, and thus absorbs the consumer of the hylo into a new hylo. Dually, the hylo-ana rule absorbs an unfold on the right of a hylo, and thus absorbs the producer to the hylo into a new hylo. We noted that there are functions such as *zip* that consume two data structures, and consequently we introduced the parallel hylo-ana rule. In this case, two producers on the right of hylo are absorbed into a hylo. One might naturally ask if there is a need for an analogous parallel cata-hylo rule. Such a rule would cover the case where a function produces a pair of data structures, which are consumed by two folds. The Haskell function $partition : (a \to Bool) \to [a] \to ([a], [a])$ is a good example of this scenario. It takes a predicate and a list, and partitions the list based on the predicate. Let $partition\ even = (\!|\mathfrak{pt}|\!)$, where the algebra $\mathfrak{pt}$ is given by,

$$\begin{aligned}
&\mathfrak{pt} : \mathsf{L}\,\mathbb{N}\,(\mu(\mathsf{L}\,\mathbb{N}), \mu(\mathsf{L}\,\mathbb{N})) \to (\mu(\mathsf{L}\,\mathbb{N}), \mu(\mathsf{L}\,\mathbb{N}))\\
&\mathfrak{pt}\,Nil = (in\,Nil, in\,Nil)\\
&\mathfrak{pt}\,(Cons\,(n, (x, y))) = \mathbf{if}\ even\ n\\
&\qquad \mathbf{then}\,(in\,(Cons\,(n, x)), y)\\
&\qquad \mathbf{else}\ \ (x, \quad in\,(Cons\,(n, y)))\ .
\end{aligned}$$

The example we will use is $((\!|\mathfrak{s}|\!) \times (\!|\mathfrak{s}|\!)) \cdot (\!|\mathfrak{pt}|\!)$; a list of natural numbers is partitioned into the odds and evens, and then these two lists are independently summed.

Now we are ready to present *parallel cata-hylo fusion:* If τ satisfies,

$$\begin{aligned}
&h_1 \times h_2 : \tau\,(a_1, a_2) \to \tau\,(b_1, b_2) : \mathbf{Alg}(\mathsf{G})\\
&\qquad \Longleftarrow \quad h_1 : a_1 \to b_1 : \mathbf{Alg}(\mathsf{F}_1) \ \wedge\ h_2 : a_2 \to b_2 : \mathbf{Alg}(\mathsf{F}_2)\ ,
\end{aligned} \tag{34}$$

then

$$((\!|a_1|\!)_{\mathsf{F}_1} \times (\!|a_2|\!)_{\mathsf{F}_2}) \cdot (\!|\tau\,(in, in) \leftarrow c|\!)_{\mathsf{G}} \ = \ (\!|\tau\,(a_1, a_2) \leftarrow c|\!)_{\mathsf{G}}\ . \tag{35}$$

To apply parallel cata-hylo fusion to our partition example, we first need to redefine $\mathfrak{pt}$ as ψ so that it satisfies the form of τ.

$$\begin{aligned}
&\psi : (\mathsf{L}\,\mathbb{N}\,x \to x, \mathsf{L}\,\mathbb{N}\,y \to y) \to (\mathsf{L}\,\mathbb{N}\,(x, y) \to (x, y))\\
&\psi\,(a_1, a_2)\,Nil = (a_1\,Nil, a_2\,Nil)\\
&\psi\,(a_1, a_2)\,(Cons\,(n, (x, y))) = \mathbf{if}\ even\ n\\
&\qquad \mathbf{then}\,(a_1\,(Cons\,(n, x)), y)\\
&\qquad \mathbf{else}\ \ (x, \quad a_2\,(Cons\,(n, y)))
\end{aligned}$$

Now we can employ parallel cata-hylo-fusion: $((\!|\mathfrak{s}|\!) \times (\!|\mathfrak{s}|\!)) \cdot (\!|\psi\,(in, in)|\!) = (\!|\psi\,(\mathfrak{s}, \mathfrak{s})|\!)$

5.5 Church and Co-Church Encodings

In the two previous sections we have studied generalisations of *foldr*/*build* and *destroy*/*unfoldr* fusion. We have noted that $(\!|-|\!)$ generalises the list function *foldr*, and, likewise, $[\![-]\!]$ generalises *unfoldr*. We have been silent, however, about their counterparts *build* and *destroy*. It is time to break that silence, and in the process, provide a fresh perspective on recursive datatypes. For simplicity, we assume that we are working in **Set**.[4]

