ESSAY

The Heat Equation and the Atiyah-Singer Index Theorem

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Introduction

The Atiyah-Singer index theorem is a milestone of twentieth century mathematics. Roughly speaking, it relates a global analytical datum of a manifold - the number of solutions of a certain linear PDE - to an integral of local topological expressions over this manifold. The index theorem provided a link between analysis, geometry and topology, paving the way for many further applications along these lines.

An operator $T : H_1 \to H_2$ between Hilbert spaces is called Fredholm if both its kernel and its cokernel are finite dimensional. The index of such an operator is defined to be the difference between these two quantities. Every elliptic differential operator between vector bundles over a compact manifold defines a Fredholm operator and therefore has a finite index. The index is a very well behaved analytical quantity which is, for example, stable under compact perturbations. This prompted Israel Gelfand in 1960 to conjecture that the index of an elliptic operator is a topological invariant and to ask for an explicit expression of this index in terms of other invariants.

The easiest example of such an 'index theorem' is the Toeplitz theorem.
Let $L^2(S^1) = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\}$ and let $\mathcal{H} := \{\sum_{n=0}^{\infty} a_n z^n \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$ be the Hardy space with projector $\Pi : L^2(S^1) \to \mathcal{H}$. For $f \in C(S^1)$, the Toeplitz operator is the map $T_f := \Pi\, M_f|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$, where $M_f : L^2(S^1) \to L^2(S^1)$ denotes the multiplication operator with f. Using basic properties of the index one can show that T_f is Fredholm if and only if f is non-vanishing and

$$\operatorname{ind}(T_f) = -\text{ winding number of } f \text{ around } 0. \tag{1}$$

If f is differentiable, we can use the logarithmic derivative and express this as $\operatorname{ind}(T_f) = -\frac{1}{2\pi i}\int_{S^1} \frac{f'}{f}\,\mathrm{d}z$.

In guise of the 'classical index theorems' - the Signature theorem, the Chern-Gauss-Bonnet theorem and the Hirzebruch-Riemann-Roch theorem - more complicated index theorems had already been known for specific elliptic operators.

In 1963, Michael Atiyah and Isadore Singer solved Gelfand's problem and announced their theorem, expressing the index of a general elliptic operator on a compact oriented manifold in terms of certain characteristic classes - subsequently dubbed 'topological index' - of this manifold.
In the following essay, we explain how this can be done for a particular class of elliptic operators - twisted Dirac operators on even dimensional compact spin manifolds - and then indicate how this solves the general problem.
We will now give a brief overview of our main results.

Spin Geometry

The Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(V)$ of a vector space V with inner product $(\cdot,\cdot)$ is the complex algebra generated by vectors $v \in V$ with relations

$$v_1 \cdot v_2 + v_2 \cdot v_1 = -2(v_1, v_2). \tag{2}$$

As a vector space, the Clifford algebra is isomorphic to the (complex) exterior algebra ΛV. Using an orthonormal basis $e_1, \ldots, e_n$ of V, this isomorphism is given by

$$\sigma : \mathrm{Cl}_{\mathbb{C}}(V) \to \Lambda V \qquad e_{i_1} \cdots e_{i_k} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_k}. \tag{3}$$

If M is a Riemannian manifold we define the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(M)_x$ at the point x to be the Clifford algebra of T_x^*M with the induced Riemmanian metric. The Levi-Civita connection on TM extends to a connection on the exterior bundle ΛT^*M which induces a connection ∇ on the Clifford algebra bundle $\mathrm{Cl}_{\mathbb{C}}(M)$.

A hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ with compatible connection $\nabla^{\mathcal{E}}$ is said to be a Clifford bundle if there is an algebra bundle homomorphism $c : \mathrm{Cl}_{\mathbb{C}}(M) \to \mathrm{End}(\mathcal{E})$ such that for a section α of T^*M and $\epsilon, \epsilon' \in \mathcal{E}$

$$(1)\ c(\alpha) \text{ swaps } \mathcal{E}^+ \text{ and } \mathcal{E}^-, \quad (2)\ \big(c(\alpha)\epsilon, \epsilon'\big) + \big(\epsilon, c(\alpha)\epsilon'\big) = 0, \quad (3)\ [\nabla^{\mathcal{E}}_X, c(\alpha)] = c(\nabla_X \alpha). \tag{4}$$

The Dirac operator D on a Clifford bundle $\mathcal{E}$ is the formally self-adjoint elliptic operator defined in terms of a local frame $e_1, \dots, e_n$ of TM (with corresponding dual frame $e^1, \dots, e^n$) as

$$D := \sum_{i=1}^{\dim(M)} c(e^i)\nabla^{\mathcal{E}}_{e_i}. \tag{5}$$

In this essay, we will focus on a specific example of a Clifford bundle - the twisted spinor bundle on an even dimensional spin manifold. A spin manifold is an oriented Riemannian manifold fulfilling certain topological conditions. On an even dimensional spin manifold, there exists a Clifford bundle $\$ = \$^+ \oplus \$^-$ such that $c : \mathrm{Cl}_{\mathbb{C}}(M) \to \mathrm{End}(\$)$ is an isomorphism. In fact, the fibrewise representation $\$_x$ of $\mathrm{Cl}_{\mathbb{C}}(M)_x$ is the unique irreducible representation of the Clifford algebra. The Dirac operator $\not{D}$ on this so called spinor bundle is formally self-adjoint and maps sections of $\$^{\pm}$ to $\$^{\mp}$. If $\mathcal{V}$ is a hermitian vector bundle with compatible connection, we can define the twisted spinor bundle

$$\mathcal{E} = \$ \oplus \mathcal{V} = \left(\$^+ \otimes \mathcal{V}\right) \oplus \left(\$^- \otimes \mathcal{V}\right) \tag{6}$$

with corresponding Dirac operator $\not{D}_{\mathcal{V}}$.
If M is even-dimensional and spin, every Clifford bundle is of this form.

The Atiyah-Singer Index Theorem

Since the Dirac operator on a twisted spinor bundle $\mathcal{E} = \$ \oplus \mathcal{V}$ is self-adjoint, its index vanishes. To get an operator with non-trivial index we split $\mathcal{E} = \left(\$^+ \otimes \mathcal{V}\right) \oplus \left(\$^- \otimes \mathcal{V}\right)$ and write

$$\not{D}_{\mathcal{V}} = \begin{pmatrix} 0 & \not{D}^+_{\mathcal{V}} \\ \not{D}^-_{\mathcal{V}} & 0 \end{pmatrix}. \tag{7}$$

The operators $\not{D}^{\pm}_{\mathcal{V}}$ are called chiral Dirac operators.

The Atiyah-Singer index theorem states that the index of the chiral Dirac operator of a twisted spinor bundle $\$ \otimes \mathcal{V}$ on an even dimensional compact spin manifold M^n is given by

$$\mathrm{ind}(\not{D}^+_{\mathcal{V}}) = (2\pi i)^{-\frac{n}{2}} \int_M \left(\widehat{A}(M) \wedge \mathrm{ch}(\mathcal{V})\right)_{[n]}. \tag{8}$$

Here, $\widehat{A}(M) := \det^{\frac{1}{2}}\left(\frac{R/2}{\sinh(R/2)}\right) \in \mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is the $\widehat{A}-$genus of M and $\mathrm{ch}(\mathcal{V}) := \exp(-K) \in \mathrm{H}^\bullet_{\mathrm{dR}}(M)$ is the Chern character of $\mathcal{V}$ with R being the Riemann curvature of M and K the curvature of $\mathcal{V}$. The map $(\cdot)_{[n]} : \mathrm{H}^\bullet_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(M)$ denotes the projection of a form to its n-form component.

The heat equation proof of this formula is based on the realisation that

$$\mathrm{ind}(\not{D}^+_{\mathcal{V}}) = \mathrm{Tr}(e^{-t\not{D}^-_{\mathcal{V}}\not{D}^+_{\mathcal{V}}}) - \mathrm{Tr}(e^{-t\not{D}^+_{\mathcal{V}}\not{D}^-_{\mathcal{V}}}) \qquad \forall t > 0. \tag{9}$$

This follows from the fact that the non-zero eigenspaces of $\not{D}^2_{\mathcal{V}}\Big|_{\$^-} = \not{D}^+_{\mathcal{V}}\not{D}^-_{\mathcal{V}}$ and $\not{D}^2_{\mathcal{V}}\Big|_{\$^+} = \not{D}^-_{\mathcal{V}}\not{D}^+_{\mathcal{V}}$ are isomorphic. Thus, the only contribution from the right hand side is $\dim(\ker(\not{D}^-_{\mathcal{V}}\not{D}^+_{\mathcal{V}})) - \dim(\ker(\not{D}^+_{\mathcal{V}}\not{D}^-_{\mathcal{V}})) = \mathrm{ind}(\not{D}^+_{\mathcal{V}})$. The graded trace $\mathrm{Tr}_S(e^{-t\not{D}^2_{\mathcal{V}}}) := \mathrm{Tr}(e^{-t\not{D}^-_{\mathcal{V}}\not{D}^+_{\mathcal{V}}}) - \mathrm{Tr}(e^{-t\not{D}^+_{\mathcal{V}}\not{D}^-_{\mathcal{V}}})$ is called the supertrace.

This reduces the calculation of the index $\operatorname{ind}(\not{D}_{\mathcal{V}}^{+})$ to the study of the heat operator $e^{-t\not{D}_{\mathcal{V}}^{2}}$. Using Sobolev theory, we will show that the heat operator has a Clifford algebra valued integral kernel $p_t(x,y)$ with Mercer's theorem implying that

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^{+}) = \operatorname{Tr}_S(e^{-t\not{D}_{\mathcal{V}}^{2}}) = \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x. \tag{10}$$

Since the left hand side is independent of t, it suffices to know the small t behaviour of the heat kernel. It has an asymptotic expansion

$$p_t(x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j B_j(x,x) \qquad (t \to 0), \tag{11}$$

where the coefficients B_j only depend on local curvature and metric terms and can be computed recursively. Therefore,

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^{+}) = \lim_{t\to 0} \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (4\pi)^{-\frac{n}{2}} \int_M \operatorname{tr}_S(B_{\frac{n}{2}})\,\mathrm{d}x. \tag{12}$$

For small n, the recursion relation determining $B_{\frac{n}{2}}$ can be solved explicitly, yielding an expression for the index. However, for arbitrary n, this direct approach becomes intractable.

Supersymmetry and Rescaling

The problem of determining the coefficient $\operatorname{tr}_S(B_{\frac{n}{2}})$ was solved by Ezra Getzler using a scaling argument based on Witten's ideas on supersymmetry.

This can be motivated from the following observation.
The Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2n)$ is generated by an orthonormal basis $e_1, \dots, e_{2n}$ together with relations

$$e_i e_j + e_j e_i = -2\delta_{ij}. \tag{13}$$

Changing basis to $q_i = \frac{1}{2}(e_i - ie_{i+n}),\ p_i := \frac{1}{2}(e_i + ie_{i+n})$ for $1 \leqslant i \leqslant n$, these relations become

$$q_i q_j + q_j q_i = 0, \qquad p_i p_j + p_j p_i = 0, \qquad q_i p_j + p_j q_i = -\delta_{ij}. \tag{14}$$

Up to a factor of $-i\hbar$, these are just the canonical anticommutation relations (CAR) describing a quantum mechanical system of fermions with n degrees of freedom. From this point of view, the isomorphism $\Lambda\mathbb{C}^{2n} \to \mathrm{Cl}_{\mathbb{C}}(2n)$ can be seen as a quantisation map, mapping the classical anticommutation relations ($e_i e_j + e_j e_i = 0$) to the CAR.

Instead of the CAR, we could equally well consider the canonical commutation relations (CCR) describing a system of n bosons

$$q_i q_j - q_j q_i = 0, \qquad p_i p_j - p_j p_i = 0, \qquad q_i p_j - p_j q_i = -\delta_{ij}. \tag{15}$$

The complex algebra generated by these relations is the Weyl algebra $\mathcal{W}(2n)$.
All results obtained for the Clifford algebra (i.e. for fermions) can be transferred to results for the Weyl algebra (i.e. for bosons). For example, while the Clifford algebra is a quantisation of the exterior algebra, the Weyl algebra is a quantisation of the symmetric algebra. The unique irreducible representation of the Clifford algebra is Clifford multiplication c on spinors. The analogous irreducible representation of the Weyl algebra is given by the vector space $\mathbb{C}[z_1, \dots, z_n]$ of complex polynomials in n variables with action

$$q_i \mapsto z_i \cdot, \qquad\qquad p_i \mapsto \frac{\partial}{\partial z_i}. \tag{16}$$

The analogies between Clifford and Weyl algebra can be pushed much further. The idea to treat fermions and bosons on a completely equal footing is called supersymmetry. From a supersymmetric point of view, Clifford multplication (fermions) and differential operators (bosons) are to be treated equivalently. The Dirac operator $D = \sum_{i=1}^{n} c(e^i)\nabla^{\mathcal{E}}_{e_i}$ - being a perfect pairing of Clifford multiplication and covariant derivative - is an example of a supersymmetric operator.

To find the expansion coefficient $\operatorname{tr}_S(B_{\frac{n}{2}})$, Getzler introduced a scaling parameter u^2 into the Clifford relations

$$v \cdot w + w \cdot v = -2u^2\,(v,w) \tag{17}$$

(which is morally just Planck's constant $\hbar$) and considered the classical limit $u \to 0$ in which the Clifford algebra degenerates to the exterior algebra. To preserve supersymmetry, he rescaled differential operators (and thus spacetime $\mathbb{R}_{>0} \times M$) accordingly.

It turns out that the rescaled heat kernel $p^u_{t=1}(x,x)$ has the term $(-2i)^{-\frac{n}{2}} \operatorname{tr}_S(B_{\frac{n}{2}})$ placed in leading order in an asymptotic expansion in the scaling parameter u. On the same time, the rescaled kernel p^u_t fulfills the heat equation of an appropriately rescaled heat operator L^u. In the $u \to 0$ limit this rescaled heat operator approaches the operator

$$L^0 = -\sum_{i=1}^{n} \left(\partial_i - \sum_{j=1}^{n} R_{ij} x_j \right)^2 + K, \tag{18}$$

which is a generalized harmonic oscillator, a matrix version of the usual harmonic oscillator $H = -\frac{d^2}{dx^2} + a^2 x^2$.

Given our quantum mechanical approach to the scaling argument, the appearance of this operator shouldn't come as too much of a surprise. It also could have been expected from a more mathematical perspective since the harmonic oscillator is (up to the constant K) a quadratic element of the Weyl algebra. The quadratic elements of both Clifford and Weyl algebra form closed Lie subalgebras and therefore occupy somewhat special positions.

Its heat kernel can be calculated explicitly (Mehler's formula). On the diagonal it is given by

$$p^0_t(x,x) = \det{}^{\frac{1}{2}} \left(\frac{tR/2}{\sinh(tR/2)} \right) \exp(-tK). \tag{19}$$

Setting $t = 1$ yields $\operatorname{tr}_S(B_{\frac{n}{2}}) = (-2i)^{\frac{n}{2}} \widehat{A}(M) \wedge \operatorname{ch}(\mathcal{V})$. The index theorem then follows from equation (12).

Applications and the General Index Theorem

Many geometrical first order differential operators can be expressed in terms of Dirac operators on Clifford bundles. For example, let X be a complex manifold, $\mathcal{V}$ be a hermitian vector bundle and consider the Dolbeault complex

$$0 \to \Omega^{0,0}(\mathcal{V}) \xrightarrow{\overline{\partial}} \Omega^{0,1}(\mathcal{V}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n}(\mathcal{V}) \to 0, \tag{20}$$

where $\Omega^{0,i}(\mathcal{V})$ denotes the space of $(0,i)-$ forms with values in the vector bundle $\mathcal{V}$. Then, the combined operator

$$\overline{\partial} + \overline{\partial}^* : \bigoplus_{j=0}^{n} \Omega^{0,j}(\mathcal{V}) \to \bigoplus_{j=0}^{n} \Omega^{0,j}(\mathcal{V}) \tag{21}$$

is a Dirac operator on the Clifford bundle $\mathcal{E} = \bigoplus_{j=0}^n \Omega^{0,j}(\mathcal{V}) = \mathcal{E}^+ \oplus \mathcal{E}^- = \bigoplus_{j \text{ even}} \Omega^{0,j}(\mathcal{V}) \oplus \bigoplus_{j \text{ odd}} \Omega^{0,j}(\mathcal{V})$.

On an even dimensional spin manifold, all Clifford bundles are in fact twisted spinor bundles, such that the index problem for all these operators is covered by (8). Since the heat equation proof is inherently local and any manifold is locally a spin manifold, the statement of (8) can easily be generalised to Dirac operators on Clifford bundles on possibly non-spin manifolds.
In the case of the Dolbeault Dirac operator (21), the index theorem yields

$$\operatorname{ind}\left(\overline{\partial} + \overline{\partial}^* \Big|_{\bigoplus_{j \text{ even}} \Omega^{0,j}(\mathcal{V})} \right) = (2\pi i)^{-\dim_{\mathbb{C}}(X)} \int_X \operatorname{td}(T^{1,0}X) \wedge \operatorname{ch}(\mathcal{V}), \tag{22}$$

where $\operatorname{ch}(\mathcal{V})$ is the chern class and $\operatorname{td}(T^{1,0}X)$ is the so called Todd class of the holomorphic tangent bundle $T^{1,0}(X)$, which is for example defined in [7].
Since the index of the operator $\partial + \overline{\partial}^* : \bigoplus_{j \text{ even}} \Omega^{0,j}(\mathcal{V}) \to \bigoplus_{j \text{ odd}} \Omega^{0,j}(\mathcal{V})$ is just the Euler characteristic of the Dolbeault complex $\chi(X,\mathcal{V}) = \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}} \left(\mathrm{H}^i(X,\mathcal{V})\right)$, this yields the Hirzebruch-Riemann-Roch theorem

$$\chi(X,\mathcal{E}) = (2\pi i)^{-\dim_{\mathbb{C}}(X)} \int_X \operatorname{td}(T^{1,0}(X)) \wedge \operatorname{ch}(\mathcal{V}). \tag{23}$$

Using a similar reasoning, the Signature theorem and the Chern-Gauss-Bonnet theorem can be derived from the Atiyah-Singer index theorem for Dirac operators on Clifford bundles.

