ALGEBRAIC K-THEORY OF FINITE FIELDS

Date: April 2014.

1. Introduction

The +-construction of the algebraic K-theory of a ring R was introduced by Daniel Quillen in 1970, to link the topological K-theory of Atiyah-Hirzbruch with the algebraic functors K_1 and K_2 developed by Milnor and others. We give a sketch of Quillen's construction in section 3.1.

In general, calculating explicitly the groups K_n of algebraic K-theory is extremely difficult. Indeed, the K-groups even of the integers are not fully known: the triviality of the groups $K_n(\mathbb{Z})$ for n a multiple of 4 is equivalent to Vandiver's conjecture in algebraic number theory ([Wei13], section 4, chapter 9).

However, in a paper of 1972 ([Qui72]), Quillen was able to calculate the K-groups corresponding to a finite field. In this essay, we describe his proof of the following main theorem:

Theorem 1. *Let k be a finite field with q elements. Then for $i \geq 1$, we have*

$$K_{2i}(k) = 0$$
$$K_{2i-1}(k) \cong \mathbb{Z}/(q^i - 1).$$

We first present some topological background, including classifying spaces, Chern classes, topological K-theory and the Adams operations, and then briefly outline Quillen's definition of higher algebraic K-theory via the +-construction. We then move to the substance of the proof.

We define the space $F\Psi^q$ as the homotopical fixed point set of the q-th Adams operation, and compute its homotopy groups using the long exact sequence of a fibration. We then show that it is homotopy equivalent to $BGL(k)^+$, the space whose homotopy groups are the K-groups required.

This is done via Whitehead's theorem: which states that a map between simple spaces which induces isomorphisms on all (integral) homology groups is a homotopy equivalence. We construct a map $BGL_nk \to F\Psi^q$, and seek to show that it induces a homology isomorphism. We begin by using the Eilenberg-Moore spectral sequence to compute the 'size' of the cohomology of $F\Psi^q$. We then find explicit generators for $H^*(F\Psi^q)$, and get the multiplicative structure. Finally, we find some generators for $H^*(GL_nk)$, and show that they are mapped to by the generators of $H^*(F\Psi^q)$ and that this is (in the limit as $n \to \infty$) a cohomology isomorphism. We then use the universal coefficient theorem for cohomology to show that the map induces a homology isomorphism with field coefficients, and then use the universal coefficient theorem for homology to show that the map induces an isomorphism on integral homology. We finally show that the map extends to one from $BGL(k)^+$ which is also a homology isomorphism, and conclude that $BGL(k)^+$ and $F\Psi^q$ are homotopy equivalent.

2. Topological background

2.1. **Classifying spaces and group cohomology.** Given a topological group G, suppose we are somehow given a space EG which is weakly contractible (i.e. $\pi_i(EG) = 0$ for all i), on which G acts freely, such that there is a fibre bundle

$$G \to EG \xrightarrow{p} EG/G = BG$$

(where π is the quotient map by the action of G). Then from this fibre bundle, we obtain (see [Hat02], p.376) a long exact sequence on homotopy

$$\cdots \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \cdots \to \pi_0(EG) \to 0.$$

Since EG is weakly contractible, every third term in this sequence is zero, so we have

$$\pi_n(BG) = \pi_{n-1}(G).$$

BG is called the *classifying space* for G. We define the *group cohomology* (resp. group homology) of the group G as the cohomology (resp. homology) of the group G.

Can we always find such an EG? Milnor in [Mil56] shows that we can. Briefly, the idea is to take the *join* of G with itself, where the join of spaces $X_1, \dots, X_n$ is given by the space of tuples $(x_1, t_1, \dots, x_n, t_n)$ with $x_i \in X_i, t_i \in \mathbb{R}^{\geq 0}, t_1 + \ldots + t_n = 1$, where $(x_1, t_1, \dots, x_n, t_n) = (x'_1, t_1, \dots, x'_n, t_n)$ if $x'_i = x_i$ whenever $t_i \neq 0$. We can define the join of infinitely many spaces similarly, specifying that $t_i = 0$ for all but finitely many i. It can be shown that this space is weakly contractible, and that the quotient by the componentwise action of G gives a fibration.

The particular case $G = U(n)$ (the group of unitary $n \times n$ matrices) is significant. In this case, a suitable EG is given by the direct limit $V_n(\mathbb{C}^\infty)$ of the complex *Stiefel manifolds* $V_n(\mathbb{C}^i)$ of n-tuples of orthonormal vectors ('frames') in $\mathbb{C}^i$, acted on by matrix multiplication.

Since action by $U(n)$ can take a frame to any other frame spanning the same subspace of $\mathbb{C}^\infty$, the quotient $EU(n)/U(n)$ is given by the *Grassmannian* $G_n(\mathbb{C}^\infty)$ of n-dimensional subspaces of $\mathbb{C}^\infty$. We can show that $V_n(\mathbb{C}^\infty)$ is contractible, and that $U(n) \to V_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ is a fibre bundle. Taking the limit $n \to \infty$ (with the obvious inclusions), we have a fibre bundle

$$U \to EU = V_\infty(\mathbb{C}^\infty) \to BU = G_\infty(\mathbb{C}^\infty). \tag{2.1}$$

2.2. **The universal bundle.** What is the most general vector bundle? It turns out that such a thing exists, and that the base space is none other than the Grassmannian of the previous section.

Define the space $E_n(\mathbb{C}^k)$ to be the space of pairs (l, v) with $l \in G_n(\mathbb{C}^k)$ (where $G_n(\mathbb{C}^k)$ is the space of n-dimensional subspaces of $\mathbb{C}^k$) and $v \in l$. Then $p : E_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ is a vector bundle, with fibre $\mathbb{C}^n$ and trivialisations defined on the open sets U_l of subspaces l' whose projection $\pi_l(l')$ onto l is of full rank. The trivialisations are given by $h : p^{-1}(U_l) \to U_l \times l \sim U_l \times \mathbb{C}^n$, with $h(l', v) = (l', \pi_l(v))$.

Taking limits first as $k \to \infty$ and then as $n \to \infty$, we have a vector bundle $E_\infty(\mathbb{C}^\infty) \to G_\infty(\mathbb{C}^\infty) = BU$, called the *universal vector bundle*, because of the following useful property:

Proposition 2. *Let $p : E \to B$ be a complex vector bundle. Then there is a map $\phi : B \to BU$, unique up to homotopy, such that p is the pullback by ϕ of the universal vector bundle.*

Proof. [Hat09] 1.16 et seq. □

2.3. **Chern classes.** Suppose that we have a fibre bundle $F \to E \to B$. Can we relate the cohomology of E to the cohomology of F and B? It seems that E is somehow a product of F and B. In the case that E is the trivial bundle, it is indeed the cross product, and the Künneth formula gives a ring isomorphism

$$H^*(E) \cong H^*(F) \otimes H^*(B).$$

In the general case, we have the Leray-Hirsch theorem:

Theorem 3. *(Leray-Hirsch) Let R be a commutative ring. Let $F \to E \xrightarrow{p} B$ be a fibre bundle such that*

(i) $H^n(F; R)$ is a free R-module for each n, and
(ii) there exist classes $c_j \in H^{k_j}(E; R)$ restricting to a basis for $H^*(F; R)$ for each fibre F.

Then $H^*(E; R)$ is a free $H^*(B; R)$-module with basis c_j, where scalar multiplication in $H^*(E; R)$ is given by $bc = p^*(b) \smile c$ for $b \in H^*(B; R)$ and $c \in H^*(E; R)$.

Proof. [Hat02], p.432. □

How can we tell whether a vector bundle is trivial? By proposition 2, this is if and only if the associated map $\phi : B \to BU$ is homotopic to the map associated to the trivial bundle, which is the constant map, so if and only if ϕ is null-homotopic. But it is in general difficult to tell whether a map is null-homotopic, so we might be interested in the weaker property of whether ϕ induces the trivial map on cohomology (that is, whether the pullback $\phi^* : H^*(BU) \to H^*(B)$ is trivial).

It is thus clearly natural to study the cohomology of the Grassmannian. An application of the Leray-Hirsch theorem gives the following result:

Proposition 4. *$H^*(BU) \cong \mathbb{Z}[\tilde{c}_1, \tilde{c}_2, \ldots]$; the $\tilde{c}_i$ (suitably chosen—see proposition 6) are called the* universal Chern classes *of dimension $2i$.*

Proof. [Hat02], p.436 □

Of course, we have not uniquely specified the $\tilde{c}_i$, as there are many choices of generators for a polynomial algebra. We will shortly find (proposition 6) that there is one preferred choice, and it is this that we will use.

In order to determine whether a complex vector bundle is trivial, we are interested in the pullback along the associated map ϕ of the universal Chern classes. This motivates the following definition:

Definition 5. *Let $E \to B$ be a complex vector bundle with associated map $\phi : B \to BU$. Then the* Chern classes *$\tilde{c}_1^E, \tilde{c}_2^E, \ldots$ are given by $\tilde{c}_i^E = \phi^*(\tilde{c}_i) \in H^{2i}(B)$. We write $\tilde{c}(E) = 1 + \tilde{c}_1^E + \tilde{c}_2^E + \ldots \in H^*(B)$, called the* total Chern class*.*

Chern classes are often helpful for calculation, because they behave very well under taking the direct sum of vector bundles:

Proposition 6. *Let $E, E' \to B$ be complex vector bundles, with total Chern classes $\tilde{c}(E), \tilde{c}(E')$ as defined above. Then, for a suitable (unique, fixed independently of E and E') choice of the $\tilde{c}_i$ in proposition 4, we have*

$$\tilde{c}(E \oplus E') = \tilde{c}(E) \cup \tilde{c}(E').$$

Proof. [Hat09], p.82. □

2.4. **Topological K-theory.** We would like to form a group from the complex vector bundles (up to isomorphism) over a space X. It is easy to see what operation we should use: the direct sum, which sends vector bundles $E_1 \xrightarrow{p_1} X$ and $E_2 \xrightarrow{p_2} X$ to $E_1 \oplus E_2 = \{(x_1, x_2) | p_1(x_1) = p_2(x_2)$. This is associative and commutative, and has an identity given by the 0-dimensional vector bundle $X \to X$.

But it doesn't give us a group, because while it tells us how to add, it doesn't tell us how to subtract (that is, we don't have inverses). So we take the *Grothendieck group* of formal differences of vector bundles over X: a typical element is of the form $[E_1] - [E_2]$, for vector bundles E_1, E_2, and we say that $[E_1] - [E_2] = [E_3] - [E_4]$ if there is some vector bundle E_5 such that $E_1 \oplus E_4 \oplus E_5 \cong E_3 \oplus E_2 \oplus E_5$. It is clear that we now have inverses: the inverse of $[E_1] - [E_2]$ is $[E_2] - [E_1]$. We define this group to be $K(X)$.