Consider again the polymorphic type of β (24) repeated below.

$$\forall A\ .\ (\mathsf{F}\,A \to A) \to (B \to A) \;\;\cong\;\; B \to (\forall A\ .\ (\mathsf{F}\,A \to A) \to A)$$

We have slightly massaged the type to bring B to the front. The universally quantified type on the right is known as the *Church encoding* of $\mu\mathsf{F}$ [6]. The type is quite remarkable as it encodes a recursive type without using recursion. One part of the isomorphism $\mu\mathsf{F} \cong \forall A\ .\ (\mathsf{F}\,A \to A) \to A$ is given by the acid rain rule (23). The following derivation, which infers the isomorphisms, makes this explicit—the initial equation is (23) with the arguments of β swapped.

$$
\begin{aligned}
& \forall a\ .\ (\!|a|\!)\,(\beta\, b\, \mathit{in}) = \beta\, b\, a \\
\iff\ & \{\ \text{change of variables } \beta\, b = \gamma\ \} \\
& \forall a\ .\ (\!|a|\!)\,(\gamma\, \mathit{in}) = \gamma\, a \\
\iff\ & \{\ \text{extensionality}\ \} \\
& \lambda a\ .\ (\!|a|\!)\,(\gamma\, \mathit{in}) = \gamma \\
\iff\ & \{\ \text{define } \mathit{toChurch}\, x = \lambda a\ .\ (\!|a|\!)\, x\ \} \\
& \mathit{toChurch}\,(\gamma\, \mathit{in}) = \gamma \\
\iff\ & \{\ \text{define } \mathit{fromChurch}\, \gamma = \gamma\, \mathit{in}\ \} \\
& \mathit{toChurch}\,(\mathit{fromChurch}\, \gamma) = \gamma
\end{aligned}
$$

The isomorphism *toChurch*, creates a function whose argument is an algebra and which folds that algebra over the given data structure. Its converse *fromChurch*, commonly called *build*, applies this function to the *in* algebra. Going back and forth, we get back the original structure: $\mathit{fromChurch}\,(\mathit{toChurch}\, s) = s$. This is the other part of the isomorphism, which follows directly from fold reflection.

The Church encoding can be readily implemented in Haskell:

$$
\begin{aligned}
&\textbf{newtype}\ \mathit{Church}\, f = \mathit{Abs}\,\{\, \mathit{app} : \forall a\ .\ (f\ a \to a) \to a \,\} \\
&\mathit{toChurch} \quad : \quad (\mathit{Functor}\, f) \Rightarrow \mu f \to \mathit{Church}\, f \\
&\mathit{toChurch}\, x \quad = \quad \mathit{Abs}\,(\lambda a \to (\!|a|\!)\, x) \\
&\mathit{fromChurch} \quad : \quad \mathit{Church}\, f \to \mu f \\
&\mathit{fromChurch}\, \gamma \quad = \quad \mathit{app}\, \gamma\, \mathit{in}\ \ .
\end{aligned}
$$

As to be expected, everything nicely dualises. The polymorphic type (29) gives rise to the *co-Church encoding.*

$$\forall C\ .\ (C \to \mathsf{F}\,C) \to (C \to D) \;\;\cong\;\; (\exists C\ .\ (C \to \mathsf{F}\,C) \times C) \to D$$

Think of the co-Church encoding $\exists C\ .\ (C \to \mathsf{F}\,C) \times C$ as the type of state machines encapsulating a transition function $C \to \mathsf{F}\,C$ and the current state C. Dually to the previous situation, this existentially quantified type is known in full as the co-Church encoding of $\nu\mathsf{F}$. Again, it encodes a recursive type without using recursion, however, now we have the greatest fixpoint of F. The

[4] The development can be generalised using ends and coends [15].