But even more is true. Let M be a compact even-dimensional spin manifold. Introducing the group $K(M)$ of equivalence classes of vector bundles on M and the group $\mathrm{Ell}(M)$ of abstract elliptic operators, it can be proven that the map $K(M) \to \mathrm{Ell}(M),\ [\mathcal{V}] \mapsto D\!\!\!/_{\mathcal{V}}$ is an isomorphism. Therefore, every elliptic operator on an even-dimensional spin manifold is generated by a twisted Dirac operators. In this sense, the class of twisted Dirac operators is fundamental among elliptic operators and the index theorem (8) actually solves the index problem for general elliptic operators on even-dimensional compact spin manifolds.

Even though it is a statement about linear differential operators, the index theorem can also be used to study non-linear PDEs. In fact, applying it to a linearised version of a non-linear partial differential operator yields the local dimension of the solution manifold of this operator. This is a pivotal technique used for example in Donaldson theory and Seiberg-Witten theory.

Outline

In chapter one, we introduce the intriguing subject of spin geometry. We provide background on Clifford algebras, spin groups and spinor representations and discuss these concepts in a geometrical setting, introducing spin manifolds and Dirac operators. Our exposition mainly follows [6] with some borrowings from [2] and [7].
The proof of the index theorem for Dirac operators - indisputably the core of this essay - is presented in chapter two. We first introduce analytical techniques such as Sobolev and Fredholm theory, mainly following [3] and [8]. Our subsequent proof of the index theorem is based on the expositions in [2] and [3] with valuable amendments both from [8] and Getzler's original paper [4].
In the final chapter, we present several applications of the index theorem, including a proof of the Riemann-Roch theorem for Riemann surfaces and a brief summary on how the index theorem is used in the study of solution spaces of non-linear PDEs. Finally, we outline how the index theorem for Dirac operators can be generalised and how it is used in the proof of the index theorem for general elliptic operators.

Chapter 1

Spin Geometry

The concept of *spin* has its roots in the early years of quantum mechanics, when Wolfang Pauli - in order to formulate his exclusion principle - introduced an additional internal degree of freedom for the electron.
From a modern point of view, this additional degree of freedom comes from the fact that the rotationally invariant electron transforms under a projective representation of the group SO_3, or equivalently under an ordinary representation of its universal covering group. Consequently, this covering group came to be known as the Spin-group.
These ideas became further consolidated, when in 1928 Paul Dirac set out to find a relativistic theory of the electron. In search for this theory, Dirac was faced with the problem of finding a linear partial differential operator which squares to the Laplacian. He realised that this was only possible if he allowed the operator $\sum_i \gamma^i \partial_i$ to have coefficients in some non-commutative algebra. Equating the square of this operator with the Laplacian

$$\left(\sum_{i=1}^{n} \gamma^i \partial_i\right)^2 = \frac{1}{2} \sum_{i,j=1}^{n} \left(\gamma^i \gamma^j + \gamma^j \gamma^i\right) \partial_i \partial_j \overset{!}{=} \Delta = -\sum_{i=1}^{n} \partial_i^2 \tag{1.1}$$

yields

$$\gamma^i \gamma^j + \gamma^j \gamma^i = -2\delta^{ij}. \tag{1.2}$$

This is the famous Clifford algebra, initially discovered by William Clifford in 1878 and rediscovered by Dirac in 1928. This algebra, liying at the heart of spin geometry will be the starting point for our subsequent discussions.

In the first half of the following chapter we will examine its algebraic properties and define the Spin and Pin group and their representations. In the second half, we establish these notions in a geometrical context and discuss spin manifolds and Dirac operators.

1.1 Clifford Algebras

1.1.1 Basic Definitions and Properties

In the following section, let V be a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $B : V \times V \to \mathbb{K}$ be a symmetric bilinear (possibly non-degenerate) form. Consider the associated quadratic form $Q : V \to \mathbb{K}$, given by $Q(v) = B(v, v)$ for $v \in V$. We can reconstruct the bilinear form B from Q by the polarisation identity

$$B(v, w) = \frac{1}{2} \left(Q(v + w) - Q(v) - Q(w)\right). \tag{1.3}$$

Thus, quadratic forms and symmetric bilinear forms are essentially the same. Abusing notation we will denote both the quadratic form and the bilinear form on V by Q.
Let $T^{\bullet}V := \bigoplus_{n=0}^{\infty} V^{\otimes n}$ denote the tensor algebra of V.

Definition 1.1. Let I_Q be the two sided ideal of $T^\bullet V$ generated by elements of the form

$$v \otimes v + Q(v) \in T^2 V \oplus T^0 V. \tag{1.4}$$

We define the *Clifford algebra* as

$$\mathrm{Cl}(V, Q) := T^\bullet V \big/ I_Q \ . \tag{1.5}$$

Due to the polarisation identity, the ideal I_Q also contains all elements of the form $\frac{1}{2}(v \otimes w + w \otimes v) + Q(v, w)$ for $v, w \in V$. Thus, the relation

$$vw + wv = -2Q(v, w) \qquad v, w \in V \tag{1.6}$$

follow from the relations $v^2 = -Q(v)$.
Given an orthogonal basis $e_1, \ldots, e_n$ of V, a corresponding basis for the Clifford algebra $\mathrm{Cl}(V, Q)$ is given by

$$\{e_I := e_{i_1} \cdots e_{i_p} \mid I = (i_1, \ldots, i_p) \text{ with } 1 \leqslant i_1 < \ldots < i_p \leqslant n \text{ and } 0 \leqslant p \leqslant n\}. \tag{1.7}$$

It follows that $\dim \mathrm{Cl}(V, Q) = 2^{\dim V}$.

One can also define Clifford algebras in terms of a universal property.

Definition 1.2. Let A be an associative unital $\mathbb{K}$-algebra.
We call a $\mathbb{K}$-linear map $\phi : V \to A$ *Clifford*, if for all $v \in V$

$$\phi(v)^2 = -Q(v)\, 1_A. \tag{1.8}$$

Note that the inclusion $V \hookrightarrow \mathrm{Cl}(V, Q)$ is an injective Clifford map.

Proposition 1.3. *For every Clifford map $\phi : V \to A$ into an arbitrary associative unital $\mathbb{K}-$algebra, there is a unique algebra homomorphism $\widetilde{\phi} : \mathrm{Cl}(V, Q) \to A$ extending ϕ, i.e. such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Cl}(V,Q) & & \\ \big\uparrow & \searrow^{\widetilde{\phi}} & \\ V & \xrightarrow{\ \phi\ } & A \end{array} \tag{1.9}$$

The Clifford algebra is uniquely (up to algebra isomorphisms) determined by this property.

Proof. By the universal property of the tensor algebra $T^\bullet V$, every linear map $\phi : V \to A$ lifts to a unique algebra homomorphism $\widetilde{\phi} : T^\bullet V \to A$. If ϕ is Clifford, the map $\widetilde{\phi}$ factors through the ideal I_Q and thus defines a unique algebra homomorphism $\mathrm{Cl}(V, Q) \to A$. ☐

Proposition 1.4. *Every linear isometry $f : (V, Q_V) \to (W, Q_W)$ between quadratic vector spaces extends to a unique algebra homomorphism $\mathrm{Cl}(f) : \mathrm{Cl}(V, Q_V) \to \mathrm{Cl}(W, Q_W)$.*

Proof. Since f is an isometry, it follows that

$$f(v)^2 = -Q_W(f(v)) 1_{\mathrm{Cl}(W,Q_W)} = -Q_V(v) 1_{\mathrm{Cl}(W,Q_W)} \qquad \forall v \in V.$$

Therefore, $f : V \to W \hookrightarrow \mathrm{Cl}(W, Q_W)$ is Clifford. By the universal property (Proposition 1.3), f extends to a unique map $\mathrm{Cl}(f) : \mathrm{Cl}(V, Q_V) \to \mathrm{Cl}(W, Q_W)$. ☐

The Clifford algebra has two important involutions.

Definition 1.5. We define the *grading automorphism* $\alpha : \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ as the extension (in the sense of Proposition 1.4) of the isometry $-\mathrm{Id}_V : v \mapsto -v$.
The *transpose* $()^t : \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ is defined as the unique anti-automorphism such that

$$\big((\omega)^t\big)^t = \omega, \ \ \forall \omega \in \mathrm{Cl}(V, Q) \qquad \text{and} \qquad v^t = v, \ \ \forall v \in V. \tag{1.10}$$

Given an orhogonal basis $\{e_1, \ldots, e_n\}$ of V with corresponding basis $\{e_{i_1} \cdots e_{i_p} \mid 1 \leqslant i_1 < \ldots < i_p \leqslant n\}$ of $\mathrm{Cl}(V, Q)$, the involutions are given by

$$\alpha(e_{i_1} \cdots e_{i_p}) = (-1)^p e_{i_1} \cdots e_{i_p}, \tag{1.11}$$

$$t(e_{i_1} \cdots e_{i_p}) = e_{i_p} \cdots e_{i_1}. \tag{1.12}$$

Before investigating Clifford algebras more thoroughly, we will briefly give some low dimensional examples.

Example 1.6. Let $V = \mathbb{R}^n$ with the euclidean quadratic form $\|v\|^2 = \sum_{i=1}^n v_i^2$. We denote the associated Clifford algebra $\mathrm{Cl}(V, \|\cdot\|^2)$ by $\mathrm{Cl}(n)$.
For $V = \mathbb{R}$ with unit basis vector $i \in V$, the algebra $\mathrm{Cl}(1)$ is spanned by the basis $\{1, i\}$ with relations

$$i^2 = -1. \tag{1.13}$$

Therefore, as real algebras $\mathrm{Cl}(1) \cong \mathbb{C}$.
For $V = \mathbb{R}^2$ with orthonormal basis $i, j \in V$, the Clifford algebra $\mathrm{Cl}(2)$ has basis $\{1, i, j, k\}$, where $k := ij$ and relations

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji. \qquad . \tag{1.14}$$

This defines the algebra $\mathbb{H}$ of *quaternions.* We have shown that $\mathrm{Cl}(2) \cong \mathbb{H}$.

1.1.2 Grading, Filtration and the Symbol Map

In the following section, we focus on the structure of $\mathrm{Cl}(V, Q)$ inherited by the tensor algebra $T^\bullet V$ and the inner product.

Consider the grading automorphism $\alpha : \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ from Definition 1.5. Since $\alpha^2 = \mathrm{Id}_{\mathrm{Cl}(V,Q)}$, it follows that (as vector spaces)

$$\mathrm{Cl}(V, Q) = \mathrm{Cl}^0(V, Q) \oplus \mathrm{Cl}^1(V, Q), \tag{1.15}$$

where

$$\mathrm{Cl}^i(V, Q) = \{u \in \mathrm{Cl}(V, Q) \mid \alpha(u) = (-1)^i u\}. \tag{1.16}$$

This defines a $\mathbb{Z}_2$-grading on $\mathrm{Cl}(V, Q)$. Indeed, since α is an algebra homomorphism

$$\mathrm{Cl}^i(V, Q) \cdot \mathrm{Cl}^j(V, Q) \subseteq \mathrm{Cl}^{i+j \pmod 2}(V, Q) \qquad i, j \in \mathbb{Z}_2. \tag{1.17}$$

We call elements of $\mathrm{Cl}^0(V, Q)$ *even* and elements of $\mathrm{Cl}^1(V, Q)$ *odd*. This owes to the fact that $\mathrm{Cl}^0(V, Q)$ is spanned by products of even numbers of elements of V, while $\mathrm{Cl}^1(V, Q)$ is spanned by an odd number.
Observe that $\mathrm{Cl}^0(V, Q)$ is a subalgebra of $\mathrm{Cl}(V, Q)$, whereas $\mathrm{Cl}^1(V, Q)$ is not.

This $\mathbb{Z}_2$-grading is a remnant of the $\mathbb{N}-$ grading of the tensor algebra

$$T^\bullet V = \bigoplus_{n \in \mathbb{N}} V^{\otimes n} = \left(\bigoplus_{n \text{ even}} V^{\otimes n} \right) \oplus \left(\bigoplus_{n \text{ odd}} V^{\otimes n} \right), \tag{1.18}$$

that factors through the ideal I_Q.

Another structure inherited from the grading of the tensor algebra is the filtration of $\mathrm{Cl}(V, Q)$. Indeed, since every graded algebra is trivially a filtered algebra we have the following filtration of $T^\bullet V$

$$\mathcal{F}^n V := \bigoplus_{i \leqslant n} V^{\otimes i} \qquad \mathcal{F}^0 V \subseteq \mathcal{F}^1 V \subseteq \ldots \subseteq T^\bullet V \qquad \mathcal{F}^n V \otimes \mathcal{F}^m V \subseteq \mathcal{F}^{n+m} V. \tag{1.19}$$

Since quotienting by the ideal I_Q can only decrease degree, this filtration induces a filtration on the Clifford algebra

$$\mathcal{F}^n \,\mathrm{Cl}(V, Q) := \left(\textstyle\bigoplus_{i \leqslant n} V^{\otimes i} \right) \Big/ I_Q \,. \tag{1.20}$$

Recall that every filtered algebra $\mathcal{F}^0 A \subseteq \ldots \subseteq A$ has an associated graded algebra $\mathcal{G}(A) = \bigoplus_{n \in \mathbb{N}} \mathcal{G}^n A$ with $\mathcal{G}^0 A := \mathcal{F}^0 A$ and $\mathcal{G}^n := \mathcal{F}^n A / \mathcal{F}^{n-1} A$ for $n > 0$. As a vector space $\mathcal{G}(A)$ is isomorphic to A, as algebras they are usually distinct.

Proposition 1.7. *The associated graded algebra of* $\mathrm{Cl}(V,Q)$ *is the exterior algebra* ΛV.

Proof. Indeed, note that

$$\mathcal{G}^n \,\mathrm{Cl}(V,Q) = \left(\bigoplus_{i\leqslant n} V^{\otimes i}/I_Q\right)\Big/\left(\bigoplus_{i\leqslant n-1} V^{\otimes i}/I_Q\right) \cong V^{\otimes n}/I = \Lambda^n V,$$

where I is the ideal of $T^\bullet V$ generated by $v\otimes v$ for $v\in V$. □

Therefore, the Clifford algebra can be seen as an enhancement of the exterior algebra. Indeed, Clifford invented his algebra as a means of incorporating the inner product in the exterior algebra. From the point of view of physics, the exterior algebra represents the classical fermionic Fock space with the classical anticommutation relation $e_i\cdot e_j + e_j\cdot e_i = 0$. Quantising this Fock space leads to an algebra with canonical anticommutation relations $e_i\cdot e_j + e_j\cdot e_i = \hbar\delta_{ij}$. Up to a sign, this is exactly the Clifford algebra. Therefore, the isomorphism $\Lambda V \to \mathrm{Cl}(V,Q)$ is often called *quantisation map*. This is discussed in greater detail in section 1.4.

Definition 1.8. The isomorphism $\sigma : \mathrm{Cl}(V,Q)\to\Lambda V$ is called the *symbol map*.

Explicitly, its inverse $\sigma^{-1} : \Lambda V \to \mathrm{Cl}(V,Q)$ is given by the linear extension of

$$v_1\wedge\cdots v_r \mapsto \frac{1}{r!}\sum_{\sigma\in S_r}\mathrm{sign}(\sigma)v_{\sigma(1)}\cdots v_{\sigma(r)} \qquad v_1,\ldots,v_r\in V. \tag{1.21}$$

Given an orthonormal basis $e_1,\ldots,e_n$ of V, this isomorphism maps

$$e_{i_1}\wedge\cdots\wedge e_{i_r}\mapsto e_{i_1}\cdots e_{i_r}, \qquad \text{for } 1\leqslant i_1<\ldots<i_r\leqslant n. \tag{1.22}$$

Since $\mathrm{Cl}(V,Q)$ and ΛV are isomorphic, we can define an action of $\mathrm{Cl}(V,Q)$ on the exterior algegra.

Definition 1.9. We define *Clifford multiplication* of $\mathrm{Cl}(V,Q)$ on the vector space ΛV by

$$c(\alpha) := \sigma\,(\alpha\,\cdot\,)\,\sigma^{-1}\in\mathrm{End}(\Lambda V) \qquad \alpha\in\mathrm{Cl}(V,Q). \tag{1.23}$$

From now on we suppose that Q is non-degenerate, such that we have an isomorphism $Q(-,\cdot): V\to V^*$.

Definition 1.10. We define the *interior product* $\iota : V\to\mathrm{End}(\Lambda V)$ as $\iota(v) = \iota_{Q(v,\cdot)}$, where ι_ω denotes contraction of a covector $\omega\in V^*$ with an element of ΛV. Explicitly,

$$\iota(v)v_1\wedge\cdots\wedge v_k = \sum_{i=1}^{k}(-1)^{i+1}Q(v_i,v)v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_k. \tag{1.24}$$

We also define the *exterior product* $\epsilon : V\to\mathrm{End}(\Lambda V)$ by

$$\epsilon(v) := v\wedge\cdot. \tag{1.25}$$

The interior product ι is adjoint to ϵ with respect to the quadratic form Q on ΛV,

$$Q(\epsilon(v)\omega,\omega') = Q(\omega,\iota(v)\omega') \qquad \omega,\omega'\in\Lambda V,\ v\in V. \tag{1.26}$$

Observe that since $v\wedge v=0$, it follows that

$$\epsilon(v)^2 = 0,\qquad \iota(v)^2=0,\qquad \epsilon(v)\iota(w)+\iota(w)\epsilon(v) = Q(v,w)\qquad \text{for } v,w\in V. \tag{1.27}$$

We can reexpress Clifford multiplication by vectors in V using interior and exterior product.