This definition can be improved, however. In particular, $K(X)$ views trivial vector bundles as non-trivial: the trivial vector bundles $\epsilon^n : X \times \mathbb{C}^n \to X$ are all distinct. We can resolve this by considering vector bundles up to *stable isomorphism*: that is, we regard E_1 and E_2 as isomorphic if there are some m, n such that $E_1 \oplus \epsilon^m \cong E_2 \oplus \epsilon^n$. We define the group after making this identification by $\tilde{K}(X)$, the *reduced K-theory of X*. Hereafter, we will often refer to it just as 'the K-theory of X'.

It turns out that after we make this identification, taking formal differences is no longer necessary, as the following proposition establishes:

Proposition 7. *Let X be a compact space. Then $\tilde{K}(X) \cong [X, BU]$.*

Proof. [Ben98], p.44. □

(recall that by 2, homotopy classes of maps from X to BU are in one-to-one correspondence with vector bundles over X, so this shows that every element of $\tilde{K}(X)$ is realised by a genuine vector bundle.

2.5. **The Adams operations.** We can in fact put further structure onto $K(X)$, so as to turn it into a ring. The natural operation on vector bundles corresponding to multiplication is the tensor product: for vector bundles E_1, E_2 as above of dimensions n_1, n_2 respectively, the tensor product $E_1 \otimes E_2$ is the disjoint union of the spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for $x \in X$, topologised by considering sets U on which E_1 and E_2 are trivialised, and forcing $p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$ to be a homeomorphism. This makes $K(X)$ into a ring with unit ϵ^1, and we have that $\epsilon^n \otimes E$ is the sum of n copies of E.

Since the elements killed by the identifications we used to produce $\tilde{K}(X)$ form an ideal, we have that $\tilde{K}(X)$ is also made into a ring by $\otimes$, but possibly without a 1.

We are now ready to define the Adams operations, which will be central to the remainder of this essay.

Theorem 8. *There exist ring homomorphisms $\Psi^k : \tilde{K}(X) \to \tilde{K}(X)$ for all $k \geq 0$, such that*

(i) $\Psi^k f^* = f^* \Psi^k$ for all maps $f : X \to Y$;
(ii) If L is a line bundle, then $\Psi^k L = L^k$;
(iii) $\Psi^k \circ \Psi^l = \Psi^{kl}$; and
(iv) For p prime, we have $\Psi^k E = E^p + pE'$ for some $E' \in \tilde{K}(X)$.

Proof. [Hat09], p.66. Briefly, we define the r-th exterior power $\wedge^r E$ of a vector bundle by the quotient of $E^{\otimes r}$ by elements representing skew-commutativity. We can then check that the map $\Psi^k E = s_k(\wedge^1 E, \wedge^2 E, \ldots, \wedge^k E)$ satisfies the conditions, for s_k a suitable polynomial defined recursively by

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \ldots + (-1)^{k-2} \sigma_{k-1} s_1 + (-1)^{k-1} k \sigma_k,$$

where σ_j is the j-th elementary symmetric polynomial. □

Note that this construction does not use any special properties of vector bundles or K-theory. It would work just as well on any setup with direct sums, tensor products and exterior powers. For instance, exactly the same construction can be applied to the space of representations of a group G over a field k, so that we can (and will) talk about Ψ^q as acting on representations as well.

3. The +-construction of algebraic K-theory

The plus construction is a functor assigning to a ring R a collection of groups $K_i(R)$. This proceeds in several stages. We first take the general linear group $GL(R)$ over R, which is formed by taking the direct limit of the groups GL_nR of invertible $n \times n$ matrices with entries in R; more concretely, GLR is the group of infinite matrices with entries in R, which are equal to the identity in all but finitely many rows and in all but finitely many columns.

We next take the classifying space $BGL(R)$ of $GL(R)$ as defined in section 2.1. We then apply the +-operation (defined below) to remove the non-commutative part of $\pi_1(BGL(R))$; more concretely, we form a new space $BGL(R)^+$, with the property that $\pi_1(BGL(R)^+) = \pi_1(BGL(R))/\pi$, where π is the commutator subgroup of $\pi_1(BGL(R))$.

3.1. **The +-operation.** The theorem establishing the +-operation basically says that for a space X, and a perfect normal subgroup π of $\pi_1(X)$ (that is, the commutator subgroup of π is equal to itself) we can glue on 2-cells to kill elements of π, and then add 3-cells to ensure that we have not created any new homology elements:

Theorem 9. *Let X be a CW-complex, and let π be a perfect normal subgroup of $\pi_1(X)$. Then there is a CW-complex $X+$, unique up to homotopy equivalence, and an inclusion $i : X \to X^+$, satisfying the following properties:*

(i) X^+ is formed from X by gluing on cells of dimension at most 3.
(ii) $i_* : \pi_1(X) \to \pi_1(X^+)$ is surjective, with $\ker i_* = \pi$.
(iii) Let $\widetilde{X^+}$ be a covering space of X^+, and $\tilde{X}$ the corresponding cover of X. Then $\tilde{i}_* : H_*(\tilde{X}) \to H_*(\widetilde{X^+})$ is an isomorphism. In particular $i_* : H_*(X) \to H_*(X^+)$ is an isomorphism.
(iv) Let $f : X \to Z$ be a map such that $f_* : \pi_1(X) \to \pi_1(Z)$ kills π. Then there is a map $f' : X^+ \to Z$ such that $f \simeq f' \circ i$.

Proof. [Ben98], p.68. □

Applying this theorem with π the commutator subgroup of $GL(R) = \pi_1(BGL(R))$, which is the subgroup E(R) generated by the elementary matrices (which differ from the identity in a single off-diagonal entry) and is perfect ([Ben98], p.70), gives the construction above.

3.2. **Our problem.** We are aiming to calculate the K-groups of a finite field $k = \mathbb{F}_q$, where q is some power of a prime p. We will perform most of our calculations for cohomology with coefficients in the field $\mathbb{F}_l$, for l prime not equal to p (and using the universal coefficient theorem at the end to draw conclusions about the integral cohomology). We define r to be minimal such that $q^r - 1$ is divisible by l.

4. The space $F\Psi^q$

We now define the space $F\Psi^q$. In the sections that follow, we will calculate its homotopy groups by expressing it as the fibre of a fibration of BU over itself, and then (after a great deal of work) show that it is homotopy equivalent to the space $BGLk^+$ whose homotopy groups we desire.

4.1. **Definition.** Let $\sigma : BU \to BU$ be the map representing the Adams operation Ψ^q in K-theory (see section 2.5). We think of $F\Psi^q$ as the 'homotopy-theoretic fixed point set of σ'. What does this mean? We define it as the set of pairs (x, γ), where γ is a path from x to $\sigma(x)$. We can equivalently view it as the fibre product $BU \times_{BU\times BU} BU^I$ (where BU^I is the space of paths in BU) defined by the following Cartesian square:

$$\begin{array}{ccc} F\Psi^q & \xrightarrow{\quad\theta\quad} & BU^I \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\Delta} \\ BU & \xrightarrow[(\mathrm{id},\sigma)]{\quad\quad} & BU \times BU \end{array} \tag{4.1}$$

(where Δ is the map sending a path in BU from x to y to the pair (x, y)).

4.2. **The homotopy of $F\Psi^q$.** As foreshadowed, we want to express $F\Psi^q$ as the fibre of a fibration of BU over itself, rather than over BU^2, as we have currently. We define $d : BU \times BU \to BU$ to be the map representing the difference operation in K-theory, with d chosen such that, for some base-point b, we have for all $x \in BU$ that $d(x, x) = b$ and $d(x, b) = x$, and $\sigma(b) = b$. Now the above diagram extends to the following:

$$\begin{array}{ccccc} F\Psi^q & \xrightarrow{\quad\quad} & BU^I & \xrightarrow{\quad m\quad} & BU^I \times_{BU} \{b\} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\Delta} & & \big\downarrow{\scriptstyle n} \\ BU & \xrightarrow[(\mathrm{id},\sigma)]{\quad\quad} & BU \times BU & \xrightarrow[d]{\quad\quad} & BU \end{array} \tag{4.2}$$

where n sends a path to its starting point, and m sends a path γ to the path joining $d\Delta(\gamma) = d(\gamma(0), \gamma(1))$ to b given by $t \mapsto d(\gamma(t), \gamma(1))$. Note that $BU^I \times_{BU} \{b\}$ means the fibre product with respect to the inclusion map in the second factor, and in the first factor the map sending a path to its endpoint. This is the space of paths in BU ending at b, and is contractible.

We now have that Δ and n are both fibrations with fibre ΩBU, the space of loops in BU based at b, which tells us

Lemma 10. *The space $F\Psi^q$ is (homotopy equivalent to) the fibre of the map $(id, \sigma) \circ d = d(id, \sigma) : BU \to BU$ representing the operation $1 - \Psi^q$ in K-theory.*

We now possess the necessary tools to compute the homotopy groups of $F\Psi^q$, using (part of) the Bott Periodicity Theorem and the long exact sequence of a fibration.

Theorem 11. *(Bott Periodicity) Let $U(n)$ be the space of $n \times n$ unitary complex matrices. Then if $U = \bigcup_{n \geq 0} U(n)$, we have*

$$\pi_i(U) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 1 \mod 2 \\ 0 & \text{if } i \equiv 0 \mod 2 \end{cases}.$$

Proof. [Hat02], p.384. □

Lemma 12. *The space $F\Psi^q$ is simple (that is, its fundamental group acts trivially on the higher homotopy groups) with homotopy groups*

$$\pi_{2i}(F\Psi^q) = 0$$
$$\pi_{2i-1}(F\Psi^q) = \mathbb{Z}/(q^i - 1).$$

Proof. By lemma 10, we have the following long exact sequence on homotopy:

$$\ldots \to \pi_i(BU) \xrightarrow{1-\Psi^q} \pi_i(BU) \xrightarrow{\partial} \pi_{i-1}(F\Psi^q) \xrightarrow{\phi} \pi_{i-1}(BU) \to \ldots. \quad (4.3)$$

By the long exact sequence corresponding to the fibration (2.1), we have that $\pi_i(BU) = \pi_{i-1}(U)$ (as EU is weakly contractible), so by theorem 11, $\pi_{2i-1}(BU) = 0$ and $\pi_{2i}(BU) = \mathbb{Z}$. Now by proposition 7, we have $\pi_{2i}(BU) = [S^{2i}, BU] \cong \tilde{K}(S^{2i})$, on which $1 - \Psi^q$ acts by multiplication by $1 - q^i$, so in particular injectively.

In the even case, we have an exact sequence

$$0 \to \pi_{2i}(F\Psi^q) \to \mathbb{Z} \xhookrightarrow{1-q^i} \mathbb{Z},$$

so $\pi_i(F\Psi^q) = 0$.