co-Church encoding can also be readily implemented in Haskell:

$$\begin{aligned}
&\textbf{data}\ \mathit{CoChurch}\,f = \forall\, s\ .\ \mathit{CC}\ \{\mathit{trans} : (s \to f\ s), \mathit{state} : s\}\\
&\mathit{toCoChurch} \quad : (\mathit{Functor}\,f) \Rightarrow \mu f \to \mathit{CoChurch}\,f\\
&\mathit{toCoChurch}\ x = \mathit{CC}\ \mathit{out}\ x\\
&\mathit{fromCoChurch} \quad : \mathit{CoChurch}\,f \to \mu f\\
&\mathit{fromCoChurch}\ \phi = [\![\mathit{trans}\ \phi]\!]\,(\mathit{state}\ \phi)
\end{aligned}$$

The datatype declarations *Church* and *CoChurch* make explicit the underlying conversions that are central to the concept of shortcut fusion. By changing representations to one with the recursion "built-in", we can write our transformations as non-recursively-defined (co-)algebras. Unlike recursive programs, compositions of these (co-)algebras can be optimised by the compiler to remove any intermediate allocations. All that remains is for the programmer to instruct the compiler to remove any unnecessary conversions, i.e. cases of $\mathit{toChurch} \cdot \mathit{fromChurch}$ and $\mathit{toCoChurch} \cdot \mathit{fromCoChurch}$. Removing these transformations preserves the semantics of the program because we can prove the isomorphism between these representations. It also, however, prevents us from producing a data structure only to immediately consume it.

This encoding underlies the original formulation of stream fusion, which we consider next.

5.6 Stream Fusion

The *foldr*/*build* flavour of fusion is fold-centric, in that it requires all functions that are intended to be fusible to be written as folds; similarly, *destroy*/*unfoldr* is unfold-centric. The boundaries of these world views are fuzzy. A *zip* can be written as a fold, the snag is that only one of the two inputs can be fused [11, §9]. Along a similar vein, a *filter* for the odd natural numbers, which we wrote before as a fold, can also be written as an unfold: $[\![\mathfrak{f}]\!]$ where

$$\begin{aligned}
&\mathfrak{f} : \mu(\mathsf{L}\,\mathbb{N}) \to \mathsf{L}\,\mathbb{N}\,(\mu(\mathsf{L}\,\mathbb{N}))\\
&\mathfrak{f}\,x = \textbf{case}\ \mathit{out}\ x\ \textbf{of}\ \mathit{Nil} \to \mathit{Nil};\ \mathit{Cons}\,(x,y) \to \textbf{if}\ \mathit{odd}\ x\ \textbf{then}\ \mathit{Cons}\,(x,y)\ \textbf{else}\ \mathfrak{f}\,y\ \ .
\end{aligned}$$

The coalgebra $\mathfrak{f}$ is recursive and thus theoretically fine, but it is also recursive in its definition and this is a practical problem. A coalgebra must be non-recursively defined for it to be fused with others. We have two definitions and are caught between two worlds; is it possible to free ourselves?

Perhaps surprisingly, the answer is yes. Let us first try to eliminate the recursion from the definition above—the rest will then fall out. The idea is to use a different base functor, one that allows us to skip list elements. We draw inspiration from stream fusion [7] here:

$$\textbf{data}\,\mathsf{S}\ a\ b = \mathit{Done} \mid \mathit{Yield}\,(a,b) \mid \mathit{Skip}\ b\ \ .$$

$$\begin{aligned}
&\textbf{instance}\ \mathit{Functor}\,(\mathsf{S}\ a)\ \textbf{where}\\
&\quad \mathit{fmap}\,f\ \mathit{Done} = \mathit{Done}\\
&\quad \mathit{fmap}\,f\ (\mathit{Skip}\ b) = \mathit{Skip}\,(f\ b)\\
&\quad \mathit{fmap}\,f\ (\mathit{Yield}\,(a,b)) = \mathit{Yield}\,(a, f\ b)
\end{aligned}$$

The *filter* coalgebra can now be written as a composition of *out* with

$$\begin{aligned}
&\mathfrak{f} : \mathsf{S}\,\mathbb{N}\,b \to \mathsf{S}\,\mathbb{N}\,b\\
&\mathfrak{f}\,\mathit{Done} = \mathit{Done}\\
&\mathfrak{f}\,(\mathit{Skip}\ y) = \mathit{Skip}\ y\\
&\mathfrak{f}\,(\mathit{Yield}\,(x,y)) = \textbf{if}\ \mathit{odd}\ x\ \textbf{then}\ \mathit{Yield}\,(x,y)\ \textbf{else}\ \mathit{Skip}\ y\ \ .
\end{aligned}$$

So, $\mathit{filter} = [\![\mathfrak{f} \cdot \mathit{out}]\!]$. Something interesting has happened: since $\mathfrak{f}$ is a natural transformation, we also have $\mathit{filter} = (\!|\mathit{in} \cdot \mathfrak{f}|\!)$. We are unstuck; *filter* is both a fold and an unfold. Moreover, it is an application of a mapping function: $\mathit{filter} = \mu\mathfrak{f}$. We are really talking about a specialised *filter*: filtering for odd numbers.