Proposition 1.11. *For* $v\in V\subseteq\mathrm{Cl}(V,Q)$ *we have that*

$$c(v) = \epsilon(v)-\iota(v)\in\mathrm{End}(\Lambda V). \tag{1.28}$$

Proof. Let $e_{i_1}\wedge\cdots\wedge e_{i_k}\in\Lambda V$, where $e_1,\ldots e_n$ is an orthonormal basis for V. Then

$$c(e_i)e_{i_1}\wedge\cdots\wedge e_{i_k} = \sigma(e_i\cdot e_{i_1}\cdots e_{i_k}) = \begin{cases} e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_k} & i\notin\{i_1,\ldots,i_k\}\\ (-1)^j e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_k} & i=i_j\end{cases}.$$

On the other hand,

$$(\epsilon(e_i)-\iota(e_i))e_{i_1}\wedge\cdots\wedge e_{i_k} = \begin{cases} e_i\wedge e_{i_1}\wedge\cdots\wedge e_{i_k} & i\notin\{i_1,\ldots,i_k\}\\ (-1)^j e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_k} & i=i_j\end{cases}.$$

□

1.2 The Spin and Pin Groups

We are now ready to introduce the groups Spin_n and Pin_n as certain multiplicative subgroups of the Clifford algebra. As a guiding principle, we will try to find a double cover of SO_n and O_n among these subgroups.

1.2.1 Subgroups of $\mathrm{Cl}(n)$

From now on we let $V = \mathbb{R}^n$ with euclidean quadratic form $\|v\|^2 := \sum_{i=1}^n v_i^2$ and denote the corresponding Clifford algebra $\mathrm{Cl}(\mathbb{R}^n, \|\cdot\|)$ by $\mathrm{Cl}(n)$.
In this section we study several (multiplicational) subgroups of the algebra $\mathrm{Cl}(n)$, eventually leading to the definition of the Pin and Spin groups.

Since $\mathrm{Cl}(n)$ is a unital algebra, a first subgroup to consider is the following.

Definition 1.12. The *group of units* of the algebra $\mathrm{Cl}(n)$ is

$$\mathrm{Cl}_n^\times := \{u \in \mathrm{Cl}(n) \mid \exists u^{-1} \in \mathrm{Cl}(n), \text{ s.t. } u^{-1}u = uu^{-1} = 1\}. \tag{1.29}$$

Since $v\frac{-v}{\|v\|^2} = 1$, it follows that an element $v \in V$ is invertible, if and only if $v \neq 0$.

Having defined a group $\mathrm{Cl}_n^\times$, we could consider its adjoint action on itself, given by $\mathrm{Ad}(u)v = uvu^{-1}$ for $u, v \in \mathrm{Cl}_n^\times$. However, we will use the grading automorphism α from Definition 1.5 to define a slightly different action.

Definition 1.13. The group of units $\mathrm{Cl}_n^\times$ acts on $\mathrm{Cl}(n)$ via the *twisted adjoint action*

$$\widetilde{\mathrm{Ad}} : \mathrm{Cl}_n^\times \to \mathrm{Aut}(\mathrm{Cl}(n)) \qquad \widetilde{\mathrm{Ad}}_u(x) := \alpha(u)xu^{-1} \quad u \in \mathrm{Cl}_n^\times,\ x \in \mathrm{Cl}(n). \tag{1.30}$$

This action is well defined since $\widetilde{\mathrm{Ad}}_u$ is invertible with inverse $\widetilde{\mathrm{Ad}}_{u^{-1}}$ and is a group homomorphism since α is.

For $v \neq 0 \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, we have that

$$\widetilde{\mathrm{Ad}}_v(w) = \alpha(v)wv^{-1} = -vw\frac{-v}{\|v\|^2} = w - 2v\frac{\langle v, w\rangle}{\|v\|^2} \in \mathbb{R}^n. \tag{1.31}$$

Thus, we see that $\widetilde{\mathrm{Ad}}_v : \mathbb{R}^n \to \mathbb{R}^n$ defines the reflection at the plane orthogonal to the line passing through v. This is the reason we considered the twisted adjoint action instead of the adjoint action.
However, for a general $\omega \in \mathrm{Cl}_n^\times$, $\widetilde{\mathrm{Ad}}_\omega(\mathbb{R}^n) \nsubseteq \mathbb{R}^n$.

Definition 1.14. We define the *Clifford group*

$$\Gamma_n := \{u \in \mathrm{Cl}_n^\times \mid \widetilde{\mathrm{Ad}}_u(v) \in \mathbb{R}^n,\ \forall v \in \mathbb{R}^n\} \tag{1.32}$$

as the subgroup of $\mathrm{Cl}_n^\times$ that stabilises $\mathbb{R}^n$.

Clearly, any product of non-zero vectors in $\mathbb{R}^n$ is contained in Γ_n. In fact, the Clifford group is the subgroup of $\mathrm{Cl}(n)$ generated by non-vanishing vectors. Therefore, the maps α and $(\cdot)^t$ restrict to an automorphism and an anti-automorphism on Γ_n. A proof of this can be found in [6].

By construction, the twisted adjoint representation $\widetilde{\mathrm{Ad}} : \mathrm{Cl}_n^\times \to \mathrm{Aut}(\mathrm{Cl}(n))$ restricts to a representation

$$\widetilde{\mathrm{Ad}} : \Gamma_n \to \mathrm{Aut}(\mathbb{R}^n) = \mathrm{Gl}_n . \tag{1.33}$$

Since Γ_n is generated by all non-zero vectors in $\mathbb{R}^n$, it follows that its image under $\widetilde{\mathrm{Ad}}$ is the set of all possible compositions of reflections. The following lemma thus shows that any composition of reflections is a rotation.

Lemma 1.15. *The image of* $\widetilde{\mathrm{Ad}} : \Gamma_n \to \mathrm{Gl}_n$ *is contained in* O_n.

Proof. Observe that for $v \in \mathbb{R}^n$, we have that $\|v\|^2 = -v \cdot v = \alpha(v)v$. Thus, for $v \in \mathbb{R}^n$, $\phi \in \Gamma_n$

$$\|\widetilde{\mathrm{Ad}}_\phi v\|^2 = \alpha(\widetilde{\mathrm{Ad}}_\phi v)\widetilde{\mathrm{Ad}}_\phi v = \phi(-v)\alpha(\phi^{-1})\alpha(\phi)v\phi^{-1} = -\phi v \cdot v\phi^{-1} = \|v\|^2.$$

Since $\widetilde{\mathrm{Ad}}_\phi$ preserves the norm $\|\cdot\|$, it is an element of O_n. □

We've seen that Γ_n is generated by non-vanishing vectors in $\mathbb{R}^n$ and that $\widetilde{\mathrm{Ad}} : \Gamma_n \to \mathrm{O}_n$ maps such vectors to the reflection at the plane orthogonal to these vectors. The following theorem from linear algebra guarantees that every rotation can be obtained from reflections.

Theorem 1.16 (Cartan-Dieudonné)**.** *Every rotation* $R \in O_n$ *can be written as a product of at most* n *reflections.*

Proof. A proof can be found in [6]. □

Combining this result with a calculation of the kernel of $\widetilde{\mathrm{Ad}}$ we find the following lemma.

Lemma 1.17. *The following is a short exact sequence*

$$1 \longrightarrow \mathbb{R}^\times \hookrightarrow \Gamma_n \xrightarrow{\widetilde{\mathrm{Ad}}} O_n \longrightarrow 1. \tag{1.34}$$

Proof. By the Cartan-Dieudonné theorem 1.16 every rotation $R \in \mathrm{O}_n$ can be written as a product of reflections $R = \rho_{v_1} \cdots \rho_{v_r}$, where ρ_v denotes reflection at the plane orthogonal to $v \in \mathbb{R}^n$.
Since $\rho_v = \widetilde{\mathrm{Ad}}_v$, it follows that $\widetilde{\mathrm{Ad}} : \Gamma_n \to \mathrm{O}_n$ is surjective.
Let's compute its kernel. Suppose $\phi \in \Gamma_n$ is in the kernel of $\widetilde{\mathrm{Ad}}$. Then $\alpha(\phi)v = v\phi$. Decomposing $\phi = \phi^+ + \phi^-$, where ϕ^+ is even and ϕ^- is odd we obtain $(\phi^+ - \phi^-)v = v(\phi^+ + \phi^-)$, or

$$\phi^+ v = v\phi^+ \qquad \phi^- v = -v\phi^- \qquad \forall v \in \mathbb{R}^n.$$

Fix an orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$.
We write $\phi^\pm = a_\pm + v_1 b_\mp$, where $a_\pm$, $b_\mp$ are elements of $\mathrm{Cl}(n)$ not containing v_1 if spanned in terms of the basis of $\mathrm{Cl}(n)$ associated to the basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$. Observe that a_+ and b_+ are even, while a_- and b_- are odd. Therefore, $a_\pm v_1 = \pm v_1 a_\pm$, $b_\pm v_1 = \pm v_1 b_\pm$ and we calculate

$$\pm v_1 a_\pm \pm b_\mp = (a_\pm + v_1 b_\mp)v_1 = \phi^\pm v_1 = \pm v_1 \phi^\pm = \pm v_1 a_\pm \mp b_\mp.$$

Therefore, $b_\pm = 0$. Repeating this argument succesively for all basis vectors, it follows that $\phi^\pm$ does not containing any v_i and is therefore a constant. Since $\phi^\pm \in \Gamma_n$, this constant is non-zero. Thus $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times$. □

1.2.2 The Groups Pin_n and Spin_n

So far we have considered the Clifford group Γ_n generated by all non-zero vectors and have obtained a $\mathbb{R}^\times$-fold covering of O_n. To obtain a double cover, we have to restrict to the group generated by unit vectors.

Definition 1.18. We define the *spinor norm* $N : \mathrm{Cl}(n) \to \mathrm{Cl}(n)$ as

$$N : u \mapsto u\alpha(u)^t. \tag{1.35}$$

The name 'spinor norm' comes from the observation that for $v \in \mathbb{R}^n$, we have that $N(v) = -v^2 = \|v\|^2$.
However, on arbitrary elements of $\mathrm{Cl}(n)$ the spinor norm has not much in common with a norm; in general it isn't even a real number. If we restrict the spinor norm to the Clifford group Γ_n we can recover its norm-like behaviour.

Lemma 1.19. *The restriction of the spinor norm N to the Clifford group Γ_n is a homomorphism*

$$N : \Gamma_n \to \mathbb{R}^\times \tag{1.36}$$

such that $N(\alpha(x)) = N(x),\ \forall x \in \Gamma_n$.

Proof. Let $x \in \Gamma_n$. Since α and $(\cdot)^t$ restrict to (anti-) automorphism on Γ_n it follows that $\alpha(x)^t \in \Gamma_n$. Therefore, $N(x) = x\alpha(x)^t \in \Gamma_n$.
We will show that $\widetilde{\mathrm{Ad}}_{N(x)} = 1$. It then follows from Lemma 1.17 that $N(x) \in \ker(\widetilde{\mathrm{Ad}}) = \mathbb{R}^\times$.
Let $v \in \mathbb{R}^n$. Since $x^t \in \Gamma_n$, it follows that $\alpha(x)^t v x^{t^{-1}} \in \mathbb{R}^n$. Applying $(\cdot)^t$ and observing that $w^t = w,\ \forall w \in \mathbb{R}^n$,

$$\alpha(x)^t v (x^{-1})^t = x^{-1} v \alpha(x).$$

Therefore,

$$v = x\alpha(x)^t v \alpha\left(x\alpha(x)^t\right)^{-1} = \widetilde{\mathrm{Ad}}_{\alpha N(x)} v = \widetilde{\mathrm{Ad}}_{N(x)} v. \qquad \square$$

We can use this norm N to restrict Γ_n to unit vectors and introduce the groups Pin_n and Spin_n.

Definition 1.20. We define the *pinor group* Pin_n as the kernel of the homomorphism $N : \Gamma_n \to \mathbb{R}^\times$ and the *spinor group* Spin_n as its even part $\mathrm{Pin}_n \cap \mathrm{Cl}^0(n)$, where $\mathrm{Cl}^0(n)$ is the even part of the Clifford algebra.
Since $\alpha|_{\mathrm{Cl}^0(n)} = \mathrm{Id}$, we don't have to distinguish between adjoint and twisted adjoint for the Spin_n group.
Therefore,

$$\mathrm{Pin}_n = \{x \in \mathrm{Cl}(n)\ \mid \alpha(x) v x^{-1} \in \mathbb{R}^n,\ \forall v \in \mathbb{R}^n,\ N(x) = 1\} \tag{1.37}$$

$$\mathrm{Spin}_n = \{x \in \mathrm{Cl}^0(n) \mid x v x^{-1} \in \mathbb{R}^n,\ \forall v \in \mathbb{R}^n,\ N(x) = 1\}. \tag{1.38}$$

The group Pin_n is the subgroup of $\mathrm{Cl}_n^\times$ generated by unit vectors of $\mathbb{R}^n$ and Spin_n is the subgroup of Pin_n generated by even products of unit vectors of $\mathbb{R}^n$.

Using the spinor norm N to restrict the short exact sequence of Γ_n to unit vectors, we find that the Spin_n and Pin_n group are indeed double covers of SO_n and O_n, respectively.

Theorem 1.21. *There are short exact sequences*

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Pin}_n \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}_n \longrightarrow 1 \tag{1.39}$$

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}_n \xrightarrow{\mathrm{Ad}} \mathrm{SO}_n \longrightarrow 1. \tag{1.40}$$

Proof. Observe that for $k \in \mathbb{R}^\times$, $N(k) = k^2$. Thus, the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^\times & \hookrightarrow & \Gamma_n & \xrightarrow{\widetilde{\mathrm{Ad}}} & O_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle (\cdot)^2} & & \downarrow{\scriptstyle N} & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^\times & \xrightarrow{=} & \mathbb{R}^\times & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

which means that $\phi := \{()^2, N, 1\}$ is a morphism of chain complexes with kernel

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Pin}_n \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}_n \longrightarrow 1.$$

Since both domain and codomain of ϕ are exact, it follows that the above sequence is exact. Restricting to $\mathrm{Cl}^0(n)$ and observing that the product of an even number of reflections is contained in SO_n, it follows that

$$1 \to \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}_n \xrightarrow{\mathrm{Ad}} \mathrm{SO}_n \longrightarrow 1$$

is exact. $\square$

From now on we will focus on the spinor group Spin_n.
As alluded to in the introduction of this chapter, one can prove that the spin group Spin_n is simply connected for $n \geqslant 3$ and therefore the universal cover of SO_3. This explains its appearance in quantum physics.

Example 1.22. We continue Example 1.6 and discuss the Spin_n-group and the adjoint map $\mathrm{Ad} : \mathrm{Spin}_n \to \mathrm{SO}_n$ in the cases $n = 1$ and $n = 2$.
For $n = 1$ we have that $\mathrm{Cl}(1) = \mathrm{span}\{1, i\} \cong \mathbb{C}$. The even subalgebra is $\mathrm{Cl}^0(1) = \mathrm{span}\{1\} \cong \mathbb{R}$. The group Spin_1 is the group generated by even products of unit vectors. Thus, Spin_1 is the group $\{1, i^2 = -1\} = \mathbb{Z}_2$.
For $n = 2$ with $\mathrm{Cl}(2) = \mathbb{H} = \mathrm{span}\{1, i, j, k = ij\}$ we have that $\mathrm{Cl}^0(2) = \mathrm{span}\{1, k\}$ with relation

$$k^2 = ijij = -1. \tag{1.41}$$

Therefore, $\mathrm{Cl}^0(2) \cong \mathbb{C}$. Under this isomorphism, for $x \in \mathrm{Cl}^0(2)$, the map Ad_x acts as

$$\mathrm{Ad}_x(v) = \frac{x^2}{|x|^2}v \qquad v \in \mathbb{R}^2 \cong \mathbb{C},$$

where $|\cdot|$ denotes the aboslute value in $\mathbb{C}$. Thus, $\mathrm{Ad}_x(\mathbb{R}^2) \subseteq \mathbb{R}^2$, which implies that

$$\mathrm{Spin}_2 = \{x \in \mathrm{Cl}^0(2) \mid N(x) = 1\} \cong \{x \in \mathbb{C} \mid |x|^2 = 1\} = S^1, \tag{1.42}$$

where we have used that $N(x) = xx^t = (x_1 + kx_2)(x_1 - kx_2) = x\overline{x} = |x|^2$.
Identifying SO_2 with S^1 acting on $\mathbb{C}$ by phase multiplication we thus find that $\mathrm{Ad} : \mathrm{Spin}_2 \cong S^1 \to \mathrm{SO}_2 \cong S^1$ acts as

$$\mathrm{Ad} : S^1 \to S^1 \qquad z \mapsto z^2. \tag{1.43}$$

1.2.3 The Lie Algebra $\mathfrak{spin}_n$

Since Spin_n is a subgroup of the algebra $\mathrm{Cl}(n)$, it follows that its Lie algebra $\mathfrak{spin}_n$ is a Lie subalgebra of $\mathrm{Cl}(n)$ with commutator $[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha$. Let $\sigma : \mathrm{Cl}(n) \to \Lambda\mathbb{R}^n$ be the symbol map, the isomorphism between the Clifford algebra and the exterior algebra from Definition 1.8.