In the odd case, we have an exact sequence

$$\mathbb{Z} \xhookrightarrow{1-q^i} \mathbb{Z} \twoheadrightarrow \pi_{2i-1}(F\Psi^q) \to 0, \quad (4.4)$$

so $\pi_{2i-1}(F\Psi^q) \cong \mathbb{Z}/(q^i - 1)$.

It remains to show that $F\Psi^q$ is simple; that is, that $\pi_1(F\Psi^q)$ acts trivially on the higher homotopy groups. Recall that the action arises in this way: given two basepoints $b, b' \in F\Psi^q$, a path γ from b to b' is a homotopy $\gamma : I \times \{\text{pt}\} \to F\Psi^q$. For any map $x : S^n \to F\Psi^q$ (with the basepoint of S^n mapping to b), by the homotopy extension property x extends to a homotopy $S^n \times I \to F\Psi^q$ covering γ (and a homotopy between maps x and x' extends in the same way). Thus γ gives a map $\pi_n(F\Psi^q, b) \to \pi_n(F\Psi^q, b')$. Since the homotopy groups are discrete, this map is well-defined up to the homotopy class of γ. Taking $b' = b$, we obtain an action of $\pi_1(F\Psi^q)$ on $\pi_n(F\Psi^q)$.

The obvious push-forwards give actions of $\pi_1(BU)$ on $\pi_i(BU)$, viewed as the base and total space of the fibration (2.1). Since BU is simply connected, these push-forwards are both trivial. These actions commute with the exact sequence (4.3), so the action on $\pi_i(F\Psi^q)$ is trivial (for instance by examining the action on the subsequence (4.4)), so $F\Psi^q$ is simple, as required. □

5. Representations to (co)homology

We are now going to study how we can use group representations to produce cohomology (or homology) classes for $F\Psi^q$. The plan is to first show that (homotopy classes of) maps to BU invariant under the action of Ψ^q are equivalent to (classes of) maps to $F\Psi^q$. We then show how group representations give rise to complex vector bundles, hence to maps to BU. We are then able to pull back (respectively push forward) cohomology (resp. homology) classes from (resp. to) $F\Psi^q$.

5.1. Ψ^q-invariant maps.

We prove the following lemma:

Lemma 13. *Let X be a space with $[X, \Omega BU] = 0$ (where ΩBU is the loop space of BU). Then the pushforward of $\phi : F\Psi^q \to BU$ gives a bijection*

$$\phi_* : [X, F\Psi^q] \to [X, BU]^{\Psi^q},$$

where $[X, BU]^{\Psi^q}$ is the space of homotopy classes invariant under composition with the map $\sigma : BU \to BU$ representing Ψ^q.

Proof. A map $X \to F\Psi^q$ may be viewed as a map $f : X \to BU$ together with a homotopy from f to σf, so it is clear that ϕ_* is surjective. On the other hand, if two maps $f, g : X \to F\Psi^q$ are equalised by ϕ_*, that is, there is a homotopy F from $\phi \circ f$ to $\phi \circ g$, we have (since ϕ is the inclusion of the fibre in a fibration) that $d(\mathrm{id}, \sigma) \circ F$ is a based loop in BU for each $x \in X$, that is, a map $X \to \Omega BU$; since $[X, \Omega BU] = 0$, we have that ϕ_* is injective. □

5.2. Representations to vector bundles.

Given a representation ρ of a finite group G on $\mathbb{C}^n$, we form a vector bundle such that an element h of $\pi_1(BG) = G$ acts on the fibres by $\rho(h)$. We take as the total space $EG \times \mathbb{C}^n / \sim$, where EG is the space in section 2.2, and $\sim$ is given by $(x, v) \sim (g(x), \rho(g)^{-1}(v))$ for all $x \in EG, v \in \mathbb{C}^n, g \in G$. By proposition 2, there is a map $BG \to BU$, unique up to homotopy, whose pullback gives this vector bundle. As a matter of notation, we denote the map associated to a representation ρ by ρ^+.

We observe the relevance of this construction to proposition 7. The proposition shows that we can extend to the Grothendieck group of *virtual* complex representations, that is, formal differences of genuine representations. The above construction takes a formal difference of representations to a formal difference of vector bundles, that is, an element of the K-group, which goes by proposition 7 to a map to BU. Once again, the map associated to a virtual complex representation ρ is denoted by ρ^+.

In either case, if the representation is Ψ^q-invariant, we have by lemma 13 that there is a corresponding map (up to homotopy class) $BG \to F\Psi^q$, which we denote by $\rho^\#$.

5.3. **Modular representations.** Suppose that our representation is not over $\mathbb{C}$, but rather over a finite field k with q elements, or its algebraic closure $\overline{k}$. If we choose an embedding into the complex numbers

$$\iota : \overline{k}^* \to \mathbb{C}^*$$

(by taking compatible embeddings of finite extensions of k), then for a representation E over $\overline{k}$ we can define the *Brauer character* χ_E of E, by

$$\chi_E(g) = \sum \iota(\lambda_i),$$

where $\{\lambda_i\}$ is the set of eigenvalues (with multiplicity) of the automorphism $E(g)$ of $\overline{k}^n$. Then we use the following theorem:

Theorem 14. *Let $x \mapsto E(x)$ be a representation of a finite group by invertible $n \times n$ matrices $E(x)$ with coefficients in a finite field $\mathbb{F}$. Suppose that $\mathbb{F}$ contains the 'latent roots' (eigenvalues) $\lambda_1(x), \dots, \lambda_n(x)$ of $E(x)$. Then if*

$$\iota : \mathbb{F}^* \to \mathbb{C}^*$$

is any homomorphic embedding, and S is any symmetric polynomial in n variables with integer coefficients, then the function

$$\chi_E(x) = S(\iota(\lambda_1(x)), \dots, \iota(\lambda_n(x)))$$

is the character of a virtual complex representation of G.

Proof. [Gre55], Theorem 1. □

Specialising to the case where $\mathbb{F} = \overline{k}$ and $S(t_1, \dots, t_n) = t_1 + \dots + t_n$, we have that χ_E is the character of a virtual complex representation.

Now if E is instead a representation over k, we can tensor with $\overline{k}$ over k to obtain a representation $\overline{E} = E \otimes_k \overline{k}$. Because the Adams operation on characters has $(\Psi^q \chi)(g) = \chi(g^q)$, and the set of eigenvalues of the automorphism $E(g)$ of k^n is invariant under the Frobenius isomorphism $x \mapsto x^q$, we have that the complex representation obtained from $\overline{E}$ is necessarily Ψ^q-invariant.

We now want to apply the earlier work of this section, but to do so we have to show (in order to satisfy the conditions of lemma 13) that $[BG, \Omega BU] = 0$. To do this, we use a part of the Atiyah completion theorem:

Theorem 15. *Let G be a compact Lie group. Then $\tilde{K}^1(BG) = 0$.*

Proof. [Ben98], p.48. □

Now the definition of $\tilde{K}^1$ is that $\tilde{K}^1(X) = \tilde{K}(S(X))$ (where $S(X)$ is the suspension of X). By proposition 7, we have that $\tilde{K}(X) = [X, BU]$. So overall we have that

$$0 = \tilde{K}^1(BG) = \tilde{K}(S(BG)) = [S(BG), BU] = [BG, \Omega BU],$$

as required (the last equality being by the general fact that $[S(X), Y] = [X, \Omega Y]$). By section 5.2, we now have that E gives rise to a map (up to homotopy class)

$$E^{\#} : BG \to F\Psi^q.$$

5.4. **The Brauer lift of $k(\mu_l)$.** Let the multiplicative group of the field extension $k(\mu_l)$ be denoted C, because it is isomorphic to the cyclic group of order $q^r - 1$ (recalling that r was defined as the minimal integer such that l divides r^q). It has a natural representation L over k, obtained by letting $C = k(\mu_l)^*$ act on $k(\mu_l)$ by multiplication, and viewing $k(\mu_l)$ as a k-vector space of dimension $r - 1$.

What is the Brauer lifting of L? We have a vector space homomorphism

$$k(\mu_l) \otimes_k \overline{k} \to \overline{k}^r$$

given by

$$z \otimes w \mapsto \left(w, z^q w, z^{q^2} w, \ldots, z^{q^{r-1}} w\right)$$

(this is a homomorphism by the property that $(z_1 + z_2)^q = z_1^q + z_2^q$ in a field of characteristic dividing q). By Galois theory, this is an isomorphism (by dimension-counting).

Viewed in the world of $\overline{k}^r$, we see that the action of an element z of $k(\mu_l)^*$ is given by componentwise multiplication by $(1, z, z^q, z^{q^2}, \ldots, z^{q^{r-1}})$. The set of eigenvalues is hence $\{1, z, z^q, z^{q^2}, \ldots, z^{q^{r-1}}\}$, and we have

$$\chi_L(z) = \sum_{i=0}^{r-1} \iota(z^{q^i}). \tag{5.1}$$

A natural choice of ι is $\iota = \zeta : C \to \mathbb{C}^*$, given by $\zeta(1) = e^{2\pi i/(q^r-1)}$, where the 1 on the left hand side is a generator of C. What complex representation then has the same character as (5.1)? It is easy to see that the answer is

$$W = \zeta \oplus \zeta^q \oplus \zeta^{q^2} \oplus \ldots \oplus \zeta^{q^{r-1}}.$$

This fact will be needed later, in section 8.

5.5. **The fundamental example.** There is one application of this section that is absolutely fundamental to the main argument of the proof. The group GL_nk has a natural n-dimensional representation over k given by matrix multiplication. By the work of this section this gives rise to a map

$$BGL_nk \to F\Psi^q.$$

In the remainder of the proof, we will aim to show that the direct limit of this map as $n \to \infty$ induces an isomorphism on cohomology, and hence on homology. We will then (after replacing $BGLk$ with the simple space $BGLk^+$ using the universal property of the +-construction) use the Whitehead theorem to argue that the homotopy groups of $BGLk^+$ are the same as those of $F\Psi^q$, which we have calculated above.

6. A rough calculation of $H^*(F\Psi^q)$

In this section, we will use the Eilenberg-Moore spectral sequence to calculate the additive structure (and most of the multiplicative structure) of $H^*(F\Psi^q)$. This is tantalisingly close to enough information about the cohomology of $F\Psi^q$, but unfortunately the generators obtained here are found in a purely abstract way, so we have no chance of knowing where they go under the map of section 5.5, so we can't tell whether that map produces a homology isomorphism.

As a result, we will have to get our hands dirty and find explicit generators, which we will do in the next section. The effort of this section will not be wasted, however, because our calculation will tell us the dimensions of the cohomology groups, so once we have found enough independent elements, we will know know that they generate.

Rather than integral cohomology, we instead consider cohomology with coefficients in $\mathbb{Q}$ and in finite fields of prime order. This will eventually allow us to use the universal coefficient theorem to show that we have an isomorphism on integral homology.