A map for streams passes on any *Skip*s it encounters, but otherwise behaves as it would over a list:

$$
\begin{aligned}
&\mathfrak{m} : (a_1 \to a_2) \to \mathsf{S}\, a_1\, b \to \mathsf{S}\, a_2\, b\\
&\mathfrak{m}\, f\ \mathit{Done} = \mathit{Done}\\
&\mathfrak{m}\, f\ (\mathit{Skip}\ b) = \mathit{Skip}\ b\\
&\mathfrak{m}\, f\ (\mathit{Yield}\ (a, b)) = \mathit{Yield}\ (f\ a, b)\ \ .
\end{aligned}
$$

Given a function $f : a_1 \to a_2$, $\mathfrak{m}\, f$ is natural transformation, meaning that we can state it as both a fold $(\!|\mathit{in} \cdot \mathfrak{m}\, f|\!)$ and an unfold $[\![\mathfrak{m}\, f \cdot \mathit{out}]\!]$.

In general, consumers are folds, transformers are maps, and producers are unfolds. An entire pipeline of these an be fused into a single hylo:

$$(\!|a|\!) \cdot \mu\alpha_1 \cdot \cdots \cdot \mu\alpha_n \cdot [\![c]\!] = (\!|a \cdot \alpha_1 \cdot \cdots \cdot \alpha_n \leftarrow c|\!)\ \ .$$

Inspecting the types, the rule is clear:

$$A \xleftarrow{(\!|a|\!)} \mu\mathsf{F}_0 \xleftarrow{\mu\alpha_1} \mu\mathsf{F}_1 \quad \cdots \quad \mu\mathsf{F}_{n-1} \xleftarrow{\mu\alpha_n} \mu\mathsf{F}_n \xleftarrow{[\![c]\!]} C\ \ .$$

In a sense, the introduction of *Skip* keeps the recursion in sync. Each transformation consumes a token and produces a token. Before, *filter* possibly consumed several tokens before producing one. We are finally in a position to deal with the example from the introduction, written in terms of the combinators we have:

$$(\!|\mathfrak{s}|\!) \cdot \mu(\mathfrak{m}\, sq) \cdot \mu\mathfrak{f} \cdot [\![\mathfrak{b}]\!] = (\!|\mathfrak{s} \cdot \mathfrak{m}\, sq \cdot \mathfrak{f} \leftarrow \mathfrak{b}|\!)\ \ .$$

Utilising streams in this fashion is an instance of data abstraction; although we wish to present the List type using $\mu(\mathsf{L}\, a)$, we intend to do all the work using $\mu(\mathsf{S}\, a)$. We have functions $\rightharpoondown\!\mathsf{S}$ and $\leftharpoondown\!\mathsf{S}$ to convert to and from streams, respectively. They are defined as an algebra and a coalgebra that allow us to consume streams using a fold and produce them using an unfold:

$$
\begin{aligned}
&\leftharpoondown\!\mathsf{S} : \mathsf{S}\, a\, (\mu(\mathsf{L}\, a)) \to (\mu(\mathsf{L}\, a))\\
&\leftharpoondown\!\mathsf{S}\ \mathit{Done} = \mathit{in}\ \mathit{Nil}\\
&\leftharpoondown\!\mathsf{S}\ (\mathit{Skip}\ xs) = xs\\
&\leftharpoondown\!\mathsf{S}\ (\mathit{Yield}\ (x, xs)) = \mathit{in}\ (\mathit{Cons}\ (x, xs))\\
&\rightharpoondown\!\mathsf{S} : \mu(\mathsf{L}\, a) \to \mathsf{S}\, a\, (\mu(\mathsf{L}\, a))\\
&\rightharpoondown\!\mathsf{S}\ (\mathit{in}\ \mathit{Nil}) = \mathit{Done}\\
&\rightharpoondown\!\mathsf{S}\ (\mathit{in}\ (\mathit{Cons}\ (x, xs))) = \mathit{Yield}\ (x, xs)\ \ .
\end{aligned}
$$