Lemma 1.23. *The Lie algebra $\mathfrak{spin}_n$ is the subalgebra $\sigma^{-1}(\Lambda^2\mathbb{R}^n)$ of $\mathrm{Cl}(n)$.*

Proof. We will first show that $\sigma^{-1}(\Lambda^2\mathbb{R}^n)$ is indeed a subalgebra of $\mathrm{Cl}(n)$.
Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, such that $\sigma^{-1}(\Lambda^2\mathbb{R}^n) = \mathrm{span}\{e_i \cdot e_j \mid i < j\}$. A computation shows that the commutator $[e_i \cdot e_j, e_k \cdot e_l]$ is again contained in $\mathrm{span}\{e_i \cdot e_j \mid i < j\}$. Thus, $\sigma^{-1}(\Lambda^2\mathbb{R}^n)$ is a subalgebra of $\mathrm{Cl}(n)$.
To prove that the subalgebra $\mathfrak{spin}_n$ equals $\sigma^{-1}(\Lambda^2\mathbb{R}^n)$ we consider the curve

$$\gamma_{ij}(t) := \cos(t)1 + \sin(t)e_i \cdot e_j \in \mathrm{Spin}_n \qquad \text{for } i < j$$

and observe that $\gamma(0) = 1,\ \dot{\gamma}(0) = e_i \cdot e_j$. Thus, the span $\mathrm{span}\{e_i \cdot e_j \mid i < j\} = \sigma^{-1}(\Lambda^2\mathbb{R}^n)$ is contained in the Lie algebra $\mathfrak{spin}_n$.
Since $\mathrm{Ad} : \mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover, it follows that $\mathfrak{spin}_n \cong \mathfrak{so}_n$. Therefore, $\dim(\mathfrak{spin}_n) = \dim(\mathfrak{so}_n) = \binom{n}{2}$. On the other hand, $\dim(\sigma^{-1}(\Lambda^2\mathbb{R}^n)) = \dim(\Lambda^2\mathbb{R}^n) = \binom{n}{2}$, which proves that $\sigma^{-1}(\Lambda^2\mathbb{R}^n) = \mathfrak{spin}_n$. ☐

The double cover $\mathrm{Ad} : \mathrm{Spin}_n \to \mathrm{SO}_n$ induces an isomorphism $\mathrm{Ad}_* : \mathfrak{spin}_n \to \mathfrak{so}_n$.
Fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. The associated standard basis of $\mathfrak{so}_n$ is given by $e_i \wedge e_j \in \mathfrak{so}_n$,

$$e_i \wedge e_j(v) := \langle v, e_j \rangle e_i - \langle v, e_i \rangle e_j \qquad v \in \mathbb{R}^n. \tag{1.44}$$

In matrix notation, every antisymmetric matrix (A_{ij}) corresponds to the element

$$A = \sum_{i<j} A_{ij} e_i \wedge e_j \in \mathfrak{so}_n. \tag{1.45}$$

The notation $e_i \wedge e_j$ is chosen since $\Lambda^2\mathbb{R}^n \to \mathfrak{so}_n$, $e_i \wedge e_j \to e_i \wedge e_j$ defines an isomorphism of Lie algebras.
Using the basis $e_i \wedge e_j$ of $\mathfrak{so}_n$, we can now describe the isomorphism $\mathrm{Ad}_* : \mathfrak{spin}_n = \sigma^{-1}(\Lambda^2\mathbb{R}^n) \to \mathfrak{so}_n$.

Proposition 1.24. *Given an orthonormal basis $e_1, \ldots, e_n \in \mathbb{R}^n$, the isomorphism $\mathrm{Ad}_* : \mathfrak{spin}_n \to \mathfrak{so}_n$ is given on basis elements by*

$$\mathrm{Ad}_* : e_i \cdot e_j \mapsto -2e_i \wedge e_j. \tag{1.46}$$

Proof. Let $\gamma(t) = \cos(t)1 + \sin(t)e_i \cdot e_j \in \mathrm{Spin}_n$. Then $\mathrm{Ad}_{\gamma(t)} \in \mathrm{SO}_n$ is given by

$$\mathrm{Ad}_{\gamma(t)} v = \gamma(t) \cdot v \cdot \gamma(t)^{-1} \qquad v \in \mathbb{R}^n,$$

with derivative at the identity

$$\left(\mathrm{Ad}_* e_i \cdot e_j\right)(v) = \left(\mathrm{Ad}_* \dot{\gamma}(0)\right)(v) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\gamma(t)} v = \left.\frac{d}{dt}\right|_{t=0} \left(\gamma(t) \cdot v \cdot \gamma(t)^{-1}\right)$$
$$= e_i \cdot e_j \cdot v - v \cdot e_i \cdot e_j.$$

This equals

$$[e_i \cdot e_j, v] = e_i \cdot e_j \cdot v + e_i \cdot v \cdot e_j + 2\langle v, e_i\rangle e_j = 2\langle v, e_i\rangle e_j - 2\langle v, e_j\rangle e_i = -2e_i \wedge e_j(v)$$ ☐

Given an antisymmetric matrix $A = (A_{ij}) \in \mathfrak{so}_n$, we have that

$$\mathrm{Ad}_*^{-1} A = \mathrm{Ad}_*^{-1}\left(\sum_{i<j} A_{ij} e_i \wedge e_j\right) = -\frac{1}{2}\sum_{i<j} A_{ij} e_i \cdot e_j = -\frac{1}{4}\sum_{i,j=1}^{n} A_{ij} e_i \cdot e_j. \tag{1.47}$$

1.3 Spinor Representations

In the following section we will study representations of the Clifford algebra $\mathrm{Cl}(n)$ and the spin group Spin_n . We will see that in even dimensions, the Clifford algebra has exactly one irreducible representations S, called the spin representation. This representation splits in two irreducible Spin_n-representations $S = S^+ \oplus S^-$.

To deal with representations it is more convenient to consider the complexification of the Clifford algebra

$$\mathrm{Cl}_{\mathbb{C}}(n) := \mathrm{Cl}(n) \otimes \mathbb{C}. \tag{1.48}$$

Observe that this complexification equals the complex Clifford algebra $\mathrm{Cl}(\mathbb{C}^n, q)$, where q is the (complex-bilinear!) form

$$q(z, w) := \sum_{i=1}^{n} z_i w_i \qquad z, w \in \mathbb{C}^n. \tag{1.49}$$

Definition 1.25. A representation of the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(n)$ is a $\mathbb{C}-$algebra homomorphism

$$\rho : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}_{\mathbb{C}}(W), \tag{1.50}$$

where W is a finite-dimensional $\mathbb{C}-$vector space.

In Definition 1.9, we have already encountered the representation $c : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}_{\mathbb{C}}(\Lambda\mathbb{C}^n)$ induced by the map

$$c : V_{\mathbb{C}} \to \mathrm{End}(\Lambda\mathbb{C}^n) \qquad c(v) := \epsilon(v) - \iota(v). \tag{1.51}$$

However, observe that $\Lambda\mathbb{C}^n$ is generated by $\epsilon(v)$ and $\iota(v)$ acting on $1 \in \Lambda\mathbb{C}^n$ for $v \in \mathbb{C}^n$, while the image of $\mathrm{Cl}_{\mathbb{C}}(n)$ is only generated by $c(v) = \epsilon(v) - \iota(v)$. Thus, $\mathrm{Cl}_{\mathbb{C}}(n)1 \subseteq \Lambda\mathbb{C}^n$ is a proper subspace and $\Lambda\mathbb{C}^n$ therefore not an irreducible representation.

However, in the even dimensional case $n = 2m$, we can use a similar construction as (1.51) to obtain an irreducible representation of $\mathrm{Cl}_{\mathbb{C}}(n)$.

Let $V = \mathbb{R}^{2m}$ and let J be a complex structure on V, i.e. a linear map $J : V \to V$ such that $J^2 = -\mathbb{1}_V$ which is compatible with the real euclidean product $(\cdot, \cdot)$ in the sense that $(Jv, Jw) = (v, w)$.
Let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ be the complexification of V. The euclidean product on V extends to the (complex-bilinear!) form q on $V_{\mathbb{C}}$ defined in equation (1.49). The complexification decomposes into

$$V_{\mathbb{C}} = P \oplus \overline{P}, \tag{1.52}$$

where P and $\overline{P}$ are the $+i$ and $-i$ eigenspaces of $J \otimes 1$.
The notation $\overline{P}$ comes from the fact that complex conjugation

$$\bar{\cdot} : V_{\mathbb{C}} \to V_{\mathbb{C}}, \qquad v \otimes \mu \mapsto v \otimes \overline{\mu} \qquad v \in V, \mu \in \mathbb{C}, \tag{1.53}$$

restricts to a real isomorphism $\bar{\cdot} : P \to \overline{P}$.
Since J is compatible with the euclidean product $(\cdot, \cdot)$, it follows that

$$q(p_1, p_2) = 0 = q(b_1, b_2) \qquad p_1, p_2 \in P,\ b_1, b_2 \in \overline{P}. \tag{1.54}$$

and that q places P and $\overline{P}$ in duality, i.e. such that the map

$$\overline{P} \to P^* \qquad b \mapsto q(b, \cdot) \tag{1.55}$$

is an isomorphism. The choice of subspace P is known as a *polarisation* of $V_{\mathbb{C}}$.

On the complex vector space $V_{\mathbb{C}}$ with bilinear form q, we define the exterior product $\epsilon : V_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Lambda V_{\mathbb{C}})$ and interior product $\iota : V_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Lambda V_{\mathbb{C}})$ as in Definition 1.10.
Observe that for $p \in P$, $b \in \overline{P}$ we have that

$$\epsilon(p) \in \mathrm{End}_{\mathbb{C}}(\Lambda P) \qquad \text{and} \qquad \iota(b) \in \mathrm{End}_{\mathbb{C}}(\Lambda P). \tag{1.56}$$

We define the map $c : V_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Lambda P)$ as

$$c(p + b) := \sqrt{2}\,(\epsilon(p) - \iota(b))\,, \qquad p + b \in V_{\mathbb{C}} = P \oplus \overline{P}. \tag{1.57}$$

It squares to

$$c(p + b)^2 = 2\left(\epsilon(p)^2 - \epsilon(p)\iota(b) - \iota(b)\epsilon(p) + \iota(b)^2\right). \tag{1.58}$$

Using relations (1.27), it follows that

$$c(p + b)^2 = -2q(p, b) = -q(p + b, p + b). \tag{1.59}$$

Thus, the map $c : V_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Lambda P)$ is Clifford and induces a map

$$c : \mathrm{Cl}(V_{\mathbb{C}}, q) = \mathrm{Cl}_{\mathbb{C}}(2m) \to \mathrm{End}_{\mathbb{C}}(\Lambda P). \tag{1.60}$$

For different choices of complex structure J we get different polarisations P. However, one can prove that the representations $c : \mathrm{Cl}_{\mathbb{C}}(2m) \to \mathrm{End}_{\mathbb{C}}(\Lambda P)$ are equivalent. For definiteness, we let $S := \Lambda P$, where P is the subspace of $\mathbb{C}^{2m}$ obtained from the standard complex structure $J = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix}$ on $\mathbb{R}^{2m}$. Explicitly,

$$P = \mathrm{span}\{e_j - i\, e_{j+m} \mid 1 \leqslant j \leqslant m\} \qquad \text{and} \qquad \overline{P} = \mathrm{span}\{e_j + i\, e_{j+m} \mid 1 \leqslant j \leqslant m\}, \tag{1.61}$$

where $e_1, \ldots, e_{2m}$ is the standard basis of $\mathbb{R}^{2m}$.

We observe that $\dim(S) = 2^m$.

Definition 1.26. The representation $c : \mathrm{Cl}_{\mathbb{C}}(2m) \to \mathrm{End}_{\mathbb{C}}(S)$ is called the *spin representation*.
The 2^m-dimensional space S is the *spin space*.

Usually, the map $c : \mathrm{Cl}_{\mathbb{C}}(2m) \to \mathrm{End}_{\mathbb{C}}(S)$ is called *Clifford multiplication*.

Proposition 1.27. *The representation*

$$c : \mathrm{Cl}_{\mathbb{C}}(2m) \to \mathrm{End}_{\mathbb{C}}(S) \tag{1.62}$$

is an isomorphism of $\mathbb{C}-$*algebras and thus irreducible.*

Proof. We start by proving that c is surjective.
Let $V = \mathbb{R}^{2m}$, $V_{\mathbb{C}} = \mathbb{C}^{2m}$ and $P, \overline{P}$ as in 1.52. The algebra $\mathrm{End}_{\mathbb{C}}(S) = \mathrm{End}_{\mathbb{C}}(\Lambda P)$ is generated by $\epsilon(p)$ for $p \in P$ and ι_ω for $\omega \in P^*$. The complex bilinear form q induces an isomorphism between P^* and $\overline{P}$ (equation (1.55)) such that for every $\omega \in P^*$, there is a $b \in \overline{P}$ with $\iota_\omega = \iota(b)$. Since $c(p) = \sqrt{2}\epsilon(p)$ and $c(b) = -\sqrt{2}\iota(b) = -\sqrt{2}\iota_\omega$, it follows that c is surjective.
It is an isomorphism, since $\dim_{\mathbb{C}}(\mathrm{Cl}_{\mathbb{C}}(2m)) = 2^{2m}$ and $\dim_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(S)) = (2^m)^2 = 2^{2m}$. □

It is proven in [6], that in even dimension $n = 2m$, the spin space S is up to equivalence the unique irreducible $\mathrm{Cl}_{\mathbb{C}}(2m)-$representation.

Proposition 1.28. *There exists a hermitian product* $\langle \cdot, \cdot \rangle$ *on* S *such that*

$$\langle c(v)s, s' \rangle = -\langle s, c(v)s' \rangle \qquad \forall s, s' \in S,\ v \in \mathbb{R}^{2m} \subseteq \mathbb{C}^{2m}. \tag{1.63}$$

Proof. We define the inner product on S using our explicit construction of $S = \Lambda P$ from above.
It suffices to define an hermitian product with properties (1.63) on P, which can then be extended to ΛP.
Let $\bar{\cdot} : P \to \overline{P}$ denote complex conjugation. We define the hermitian product

$$\langle p, p' \rangle := q(p, \overline{p}') \qquad p, p' \in P. \tag{1.64}$$

Let $v \in V = \mathbb{R}^{2m}$. Since v is real, it is of the form $v = w + \overline{w}$ for some $w \in P$. Thus, $c(v) = \sqrt{2}\,(\epsilon(w) - \iota(\overline{w}))$ and

$$\langle c(v)p, p' \rangle = \sqrt{2}\langle (\epsilon(w) - \iota(w))p, p' \rangle = \sqrt{2}q\left(\epsilon(w) - \iota(\overline{w})p, \overline{p'}\right).$$

Since ι is adjoint to ϵ with respect to q (equation 1.26), it follows that

$$\langle c(v)p, p' \rangle = \sqrt{2}\, q\left(p, \big(\iota(w) - \epsilon(\overline{w})\big)\,\overline{p}'\right) = -\sqrt{2}\, q\left(p, \overline{\big(\epsilon(w) - \iota(\overline{w})\big)\, p'}\right) = -\langle p, c(v)p' \rangle. \qquad \square$$

If we restrict this representation of $\mathrm{Cl}_{\mathbb{C}}(2m)$ to the Spin group, we obtain a representation of Spin_{2m}.

Definition 1.29. The complex *spin representation* of Spin_{2m} is the homomorphism

$$\pi_S : \mathrm{Spin}_{2m} \to \mathrm{Gl}_{\mathbb{C}}(S), \tag{1.65}$$

obtained by restricting the irreducible spin representation of $\mathrm{Cl}_{\mathbb{C}}(2m)$ to $\mathrm{Spin}_{2m} \subseteq \mathrm{Cl}^0(2m) \subseteq \mathrm{Cl}_{\mathbb{C}}(2m)$.

Using equation (1.63) and the fact that Spin_n is generated by an even number of unit vectors in $\mathrm{Cl}(n)$, we see that the spin representation π_S is unitary with respect to the hermitian product $\langle \cdot, \cdot \rangle$ on S.

The representation π_S is not irreducible. This can be seen by the following.

Definition 1.30. We define the *complex volume element* $\omega_{\mathbb{C}} \in \mathrm{Cl}_{\mathbb{C}}(2m)$ as

$$\omega_{\mathbb{C}} := i^m e_1 \cdots e_{2m}, \tag{1.66}$$

where $e_1, \ldots, e_{2m}$ is a positively oriented orthonormal basis of $\mathbb{R}^{2m}$.

A calculation shows that $\omega_{\mathbb{C}}^2 = 1$. Thus, we can split every $\mathrm{Cl}_{\mathbb{C}}(2m)$-representation W into a direct sum of ± 1 eigenspaces of $\omega_{\mathbb{C}}$, i.e.

$$W = W^+ \oplus W^-, \qquad W^\pm := \frac{1}{2}\big(1 \pm \rho(\omega_{\mathbb{C}})\big)W. \tag{1.67}$$

Noticing that

$$\omega_{\mathbb{C}}\alpha = \alpha\omega_{\mathbb{C}} \qquad \text{for all } \alpha \in \mathrm{Cl}^0_{\mathbb{C}}(2m), \tag{1.68}$$

we see that each of $W^\pm$ is invariant under the even subalgebra $\mathrm{Cl}^0_{\mathbb{C}}(2m)$ and defines a subrepresentation for $\mathrm{Cl}^0_{\mathbb{C}}(2m)$.

Given the spin representation S, we can therefore decompose $S = S^+ \oplus S^-$ in Spin_{2m}-subrepresentations.
In terms of our explicit construction $S = \Lambda P$, these subrepresentations are $S^+ = \Lambda^{\text{even}} P$ and $S^- = \Lambda^{\text{odd}} P$.
The Spin_{2m} representations S^+ and S^- are inequivalent and irreducible. In fact, they are up to equivalence the only irreducible Spin_{2m} representations. A proof of these facts can be found in [6].

Definition 1.31. The complex irreducible unitary representations $\pi_{S^\pm} : \mathrm{Spin}_{2m} \to \mathrm{Gl}_{\mathbb{C}}(S^\pm)$ are called *half spin representations*.

With respect to the hermitian product $\langle \cdot, \cdot \rangle$ from Definition 1.28, the spaces S^+ and S^- are orthogonal to each other.