6.1. **Coefficients in $\mathbb{Q}$ or $\mathbb{F}_p$.** We are able to rapidly dispose of these cases, using the Hurewicz theorem:

Theorem 16. *Let k' be a field, and suppose that $k' \otimes_{\mathbb{Z}} \pi_i(X) = 0$ for all $i < n$. Then $H_i(X; k') = 0$ for $0 < i < n$.*

Proof. [Ben98], p.11, quoting [MT68]. □

By lemma 12, we have that $k' \otimes_{\mathbb{Z}} \pi_i(F\Psi^q) = 0$ for $k' = \mathbb{Q}$ or $\mathbb{F}_p$, so $H_i(F\Psi^q; k') = 0$.

From now on, we will work with coefficients in $\mathbb{F}_l$ for l any prime other than p.

6.2. **Spectral sequences.** A spectral sequence is like a souped-up version of an exact sequence. It consists (for the type of sequence considered here) of a series of *pages* E_n, each consisting of a two-dimensional array of groups $E_n^{i,j}$, together with *differentials*

$$d_n : E_n^{p,q} \to E_n^{p+n,q-n+1},$$

with the property that $d_n^2 = 0$.

The pages are related by the property that each is the homology of the previous one, that is,

$$E_{n+1}^{p,q} = \ker d_n / \mathrm{Im} d_n \text{ at } E_n^{p,q}.$$

The theorem establishing the spectral sequence will generally tell us the entries of the first page $E_2^{*,*}$, and also that each element eventually converges to some stable element $E_\infty^{i,j}$, which will usually have some significance; in some sense, this is the 'output' of the spectral sequence.

6.3. **The Eilenberg-Moore spectral sequence.** The Eilenberg-Moore spectral sequence is in some sense a more sophisticated version of the Künneth formula: it tells us not just about the cohomology of the direct product of two spaces $X \times Y$ but about any fibre product $X \times_B Y$.

Of course, with a more complicated input we should expect a more complicated output, and indeed where the Künneth formula uses the tensor product, the Eilenberg-Moore spectral sequence uses a more complicated object: the Tor functor.

6.3.1. *The Tor functor.* Let R be a commutative ring, and A and B be graded R-modules. Take a free resolution

$$\ldots \to F_1 \to F_0 \to A \to 0$$

of A (that is, an exact sequence where all the modules except for A are free). Throw away A and tensor over R with B to obtain a chain complex:

$$\ldots \to F_1 \otimes_R B \to F_0 \otimes_R B \to 0. \tag{6.1}$$

We define $\mathrm{Tor}_n^R(A,B)$ to be the nth homology group of this chain complex.

In fact, since A and B were graded, $\mathrm{Tor}_n^R(A,B)$ is itself graded, and we define $\mathrm{Tor}_{n,q}^R(A,B)$ to be the qth graded subgroup of $\mathrm{Tor}_n^R(A,B)$.

Note that we could have performed this construction exchanging the roles of A and B, and we would in fact obtain the same groups $\mathrm{Tor}_{n,q}^R(A,B)$.

We now state two properties of Tor that will be useful later:

Lemma 17. *Let R and S be rings, and let M, N and P, Q be graded R- and S-modules respectively. Then*

$$Tor_*^{R\otimes S}(M \otimes P, N \otimes Q) = Tor_*^R(M,N) \otimes Tor_*^S(P,Q).$$

Proof. This essentially derives from the Künneth formula. See [May11], p.14. □

Lemma 18. *Let* $R = k[x_1, x_2, \ldots]$*. Then* $Tor_*^R(k,k) = \bigwedge[y_1, y_2, \ldots]$*, where* y_i *is in the first homology group of the chain complex 6.1, and has grading within that group equal to the grading of* x_i*, so* $y_i \in Tor^R_{-1,|x_i|}(k,k)$*.*

Proof. Ibid, p.16. □

6.3.2. *The spectral sequence.* The Eilenberg-Moore spectral sequence is as follows:

Theorem 19. *Let* $X \to B$ *be a map and* $Y \to B$ *a fibration, with* B *simply connected and all the cohomology groups of* X, Y *and* B *finite-dimensional. Then (where all cohomology is taken with coefficients in a field* k*) there is a spectral sequence with*

$$E_2^{p,q} = Tor_{p,q}^{H^*(B)}(H^*(X), H^*(Y)),$$

such that for fixed n *the direct sum of the stable terms* $E_\infty^{p,n-p}$ *is isomorphic to* $H^n(X \times_B Y)$*.*

Proof. [Hat04], Theorem 3.2. □

6.4. **Calculation of** $H^*(F\Psi^q)$**.** It is clear that the Eilenberg-Moore spectral sequence is relevant to our problem: in diagram (4.1), we have expressed $F\Psi^q$ as the fibre product $BU \times_{BU^2} BU^I$. Denoting by $c_i \in H^{2i}(BU)$ the mod-l reduction of the universal Chern classes of proposition 4. Then we have (where P denotes a polynomial algebra)

$$\begin{aligned} H^*(BU) &= P[c_1, c_2, \ldots] \\ H^*(BU^2) &= P[c_1', c_2', \ldots, c_1'', c_2'', \ldots], \end{aligned}$$

where c_i' and c_i'' are the pullbacks of c_i with respect to projection onto the first and second factors respectively.

Let $A = H^*(BU^2)$. Now since $\Delta^* c_i' = \Delta^* c_i'' = c_i$, the elements of $H^*(BU^I)$ that are killed by the pullback with respect to Δ are generated by those of the form $c_i' - c_i''$; let this ideal be I, so $H^*(BU^I) = A/I$. We also have $(\mathrm{id}, \sigma)^* c_i' = c_i$ and $(\mathrm{id}, \sigma)^* c_i'' = q^i c_i$, so the elements of $H^*(BU)$ killed by the pullback with respect to (id, σ) are generated by those of the form $q^i c_i' - c_i''$; let this ideal be J, so $H^*(BU) = A/J$.

Trivial operations allow us to express the generating set for A as comprising elements of four types:

$$\begin{aligned} (1) c_i' - c_i'' &\text{ for } i \equiv 0 \pmod r \\ (2) c_i' - c_i'' &\text{ for } i \not\equiv 0 \pmod r \\ (3) q^i c_i' - c_i'' &\text{ for } i \not\equiv 0 \pmod r \\ (4) c_i' &\text{ for } i \equiv 0 \pmod r \end{aligned}$$

(noting that q^i is invertible mod l and $q^i \not\equiv 1 \pmod{l}$ if $i \not\equiv 0 \pmod{r}$). Let the subrings of A generated by these classes of elements be denoted A_1, A_2, A_3 and A_4 respectively; they are all polynomial rings, and we have

$$\begin{aligned} A &\cong A_1 \otimes A_2 \otimes A_3 \otimes A_4 \\ A/I &\cong k \otimes k \otimes A_3 \otimes A_4 \\ A/J &\cong k \otimes A_2 \otimes k \otimes A_4. \end{aligned}$$

Now (since BU is simply connected and BU, BU^2 and BU^I all have finite-dimensional cohomology in each degree), by theorem 19 there is an Eilenberg-Moore spectral sequence with

$$E_2 = \mathrm{Tor}^{H^*(BU)}(H^*(BU^I), H^*(BU)) = \mathrm{Tor}^A(A/J, A/I).$$

By lemma 17 (noting that for any ring R and R-module M we have $\mathrm{Tor}^R_*(R, M) = M$, by taking a trivial free resolution for R as an R-module), we have

$$\mathrm{Tor}^A(A/J, A/I) = \mathrm{Tor}^{A_1}(k, k) \otimes k \otimes k \otimes A_4.$$

Using lemma 18, we then have that

$$E_2 \cong P[c_r, c_{2r}, \ldots] \otimes \wedge[e_r, e_{2r}, \ldots],$$

with $c_{jr} \in E_2^{0,2jr}$ and $e_{jr} \in E_2^{-1,2jr-1}$.

In fact, the spectral sequence also has some multiplicative structure, which is enough to tell us that since the c_{jr}, e_{jr} generate E_2, the only non-zero entries of E_2 are in the top-left (first co-ordinate ≤ 0, second co-ordinate ≥ 0) region. Since the differentials d_2 take c_{jr} and e_{jr} to points in the top-*right* region, we have that the sequence is instantly stable, i.e. $E_2 = E_\infty$, so we can deduce the theorem we wanted:

Theorem 20. *There is an isomorphism* of additive groups

$$H^*(F\Psi^q) \cong P[c_r, c_{2r}, \ldots] \otimes \wedge[e_r, e_{2r}, \ldots],$$

with deg$(c_{jr}) = 2jr$ *and deg*$(e_{jr}) = 2jr - 1$ *for* $j \geq 1$.

In fact, the multiplicative structure of the spectral sequence alluded to above is almost enough to tell us that this is actually an algebra isomorphism: we have that the multiplicative structures are the same, except that we only know that the e_i square to an element in the span of the other elements, rather than to zero (see [Ben98], p.65. Of course, even if we knew that this was an isomorphism, that still wouldn't be enough for the Whitehead theorem: we have to show not just that the homology of $F\Psi^q$ is the same as that of GLk, but that this isomorphism is realised by a map $GLk \to F\Psi^q$.

7. Generators for $H^*(F\Psi^q)$

In this section we will get our hands dirty and find explicit generators c_{jr} and e_{jr} for $H^*(F\Psi^q)$. The c_{jr} will arise (as the notation suggests) from pullbacks of the universal Chern classes, but defining the e_{jr} will require more work. The reader is advised to don their apron.

We are going to work with cohomology with coefficients in the field $\mathbb{F}_l$, with $l \neq p$—hereafter, whenever cohomology coefficients are not specified, they will be in $\mathbb{F}_l$. We are also going to assume that $l \neq 2$. The case $l = 2$ introduces some technicalities that are not included here for reasons of space, but can be found in Quillen's paper.

7.1. **Technical preliminaries.** We work with coefficients in a general ring A.