We must prove that our stream implementations, together with these conversion functions, fulfil the same specification as the analogous functions over $\mu(\mathsf{L}\, a)$ (*cf.* Lemma 1 and Theorem 3 in [24]). This is called the *data abstraction property*. In our framework, this obligation is expressed as a simple equality between a conventional list function definition and its associated stream version composed with our conversion functions. For example, for *filter* we must prove

$$\mathit{filter} = (\!|\leftharpoondown\!\mathsf{S}|\!) \cdot \mu\mathfrak{f} \cdot [\![\rightharpoondown\!\mathsf{S}]\!]\ \ .$$

Because we can phrase these functions as folds, unfolds, and natural transformations, the proof is straightforward, using the laws we have set out in previous sections.

It is simple to see by case analysis that $\leftharpoondown\mathsf{S} \cdot \rightharpoonup\mathsf{S} = id$ and therefore $(\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\rightharpoonup\mathsf{S}]\!] = id$.[5] We can use this fact to simplify our obligation:

$$
\begin{aligned}
& \mathit{filter} = (\!|\leftharpoondown\mathsf{S}|\!) \cdot \mu\mathfrak{f} \cdot [\![\rightharpoonup\mathsf{S}]\!] \\
\Longleftrightarrow\quad & \{\ (\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\rightharpoonup\mathsf{S}]\!] = id\ \} \\
& \mathit{filter} \cdot (\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\rightharpoonup\mathsf{S}]\!] = (\!|\leftharpoondown\mathsf{S}|\!) \cdot \mu\mathfrak{f} \cdot [\![\rightharpoonup\mathsf{S}]\!] \\
\Longleftarrow\quad & \{\ \text{composition}\ \} \\
& \mathit{filter} \cdot (\!|\leftharpoondown\mathsf{S}|\!) = (\!|\leftharpoondown\mathsf{S}|\!) \cdot \mu\mathfrak{f} \\
\Longleftrightarrow\quad & \{\ \text{functor fusion}\ \} \\
& \mathit{filter} \cdot (\!|\leftharpoondown\mathsf{S}|\!) = (\!|\leftharpoondown\mathsf{S} \cdot \mathfrak{f}|\!) \\
\Longleftarrow\quad & \{\ \text{fold fusion}\ \} \\
& \mathit{filter} \cdot \leftharpoondown\mathsf{S} = \leftharpoondown\mathsf{S} \cdot \mathfrak{f} \cdot (\mathsf{S}\ a)\,(\mathit{filter})
\end{aligned}
$$

Our obligation has been simplifed to requiring that a filter step is equivalent regardless of whether we perform it over streams and then convert to a list or vice versa. This is easily accomplished by case analysis. Although we used *filter* in our example, this proof generalises to any other transformer functions.

We must also, however, deal with functions that are producers or consumers, but not both. As with transformers, we can start from the definition principle for consumers:

$$\mathit{consL} = \mathit{conS} \cdot [\![\rightharpoonup\mathsf{S}]\!]$$

and producers

$$\mathit{prodL} = (\!|\leftharpoondown\mathsf{S}|\!) \cdot \mathit{prodS}\ \ .$$

For consumers, we again need to make use of the relationship between $\rightharpoonup\mathsf{S}$ and $\leftharpoondown\mathsf{S}$ to apply the fusion laws:

$$
\begin{aligned}
& \mathit{consL} = \mathit{conS} \cdot [\![\rightharpoonup\mathsf{S}]\!] \\
\Longleftrightarrow\quad & \{\ (\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\rightharpoonup\mathsf{S}]\!] = id\ \} \\
& \mathit{consL} \cdot (\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\rightharpoonup\mathsf{S}]\!] = \mathit{consS} \cdot [\![\rightharpoonup\mathsf{S}]\!] \\
\Longleftarrow\quad & \{\ \text{composition}\ \} \\
& \mathit{consL} \cdot (\!|\leftharpoondown\mathsf{S}|\!) = \mathit{consS} \\
\Longleftarrow\quad & \{\ \text{fold fusion}\ \} \\
& \mathit{consL} \cdot \leftharpoondown\mathsf{S} = \mathit{conS} \cdot (\mathsf{S}\ a)\,(\mathit{consL})
\end{aligned}
$$

We can again proceed by case analysis.