Example 1.32. Continuing Example 1.6 and 1.22, we discuss the spinor representation of $\mathrm{Spin}_2 \cong S^1$.
Consider $\mathbb{R}^2$ with the standard complex structure $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\mathbb{R}^2 \otimes \mathbb{C} = \mathbb{C}^2 = P \oplus \overline{P}$, where $P = \mathrm{span}\{f := \frac{1}{\sqrt{2}}(e_1 - ie_2)\}$ and $\overline{P} = \mathrm{span}\{\overline{f} := \frac{1}{\sqrt{2}}(e_1 + i\, e_2)\}$ and where e_1, e_2 is the standard basis of $\mathbb{R}^2$.
The spin space is then $S = \Lambda P = \mathbb{C} \oplus P$. We decompose

$$e_1 = \frac{1}{2}(e_1 - ie_2) + \frac{1}{2}(e_1 + ie_2) = \frac{1}{\sqrt{2}} f + \frac{1}{\sqrt{2}} \overline{f} \text{ and } e_2 = \frac{i}{2}(e_1 - ie_2) - \frac{i}{2}(e_1 + ie_2) = \frac{i}{\sqrt{2}}(f - \overline{f}) \quad (1.69)$$

and obtain that $c : \mathrm{Cl}_{\mathbb{C}}(2) \to \mathrm{End}_{\mathbb{C}}(S) = \mathrm{End}_{\mathbb{C}}(\mathbb{C} \oplus \mathrm{span}\{f\})$ is given as the complex linear extension of

$$c(e_1)(a + b\, f) = \big(\epsilon(f) - \iota(\overline{f})\big)\,(a + b\, f) = -b + a\, f \quad (1.70)$$

$$c(e_2)(a + b\, f) = i\,\big(\epsilon(f) + \iota(\overline{f})\big)\,(a + b\, f) = ib + ia\, f. \quad (1.71)$$

Here, we have used that $\iota(\overline{f}) f = q(\overline{f}, f) = 1$.
The half spin spaces are $S^+ = \Lambda^{\text{even}} P = \mathbb{C} = \mathrm{span}\{1\}$ and $S^- = \Lambda^{\text{odd}} P = P = \mathrm{span}\{f\}$.
The spin representation π_S is given by Clifford multiplication restricted to Spin_2.
Therefore, for $\cos(\phi) + \sin(\phi) e_1 e_2 \in \mathrm{Spin}_2$,

$$\pi_S\Big(\cos(\phi) + \sin(\phi) e_1 e_2\Big)\,(a + b\, f) = \Big(\cos(\phi) - i \sin(\phi)\Big)\, a + \Big(\cos(\phi) + i \sin(\phi)\Big)\, b\, f. \quad (1.72)$$

Identifying $S^1 \cong \mathrm{Spin}_2$ via $e^{i\phi} \mapsto \cos(\phi) + \sin(\phi) e_1 e_2$ and $S = S^+ \oplus S^- = \mathrm{span}\{1\} \oplus \mathrm{span}\{f\} \cong \mathbb{C} \oplus \mathbb{C}$, we find that

$$\pi_{S^+} : S^1 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}) \qquad z \mapsto z^{-1} \quad (1.73)$$

$$\pi_{S^-} : S^1 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}) \qquad z \mapsto z \quad (1.74)$$

$$\pi_S : S^1 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2) \qquad z \mapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}. \quad (1.75)$$

1.4 Fermions and Bosons

When we discussed the isomorphism between Clifford algebra and exterior algebra, we mentioned that the Clifford algebra can be seen as representing the *canoncial anticommutation relations (*CAR).
The usual presentation of the CAR algebra describing a quantum mechanical system of *fermions* with n degrees of freedom is slightly different. It is generated by position operators $q_1, \ldots, q_n$ and momentum operators $p_1, \ldots, p_n$ with relations

$$q_i q_j + q_j q_i = 0, \qquad p_i p_j + p_j p_i = 0, \qquad q_i p_j + p_j q_i = -\delta_{ij} \qquad 1 \leqslant i, j \leqslant n, \quad (1.76)$$

(modulo a factor of $-i\hbar$). However, this algebra is equivalent to the complex Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2n)$ in $2n$ generators. In fact, choosing a polarisation of $\mathbb{R}^{2n}$, e.g.

$$(\mathbb{R}^{2n})_{\mathbb{C}}\ (\cong \mathbb{C}^{2n}) = P \oplus \overline{P} = \mathrm{span}\{e_j - ie_{j+n} \mid 1 \leqslant j \leqslant n\} \oplus \mathrm{span}\{e_j + ie_{j+n} \mid 1 \leqslant j \leqslant n\} \quad (1.77)$$

(where $e_1, \ldots, e_{2n}$ is an ONB of $\mathbb{R}^{2n}$, see (1.61)), we find that the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2n)$ is generated by vectors $q_i = \frac{1}{2}(e_i - ie_{j+n})$ and $p_i = \frac{1}{2}(e_i + ie_{i+n})$ with relations

$$q_i q_j + q_j q_i = 0 \qquad p_i p_j + p_j p_i = 0 \qquad q_i p_j + p_j q_i = -\delta_{ij} \qquad 1 \leqslant i, j \leqslant n. \quad (1.78)$$

Therefore, the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2n)$ is in fact equivalent to the CAR algebra.

Completely analogously, we can consider the *canonical commutation relations (*CCR*)*

$$q_i q_j - q_j q_i = 0 \qquad p_i p_j - p_j p_i = 0 \qquad q_i p_j - p_j q_i = -\delta_{ij} \qquad 1 \leqslant i,j \leqslant n, \tag{1.79}$$

describing a system of *bosons* with n degrees of freedom. Similar to the Clifford algebra, there is a coordinate independent definition of this algebra.
Let V be an even dimensional complex vector space with a symplectic form ω (i.e. an antisymmetric non-degenerate bilinear form $\omega : V \times V \to \mathbb{C}$). We define the *Weyl algebra*

$$\mathcal{W}(V, \omega) := T^{\bullet}V / I_{\omega} \ , \tag{1.80}$$

where $T^{\bullet}V$ is the tensor algebra and I_{ω} is the ideal generated by $v \otimes w - w \otimes v + \omega(v,w)1$ for $v, w \in V$.
Observe that in contrary to the Clifford algebra, the Weyl algebra is only defineable in even dimensions.
Similar to (1.54), we define a polarisation of V with respect to ω to be a choice of subspaces $V = Q \oplus P$ such that

$$\omega(q_1, q_2) = 0 = \omega(p_1, p_2) \qquad \forall q_1, q_2 \in Q,\ p_1, p_2 \in P. \tag{1.81}$$

Choosing a basis $q_1, \ldots, q_n$ of Q and $p_1 \ldots p_n$ of P such that $\omega(q_i, p_j) = \delta_{ij}$, the Weyl algebra is generated by the relations

$$q_i q_j - q_j q_i = 0 \qquad p_i p_j - p_j p_i = 0 \qquad q_i p_j - p_j q_i = -\omega(q_i, p_j) = -\delta_{ij}. \tag{1.82}$$

Therefore, the Weyl algebra is equivalent to the CCR-algebra describing a bosonic system with n degrees of freedom.

We can transfer all results obtained for the Clifford algebra to corresponding results for the Weyl algebra.
While the Clifford algebra is a quantisation of the exterior algebra ΛV (i.e. ΛV is the associated graded algebra to the filtered algebra $\mathrm{Cl}(V)$), the Weyl algebra is a quantisation of the symmetric algebra $\mathrm{Sym}(V)$.
The Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(2n)$ is defined via an inner product and therefore related to the group SO_{2n}. In the same way, the Weyl algebra is defined via a symplectic form and is related to the symplectic group Sp_{2n}.
For example, we found in Lemma 1.23 that the quadratic elements $\sigma^{-1}(\Lambda^2\mathbb{C}^{2n})$ of the Clifford algebra are canonically isomorphic to the Lie algebra of the Spin group Spin_{2n} (here, $\sigma : \mathrm{Cl}_{\mathbb{C}}(2n) \to \Lambda\mathbb{C}^{2n}$ is the symbol map).
Analogously, we can show that the quadratic elements $\mu^{-1}(\mathrm{Sym}^2\,\mathbb{C}^{2n})$ of the Weyl algebra $\mathcal{W}(2n)$ are canonically isomorphic to the Lie algebra of the double cover of the symplectic group (here $\mu : \mathcal{W}(2n) \to \mathrm{Sym}(\mathbb{C}^{2n})$ is the isomorphism between symmetric and Weyl algebra). This double cover of the symplectic group is known as the *metaplectic group* Mp_{2n}.

We can push these analogies between Clifford and Weyl algebra much further. The idea to treat fermions and bosons - or Clifford and Weyl algebra - on a completely equal footing is called *supersymmetry*.

The utility of supersymmetry becomes apparent if we consider the bosonic analogon of Clifford multiplication (the spin representation) $c : \mathrm{Cl}_{\mathbb{C}}(2n) \to \mathrm{End}_{\mathbb{C}}(S)$ from Definition 1.27.
Following equation (1.57), we define the representation $e : \mathcal{W}(2n) \to \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}\,Q)$ as the extension of

$$e : \mathbb{C}^{2n} = Q \oplus P \to \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}\,Q) \qquad e(q+p) := \tau(q) + \kappa(p), \tag{1.83}$$

where $\tau : Q \to \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}\,Q)$ is the symmetric product with elements of Q and κ its formal adjoint.
Observe that in comparison to (1.57), we ommited a factor $\sqrt{2}$, which is due to the missing factor of 2 in the defining relation $v \otimes w - w \otimes v = -\omega(v,w)1$ of the Weyl algebra.

The algebra $\mathrm{Sym}\,Q$ is the algebra of polynomial expressions in vectors of Q (e.g. $q_1^2 + q_2$) and can thus be identified with the algebra $\mathbb{C}[Q^*]$ of polynomial functions on the vector space Q^*. Fixing a basis $q_1, \ldots, q_n$ of Q, this yields an identification $\mathrm{Sym}\,Q \cong \mathbb{C}[q_1, \ldots, q_n]$. Under this identification, the symmetric product $\tau(q)$ becomes multiplication with q, such that

$$e(q_i) : \mathbb{C}[q_1, \ldots, q_n] \to \mathbb{C}[q_1, \ldots, q_n] \qquad (e(q_i)f)\,(q_1, \ldots, q_n) = q_i f(q_1, \ldots, q_n). \tag{1.84}$$

A computation shows that the formal adjoint $\kappa(p_i)$ corresponds to the derivative with respect to q_i, such that

$$e(p_i) : \mathbb{C}[q_1, \ldots, q_n] \to \mathbb{C}[q_1, \ldots, q_n] \qquad (e(p_i)f)\,(q_1, \ldots, q_n) = \frac{\partial f}{\partial q_i}(q_1, \ldots, q_n). \tag{1.85}$$

Since $e(q_i)e(p_j) - e(p_j)e(q_i) = -\frac{\partial q_i}{\partial q_j} = -\delta_{ij} = -\omega(q_i, p_j)$, this indeed defines a representation of the Weyl algebra. In the same way as the spinor representation S is the unique irreducible representation of the Clifford algebra, the representation $e : \mathcal{W}(2n) \to \mathrm{End}(\mathbb{C}[q_1, \ldots, q_n])$ is the unique irreducible representation of the Weyl algebra - a fact known as the Stone-von Neumann theorem.

Therefore, in the same way that fermions are represented by Clifford multplication on spinors, bosons correspond to (polynomial) differential operators on Q. The power of supersymmetry is that it relates these two concepts.

In the following, we will mainly focus on fermions and spinors and come back to the idea of supersymmetry only in the final step of the proof of the index theorem.

1.5 Spin Geometry

Having discussed algebraic properties of Clifford algebras and spin groups we can now place them in a geometrical context.

1.5.1 Differential Geometry

In the following, we will briefly recapitulate important concepts from differential geometry.
Let M be a smooth manifold. For a vector bundle $\mathcal{E}$, we denote its space of sections by $\Gamma(\mathcal{E})$.

Definition 1.33. Let G be a Lie group. A *principal $G-$bundle* $P_G \to M$ over M is a $G-$fibre bundle with a free and transitive right action of G on P_G which preserves fibres.

The main example of a ($\mathrm{Gl}_r -$)principal bundle is the frame bundle $P_{\mathrm{Gl}_r}\mathcal{E}$ of a rank r vector bundle $\mathcal{E}$. Its fibre $(P_{\mathrm{Gl}_r}(\mathcal{E}))_x$ at $x \in M$ is given by the set of all ordered bases of $\mathcal{E}_x$ with Gl_r acting on them by change of basis.

To every principal $G-$bundle and every G-representation, we can associate a vector bundle.

Definition 1.34. Given a principal $G-$bundle $P_G \to M$ and a representation $\rho : G \to \mathrm{Gl}(V)$ we define the *associated vector bundle*

$$P_G \times_\rho V := (P_G \times V)/\sim \,, \tag{1.86}$$

where $(p.g, v) \sim (p, \rho(g)v), \ \forall g \in G$.

Every rank r vector bundle $\mathcal{E}$ is associated to its frame bundle under the fundamental representation

$$\rho := \mathrm{Id} : \mathrm{Gl}_r \to \mathrm{Gl}(\mathbb{R}^r) = \mathrm{Gl}_r \,. \tag{1.87}$$

All other bundles usually associated to $\mathcal{E}$ can be reconstructed as associated to its frame bundle. For example,

$$\mathcal{E} = P_{\mathrm{Gl}_r}(\mathcal{E}) \times_\rho \mathbb{R}^r, \quad \mathcal{E}^* = P_{\mathrm{Gl}_r}(\mathcal{E}) \times_{\rho^*} \mathbb{R}^r, \quad \Lambda^k\mathcal{E} = P_{\mathrm{Gl}_r}(\mathcal{E}) \times_{\Lambda^k\rho} \left(\Lambda^k\mathbb{R}^r\right), \quad \otimes^k\mathcal{E} = P_{\mathrm{Gl}_r}(\mathcal{E}) \times_{\otimes^k\rho} \left(\otimes^k\mathbb{R}^r\right),$$

where $\rho^*, \ \Lambda^k\rho, \ \otimes^r \rho$ are the induced dual, exterior power and tensor product representations.

Let $\mathcal{E}$ be a rank r vector bundle with transition functions $\phi_{ij} : U_i \cap U_j \to \mathrm{Gl}_r$ and let $G \subseteq \mathrm{Gl}_r$. We say that $\mathcal{E}$ has a $G-$structure, if its transition functions can be chosen to map into G, i.e. if $\phi_{ij} : U_i \cap U_j \to G \subseteq \mathrm{Gl}_r$.
For example, if $\mathcal{E}$ is orientable its transition functions can be chosen to lie in Gl_r^+. Similarly, if $\mathcal{E}$ has an inner product, then they can be chosen to lie in O_r. We can then consider its oriented or orthonormal frame bundle $P_{\mathrm{Gl}_r^+}(\mathcal{E})$ or $P_{\mathrm{O}_r}(\mathcal{E})$ of oriented or othonormal frames, respectively. This can be generalised to arbitrary principal bundles.

Definition 1.35. Let $P_G \to M$ be a principal G-bundle over a manifold M and let $H \xrightarrow{\phi} G$ be a group homomorphism. The bundle P_G *reduces/lifts along* ϕ if there exists a trivialisation $\{U_i\}$ of P_G with transition maps $g_{ij} : U_i \cap U_j \to G$, such that there exist maps $G_{ij} : U_i \cap U_j \to H$ fulfilling the cocycle condition $G_{ij} = G_{ik}G_{kj}$ (i.e. such that they also define a bundle) and such that $g_{ij} = \phi(G_{ij})$.
The principal bundle P_H induced by the transition functions G_{ij} is called a *reduction/lift of* P_G *along* ϕ.

If P_H is a reduction/lift of P_G along $\phi : H \to G$, then there exists a bundle map $\xi : P_H \to P_G$, such that

$$\xi(p.h) = \xi(p).\phi(h) \qquad \forall h \in H. \tag{1.88}$$

In most cases $H \to G$ is taken to be an inclusion of a subgroup, which is why we usually speak about a reduction of a $G-$bundle. However, in the following essay we will be considering the double cover $\mathrm{Spin}_n \to SO_n$, which makes it more natural to talk about a lift of the structure group.

Many geometrical notions can be restated in terms of reductions of specific principal bundles.
A Riemannian structure on a manifold M^n is defined to be a reduction of its frame bundle $P_{\mathrm{Gl}_n}(TM)$ along the inclusion $\mathrm{O}_n \hookrightarrow \mathrm{Gl}_n$. Each Riemannian structure corresponds to a choice of a specific (isometry class of) Riemannian metrics on TM.
An orientation of M is a reduction of the frame bundle $P_{\mathrm{Gl}_n}(TM)$ along $\mathrm{Gl}_n^+ \hookrightarrow \mathrm{Gl}_n$. If such a reduction exists, we say that M is orientable.

Example 1.36. To prepare for our discussion of the existence of spin structures, we briefly investigate the notion of orientability of a manifold M. Consider the short exact sequence

$$1 \to \mathrm{Gl}_n^+ \hookrightarrow \mathrm{Gl}_n \overset{\mathrm{sign(det)}}{\to} \mathbb{Z}_2 \to 1. \tag{1.89}$$

This induces a long exact sequence in cohomology

$$\mathrm{H}^1(M, \mathrm{Gl}_n^+) \to \mathrm{H}^1(M, \mathrm{Gl}_n) \overset{w_1}{\to} \mathrm{H}^1(M, \mathbb{Z}_2). \tag{1.90}$$

We use Čech-cohomology to identify isomorphism classes of principal G-bundles with $\mathrm{H}^1(M, G)$. A principal Gl_n-bundle P is orientable if it reduces to a Gl_n^+-bundle, i.e. if it lies in the image of $\mathrm{H}^1(M, \mathrm{Gl}_n^+) \to \mathrm{H}^1(M, \mathrm{Gl}_n)$. By exactness of (1.90), we have thus obtained that a principal Gl_n $-$bundle is orientable if and only if $w_1(P) = 0$. The class $w_1(P) \in \mathrm{H}^1(M, \mathbb{Z}_2)$ is called the *first Stieffel-Whitney class of* P.

A local section $s : U \to P$ of a principal bundle P yields trivialisations of P and all its associated bundles over U. Indeed, the map $(x, g) \mapsto s(x).g$ identifies $U \times G \cong P|_U$, trivialising P. Given an associated bundle $\mathcal{E} = P \times_\rho \mathbb{R}^r$, the map $(x, v) \mapsto [(s(x).g, v)] = [(s(x), \rho(g)v)]$ identifies $U \times \mathbb{R}^r \cong \mathcal{E}|_U$.
In particular, every local section $s : U \to P$ induces a local frame for every associated bundle.

One of the main advantages of the use of principal bundles is to facilitate the study of connections.

Definition 1.37. A *covariant derivative* on a vector bundle $\mathcal{E} \to M$ is a map

$$\nabla : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E}), \tag{1.91}$$

fulfilling the Leibniz rule

$$\nabla(fX) = df \otimes X + f\nabla X \qquad f \in C^\infty(M),\ X \in \Gamma(\mathcal{E}). \tag{1.92}$$

Given a local frame $(e_1, \ldots, e_r)$ of $\mathcal{E}|_U$, we define the connection one-form as the $r \times r$ matrix of one-forms $\tilde{\omega}^i{}_j \in \Omega^1(U)$ such that

$$\nabla e_i = \sum_{k=1}^r \tilde{\omega}^k{}_i \otimes e_k. \tag{1.93}$$

The connection one-forms $\tilde{\omega} \in \Omega^1(U, \mathrm{End}(\mathbb{R}^r)) = \Omega^1(U, \mathfrak{gl}_r)$ are only locally defined and depend on the specific trivialisation of $\mathcal{E}|_U$. Using principal bundles we can organise them into globally defined one-forms independent of the trivialisation.