7.1.1. *The mapping cylinder.* Let $f : X \to Y$ be a map, and define its *mapping cylinder*

$$\mathrm{Cyl}(f) = (X \times I) \cup_f Y = (X \times I) \sqcup Y/(x, 0) \sim f(x).$$

Defining the relative cohomology

$$H^*(f, A) = H^*(\mathrm{Cyl}(f), X \times \{0\}; A),$$

and observing that $\mathrm{Cyl}(f)$ retracts to Y, we have the long exact sequence of a pair gives the following long exact sequence of cohomology:

$$\ldots \to H^{i-1}(X; A) \xrightarrow{\delta} H^i(f; A) \xrightarrow{j} H^i(Y; A) \xrightarrow{f^*} H^i(X; A) \to \ldots. \tag{7.1}$$

7.1.2. *The Bockstein exact sequence.* We have an exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0,$$

with the first map given by multiplication by n, and the second by reduction modulo n. If $C_\bullet(?)$ is a chain complex (either the ordinary singular chain complex of a space, or the relative complex arising from a mapping cylinder as above), we can apply to the above exact sequence the covariant functor $\mathrm{Hom}(C_\bullet(?), -)$ sending each group to the homomorphisms from $C_\bullet$ to that group to obtain a short exact sequence of cochain complexes

$$0 \to C^\bullet(?; \mathbb{Z}) \to C^\bullet(?; \mathbb{Z}) \to C^\bullet(?; \mathbb{Z}/n) \to 0,$$

which gives rise to a long exact sequence of cohomology, the *Bockstein exact sequence*:

$$\ldots \to H^{i-1}(?; \mathbb{Z}/n) \xrightarrow{\beta_n} H^i(?; \mathbb{Z}) \xrightarrow{n} H^i(?; \mathbb{Z}) \xrightarrow{\rho_n} H^i(?; \mathbb{Z}/n) \to \ldots. \tag{7.2}$$

We now prove the following lemma:

Lemma 21. *Let $u \in H^i(f;\mathbb{Z})$ satisfy $ju = nc$ with $c \in H^i(Y;\mathbb{Z})$. By exactness, we may choose $e \in H^{i-1}(X;\mathbb{Z}/n)$ such that $\delta e = \rho_n u$. Then $\beta_n e \equiv -f^* c \pmod{\beta_n f^* H^{i-1}(Y;\mathbb{Z}/n)}$.*

Proof. Since e was determined mod $f^* H^{i-1}(Y,\mathbb{Z}/n)$, it is enough to show that there is one possible choice of e such that $\beta_n e = -f^* c$. We may assume that f is an embedding, so that $H^*(f;A)$ can be computed using the relative cochain complex $C^\bullet(Y,X;A)$ of cochains on Y vanishing on X. Let $x \in C^i(Y,X;\mathbb{Z})$ and $y \in C^i(Y,\mathbb{Z}$ be cocycles representing u and c respectively, i.e. $x = ny + dz$ for some $z \in C^{i-1}(Y,\mathbb{Z})$.

Now if we denote reduction of cochains mod n by ρ_n, we have $\rho_n(ny) = 0$, so $\rho_n x = \rho_n dz$. Then we have $\rho_n x = \rho_n dz = d\rho_n z$. Now x vanishes on X, so we have $f^*\rho_n x = 0 = f^* d\rho_n z = df^*\rho_n z$, so $f^*\rho_n z$ is coclosed, so represents an element $e \in H^{i-1}(X;\mathbb{Z}/n)$. To compute δe we take $f^*\rho_n z$, undo f^* to get $\rho_n z$, apply d to get $d\rho_n z = \rho_n x \in C^i(Y;\mathbb{Z}/n)$, and undo j to get $\rho_n x \in C^i(Y,X;\mathbb{Z}/n)$. Recalling that x represents u, we have that $\delta e = \rho_n u$.

Now to compute $\beta_n e$, we take $f^*\rho_n z$ and lift to an integral cochain $f^* z$, apply d to get $df^* z = f^*(x - ny) = -nf^* y$, and divide by n to get $-f^* y$. So (since y represents c) we conclude that $\beta_n e = -f^* c$, as required. □

7.1.3. *Morphisms between maps.*

Definition 22. *Let $f : X \to Y$ and $f' : X' \to Y'$ be maps. A* morphism *$g : f \to f'$ is a pair of maps $g_1 : Y \to Y'$ and $g_2 : X \to X'$, such that the following square commutes:*

$$\begin{array}{ccc} X & \xrightarrow{g_2} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g_1]{} & Y' \end{array}, \tag{7.3}$$

that is, $g_1 f = f' g_2$.

The pullback of such a morphism yields a map of the mapping cylinder exact sequences from (7.1):

$$\begin{array}{ccccccc} H^{i-1}(X') & \xrightarrow{\delta'} & H^i(f') & \xrightarrow{j'} & H^i(Y') & \xrightarrow{f'^*} & H^i(X') \\ \downarrow{\scriptstyle g_2^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g_1^*} & & \downarrow{\scriptstyle g_2^*} \\ H^{i-1}(X) & \xrightarrow[\delta]{} & H^i(f) & \xrightarrow[j]{} & H^i(Y) & \xrightarrow[f^*]{} & H^i(X) \end{array}, \tag{7.4}$$

where all cohomology is with coefficients in an arbitrary ring A.

Consider the map

$$(f'^*, g_1^*) : H^i(Y') \to H^i(X') \oplus H^i(Y).$$

If we have an element u of $\ker(f'^*, g_1^*)$ then in particular $u \in \ker(f'^*)$ so by exactness $u = j'w$ for some w. Now since also $u \in \ker(g_1^*)$ we have by the commutivity of the middle square that $g^*w \in \ker(j)$, so by exactness $g^*w = \delta a$ for some $a \in H^{i-1}(X)$, well-defined up to an element in the image of f^*, that is, up to an element of $f^*H^{i-1}(Y)$. But also w was only well-defined up to an element of the image of δ', so a can also vary by an element of $g_2^*H^{i-1}(X')$. We thus associate to g a homomorphism

$$D_g : \ker(f'^*, g_1^*) \to H^{i-1}(X)/(f^*H^{i-1}(Y) + g_2^*H^{i-1}(X')). \tag{7.5}$$

We observe

Lemma 23. *(i) D_g is an $H^*(Y')$-module homomorphism; that is, if $v \in H^a(Y')$ and $u \in \ker(f'^*, g_1^*)$ then $D_g(vu) = (-1)^a(f^*g_1^*v) \cdot D_gu$.*
(ii) If $u \in \ker f'^*$ and $v \in \ker g_1^*$, then $D_g(uv) = 0$.

Proof. (i) Note that we have maps from all the relevant spaces to Y', so the pullbacks make all the cohomology groups into $H^*(Y')$-modules. Since all the maps in diagram (7.4) are $H^*(Y')$-homomorphisms, so is D_g.
(ii) Let $u = j'w$. Then by part (i) we have that $j'(wv) = uv$, and $g^*(wv) = g^*w \cdot g_1^*v = 0$.

□

7.2. **Defining the classes e_{jr}.** The observant reader will have noticed the similarity between the diagrams in (7.3) and (4.1), and the bright one may have thought of the idea of viewing the diagram (4.1) as a morphism $\Gamma : \phi \to \Delta$ and applying the construction of section 7.1.3. Then we have the map from (7.4) of long exact sequences of cohomology:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & H^{2i}(\Delta; A) & \xrightarrow{j'} & H^{2i}(BU \times BU; A) & \xrightarrow{\Delta^*} \\
 & & \downarrow & & \downarrow \Gamma^* & & \downarrow (\mathrm{id}, \sigma)^* & \\
0 & \longrightarrow & H^{2i-1}(F\Psi^q; A) & \xrightarrow[\delta]{} & H^{2i}(\phi; A) & \xrightarrow[j]{} & H^{2i}(BU; A) & \xrightarrow[\phi^*]{}
\end{array}, \tag{7.6}$$

where the zeros arise from the fact (proposition 4) that the cohomology of BU is zero in all odd dimensions.

Recall that in proposition 4 we defined $\tilde{c}_i \in H^{2i}(BU; \mathbb{Z})$ as the i-th universal Chern class. Let $c_i \in H^{2i}(BU)$ represent the mod-l reduction of $\tilde{c}_i$. We can also consider the images of the c_i under the map ϕ^*, which we will also denote $c_i \in H^{2i}(F\Psi^q)$.

Now by definition of Δ we have $\Delta^*(\tilde{c}_i \otimes 1 - 1 \otimes \tilde{c}_i) = 0$ so by exactness, there is some (unique) $z \in H^{2i}(\Delta; \mathbb{Z})$ such that $j'z = \tilde{c}_i \otimes 1 - 1 \otimes \tilde{c}_i$. Now $j(\Gamma^* z) = (\mathrm{id}, \sigma)^*(\tilde{c}_i \otimes 1 - 1 \otimes \tilde{c}_i) = (1 - q^i)\tilde{c}_i$, so moving to cohomology with coefficients in $\mathbb{Z}/(q^i - 1)$, we have $\Gamma^* z \in \ker j$. Hence there is a unique $\tilde{e}_i \in H^{2i-1}(F\Psi^q; \mathbb{Z}/(q^i - 1))$ such that $\delta \tilde{e}_i = \Gamma^* z$.

Observe that by lemma 21, we have

$$\beta_{q^i - 1}(\tilde{e}_i) = \phi^*(\tilde{c}_i). \tag{7.7}$$

We are now in a position to define e_{jr}:

Definition 24. *$e_{jr} \in H^{2jr-1}(F\Psi^q), j \geq 0$, is the image of $\tilde{e}_{jr}$ under reduction of coefficients mod l (recalling that by definition of r we have that $q^{rj} - 1$ is divisible by l, so we can reduce from coefficients in $\mathbb{Z}/(q^{rj} - 1)$ to coefficients in $\mathbb{F}_l$).*

Observe that by the definition of D_g for a morphism g, we have

$$e_{jr} = D_\Gamma(c_{jr} \otimes 1 - 1 \otimes c_{jr}), \tag{7.8}$$

and indeed we could have defined e_{jr} in this way.

Since the e_{jr} are of odd cohomological degree, by the skew-commutativity of the cup product (and since we are working with coefficients in $\mathbb{F}_l$ with $l \neq 2$), we have that $e_{jr}^2 = 0$ for all j. So by the work of the previous section, we will know that we have mastered the cohomology of $F\Psi^q$ if we can show that the monomials formed out of the c_{jr} and the e_{jr} are linearly independent:

Lemma 25. *We have $e_{jr}^2 = 0$ for all j, and the monomials*

$$c_r^{\alpha_1} c_{2r}^{\alpha_2} \dots e_r^{\beta_1} e_{2r}^{\beta_2} \dots$$

with $0 \leq \alpha_j$ and $0 \leq \beta_j \leq 1$ are linearly independent.

7.3. **The plan.** Our strategy is to use section 5 to map these cohomology classes into the cohomology of the group C^m (the direct sum of cyclic groups of order $q^r - 1$), and show that they are linearly independent there.

Given a Ψ^q-invariant representation E of a group G, we denote by $\tilde{c}_i(E), c_i(E), \tilde{e}_i(E)$ and $e_{jr}(E)$ the images of $\tilde{c}_i, c_i, \tilde{e}_i$ and e_{jr} under the pullback of the map $E^\# : BG \to F\Psi^q$ from section 5.2.

The representation of C^m we are going to use is W^m, with W as defined in section 5.4.

Step 1 is to understand the classes $c_i(W)$ and $e_{jr}(W)$ in $H^*(BC)$. We then need to understand how these classes behave under the addition of representations. Step 2 is to define the maps $\mu : BU^2 \to BU$ and $\nu : (F\Psi^q)^2 \to F\Psi^q$ corresponding to addition of representations. Step 3 is to show that these maps commute with D_Γ in the way we would expect (explained in more detail below). Step 4 is to calculate the action of D on Chern classes. Step 5 is then to derive the formula for direct sums of representations and show that this leads to the conclusion.