For producers, we cannot simply appeal to the fold fusion laws. We must instead use the fact that we are working with hylomorphisms to find a suitable obligation:

$$
\begin{aligned}
& [\![\mathit{prodL}]\!] = (\!|\leftharpoondown\mathsf{S}|\!) \cdot [\![\mathit{prodS}]\!] \\
\Longleftrightarrow\quad & \{\ \text{fold/unfold law}\ \} \\
& [\![\mathit{prodL}]\!] = (\!|\leftharpoondown\mathsf{S} \leftarrow \mathit{prodS}|\!) \\
\Longleftrightarrow\quad & \{\ \text{uniqueness property of hylomorphisms}\ \} \\
& [\![\mathit{prodL}]\!] = \leftharpoondown\mathsf{S} \cdot (\mathsf{S}\ a)\,[\![\mathit{prodL}]\!] \cdot \mathit{prodS}
\end{aligned}
$$

We now have an equation of suitable form, where we only need to verify a single step in the recursion process by case analysis in order to complete the proof.

Just as for lists, every datatype can be extended with a *Skip*. Although stream fusion is the first to make use of this augmentation, we note its relation to Capretta's representation of general recursion in type theory [2], which proposes adding a "computation step" constructor to coinductive types.

---

[5] The inverse is not true, however, e.g. $(\rightharpoonup\mathsf{S} \cdot \leftharpoondown\mathsf{S})\,(\mathit{Skip\ Nil}) == \mathit{Done} \neq \mathit{Skip\ Nil})$.

6 Related Work

Wadler first introduced the idea of simplifying the fusion problem with his deforestation algorithm [23]. This was limited to so-called *treeless* programs, a subset of first-order programs. The fusion transformation proposed by Chin [4] generalises Wadler's deforestation. It uses a program annotation scheme to recognise the terms that can be fused and skip the terms that cannot.

Sheard and Fegaras focus on the use of folds over algebraic types as a recursion scheme [18]. Their algorithm for normalising the nested application of folds is based on the fold fusion law. Their recursion schemes are suitably general to handle functions such as *zip* that recurse over multiple data structures simultaneously [8]. Gill et al. first introduced the notion of shortcut fusion with *foldr*/*build* fusion [11] for Haskell. This allowed programs written as folds to be fused. It was subsequently introduced into the List library for Haskell in GHC. Takano and Meijer [20] provided a calculational view of fusion and generalised it to arbitrary data structures. It generalised the fusion law by using hylomorphisms and also noted the possibility of dualising *foldr*/*build* fusion. They worked in the setting of **Cpo**, however, where hylomorphisms do not have unique solutions, only canonical ones. Takano and Meijer claimed that, even when restricted to lists, their method is more powerful than that of Gill et al. as theirs could fuse both parameters of *zip*. This was incorrect, and the need for an additional parallel rule for *zip* was pointed out later by Hu et al. [12]. Their extension is what we present as the parallel hylo-ana rule. While it is an obvious duality, they did not follow on to present what we call the parallel cata-hylo rule. To our knowledge, ours is the first presentation, and Chitil is the only one to mention fusion of this nature [5]. Svenningson provided an actual implementation of *destroy*/*unfoldr* fusion [19], where he showed how *filter*-like functions could be expressed as unfolds. Svenningson did not, however, solve the issue of recursion in the coalgebras of such functions, which could therefore not be fused even though they could be written as unfolds. This was addressed by Coutts et al., who presented stream fusion [7], which introduced the *Skip* constructor as a way to encode non-productive computation steps, similar to Capretta's work on encoding general recursion in type theory [2].

The correctness and generalisation of fusion has been explored in many different settings. In addition to the work of Takano and Meier, Ghani et al. generalised *foldr*/*build* to work with datatypes "induced by inductive monads". Johann and Ghani further showed how to apply initial algebra semantics, and thus *foldr*/*build* fusion, to nested datatypes [13]. Voigtländer has also used free theorems to show correctness, specifically of the *destroy*/*build* rule [21].