Definition 1.38. A *principal connection* on a principal $G-$bundle $P \to M$ is an element $\omega \in \Omega^1(P, \mathfrak{g})$ fulfilling several technical conditions, given for example in [2].

Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a principal connection on P and let $\mathcal{E}$ be a vector bundle associated to P via a representation $\rho : G \to \mathrm{Gl}_r$. We define a covariant derivative on $\mathcal{E}$ as follows.
Let $U \subseteq M$ be an open subset and $s : U \to P$ be a local section of P trivialising P and inducing a local frame $e_1, \dots, e_n$ of $\mathcal{E}|_U$.
Then $s^*\omega \in \Omega^1(U, \mathfrak{g})$ and we define $\widetilde{\omega} := \rho_* s^*\omega \in \Omega^1(U, \mathfrak{gl}_r)$, where $\rho_* : \mathfrak{g} \to \mathfrak{gl}_r$ is the tangent map of ρ. This defines local connection forms $\widetilde{\omega}^i{}_j = e^i\left(\widetilde{\omega} e_j\right)$ and a covariant derivative

$$\nabla = d + \widetilde{\omega} : C^\infty(U, \mathbb{R}^r) \to \Omega^1(U, \mathbb{R}^r). \tag{1.94}$$

One can prove that this definition of ∇ is independent of the choice of trivialisation.

We can now define the notion of curvature of a connection.

Definition 1.39. Given a covariant derivative ∇ on a vector bundle $\mathcal{E}$, we define its *curvature tensor* $K^\nabla \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ by

$$K^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \qquad X, Y \in \Gamma(TM). \tag{1.95}$$

Given a local frame $e_1, \dots, e_r$ over $U \subseteq M$, we can trivialise K^∇ to obtain the local curvature form $K_U^\nabla \in \Omega^2(U, \mathfrak{gl}_r)$ defined by

$$\left(K_U^\nabla(X, Y)\right)^i{}_j = e^i\left(K^\nabla(X, Y) e_j\right) \qquad X, Y \in \Gamma(TM|_U) \tag{1.96}$$

Having defined the curvature of a covariant derivative, we can define the associated notion for a principal connection ω on a principal bundle.

Definition 1.40. Given a principal connection $\omega \in \Omega^1(P, \mathfrak{g})$ on a principal bundle $P \to M$, we define its *principal curvature*

$$\Omega^\omega = d\omega + \omega \wedge \omega \in \Omega^2(P, \mathfrak{g}). \tag{1.97}$$

Let $\mathcal{E}$ be a vector bundle associated to P via a representation $\rho : G \to \mathrm{Gl}_r$ and let ω be a principal connection on P with induced covariant derivative ∇ on $\mathcal{E}$. Let $U \subseteq M$ be an open subset and $s : U \to P$ be a local section of P trivialising $P|_U$ and $\mathcal{E}|_U$ with induced local frame $e_1, \dots, e_n$ of $\mathcal{E}$.
In this trivialisation we have that $s^*\Omega^\omega \in \Omega^2(U, \mathfrak{g})$ is related to the trivialised curvature $K_U^\nabla \in \Omega^2(U, \mathfrak{gl}_n)$ by

$$\rho_*(s^*\Omega^\omega) = K_U^\nabla. \tag{1.98}$$

We will now focus on the tangent bundle TM of a Riemannian manifold (M, g).
The *Levi-Civita connection* ∇ on $\Gamma(TM)$ is the unique connection that is *torsion free*, i.e. such that

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \text{for all } X, Y \in \Gamma(TM) \tag{1.99}$$

and *compatible with the metric*

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad \text{for all } X, Y, Z \in \Gamma(TM). \tag{1.100}$$

Its curvature tensor $R \in \Omega^2(M, \mathrm{End}(TM))$ is the *Riemann curvature tensor*. Given a local frame $e_1, \dots, e_n$ of $TM|_U$, we let

$$R_{ij}{}^k{}_l = \left(R_U(e_i, e_j)\right)^k_l = e^k(R(e_i, e_j)e_l). \tag{1.101}$$

Usually, we lower the index k using the metric and define

$$R_{ijkl} = \sum_r g_{rk} R_{ij}{}^r{}_l = g(e_k, R(e_i, e_j)e_l) \qquad \text{where } g_{rk} = g(e_r, e_k). \tag{1.102}$$

The Riemann curvature tensor has the following symmetries:

$$R_{ijkl} + R_{ijlk} = 0, \tag{1.103}$$
$$R_{ijkl} + R_{jikl} = 0, \tag{1.104}$$
$$R_{ijkl} - R_{klij} = 0, \tag{1.105}$$
$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{1.106}$$

The last identity is known as the *Bianchi identity*.
We define the *Ricci tensor* as the contraction of the Riemann tensor $\mathrm{Ric}_{ij} := \sum_{k=1}^{n} g^{kl} R_{ikjl}$. The symmetries of the Riemann tensor imply that the Ricci tensor is symmetric; $\mathrm{Ric}_{ij} = \mathrm{Ric}_{ji}$.
Finally, we define the scalar curvature of M as

$$r_M := \sum_{i,j} \mathrm{Ric}_{ij}\, g^{ij} = \sum_{ilk} R_{ik}{}^{i}{}_{l} g^{kl}. \tag{1.107}$$

This is clearly independent of the local frame and defines a global function $r_M \in C^\infty(M)$.

1.5.2 Spin Manifolds

In the following let (M^n, g) be an $n-$dimensional oriented Riemannian manifold. We denote its positively oriented orthonormal coframe bundle by $P^*_{\mathrm{SO}_n}$.

Definition 1.41. A *spin structure* $P^*_{\mathrm{Spin}_n}$ on an oriented Riemannian manifold (M^n, g) is a lift of $P^*_{\mathrm{SO}_n}$ to a principal Spin_n-bundle along the double cover $\mathrm{Ad} : \mathrm{Spin}_n \to \mathrm{SO}_n$. An oriented Riemannian manifold with a spin structure is called a *spin manifold*. We denote the induced bundle map by $\xi : P^*_{\mathrm{Spin}_n} \to P^*_{\mathrm{SO}_n}$.

In the literature, spin structures are either defined as a lift of the frame bundle P_{SO_n} (as for example in [6]) or of the coframe bundle (see [2]). Here, we have chosen the coframe appraoch to avoid the exterior algebra ΛTM in favour of the much more geometrical ΛT^*M.
Since the metric g induces a canonical isomorphism between TM and T^*M , both definitions are equivalent.

Similar to our discussion of orientability in Example 1.36, we can find topological obstructions to the existence of spin structures. Let's consider the exact sequence

$$1 \to \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}_n \overset{\mathrm{Ad}}{\to} \mathrm{SO}_n \to 1 \tag{1.108}$$

with induced long exact sequence

$$\mathrm{H}^1(X, \mathrm{Spin}_n) \overset{\mathrm{Ad}}{\to} \mathrm{H}^1(X, \mathrm{SO}_n) \overset{w_2}{\to} \mathrm{H}^2(X, \mathbb{Z}_2). \tag{1.109}$$

Thus, the oriented orthonormal coframe bundle $P^*_{\mathrm{SO}_n}$ lies in the image of Ad if and only if $w_2(P^*_{\mathrm{SO}_n}) = 0$.
This shows that a manifold M admits a spin structure if and only if the *second Stieffel Whitney class* of its tangent bundle $w_2(TM)$ vanishes.

Definition 1.42. Let $M^{n=2m}$ be an even dimensional spin manifold. We define the (complex) *spinor bundle* $\$$ as the associated bundle

$$\$:= P^*_{\mathrm{Spin}_n} \times_{\pi_S} S, \tag{1.110}$$

where $\pi_S : \mathrm{Spin}_n \to \mathrm{Gl}_{\mathbb{C}}(S)$ is the spin representation. Its (local) sections are called *spinors*.
The *halfl spinor bundles* $\$^\pm$ are defined as $\$^\pm := P^*_{\mathrm{Spin}_n} \times_{\pi_{S^\pm}} S^\pm$ with sections called *half spinors*.

As vector bundles $\$$ decomposes as the direct sum $\$ = \$^+ \oplus \$^-$. The hermitian product on S induces a hermitian metric on $\$$ with respect to which the bundles $\$^\pm$ are orthogonal.

Having defined a spinor bundle it is a natural next step to ask for a bundle of Clifford algebras acting on $\$$.

To define such a bundle, we observe that any orthogonal transformation $R \in O_n$ induces a map of the tensor algebra to itself which preserves the ideal generated by $v \otimes v + q(v)$. Therefore, it induces an orthogonal transformation $\mathrm{Cl}(R)$ on $\mathrm{Cl}(n)$ (see Proposition 1.4). This defines a representation

$$\mathrm{Cl} : \mathrm{SO}_n \to \mathrm{Aut}(\mathrm{Cl}_n). \tag{1.111}$$

Definition 1.43. Let (M^n, g) be an $n-$dimensional oriented Riemannian manifold. We define the *Clifford bundle* on M to be the associated bundle

$$\mathrm{Cl}(M) := P^*_{\mathrm{SO}_n} \times_{\mathrm{Cl}} \mathrm{Cl}(n). \tag{1.112}$$

The Clifford bundle has fibres $\mathrm{Cl}(M)_x$ over $x \in M$ given by the Clifford algebra $\mathrm{Cl}(T^*_x M)$ associated to $T^*_x M$. We note that in contrast to the spinor bundle, we do not need a spin structure on M to define a Clifford bundle.

We define the *complex Clifford bundle* as the complexification $\mathrm{Cl}_{\mathbb{C}}(M) := \mathrm{Cl}(M) \otimes \mathbb{C}$.

Definition 1.44. We define the *symbol map*

$$\sigma : \mathrm{Cl}(M) \to \Lambda T^* M \tag{1.113}$$

as the vector bundle isomorphism induced by the fibrewise symbol maps $\sigma_x : \mathrm{Cl}(T^*_x M) \to \Lambda T^*_x M$ from Definition 1.8.

Proposition 1.45. *Let $M^{n=2m}$ be an even dimensional spin manifold with spinor bundle $\$$. Clifford multiplication $c : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ induces an algebra-bundle (iso)morphism*

$$c : \mathrm{Cl}_{\mathbb{C}}(M) \to \mathrm{End}_{\mathbb{C}}(\$). \tag{1.114}$$

Proof. A proof can be found in [6]. □

As in equation (1.67) we can use a volume form in $\mathrm{Cl}_{\mathbb{C}}(M)$ to define the projections of $\$$ to $\$^{\pm}$ in terms of Clifford multiplication.

Definition 1.46. For an even dimensional spin manifold $M^{n=2m}$ with volume form $\omega \in \Gamma(\Lambda^n T^* M)$ induced by the metric, we let $\omega^{\mathbb{C}} := i^{\frac{n}{2}} \sigma^{-1}(\omega) \in \Gamma(\mathrm{Cl}_{\mathbb{C}}(M))$ be the *complex volume form*.

We then obtain that

$$\$^{\pm} = \left(\mathbb{1}_{\$} \pm c(\omega^{\mathbb{C}})\right) \$. \tag{1.115}$$

1.5.3 Spin Connection

Since the Levi-Civita connection on M is compatible with the Riemannian structure it defines a principal connection $\tau \in \Omega^1(P^*_{\mathrm{SO}_n}, \mathfrak{so}_n)$ on $P^*_{\mathrm{SO}_n}$. Under the spin structure $\xi : P^*_{\mathrm{Spin}_n} \to P^*_{\mathrm{SO}_n}$, τ lifts to a connection $\tau' = \xi^* \tau \in \Omega^1(P^*_{\mathrm{Spin}_n}, \mathfrak{spin}_n)$ on $P^*_{\mathrm{Spin}_n}$.

Definition 1.47. Let $M^{n=2m}$ be an even dimensional spin manifold with Levi-Civita connection τ and let $\tau' = \xi^* \tau$ be the lift of τ to $P^*_{\mathrm{Spin}_n}$. The induced covariant derivative

$$\nabla^{\$} : \Gamma(\$) \to \Gamma(T^* M \otimes \$) \tag{1.116}$$

is called *spin connection*.

Trivialised by a local orthonormal frame $(e_1, \ldots, e_n)$ of $TM|_U$ we can write the Levi-Civita connection on TM as

$$\nabla_{\partial_i} = \partial_i + \Gamma_i : \; C^{\infty}(U, \mathbb{R}^n) \to C^{\infty}(U, \mathbb{R}^n), \tag{1.117}$$

where $\Gamma_i := \Gamma(\partial_i)$ and $\Gamma \in \Omega^1(U, \mathfrak{so}_n)$ are the Christoffel symbols, the local one-form of the Levi-Civita connection. In terms of our frame, they are given by $(\Gamma_i)^\alpha{}_\beta = e^\alpha\left(\nabla_{\partial_i} e_\beta\right) = g\left(e_\alpha, \nabla_{\partial_i} e_\beta\right)$.
Since $g(e_\alpha, \nabla_{\partial_i} e_\beta) = -g(e_\beta, \nabla_{\partial_i} e_\alpha)$, it follows that $\Gamma_i \in \mathfrak{so}_n$. Using the basis $e^i \wedge e^j$ of $\mathfrak{so}_n$ introduced in equation (1.44), we can write them as

$$\Gamma_i = \sum_{\alpha,\beta=1}^{n} (\Gamma_i)^\alpha{}_\beta e_\alpha \otimes e^\beta = \sum_{\alpha<\beta} (\Gamma_i)^\alpha{}_\beta e^\alpha \wedge e^\beta. \tag{1.118}$$

Lets compare this to the spin connection $\nabla^{\$}$.
The (dual of the) local orthonormal frame $s = (e^1, \ldots, e^n)$ defines a local section of $P^*_{\mathrm{SO}_n}$. Using this section, we can relate the principal Levi-Civita connection τ to the Christoffel symbols Γ, via

$$\Gamma = -s^*\tau, \tag{1.119}$$

where the minus comes from the fact that τ defines a connection on the cotangent bundle.
Let $\tilde{s}$ be a section of $P^*_{\mathrm{Spin}_n}$ such that $\xi(\tilde{s}) = s$ (since ξ is a double cover, there are two sections with this property). This section $\tilde{s}$ trivialises $P^*_{\mathrm{Spin}_n}$ and therefore also $\$$. In this trivialisation, the local connection one-form of the spin-connection is $\Gamma^{\$} = \pi_{S*}\tilde{s}^*\tau' \in \Omega^1(U, \mathfrak{gl}(S))$, where $\tau' \in \Omega^1(P^*_{\mathrm{Spin}_n}, \mathfrak{spin}_n)$ is the lift of the Levi-Civita connection and $\pi_S : \mathrm{Spin}_n \to \mathrm{Gl}_{\mathbb{C}}(S)$ is the spin representation. Since $\tau' = \xi^*\tau$, it follows that

$$\Gamma^{\$} = \pi_{S*}\tilde{s}^*\xi^*\tau = \pi_{S*}\,\mathrm{Ad}_*^{-1}(\xi \circ \tilde{s})^*\tau = \pi_{S*}\,\mathrm{Ad}_*^{-1}\, s^*\tau = (\pi_S \circ \mathrm{Ad}^{-1})_*(-\Gamma), \tag{1.120}$$

where $\mathrm{Ad} : \mathrm{Spin}_n \to \mathrm{SO}_n$ and we have used that $\tilde{s}^*\xi^* = \mathrm{Ad}_*^{-1}(\xi \circ \tilde{s})^* : \Omega^1(P^*_{\mathrm{SO}_n}, \mathfrak{so}_n) \to \Omega^1(U, \mathfrak{spin}_n)$.
By definition, the map $\pi_S : \mathrm{Spin}_n \to \mathrm{Gl}_{\mathbb{C}}(S)$ is simply the restriction of Clifford multiplication $c : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ to Spin_n, such that

$$\nabla^{\$}_{\partial_i} = \partial_i + \Gamma^{\$}_i = \partial_i - c\left(\mathrm{Ad}_*^{-1}(\Gamma_i)\right). \tag{1.121}$$

Writing $\Gamma_i = \sum_{j<k} (\Gamma_i)^j{}_k e^j \wedge e^k$ and using (1.47), it follows that

$$\Gamma^{\$}_i = -\frac{1}{4} \sum_{\alpha,\beta=1}^{n} (\Gamma_i)^\alpha{}_\beta c(e^\alpha)c(e^\beta).$$

Ommiting the local coordinate system x^i of M, we summarise our findings in the following proposition.

Proposition 1.48. *Let $e_1, \ldots, e_n$ be a local orthonormal frame of $TM|_U$. Let $s = (e^1, \ldots, e^n)$ be the corresponding local section of $P^*_{\mathrm{SO}_n}$ and let $\tilde{s}$ be a local section of $P^*_{\mathrm{Spin}_n}$ in the preimage of s under the spin structure $\xi : P^*_{\mathrm{Spin}_n} \to P^*_{\mathrm{SO}_n}$ (there are two choices $\pm\tilde{s}$ for $\tilde{s}$). In the trivialisation of $\$$ induced by $\tilde{s}$ we have that*

$$\nabla^{\$} = d + \frac{1}{4} \sum_{\alpha,\beta=1}^{n} (\Gamma)^\alpha{}_\beta c(e^\alpha)c(e^\beta) \qquad \text{where } (\Gamma)^\alpha{}_\beta = g(e_\alpha, \nabla e_\beta) \in \Omega^1(U). \tag{1.122}$$

With a completely analogous discussion we can relate the trivialised curvature of the Levi-Civita connection to the curvature of the spin connection.

Proposition 1.49. *Let $e_1, \ldots, e_n$ be a local orthonormal frame of $TM|_U$.
The curvature of the spin connection $R^{\$} \in \Omega^2(M, \mathrm{End}(\$))$ is*

$$R^{\$}(e_i, e_j) = -\frac{1}{4} \sum_{k,l=1}^{n} R_{ijkl}\, c(e^k)c(e^l) = -\frac{1}{2} \sum_{k<l} R_{ijkl} c(e^k)c(e^l) \quad \text{where } R_{ijkl} = g(e_k, R(e_i, e_j)e_l) \tag{1.123}$$

and where $R \in \Omega^2(M, \mathrm{End}(TM))$ is the Riemann curvature tensor.

Proof. This proof is very similar to the proof of the previous proposition. Let s and $\tilde{s}$ be as in Proposition 1.48. Let $\Omega^\tau \in \Omega^2(P^*_{\mathrm{SO}_n}, \mathfrak{so}_n)$ be the principal curvature of the Levi-Civita connection τ, such that the local curvature (see equation (1.96)) is $R_U = s^*\Omega^\tau \in \Omega^2(U, \mathfrak{so}_n)$.