7.4. **Step 1: $c_i(W)$ and $e_{jr}(W)$.**

Lemma 26. *Let C, ξ and W be defined as in section 5.4. Let $v \in H^1(BC)$ be the class of the homomorphism $C \to \mathbb{F}_l$ sending 1 to 1, and let $u = c_1(\xi) \in H^2(BC)$. Then*

$$c_i(W) = \begin{cases} 1 & i = 0 \\ (-1)^{r-1}u^r & i = r \\ 0 & \textit{otherwise} \end{cases}$$

$$e_{jr}(W) = \begin{cases} (-1)^{r-1}u^{r-1}v & j = 1 \\ 0 & j \neq 1 \end{cases}.$$

Proof. Let $\tilde{v} \in H^1(BC; \mathbb{Z}/(q^r - 1))$ be the class of the identity homomorphism, and let $\tilde{u} = \tilde{c}_1(\xi) \in H^2(BC; \mathbb{Z})$.

Since ξ is a dimension 1 representation, the corresponding vector bundle is a line bundle. We have ([Tot14], p.5) that $H^*(BC; \mathbb{Z}) = \mathbb{Z}[\tilde{u}]/(q^r - 1)\tilde{u}$. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(q^r - 1) \to 0$ gives a Bockstein exact sequence

$$H^1(BC; \mathbb{Z}) = 0 \to H^1(BC; \mathbb{Z}/(q^r - 1)) \xrightarrow{\beta_{q^r-1}} H^2(BC; \mathbb{Z}) \xrightarrow{\cdot(q^r-1)} H^2(BC; \mathbb{Z}).$$

It is clear that $q^r - 1 : H^2(BC; \mathbb{Z}) \xrightarrow{q^r-1} H^2(BC; \mathbb{Z})$ is the zero map, so clearly β_{q^r-1} is an isomorphism, and in fact $\beta_{q^r-1}(\tilde{v}) = \tilde{u}$.

Now W is an r-dimensional representation, so the corresponding vector bundle is rank r, so we have $\tilde{c}_i(W) = 0$ for $i > r$. By the product formula for Chern classes (proposition 6), we have (recalling that the vector bundle associated to ξ is a line bundle so its Chern classes are trivial after the first)

$$\tilde{c}_r(W) = \prod_{a=0}^{r-1} \tilde{c}_1(\xi^{q^a}) = \prod_{a=0}^{r-1} q^a \tilde{u} = q^{r(r-1)/2}\tilde{u}^r.$$

Observing that $q^{r(r-1)/2} \equiv (-1)^{r-1} \pmod{l}$, these reduce mod l to give the required formulae for $c_0(W)$ and $c_r(W)$. If $i \not\equiv 0 \pmod{r}$, then $q^i \not\equiv 1 \pmod{l}$, so $c_i(W) = c_i(\Psi^q W) = q^i c_i(W)$ implies that $c_i(W) = 0$.

Now the fact that BC has no odd degree cohomology (see later) means that by the Bockstein exact sequence (7.2) that $\beta_n : H^{2i-1}(BC; \mathbb{Z}/n) \to H^{2i}(BC; \mathbb{Z})$ is injective for all n. This allows equation (7.7) to come into its own: since

$$\beta_{q^i-1}\tilde{e}_i(W) = \tilde{c}_i(W) = \begin{cases} 0 & i > r \\ \beta_{q^r-1}(q^{r(r-1)/2}\tilde{u}^{r-1}\tilde{v}) & i = r \end{cases}$$

(since $\beta_{q^r-1}(\tilde{v}) = \tilde{u}$ and β_n is an $H^*(BC; \mathbb{Z})$-homomorphism), we can deduce that $\tilde{e}_i(W) = 0$ for $(i > r)$ and $\tilde{e}_r(W) = q^{r(r-1)/2}\tilde{u}^{r-1}\tilde{v}$ which reduces mod l as before to give the required formulae. $\square$

7.5. **Step 2: the maps μ and ν.** Let $\mu : BU^2 \to BU$ be the map representing addition in K-theory. We wish to show that there is a map

$$\nu : (F\Psi^q)^2 \to F\Psi^q$$

covering μ, that is, so that $\phi\nu = \mu\phi^2$. The content of this statement is to show that $\mu\phi^2$ is homotopic to its image under the map σ representing Ψ^q, and this homotopy will be $\theta\nu$.

Because Ψ^q is additive on K-theory ([Hat09], p.64), we have that $\mu\sigma^2$ is homotopic to $\sigma\mu$; call this homotopy h. By the definition of ϕ, we have that θ is a homotopy from ϕ to $\sigma\phi$, so overall we have that $\mu\phi^2$ is homotoped by $\mu \cdot \theta^2$ to $\mu\sigma^2\phi^2$, which is homotoped by $h \cdot \phi^2$ to $\sigma\mu\phi^2$, which is what we wanted.

7.6. **Step 3: commutivity of D.** We can take the product of the square (4.1) with itself to obtain a square

$$\begin{array}{ccc} (F\Psi^q)^2 & \longrightarrow & (BU^2)^I \\ {\scriptstyle \phi^2}\downarrow & & \downarrow{\scriptstyle \Delta^2} \\ BU^2 & \xrightarrow[(\mathrm{id},\sigma^2)]{} & BU^2 \times BU^2 \end{array}, \tag{7.9}$$

where we have sometimes exchanged the order the direct product and another operation by making obvious identifications (e.g. between $(BU \times BU)^2$ and $BU^2 \times BU^2$). As in section 7.1.3, we can view this as a morphism $\Gamma^2 : \phi^2 \to \Delta^2$, and as before this gives rise to a map

$$D_{\Gamma^2} : H^{2i}(BU^2 \times BU^2) \supset \ker((\Delta^2)^*, (\mathrm{id}, \sigma^2)) \to H^{2i-1}((F\Psi^q)^2)$$

(recalling that BU, and hence also by the Künneth formula $BU^2 \times BU^2$, has no odd-degree cohomology).

We are trying to investigate the effect of ν^* (arising from adding representations) on $e_{jr} = D_\Gamma(c_{jr} \otimes 1 - 1 \otimes c_{jr})$. We would like to transform this into a question about the action of addition on Chern classes themselves, because we know what that is by proposition 6. The most natural way to try to take the ν^* inside the D_Γ is the following:

Lemma 27. *$\nu^* D_\Gamma = D_{\Gamma^2}(\mu \times \mu)^*$.*

Proof. Because D_g is derived from the map of cohomology long exact sequences induced by g, and hence depends only on the homotopy class of g, it is enough to show that the following square commutes up to homotopy:

$$\begin{array}{ccc} \phi^2 & \xrightarrow{\Gamma^2} & \Delta^2 \\ {\scriptstyle (\mu,\nu)}\downarrow & & \downarrow{\scriptstyle (\mu\times\mu,\, \mu^I)} \\ \phi & \xrightarrow[\Gamma]{} & \Delta \end{array}, \tag{7.10}$$

where the pairs by the vertical arrows are the pairs (g_1, g_2) defining morphisms as in section 7.1.3. We consider the morphisms Γ^2 and $(\mu \times \mu, \mu^I)$:

$$\begin{array}{ccccc}
(F\Psi^q)^2 & \xrightarrow{\theta^2} & (BU^2)^I & \xrightarrow{\mu^I} & BU^I \\
\phi^2 \downarrow & & \downarrow \Delta^2 & & \downarrow \Delta \\
BU^2 & \xrightarrow[(\mathrm{id}, \sigma^2)]{} & BU^2 \times BU^2 & \xrightarrow{\mu \times \mu} & BU \times BU
\end{array},$$

Now there is a homotopy j from $(\mu \times \mu)(\mathrm{id}, \sigma^2)$ to $(\mathrm{id}, \sigma)\mu$, obtained by keeping the first factor constant and applying to the second the homotopy h from $\mu\sigma^2$ to $\sigma\mu$ from section 7.5. By the homotopy lifting property, this lifts to a homotopy starting with $\mu^I\theta^2$, and ending with a homotopy $\theta\nu'$, where ν' covers μ. We can 'unravel' this lifting homotopy, viewing it as a square telling us that the first factor of j (the constant homotopy) is the same as $\mu \cdot \theta^2$, followed by the second factor of j (which is $h \cdot \theta^2$), followed by $\theta\nu'$ backwards. This tells us that $\theta\nu'$ is homotopic to the composition of $\mu \cdot \theta^2$ and $h \cdot \phi^2$, which is $\theta\nu$. We deduce that ν and ν' are homotopic as maps covering μ. Since the construction makes clear that $\Gamma(\mu, \nu')$ and $(\mu \times \mu, \mu^I)\Gamma^2$ are homotopic, we have the lemma. □

7.7. **Step 4: the action of D_{Γ^2} on Chern classes.** We denote the pullbacks by the projections $BU^2 \to BU$ onto the first and second factor by single and double primes respectively.

Lemma 28. *Suppose* $a + b \equiv 0 \pmod r$. *Then*

$$D_{\Gamma^2}(c'_a c''_b \otimes 1 - 1 \otimes c'_a c''_b) = \begin{cases} e'_a(\phi^* c_b)'' + (\phi^* c_a)' e''_b & \text{if } a \equiv 0 \pmod r \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We deal with the first case first. We have

$$D_{\Gamma^2}(c'_a c''_b \otimes 1 - 1 \otimes c'_a c''_b) = D_{\Gamma^2}((c'_a \otimes 1 - 1 \otimes c'_a)(c''_b \otimes 1) + (1 \otimes c'_a)(c''_b \otimes 1 - 1 \otimes c''_b)).$$

By lemma 23.(i) (with $g = \Gamma^2$, so $g_1^* = (\mathrm{id}, \sigma^2)^*$ and $f^* = \phi^*$) we have

$$D_{\Gamma^2}((c'_a \otimes 1 - 1 \otimes c'_a)(c''_b \otimes 1)) = D_{\Gamma^2}(c'_a \otimes 1 - 1 \otimes c'_a)(\phi^* c_b)'',$$

and by naturality of the D-operation,

$$D_{\Gamma^2}(c'_a \otimes 1 - 1 \otimes c'_a) = (D_\Gamma(c_a \otimes 1 - 1 \otimes c_a))' = e'_a,$$

giving the first term in the form required. The second term is similar, whence the result.

In the second case ($a \not\equiv 0 \pmod r$), we have the identity

$$\begin{aligned}(q^a - 1)(c'_a c''_b \otimes 1 - 1 \otimes c'_a c''_b) &= (q^a c'_a \otimes 1 - 1 \otimes c'_a)(c''_b \otimes 1 - 1 \otimes c''_b) \\ &\quad - q^a(q^b c''_b \otimes 1 - 1 \otimes c''_b)(c'_a \otimes 1 - 1 \otimes c'_a)\end{aligned}$$

(just by multiplying out, using the fact that $q^a q^b = q^{a+b} \equiv 1 \pmod l$).