7 Conclusions

We have presented a framework that has allowed us to bring three fusion techniques into the same setting. We have exploited recursive coalgebras and hylomorphisms as 'the rug that ties the room together'. This enabled us to formally describe and reason about these fusion techniques. In doing so, we have exposed their underlying foundations, including the importance of Church and co-Church encodings. The fact that our hylomorphisms have unique solutions plays a central rôle. The knock-on effect is that we gain clear, short proofs thanks to the calculational properties available to us.

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A Category Theory Refresher

This appendix contains supplementary material. It is intended primarily as a reference, so that the reader can re-familiarise themselves with the category theory that is utilised in this paper.

Categories A category consists of objects and arrows between them. We let $\mathbb{C}$, $\mathbb{D}$ etc. range over categories.

Objects In common parlance, a category is often identified with its class of objects. For instance, we say that **Set** is the category of sets. In the same spirit, we write $A : \mathbb{C}$ to express that A is an object of $\mathbb{C}$. We let A, B etc range over objects.

Arrows However, equally, if not more important, are the arrows of a category. The category **Set** is really the category of sets and total functions. (There is also **Rel**, the category of sets and relations.) If the objects have additional structure (monoids, groups etc) then the arrows are typically structure-preserving maps. For every pair of objects $A, B : \mathbb{C}$ there is a class of arrows from A to B, denoted $\mathbb{C}(A, B)$. When referring to a specific arrow $f : \mathbb{C}(A, B)$, we use $f : A \to B : \mathbb{C}$, and if $\mathbb{C}$ is obvious from the context, we abbreviate with $f : A \to B$. We shall also loosely speak of $A \to B$ as the type of f. We let f, g etc range over arrows.

For every object $A : \mathbb{C}$ there is an arrow $id_A : A \to A$, called the identity. Two arrows can be composed if their types match: if $f : A \to B$ and $g : B \to C$, then $g \cdot f : A \to C$. We require composition to be associative with identity as its neutral element.

Initial and Final Objects An object is called initial if for each object $B \in \mathbb{C}$ there is exactly one arrow from the initial object to B. Any two initial objects are isomorphic, which is why we usually speak of *the* initial object. Dually, 1 is a final object if for each object $A \in \mathbb{C}$ there is a unique arrow from A to 1.

Functors Every structure comes equipped with structure-preserving maps, and for categories these maps are called *functors*. Since a category consists of two parts, objects and arrows, a functor $\mathsf{F} : \mathbb{C} \to \mathbb{D}$ consists of a mapping on objects and a mapping on arrows. It is common practise to denote both mappings by the same symbol. The action on arrows has to respect the types: if $f : A \to B$, then $\mathsf{F}\,f : \mathsf{F}\,A \to \mathsf{F}\,B$. Furthermore, F has to preserve identity and composition:

$$\mathsf{F}\,id_A = id_{\mathsf{F}\,A} \ , \tag{36}$$

$$\mathsf{F}\,(g \cdot f) = \mathsf{F}\,g \cdot \mathsf{F}\,f \ . \tag{37}$$

The force of functoriality lies in the action on arrows and in the preservation of composition. We let F, G etc range over functors.

Natural Transformations Let $\mathsf{F}, \mathsf{G} : \mathbb{C} \to \mathbb{D}$ be two functors. *A natural transformation* $\alpha : \mathsf{F} \dot{\to} \mathsf{G}$ is a collection of arrows, so that for each object $A : \mathbb{C}$ there is an arrow $\alpha\,A : \mathsf{F}\,A \to \mathsf{G}\,A : \mathbb{D}$ such that

$$\mathsf{G}\,h \cdot \alpha\,A_1 \;=\; \alpha\,A_2 \cdot \mathsf{F}\,h, \tag{38}$$

for all arrows $h : A_1 \to A_2 : \mathbb{C}$. Given α and h, there are essentially two ways of turning $\mathsf{F}\,A_1$ things into $\mathsf{G}\,A_2$ things. The coherence condition (38) demands that they are equivalent. We let α, β etc range over natural transformations.

Subcategories A *subcategory* $\mathbb{S}$ of a category $\mathbb{C}$ is a collection of some of the objects and some of the arrows of $\mathbb{C}$, such that identity and composition are preserved to ensure $\mathbb{S}$ constitutes a category. In a full subcategory, $\mathbb{S}(A, B) = \mathbb{C}(A, B)$, for all objects A and B.