We use the basis (1.44), to write $R_U(X,Y) = \sum_{\alpha<\beta} (R_U(X,Y))^\alpha{}_\beta e^\alpha \wedge e^\beta$.
By formula (1.98), $R_U^{\$} = \pi_{S*}\tilde{s}^*\Omega^{\tau'}$, where $\pi_S : \mathrm{Spin}_n \to \mathrm{Gl}_{\mathbb{C}}(S)$ is the spin representation and τ' is the spin connection. Since $\tau' = \xi^*\tau$, it follows that $\Omega^{\tau'} = \xi^*\Omega^\tau$ and thus that $R_U^{\$} = \pi_{S*}\tilde{s}^*\Omega^{\tau'} = \pi_{S*}\,\mathrm{Ad}_*^{-1}(R_U)$.
Therefore,

$$R_U^{\$}(X,Y) = -\frac{1}{4}\sum_{\alpha,\beta=1}^{n} R_U(X,Y)^\alpha{}_\beta c(e^\alpha)c(e^\beta).$$

The claim follows, since $R_U(e_i,e_j)^\alpha{}_\beta = e^\alpha(R(e_i,e_j)e_\beta) = g(e_\alpha, R(e_i,e_j)e_\beta) = R_{ij\alpha\beta}$. □

Since $\mathrm{Cl}_{\mathbb{C}}(M)$ is associated to $P^*_{\mathrm{SO}_n}$, the Levi-Civita connection induces a connection ∇ on $\mathrm{Cl}_{\mathbb{C}}(M)$. The spin connection is compatible with the structure on $\$$ defined so far.

Proposition 1.50.

(1) *The spin connection $\nabla^{\$}$ is compatible with the hermitian product $(\cdot,\cdot)$ on $\$$, in the sense that*

$$X(s,s') = (\nabla^{\$}_X s, s') + (s, \nabla + X^{\$} s') \in T^*M \qquad s,s' \in \Gamma(\$),\ X \in \Gamma(TM). \tag{1.124}$$

(2) *The spin connection $\nabla^{\$}$ is compatible with Clifford multiplication in the sense that*

$$\nabla^{\$}_X(c(\alpha)s) = c(\nabla_X\alpha)s + c(\alpha)\nabla^{\$}_X s \qquad X \in TM, s \in \Gamma(\$),\ \alpha \in \Gamma(\mathrm{Cl}_{\mathbb{C}}(M)). \tag{1.125}$$

Proof. A proof can be found in [6]. □

Having discussed the spin connection, we can now define the central object of this essay.

Definition 1.51. The *Dirac operator $D\!\!\!/$* on an even dimensional spin manifold $M^{n=2m}$ is the operator

$$D\!\!\!/ :\ \Gamma(\$) \xrightarrow{\nabla^{\$}} \Gamma(T^*M \otimes \$) \xrightarrow{c} \Gamma(\$). \tag{1.126}$$

Given a local orthonormal frame $e_1,\dots,e_n$ of $TM|_U$ with corresponding dual frame $e^1,\dots,e^n$ of $T^*M|_U$, the Dirac operator can be written as

$$D\!\!\!/ = \sum_{i=1}^{n} c(e^i)\nabla^{\$}_{e_i}. \tag{1.127}$$

So far, we have defined the Dirac operator on the spinor bundle $\$$. By twisting the spinor bundle with a vector bundle $\mathcal{V}$ and defining an associated Dirac operator on it, we can vastly increase the generality of our construction. Indeed, on a spin manifold most geometric operators are related to Dirac operators of twisted bundles (a precise statement can be found in Section 3.4).

Definition 1.52. Let $\mathcal{V} \to M$ be a hermitian vector bundle with compatible connection $\nabla^{\mathcal{V}}$. On the bundle $\mathcal{E} := \$ \otimes \mathcal{V}$, we define the connection $\nabla^{\mathcal{E}} = \nabla^{\$} \otimes \mathbb{1} + \mathbb{1} \otimes \nabla^{\mathcal{V}}$ and the Clifford action $c : \mathrm{Cl}_{\mathbb{C}}(M) \to \mathrm{End}(\mathcal{E})$ by

$$c(\alpha)\,(\sigma \otimes v) := (c(\alpha)\sigma) \otimes v \qquad \alpha \in \Gamma(\mathrm{Cl}_{\mathbb{C}}(M)),\ \sigma \in \Gamma(\$), v \in \Gamma(\mathcal{V}). \tag{1.128}$$

The bundle $\mathcal{E}$ is called a *twisted spinor bundle* with Dirac operator $D\!\!\!/_{\mathcal{V}} := \sum_i c(e^i)\nabla^{\mathcal{E}}_{e_i}$.

Definition 1.53. Given a hermitian vector bundle $\mathcal{V} \to M$ with compatible connection $\nabla^{\mathcal{V}}$, we define its *Clifford curvature* $F^{\mathcal{V}} \in \Gamma(\mathrm{End}(\$ \otimes \mathcal{V}))$ by

$$F^{\mathcal{V}} = \sum_{i<j} c(e^i)c(e^j)K^{\mathcal{V}}(e_i,e_j) = \frac{1}{2}\sum_{i,j=1}^{n} c(e^i)c(e^j)K^{\mathcal{V}}(e_i,e_j), \tag{1.129}$$

where $K^{\mathcal{V}} \in \Omega^2(M, \mathrm{End}(\mathcal{V}))$ is the curvature of $\nabla^{\mathcal{V}}$ and $e_1,\dots,e_n$ is a local orthonormal frame of TM.

This definition is independent of the choice of frame and defines a global section $F^{\mathcal{V}} \in \Gamma(\mathrm{End}(\not{S} \otimes \mathcal{V}))$.

Definition 1.54. Let $\mathcal{V} \to M$ be a vector bundle with connection $\nabla^{\mathcal{V}}$. Given a local orthonormal frame $e_1, \ldots, e_n$ of TM, we define the *connection Laplacian* $\Delta_{\mathcal{V}}$ to be

$$\Delta_{\mathcal{V}} := -\sum_{i=1}^{n} \left(\nabla^{\mathcal{V}}_{e_i} \nabla^{\mathcal{V}}_{e_i} - \nabla^{\mathcal{V}}_{\nabla_{e_i} e_i} \right) : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}). \tag{1.130}$$

This definition is independent of the choice of local frame. We will redefine the Laplacian later in a nicer, frame independent form in the context of formal adjoints (Definition 1.62).

Theorem 1.55 (Lichnerowicz)**.** *Let $\mathcal{E} = \not{S} \otimes \mathcal{V}$ be a twisted spinor bundle over an even dimensional spin manifold. Then*

$$\not{D}_{\mathcal{V}}^2 = \Delta_{\mathcal{E}} + \frac{1}{4} r_M + F^{\mathcal{V}}, \tag{1.131}$$

where $\Delta_{\mathcal{E}}$ is the connection Laplacian on $\mathcal{E}$, $F^{\mathcal{V}} \in \Gamma(\mathrm{End}(\mathcal{E}))$ is the Clifford curvature of $\nabla^{\mathcal{V}}$ and $r_M \in C^\infty(M)$ is the scalar curvature of M.

Proof. Fix a point $x \in M$. We use a local orthonormal frame $e_1, \ldots, e_n$ of TM such that $(\nabla e_i)_x = 0$ (e.g. using geodesic coordinates, see Lemma 2.44) and denote its dual frame by $e^1, \ldots, e^n$.
We abbreviate $c^i := c(e^i)$ and $\nabla_i := \nabla^{\mathcal{E}}_{e_i}$. Then at the point $x \in M$, we have that $[\nabla_i, c^j] = c(\nabla_i e^j) = 0$ and thus

$$\not{D}_{\mathcal{V}}^2 = \sum_{i,j=1}^{n} c^i \nabla_i c^j \nabla_j = \sum_{i,j=1}^{n} c^i c^j \nabla_i \nabla_j = \left(\sum_{i=1}^{n} (c^i)^2 \nabla_i \nabla_i \right) + \left(\sum_{i<j} c^i c^j [\nabla_i, \nabla_j] \right).$$

Using $(c^i)^2 = c((e^i)^2) = -1$, the first term becomes $\sum_{i=1}^{n} (c^i)^2 \nabla_i \nabla_i = -\sum_{i=1}^{n} \nabla_i \nabla_i = \Delta_{\mathcal{E}}|_x$. The second term is

$$\sum_{i<j} c^i c^j [\nabla_i, \nabla_j] = \sum_{i<j} c^i c^j K^{\mathcal{E}}(e_i, e_j)$$

where $K^{\mathcal{E}} \in \Omega^2(U, \mathrm{End}(\mathcal{E}))$ is the curvature of the connection $\nabla^{\mathcal{E}}$ and we have used that

$$[e_i, e_j]|_x = \left(\nabla_{e_i} e_j - \nabla_{e_j} e_i \right)\big|_x = 0.$$

Observe that

$$K^{\mathcal{E}} = R^{\not{S}} \otimes \mathbb{1} + \mathbb{1} \otimes K^{\mathcal{V}},$$

where $R^{\not{S}} \in \Omega^2(M, \mathrm{End}(\not{S}))$ is the curvature of the spin connection $\nabla^{\not{S}}$ and $K^{\mathcal{V}} \in \Omega^2(M, \mathrm{End}(\mathcal{V}))$ is the curvature of $\nabla^{\mathcal{V}}$.
Therefore, we have that at $x \in M$

$$\not{D}_{\mathcal{V}}^2 = \Delta_{\mathcal{E}} + \sum_{i<j} c^i c^j R^{\not{S}}(e_i, e_j) + \sum_{i<j} c^i c^j K^{\mathcal{V}}(e_i, e_j).$$

By definition, $\sum_{i<j} c^i c^j K^{\mathcal{V}}(e_i, e_j) = F^{\mathcal{V}}$. We are left with the term containing the spin curvature $R^{\not{S}}$. By Proposition 1.49, it follows that $R^{\not{S}}(e_i, e_j) = -\frac{1}{4} \sum_{k,l=1}^{n} R_{ijkl} c^k c^l$. Therefore,

$$\sum_{i<j} c^i c^j R^{\not{S}}(e_i, e_j) = -\frac{1}{8} \sum_{i,j,k,l=1}^{n} R_{ijkl} c^i c^j c^k c^l.$$

Note that the Bianchi identity implies that

$$0 = (R_{ijkl} + R_{iklj} + R_{iljk})\, c^j c^k c^l = R_{ijkl} c^j c^k c^l + R_{iklj} c^k c^l c^j + R_{iljk} c^l c^j c^k.$$

Therefore and since $R_{ijkl} = -R_{ijlk}$, it follows that

$$\sum_{j=1}^{n} \sum_{k \neq j, l \neq j} R_{ijkl} c^j c^k c^l = 2 \sum_{j<k<l} \left(R_{ijkl} c^j c^k c^l + R_{iklj} c^k c^l c^j + R_{iljk} c^l c^j c^k \right) = 0.$$

Thus,

$$\sum_{j,k,l=1}^{n} R_{ijkl} c^j c^k c^l = \sum_{j,l} R_{ijjl} c^j c^j c^l + \sum_{jk} R_{ijkj} c^j c^k c^j = 2 \sum_{l} \mathrm{Ric}_{il}\, c^l,$$

where $\mathrm{Ric}_{ij} = \sum_k R_{ikjk}$ is the Ricci curvature tensor.
By symmetry of the Ricci tensor,

$$\sum_{i,j,k,l} R_{ijkl} c^i c^j c^k c^l = 2 \sum_{ij} \mathrm{Ric}_{ij}\, c^i c^l = 2 \sum_{i} \mathrm{Ric}_{ii} (c^i)^2 = -2 r_M.$$

Putting everthing together, this means that

$$\sum_{i<j} c^i c^j R^{\$}(e_i, e_j) = \frac{1}{4} r_M,$$

which proves the claim. □

We will now investigate how the Dirac operator behaves with respect to the subbundles $\$^{\pm}$.
For a twisted Dirac bundle $\mathcal{E} := \$ \otimes \mathcal{V}$, we let $\mathcal{E}^{\pm} := \$^{\pm} \otimes \mathcal{V}$.

Lemma 1.56. *Let $\omega^{\mathbb{C}}$ be the complex volume form (Definition 1.46). Then,*

$$\not{D}_{\mathcal{V}} c(\omega^{\mathbb{C}}) = -c(\omega^{\mathbb{C}}) \not{D}_{\mathcal{V}}. \tag{1.132}$$

Proof. Let $e_1, \ldots, e_n$ be a local orthonormal frame of TM and let $\omega = e^1 \wedge \cdots \wedge e^n \in \Gamma(\Lambda^n T^*M)$ be the (real) volume form on M. We claim that $\nabla \omega = 0$.
Indeed, $\nabla_X \omega = \sum_{i=1}^n e^1 \wedge \cdot \wedge \nabla_X e^i \wedge \cdot \wedge e^n = \sum_{i=1}^n \left(\nabla_X e^i\right)(e_i)\; e^1 \wedge \cdots \wedge e^n$. By metric compatibility, $(\nabla_X e^i) e_i = g(\nabla_X e^i, e^i) = -g(e^i, \nabla_X e^i)$. It follows that $\left(\nabla_X e^i\right)(e_i) = 0$, which proves that $\nabla \omega = 0$.
Since $[\nabla^{\mathcal{E}}_X, c(\omega^{\mathbb{C}})] = c(\nabla_X \omega^{\mathbb{C}}) = 0$, it follows that

$$\not{D}_{\mathcal{V}} c(\omega^{\mathbb{C}}) = \sum_{i=1}^n c(e^i) \nabla^{\mathcal{E}}_{e_i} c(\omega^{\mathbb{C}}) = \sum_{i=1}^n c(e^i \omega^{\mathbb{C}}) \nabla^{\mathcal{E}}_{e_i} = \sum_{i=1}^n -c(\omega^{\mathbb{C}} e^i) \nabla^{\mathcal{E}}_{e_i} = -c(\omega^{\mathbb{C}}) \not{D}_{\mathcal{V}}.$$ □

Since $\mathcal{E}^{\pm} = \left(\mathbb{1}_{\mathcal{E}} \pm c(\omega^{\mathbb{C}})\right) \mathcal{E}$, the Dirac operator thus restricts to operators $\not{D}_{\mathcal{V}}\big|_{\mathcal{E}^{\pm}} : \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp})$.

Definition 1.57. The operators $\not{D}^{\pm}_{\mathcal{V}} := \not{D}_{\mathcal{V}}\big|_{\mathcal{E}^{\pm}} \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp})$ are called *chiral Dirac operators*.

Using a block matrix notation corresponding to the decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, we can write $\not{D}_{\mathcal{V}} = \begin{pmatrix} 0 & \not{D}^-_{\mathcal{V}} \\ \not{D}^+_{\mathcal{V}} & 0 \end{pmatrix}$.

1.5.4 Formal Adjoints

To make contact with the analytical part of the following discussions, we will now discuss the notion of formal adjoints.

Let M be a compact Riemannian manifold. To improve readability, we will from now on denote its volume form by $\mathrm{d}x$, remembering that this is just a notational convenience and does not indicate that the volume form is exact. Let $\mathcal{E} \to M$ be a hermitian vector bundle with inner product $(\cdot, \cdot)$. On $\Gamma(\mathcal{E})$, we define the inner product

$$\langle \psi, \phi \rangle := \int_M (\phi(x), \psi(x)) dx \qquad \psi, \phi \in \Gamma(\mathcal{E}). \tag{1.133}$$

Definition 1.58. Let $\mathcal{E}$ and $\mathcal{F}$ be hermitian vector bundles. We say that a linear operator $T^* : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{E})$ is *formally adjoint* to $T : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ if

$$\langle T\psi, \phi \rangle = \langle \psi, T^* \phi \rangle \qquad \forall \psi \in \Gamma(\mathcal{E}),\ \phi \in \Gamma(\mathcal{F}). \tag{1.134}$$

By local considerations, the formal adjoint of a k-th order linear differential operator $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ exists and is again a k-th order linear differential operator $P^* : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{E})$.

Definition 1.59. We define the *divergence* of a vector field $X \in \Gamma(TM)$ as the unique scalar function $\operatorname{div}(X)$ such that

$$\operatorname{div}(X)\,\mathrm{d}x = d(\iota_X\,\mathrm{d}x). \tag{1.135}$$

Lemma 1.60. *Let $e_1, \dots, e_n$ be a local frame and $X \in \Gamma(TM)$. Then*

$$\operatorname{div}(X) = \sum_{i=1}^{n} e^i(\nabla_{e_i} X). \tag{1.136}$$

Proof. A proof can for example be found in [8]. □

Lemma 1.61. *Let $\mathcal{E}$ be a hermitian vector bundle with compatibe connection $\nabla^{\mathcal{E}}$.*
The formal adjoint of $\nabla^{\mathcal{E}} : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ is the operator $\nabla^{\mathcal{E}*} : \Gamma(T^*M \otimes \mathcal{E}) \to \Gamma(\mathcal{E})$ given by

$$\nabla^{\mathcal{E}*}(X^\flat \otimes e) = -\nabla^{\mathcal{E}}_X e - \operatorname{div}(X) e \qquad X \in \Gamma(TM),\ e \in \Gamma(\mathcal{E}), \tag{1.137}$$

where $\flat : TM \to T^*M$ is the musical isomorphism induced by the Riemmanian metric.