Now, in each term of the above, the first factor is killed by $(\mathrm{id}, \sigma^2)^*$ and the second by $(\Delta^2)^*$. Hence by lemma 23.(ii), the whole thing is killed by by D_{Γ^2}. Since $a \not\equiv 0 \pmod r$, we have that $(q^a - 1)$ is invertible mod l, and the result follows. □

7.8. **Step 5: the product formula.** We are now able to combine the previous two lemmas, and the product formula for Chern classes (proposition 6), to obtain the product formula for the e-classes:

Lemma 29. $\nu^*(e_{jr}) = \sum_{a+b=j} e'_{ar}(\phi^* c_{br})'' + (\phi^* c_{ar})' e''_{br}$.

Proof. We have

$$\begin{aligned}\nu^*(e_{jr}) &= \nu^* D_\Gamma(c_{jr} \otimes 1 - 1 \otimes c_{jr}) \\ &= D_{\Gamma^2}(\mu \times \mu)^*(c_{jr} \otimes 1 - 1 \otimes c_{jr}) \\ &= D_{\Gamma^2}\left(\sum_{a+b=jr} c'_a c''_b \otimes 1 - 1 \otimes c'_a c''_b\right),\end{aligned}$$

where the first equality comes from equation (7.8), the second from lemma 27, the third from the product formula for Chern classes (proposition 6), and the conclusion from lemma 28. □

We can concisely write the product formula for Chern classes as

$$\nu^* c_{jr} = \sum_{a+b=j} c_{ar} \otimes c_{br},$$

and the formula from lemma 29 as

$$\nu^* e_{jr} = \sum_{a+b=j} c_{ar} \otimes e_{br} + e_{ar} \otimes c_{br}.$$

We can combine these by the algebraic trick of writing

$$c_{ts} = 1 + \sum_{j \geq 1} c_{jr} t^j + e_{jr} t^{j-1} s \in H^*(F\Psi^q; \mathbb{F}_l)[t, s]/(s^2),$$

and the two formulae above are then combined in the single formula

$$\nu^* c_{ts} = c_{ts} \otimes c_{ts}. \tag{7.11}$$

7.9. **Conclusion.** By lemma 26, writing $(-1)^{r-1} u^r = x$ and $(-1)^{r-1} u^{r-1} v$, we have

$$c_{ts}(W) = 1 + xt + ys. \tag{7.12}$$

We of course have the map

$$((W^m)^\#)^* : H^*(F\Psi^q) \to H^*(BC^m)$$

associated to the representation $W^m = W_1 \oplus \ldots \oplus W_m$. We are going to show that monomials of the form in lemma 25 formed only from generators c_{jr}, e_{jr} with $j \leq m$ map to independent elements in $H^*(BC^m)$, and hence are independent in $H^*(F\Psi^q)$.

By (7.11), we have

$$c_{ts}(W^m) = \prod_1^m c_{ts}(W_i) = \prod_1^m (1 + x_i t + y_i s),$$

so (by the definition of c_{ts}), we have

$$c_{jr}(W^m) = \sum_{i_1<\ldots<i_j} x_{i_1}\ldots x_{i_j} = \sigma_j$$

$$e_{jr}(W^m) = \sum_{i_1<\ldots<i_j} \sum_{k=1}^{j} x_{i_1}\ldots \hat{x}_{i_k}\ldots x_{i_j} y_{i_k} = d\sigma_j,$$

where σ_j is the jth elementary symmetric polynomial in the x_i, and where we associate y_i with the differential dx_i.

Since C is cyclic, we have that

$$H^*(BC) = P[u] \otimes \wedge[v]$$

(see [Ben91], p.67), so (by the Künneth formula), we have that the subring of $H^*(BC^m)$ generated by the x_i, y_i is isomorphic to $P[x_1, \ldots, x_m] \otimes \wedge[y_1, \ldots, y_m]$. So we just have to prove that the σ_i and $d\sigma_i$ are algebraically independent (in the sense of being isomorphic to the tensor product of the polynomial and alternating algebras on those symbols).

It is well-known that the symmetric polynomials are algebraically independent (see [Lan02], p.192). By the explicit form of $d\sigma_i$ (see the second equality above), we have that the possible coefficients for dx_I are the symmetric polynomials in the $x_i, i \notin I$, so by the independence of the symmetric polynomials, we are done.

We have now proved lemma 25 (because if there were a counterexample to that lemma, then we apply $((W^m)^\#)^*$ with m much larger than j for all the e_{jr}, c_{jr} involved). By theorem 20, which tells us the dimensions of the cohomology groups of $F\Psi^q$, we therefore have that the monomials in lemma 25 are a basis for $H^*(F\Psi^q)$.

Recalling that (since we are assuming that $l \neq 2$), we have that $e_{jr}^2 = 0$ by skew-commutivity of the cup product, we have now proved

Theorem 30. *There is an algebra isomorphism*

$$P[c_r, c_{2r}, \ldots] \otimes \wedge[e_r, e_{2r}, \ldots] \to H^*(F\Psi^q).$$

8. The cohomology of $GL_n K$

In this section, we compute the mod-l cohomology of $GL_n K$ (for l an odd prime not dividing p). We embed a product C^m of cyclic groups, and show that it detects the cohomology of $GL_n K$.

8.1. **Embedding $k(\mu_l)^*$.** We consider the group $C = k(\mu_l)^*$, the multiplicative group of the cyclotomic field $k(\mu_l)$, which is the cyclic group on $q^r - 1$ elements. This acts naturally on $k(\mu_l)$ by multiplication, and considering $k(\mu_l)$ as a k-vector space of dimension r, we have a representation of C. By choosing a k-basis for $k(\mu_l)$, we obtain an embedding $C \hookrightarrow GL_r k$, unique up to change of basis, which corresponds to conjugation in $GL_r k$. This pulls back to a map on cohomology

$$H^*(GL_r k) \to H^*(C).$$

The group $\pi = \mathrm{Gal}(k(\mu_l)/k) \cong \mathbb{Z}_r$ acts on C via the Frobenius isomorphism $z \mapsto z^q$. It is clear that we can extend our representation of C to $\pi \ltimes C$. Since the action of an element of π corresponds to a change of k-basis for $k(\mu_l)$, the corresponding action inside $GL_r k$ is given by conjugation. Since inner automorphisms induce the identity on cohomology, we have that the elements hit by the map are invariant under the action of π, so we have a map to the *invariants* $H^*(C)^\pi$ for the action of π:

$$i^* : H^*(GL_r k) \to H^*(C)^\pi.$$

The cohomology $H^*(C)^\pi$ is generated additively by elements c'_{ir} and e'_{ir} of degrees $2ir$ and $2ir-1$ respectively (see [Tot14], p.5). By section 5.4, the representation L of C used in this section lifts to the representation W used in section 7, so it is clear that under the map on cohomology induced by L these elements are hit by the elements c_{ir} and e_{ir} respectively in $H_*(F\Psi^q)$.

Because the map $H^*(F\Psi^q) \to H^*(C)^\pi$ induced by L factors through i^*, and the former map is surjective (because the generators c'_{ir} and e'_{ir} are hit by elements from $H^*(F\Psi^q)$), it follows that i^* is surjective.

8.2. **The direct product.** Let $n = rm + k$, with $k < r$. If we take the m-direct product of our representation above of $\pi \ltimes C$, we obtain a representation of $(\pi \ltimes C)^m$. Σ_m acts on this group by permuting the factors, so we can extend to a representation of $\Sigma_m \ltimes (\pi \ltimes C)^m$. Thus (taking a direct product with a trivial representation of dimension k to increase the dimension if necessary), we have an embedding

$$\Sigma_m \ltimes (\pi \ltimes C)^m \to GL_n k,$$

unique up to change of basis. Since the action of $\Sigma_m \ltimes \pi^m$ on C^m corresponds to a change of basis for $(k(\mu_l))^m$, we again restrict to invariants in cohomology to obtain a map

$$i_m^* : H^*(GL_n K) \to H^*(C^m)^{\Sigma_m \ltimes \pi^m}. \tag{8.1}$$

8.3. **Calculation of $H^*(C^m)^{\Sigma_m \ltimes \pi^m}$.** By the Künneth formula, $H^*(C^m) \cong (H^*(C))^{\otimes m}$. Restricting to π-invariants commutes with the tensor products, so $H^*(C^m)^{\pi^m}$ has a basis of elements of the form $a_1 \otimes \ldots \otimes a_m$, where $a_i \in \{c'_0, c'_r, \ldots, e'_0, e'_r, \ldots\}$.

Restricting to Σ_m invariants tells us that the order of the C_m factors does not matter, i.e. the order of the elements in the tensor product is insignificant, so $H^*(C^m)^{\Sigma_m \ltimes \pi^m}$ has a basis of elements of the form

$$\left(c_0'^{\otimes i_0}\right) \otimes \left(c_r'^{\otimes i_1}\right) \otimes \ldots \left(e_0'^{\otimes j_1}\right) \otimes \ldots,$$

where we have a total of m elements in the product (i.e. $i_0 + i_1 + \ldots + j_0 + j_1 + \ldots = m$.

To complete our calculation of $H^*(GL_n k)$, we just have to show that the map in (8.1) above is an isomorphism. By the final remark of section 8.1, it is certainly surjective, so it only remains for us to show that it is injective.

8.4. **Injectivity of i_m^*.** We want to show that the map i_m^* in (8.1) above is injective; since the map $i_m^* : H^*(GL_n k) \to H^*(C^m)$ restricts to $H^*(C^m)^{\Sigma_m \ltimes \pi^m}$, it is enough to show that this map is injective.

Given groups $H \leq G$, we say that H *detects* the cohomology of G if the restriction map $H^*(G) \to H^*(H)$ is injective. When does this occur? The following theorem of Cartan & Eilenberg gives one condition.

Theorem 31. *(Cartan-Eilenberg) Let H be a Sylow p-subgroup of a group G, and let M be a G-module. Then the restriction map*

$$i^* : H^*(G, M, p) \to H^*(H, M)$$

is an injection, where $H^*(G, M, p)$ means the p-primary component of $H^*(G, M)$ (that is, those elements of order a power of p).

Proof. [CE56], XII.10. □

Here, we have $M = \mathbb{F}_l$, so all cohomology classes have order l, so the theorem tells us that the mod-l cohomology of $GL_n k$ is detected by a Sylow l-subgroup.

8.5. **The subgroup $\Sigma_m \ltimes C^m$.** By taking the subgroup of $\Sigma_m \ltimes (\pi \ltimes C^m)$ with trivial elements of π, we have an embedding of the group $\Sigma_m \ltimes C^m$ into $GL_n k$. Our first goal is

Lemma 32. *$\Sigma_m \ltimes C_m \hookrightarrow GL_n k$ detects the mod-l cohomology of $GL_n k$.*

Proof. By theorem 31, it is enough to show that $\Sigma_m \ltimes C_m$ contains a Sylow l-subgroup of $GL_n k$. Since the order of a Sylow p-subgroup of a group G is the highest power of p dividing $|G|$, it is enough to show that the highest power of l to divide $|GL_n k|$ also divides $|\Sigma_m \ltimes C_m|$ (equivalently, that the index of $\Sigma_m \ltimes C_m$ in $GL_n k$ is coprime to l). Denote the highest power of l to divide a number a by $v_l(a)$: that is, $v_l(a)$ is maximal such that $l^{v_l(a)}$ divides a (the 'l-adic valuation'). Then what we want to show is that $v_l(|GL_n k|) \leq v_l\left(|\Sigma_m \ltimes C_m|\right)$.

To compute the order of GL_nk, we observe that once the first i columns of an element have been specified, the $(i+1)$th column must be outside their span, giving $q^n - q^i = q^i(q^{n-i}-1)$ choices, so we have

$$|GL_nk| = \prod_{i=0}^{n-1} q^i(q^{n-i}-1) = q^{\sum_{i=0}^{n-1} i} \prod_{i=0}^{n-1}(q^{n-i}-1)$$
$$= q^{n(n-1)/2}\prod_{j=1}^{n}(q^j-1).$$

Clearly l does not divide $q^{n(n-1)/2}$, and by definition of r (minimal such that $l|q^r-1$), l divides only those terms in the product with $j = ir$ for some $1 \le i \le m$ (recalling the bounds on m and k at the beginning of this section).

The order of $\Sigma_m \ltimes C^m$ is clearly

$$m!(q^r-1)^m.$$

Matching factors, it is enough to show that

$$v_l(q^{ir}-1) \le v_l(i(q^r-1)).$$

By definition of r, let $q^r = 1 + al$. By inductively removing powers of l one at a time, we may assume that l^2 does not divide i; that is, either l does not divide i, or $i = cl$ for some c not divisible by l.

In the first case, we have

$$v_l(q^{ir}-1) = v_l((1+al)^c - 1) = v_l\left(cal + \binom{c}{2}(al)^2 + \ldots + (al)^c\right)$$
$$= v_l(al) = v_l(i(q^r-1))$$

(the second equality coming from cancelling the 1s, and the third equality from the fact that the second and subsequent terms are all divisible by $(al)^2$, and c is coprime to l).

In the second case, we similarly have

$$v_l(q^{ir}-1) = v_l((1+al)^{cl}-1)$$
$$= v_l\left(clal + \binom{cl}{2}(al)^2 + \binom{cl}{3}(al)^3 + \ldots + (al)^{cl}\right)$$
$$= v_l(cal^2) = 1 + v_l(al) = v_l(i(q^r-1))$$

(the third equality making use of the assumption that $l \neq 2$, so that $\binom{cl}{2}$ is divisible by l). □

We now show that it follows that the cohomolgy of GL_nk is detected by C^m using the following purely algebraic lemma:

Lemma 33. *Any abelian subgroup of GL_nk of exponent dividing l^a for some a (i.e. $g^{l^a} = 1$ for all g in the subgroup) is conjugate to a subgroup of C^m.*

Proof. Omitted. See Quillen's paper, lemma 12. □

The plan is to show that the cohomology of $GL_n k$ is indeed detected by abelian subgroups of exponent dividing l^a. This shows (recalling that conjugation acts trivially on homology) that any cohomology class restricting to zero in $H^*(C^m)$ is trivial. We use the following fact:

Lemma 34. *Let G be a group whose mod-l cohomology is detected by abelian subgroups of exponent dividing l^a with $a \geq 1$. Then the semi-direct product $\Sigma_m \ltimes G^m$ has the same property.*

Proof. [Qui71], Proposition 3.4. □

We are now done: lemma 32 shows that the cohomology of $GL_n k$ is detected by that of $\Sigma_m \ltimes C^m$. The mod-l cohomology of C^m is detected by Sylow l-subgroups (by theorem 31), which are abelian since C^m is. By lemma 34 the cohomology of $\Sigma_m \ltimes C^m$ is therefore also detected by abelian subgroups of exponent dividing l^a, which by lemma 33 are conjugate to subgroups of C^m. Hence the mod-l cohomology of $GL_n k$ is detected by C^m. We have established the following theorem, the goal of this section

Theorem 35. *The restriction map*

$$i_m^* : H^*(GL_n k) \to H^*(C^m)^{\Sigma_m \ltimes \pi^m}$$

is an algebra isomorphism.

9. Conclusion

We now have maps

$$H^*(F\Psi^q) \xrightarrow{f^*} H^*(GL_n k) \xrightarrow{i_m^*} H^*(C^m)^{\Sigma_m \ltimes \pi^m},$$

where f^* is the pullback of the map $BGL_n k \to F\Psi^q$ derived from the standard representation of $GL_n k$. By theorem 35, i_m^* is an isomorphism. By theorem 30, the composition $i_m^* f^*$ is surjective, and also every element of $H^*(F\Psi^q)$ maps to a non-zero element for sufficiently large m. We therefore deduce, taking $m, n \to \infty$, that

$$f^* : H^*(F\Psi^q) \to H^*(GLk)$$

is an isomorphism.

We now show that this means that f also induces an isomorphism on homology:

Lemma 36. *Let k be a field, and let X and Y be spaces with finitely generated cohomology groups $H^n(X;k), H^n(Y;k)$. Suppose $g : X \to Y$ is a map with $g^* : H^*(Y;k) \to H^*(X;k)$ an isomorphism. Then $g_* : H_*(X;k) \to H_*(Y;k)$ is an isomorphism.*

Proof. By passing to a mapping cyclinder, we may assume that f is an inclusion, so it is enough to show that if $H^*(Y,X;k) = 0$ then $H_*(Y,X;k) = 0$. The universal coefficient theorem for cohomology ([Hat02], p.195) gives a split exact sequence (with integral coefficients where unspecified)

$$0 \to \mathrm{Ext}(H_{n-1}(Y,X),k) \to H^n(Y,X;k) \to \mathrm{Hom}(H_n(Y,X),k) \to 0,$$

so in particular $\mathrm{Hom}(H_n(Y,X),k)$ is a direct summand of $H^n(Y,X;k)$, so if $H^n(Y,X;k) = 0$ then $\mathrm{Hom}(H_n(Y,X),k) = 0$, so $H_n(Y,X;k) = 0$. □

Since our calculation was for cohomology with coefficients in the field $\mathbb{F}_l$, for l a prime not dividing p, this lemma tells us that $f_* : H_*(GLk) \to H_*(F\Psi^q)$ is an isomorphism. But to apply Whitehead's theorem, we need an isomorphism on *integral* homology. Fortunately, we have the following consequence of the universal co-efficient theorem for homology:

Theorem 37. *A map $f : X \to Y$ induces isomorphisms on integral homology if and only if it induces isomorphisms on homology with co-efficients in $\mathbb{Q}$ and in $\mathbb{F}_{p'}$ for all primes p'.*

Proof. [Hat02], p.266. □

The case l not dividing p is above, the main work of this essay (although the technicalities of the case $l = 2$ were not dealt with). In section 6.1, we showed that the homology of $F\Psi^q$ with coefficients in $\mathbb{Q}$ and $\mathbb{F}_p$ is zero. Also, GLk is the union of finite groups GL_nk, and hence has zero rational homology (because for a finite group G, the Eilenberg-Maclane space $K(G,1)$ has contractible $|G|$-sheeted universal cover, so the cohomology of $K(G,1)$ only has elements of torsion dividing $|G|$—see [Hat02], p.321—so has zero rational homology). It can also be shown that GL_nk has no mod p homology—it is in section 11 of Quillen's paper, but is omitted here for reasons of space. We can now apply the above theorem to conclude that f_* is an isomorphism on integral homology.

We are hoping to apply Whitehead's theorem:

Theorem 38. *Let $f : X \to Y$ be a map between path-connected simple CW-complexes, such that $f_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is an isomorphism. Then f is a homotopy equivalence (i.e. $f_* : \pi_*(X) \to \pi_*(Y)$) is an isomorphism.*

Proof. [Ben98], p.12&15. □

Currently, our map f goes from $BGLk$ rather than $BGLk+$, as we want. But help is at hand: since $\pi_1(BGLk) = GLk$ and $\pi_1(F\Psi^q) = \mathbb{Z}/(q-1)$ and f_* is a homology isomorphism, we have that f_* on π_1 is the determinant map, so kills the commutator subgroup of $\pi_1(BGLk)$. Part

(iv) of theorem 9 then tells us that f extends to a map $f : BGLk^+ \to F\Psi^q$ which is a homology isomorphism. $BGLk^+$ and $F\Psi^q$ are simple, so Whitehead's theorem tells us that f is a homotopy equivalence, and theorem 1 is proved. □

References

[Ben91] David J Benson, *Representations and cohomology: Volume 1, basic representation theory of finite groups and associative algebras*, Cambridge University Press, 1991.

[Ben98] ———, *Representations and cohomology: Volume 2, cohomology of groups and modules*, Cambridge University Press, 1998.

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Mathematical Series, no. 19, Princeton University Press, 1956.

[Gre55] James A Green, *The characters of the finite general linear groups*, Transactions of the American Mathematical Society (1955), 402–447.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hat04] ———, *Spectral sequences in algebraic topology*, WWW, 2004, Unfinished; generously made available from the author's webpage `http://www.math.cornell.edu/~hatcher/`.

[Hat09] ———, *Vector bundles and k-theory*, WWW, 2009, Unfinished; generously made available from the author's webpage `http://www.math.cornell.edu/~hatcher/`.

[Lan02] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, no. 211, Springer-Verlag, 2002.

[May11] JP May, *Notes on tor and ext*, `http://www.math.uchicago.edu/~may/MISC/TorExt.pdf`, 2011.

[Mil56] John Milnor, *Construction of universal bundles, ii*, Annals of Mathematics (1956), 430–436.

[MT68] Robert E Mosher and Martin C Tangora, *Cohomology operations and applications in homotopy theory*, Courier Dover Publications, 1968.

[Qui71] Daniel Quillen, *The adams conjecture*, Topology **10** (1971), no. 1, 67–80.

[Qui72] ———, *On the cohomology and k-theory of the general linear group over a finite field*, Ann. Math **96** (1972), 552–586.

[Tot14] Burt Totaro, *Group cohomology and algebraic cycles*, Cambridge University Press, 2014.

[Wei13] Charles A Weibel, *The k-book: An introduction to algebraic k-theory*, vol. 145, American Mathematical Soc., 2013.