Proof. Fix a vector field $Z \in \Gamma(TM)$ and let $X, Y \in \Gamma(\mathcal{E})$. Then

$$\langle \nabla_Z X, Y \rangle = \int_M (\nabla_Z X, Y)\,\mathrm{d}x = \int_M Z(X,Y)\,\mathrm{d}x - (X, \nabla_Z Y)\,\mathrm{d}x = \int_M Z(X,Y)\,\mathrm{d}x - \langle X, \nabla_Z Y \rangle.$$

Let $f_{XY} := (X,Y) \in C^\infty(M)$. Observe that $Z(f_{XY}) = \iota_Z\,(df_{XY})$ and thus that

$$0 = \iota_Z\,(df_{XY} \wedge \mathrm{d}x) = \iota_Z\,(df_{XY})\,\mathrm{d}x - df_{XY} \wedge \iota_Z(\mathrm{d}x).$$

This means that $\int_M \iota_Z(df_{XY})\,\mathrm{d}x = \int_M df_{XY} \wedge \iota_Z(\mathrm{d}x)$. Using Stokes theorem on the closed manifold M,

$$0 = \int_M d\left(f_{XY}\iota_Z(\mathrm{d}x)\right) = \int_M df_{XY} \wedge \iota_Z(\mathrm{d}x) + \int_M f_{XY} d\left(\iota_Z\,(\mathrm{d}x)\right),$$

such that $\int_M df_{XY} \wedge \iota_Z(\mathrm{d}x) = -\int_M f_{XY} \operatorname{div}(Z)\,\mathrm{d}x$.
Putting everthing together, we obtain that $\int_M Z(X,Y)\,\mathrm{d}x = -\int_M (X,Y) \operatorname{div}(Z)\,\mathrm{d}x$ and therefore that

$$\langle \nabla_Z X, Y \rangle = \langle X, (-\operatorname{div}(Z) - \nabla_Z)\, Y \rangle,$$

which means that the formal adjoint of $\nabla_Z : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is $\nabla_Z^* = -\operatorname{div}(Z) - \nabla_Z$.
To compute the formal adjoint of $\nabla : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ let $e, f \in \Gamma(\mathcal{E})$ and $Z \in \Gamma(TM)$. Then, by definition of the induced inner product on $T^*M \otimes \mathcal{E}$,

$$\langle f, \nabla^*(Z^\flat \otimes e) \rangle = \langle \nabla f, Z^\flat \otimes e \rangle = \langle \nabla_Z f, e \rangle = \langle f, (-\operatorname{div}(Z) - \nabla_Z)\, e \rangle.$$

This proves the claim. □

We can now give a new and frame independent definition of the Laplacian.

Definition 1.62. The *connection Laplacian* is defined to be

$$\Delta_{\mathcal{E}} := \nabla^{\mathcal{E}*} \nabla^{\mathcal{E}} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}). \tag{1.138}$$

In this new form, the self adjointness of $\Delta_{\mathcal{E}}$ is apparent.
The following proposition proves that the new definition agrees with our old Definition 1.54.

Proposition 1.63. *Given a local frame $e_1, \dots, e_n$ of TM we can write this as*

$$\Delta_{\mathcal{E}} = -\sum_{i,j=1}^{n} g^{ij} \left(\nabla^{\mathcal{E}}_{e_i} \nabla^{\mathcal{E}}_{e_j} - \nabla^{\mathcal{E}}_{\nabla_{e_i} e_j} \right), \tag{1.139}$$

where $g^{ij} = g(e^i, e^j)$ and g is the induced metric on T^*M and where ∇ is the Levi-Civita connection.
In particular, in a local orthonormal frame we have that

$$\Delta_{\mathcal{E}} = -\sum_{i=1}^{n} \left(\nabla^{\mathcal{E}}_{e_i} \nabla^{\mathcal{E}}_{e_i} - \nabla^{\mathcal{E}}_{\nabla_{e_i} e_i} \right). \tag{1.140}$$

Proof. For the proof we will denote both connections $\nabla^{\mathcal{E}}$ and ∇ by ∇.
It suffices to prove the formula for an orthonormal frame. Let $X \in \Gamma(\mathcal{E})$ and let $e_1, \dots, e_n$ be a local orthonormal frame of TM. Since it is orthonormal, we have that $e^i = e_i^\flat$ and thus that

$$\Delta_{\mathcal{E}} X = \nabla^*(\nabla X) = \nabla^*(\sum_{i=1}^n e^i \otimes \nabla_{e_i} X) = \sum_{i=1}^n \nabla^* \left(e_i^\flat \otimes \nabla_{e_i} X \right).$$

Using Lemma 1.61, we find that

$$\Delta_{\mathcal{E}} X = \sum_{i=1}^n (-\operatorname{div}(e_i) - \nabla_{e_i}) \nabla_{e_i} X = -\sum_i (\nabla_{e_i} \nabla_{e_i} X + \operatorname{div}(e_i) \nabla_{e_i} X).$$

Since $\operatorname{div}(e_i) = \sum_{j=1}^n e^j(\nabla_{e_j} e_i) = \sum_j g(e_j, \nabla_{e_j} e_i) = -\sum_j g(\nabla_{e_j} e_j, e_i) = -e^i(\sum_j \nabla_{e_j} e_j)$ it follows that

$$\sum_i \operatorname{div}(e_i) \nabla_{e_i} X = -\sum_i e^i (\sum_j \nabla_{e_j} e_j) \nabla_{e_i} X = -\sum_j \nabla_{\nabla_{e_j} e_j} X.$$

This proves the claim. □

We can use this result about the adjoint of the connection to prove the following.

Proposition 1.64. *Let $\mathcal{E} = \$ \otimes \mathcal{V}$ be a twisted spinor bundle on an even dimensional spin manifold. Then, the Dirac operator $D\!\!\!/_{\mathcal{V}} : \Gamma(\$ \otimes \mathcal{V}) \to \Gamma(\$ \otimes \mathcal{V})$ is formally self-adjoint.*

Proof. Let $e_1, \dots, e_n$ be a local orthonormal frame of TM and write $D\!\!\!/_{\mathcal{V}} = \sum_{i=1}^n c(e^i) \nabla^{\mathcal{E}}_{e_i}$.
Proposition 1.28 implies that for $\phi, \psi \in \Gamma(E)$, $(c(e^i)\phi, \psi)_{\mathcal{E}_x} = -(\phi, c(e^i)\psi)_{\mathcal{E}_x}$, where $(\cdot, \cdot)_{\mathcal{E}_x}$ is the hermitian product on $\mathcal{E}_x$. Thus,

$$\langle D\!\!\!/^2_{\mathcal{V}} \phi, \psi \rangle = \sum_i \langle c(e^i) \nabla^{\mathcal{E}}_{e_i} \phi, \psi \rangle = -\sum_i \langle \nabla^{\mathcal{E}}_{e_i} \phi, c(e^i)\psi \rangle = -\sum_i \langle \phi, \nabla^{\mathcal{E}*}_{e_i} c(e^i) \psi \rangle.$$

Lemma 1.61 implies that

$$\sum_i \nabla^{\mathcal{E}*}_{e_i} c(e^i)\psi = -\sum_i \left(\operatorname{div}(e_i) c(e^i) + \nabla^{\mathcal{E}}_{e_i} c(e^i) \right) \psi = -c \left(\sum_i \left(\operatorname{div}(e_i) e^i + \nabla_{e_i} e^i \right) \right) \psi - \sum_{i=1}^n c(e^i) \nabla^{\mathcal{E}}_{e_i} \psi.$$

We have shown in the proof of Lemma 1.63 that $\operatorname{div}(e_i) = -e^i \left(\sum_j \nabla_{e_j} e_j \right)$.
It follows that $\sum_{i=1}^n \operatorname{div}(e_i) e_i = -\sum_{j=1}^n \nabla_{e_j} e_j$ and therefore that $\sum_{i=1}^n \nabla^{\mathcal{E}*}_{e_i} c(e^i)\psi = -\sum_{i=1}^n c(e^i) \nabla^{\mathcal{E}}_{e_i} \psi$. This implies that

$$\langle D\!\!\!/_{\mathcal{V}} \phi, \psi \rangle = \langle \phi, D\!\!\!/_{\mathcal{V}} \psi \rangle,$$

which proves the formal self adjointness of $D\!\!\!/_{\mathcal{V}}$. □

This yields the following corollary for the chiral Dirac operator.

Corollary 1.65. *The chiral Dirac operators $D\!\!\!/^{\pm}_{\mathcal{V}} : \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp})$ are formally adjoint to one another, i.e.*

$$\left(D\!\!\!/^{+}_{\mathcal{V}} \right)^* = D\!\!\!/^{-}_{\mathcal{V}}. \tag{1.141}$$

Proof. Let $\phi \in \Gamma(\mathcal{E}^+)$ and $\psi \in \Gamma(\mathcal{E}^-)$. Then, $\langle D\!\!\!/^{+}_{\mathcal{V}} \phi, \psi \rangle = \langle D\!\!\!/_{\mathcal{V}} \phi, \psi \rangle = \langle \phi, D\!\!\!/_{\mathcal{V}} \psi \rangle = \langle \phi, D\!\!\!/^{-}_{\mathcal{V}} \phi \rangle$. □

Chapter 2

The Atiyah-Singer Index Theorem

The index of an elliptic operator is defined to be the difference between the dimension of its kernel and its cokernel. Therefore, the index of any formally self-adjoint operator vanishes. An important example of an elliptic operator that is not self-adjoint is given by the chiral Dirac operator $\not{D}^+_{\mathcal{V}}$ (Definition 1.57). The following chapter will be devoted to the calculation of its index on an even dimensional compact spin manifold.

We first given an introduction to Sobolev spaces and Fredholm theory and use them to study analytical properties of the chiral Dirac operator. Thereafter, we will engage in the actual calculation of the index. We follow the 'heat equation proof', initially due to Atiyah, Bott and Patodi with a crucial last step due to Getzler.

2.1 Sobolev Spaces

The main goal of the following section is to prove Theorem 2.13, which states that the space of square integrable sections of a twisted spinor bundle has an orthonormal basis of smooth eigenfunctions of $\not{D}^2_{\mathcal{V}}$.
To reach this goal, we will introduce the theory of Sobolev spaces on manifolds, making our operators accessible to methods from functional analysis on Hilbert spaces.

In the following, let M be a closed Riemannian manifold with associated volume form $\mathrm{d}x$. Let $E \to M$ be a hermitian vector bundle with hermitian product $(\cdot,\cdot)$ and space of smooth sections denoted by $\Gamma(E)$.
Let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection on E, compatible with the hermitian product. Together with the Levi-Civita connection this induces a connection $\nabla : \Gamma((T^*M)^k \otimes E) \to \Gamma((T^*M)^{k+1} \otimes E)$.

In equation (1.133), we defined the product $\langle \phi, \psi \rangle = \int_M (\phi(x), \psi(x))\, \mathrm{d}x$, which makes $\Gamma(E)$ into a non-complete inner product space. To be able to use tools from functional analysis we will consider its completion.

Definition 2.1. The space $L^2(E)$ of *square integrabel sections of* E is defined to be the completion of the space $\Gamma(E)$ with respect to the inner product

$$\langle \phi, \psi \rangle_{L^2} := \int_M (\phi(x), \psi(x))\ \mathrm{d}x. \tag{2.1}$$

For $l \in \mathbb{N}_0$, the *Sobolev spaces* $H_l(E)$ are defined to be the completion of $\Gamma(E)$ with respect to the inner product

$$\langle \phi, \psi \rangle_{H_l} := \int_M (\phi(x), \psi(x)) + (\nabla\phi(x), \nabla\psi(x)) + \ldots + \left(\nabla^l\phi(x), \nabla^l\psi(x)\right)\ \mathrm{d}x, \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the inner product on $E \otimes (T^*M)^k$ induced by $(\cdot,\cdot)$ on E and the metric g on TM.

We remark that $H_0(E) = L^2(E)$ and we will use both notations interchangeably.
From the definition, we deduce that

$$\|\psi\|_{H_l} \leqslant \|\psi\|_{H_l'} \qquad \text{for } l \leqslant l' \text{ and } \psi \in \Gamma(E). \tag{2.3}$$

The inner product $\langle\cdot,\cdot\rangle_{H_l}$ turns all Sobolev spaces $H_l(E)$ into Hilbert spaces. However, usually we want to work with the product $\langle\cdot,\cdot\rangle_{L^2}$ instead of $\langle\cdot,\cdot\rangle_{H_l}$. To extend this product from $\Gamma(E)$ to the Sobolev space $H_l(E)$ we make the following definition.

Definition 2.2. For $l \in \mathbb{N}_0$ and $f \in \Gamma(E)$ we let

$$\|f\|_{H_{-l}} := \sup \{|\langle f,\psi\rangle_{L^2}| \mid \psi \in \Gamma(E),\ \|\psi\|_{H_l} = 1\} \tag{2.4}$$

and define the Sobolev space H_{-l} as the completion of $\Gamma(E)$ with respect to $\|\cdot\|_{H_{-l}}$.

Since

$$\|f\|_{H_{-0}} = \sup\{|\langle f,\psi\rangle_{L^2}| \mid \psi \in \Gamma(E),\ \|\psi\|_{L^2} = 1\} = \|f\|_{L^2}, \tag{2.5}$$

it follows that $H_{-0}(E) = L^2(E) = H_0(E)$.

From the definition of $\|\cdot\|_{H_{-l}}$ we conclude that for $\phi, \psi \in \Gamma(E)$

$$|\langle\phi,\psi\rangle_{L^2}| \leqslant \|\phi\|_{H_{-l}}\|\psi\|_{H_l}. \tag{2.6}$$

Hence, $\langle\cdot,\cdot\rangle_{L^2}$ extends to a non-degenerate sesquilinear pairing

$$\langle\cdot,\cdot\rangle_{L^2} : H_{-l}(E) \otimes \overline{H_l}(E) \to \mathbb{C}. \tag{2.7}$$

The most important basic properties of Sobolev spaces are summarised in the following proposition.

Proposition 2.3.

(1) *There are bounded inclusions $H_{l'}(E) \hookrightarrow H_l(E)$ for $l' > l \in \mathbb{Z}$.*

(2) *The covariant derivative ∇ extends to a bounded map $\nabla : H_l(E) \to H_{l-1}(T^*M \otimes E)$ for all $l \in \mathbb{Z}$.*

(3) *A k-th order differential operator $P : \Gamma(E) \to \Gamma(F)$ between vector bundles E and F extends to a bounded map $P : H_l(E) \to H_{l-k}(F)$ for all $l \in \mathbb{Z}$.*

Proof. (1) The inclusions $H_{l'}(E) \hookrightarrow H_l(E)$ are a direct consequence of inequality (2.3) for $l' > l \geqslant 0$. By continuity and inequality (2.6), it follows that

$$|\langle f,\phi\rangle_{L^2}| \leqslant \|f\|_{H_{-l}}\|\phi\|_{H_l} \leqslant \|f\|_{H_{-l}}\|\phi\|_{H_{l+k}} \qquad \text{for } k,l \geqslant 0 \text{ and } f \in \Gamma(E),\ \phi \in H_l(E).$$

Therefore, $\|f\|_{H_{-l-k}} \leqslant \|f\|_{H_{-l}}$ which implies that $H_{-l}(E) \subseteq H_{-l-k}(E)$ for all $l,k \geqslant 0$.
Thus, $H_{l'}(E) \subseteq H_l(E)$ for $0 \geqslant l' \geqslant l$. For $l' \geqslant 0 \geqslant l$ we note that $H_l(E) \supseteq H_0(E) \supseteq H_{l'}(E)$. This proves (1).
(2) For $l \geqslant 0$, the statement follows from the inequality

$$\|\nabla\psi\|_{H_l} \leqslant \|\psi\|_{H_{l+1}} \qquad \text{for } l \geqslant 0 \text{ and } \psi \in \Gamma(E) \tag{2.8}$$

which is a direct consequence of the definition of $\|\cdot\|_{H_l}$.
For $l < 0$, we observe that the formal adjoint $\nabla^* : \Gamma(T^*M \otimes E) \to \Gamma(E)$ can be written as $\nabla^* = L_1 \circ \nabla + L_0$, where L_1 is a section of $\mathrm{Hom}(T^*M \otimes E, E)$ and L_0 is a section of $\mathrm{End}(E)$ (see (1.137)).
Therefore, ∇^* extends to a bounded map $\nabla^* : H_{l+1}(T^*M \otimes E) \to H_l(E)$ for all $l \geqslant 0$, which means that there is a constant $C > 0$ such that $\|\nabla^*\phi\|_{H_l} \leqslant C\|\phi\|_{H_{l+1}}$ for all $\phi \in H_{l+1}(T^*M \otimes E)$.
Thus, for $\psi \in \Gamma(E)$ and $\phi \in H_{l+1}(T^*M \otimes E)$ it follows that

$$|\langle\nabla\psi,\phi\rangle_{L^2}| = |\langle\psi,\nabla^*\phi\rangle_{L^2}| \leqslant \|\psi\|_{H_{-l}}\,\|\nabla^*\phi\|_{H_l} \leqslant \|\psi\|_{H_{-l}}C\|\phi\|_{H_{l+1}}.$$

Consequently, $\|\nabla\psi\|_{H_{-l-1}} \leqslant C\|\psi\|_{H_{-l}}$, implying that ∇ extends to a bounded operator

$$\nabla : H_{-l}(T^*M \otimes E) \to H_{-l-1}(E) \qquad \text{for all } l \geqslant 0.$$

(3) This statement is a consequence of the fact, that any k-th order differential operator $P : \Gamma(E) \to \Gamma(F)$ can be written as $P = L_k \circ \nabla^k + \ldots L_1 \circ \nabla + L_0$, where L_i are sections of $\mathrm{Hom}((T^*M)^i \otimes E, F)$ for $0 \leqslant i \leqslant k$.
By (2), P extends to a bounded map $P : H_l(E) \to H_{l-k}(F)$ for all $l \in \mathbb{Z}$. □

The reason we can relate results obtained via Sobolev spaces back to the space of smooth functions is the following.

Theorem 2.4.

(1) *(Rellich). The inclusion $H_l(E) \hookrightarrow H_{l-k}(E)$ is compact for all $l \geqslant k$.*

(2) *(Sobolev). If $l - \frac{n}{2} > k$, then $H_l(E) \subseteq C^k(E)$, where $C^k(E)$ denotes the space of C^k-sections of E.*

Proof. Proofs can be found in [7] and [3]. The usual approach to theorems of this kind is to use a partition of unity to divide M into chart domains, which one can take to lie on a torus. On these domains one can then use standard techniques from Sobolev theory such as Fourier analysis. □

By Proposition 2.3, it follows that

$$\ldots \supseteq H_{-1}(E) \supseteq L^2(E) = H_0(E) \supseteq H_1(E) \supseteq \ldots \supseteq \Gamma(E). \tag{2.9}$$

Combining this with the Sobolev embedding theorem 2.4 implies that

$$\bigcap_{l \geqslant k} H_l(E) = \Gamma(E) \qquad \text{for any } k \in \mathbb{Z}. \tag{2.10}$$

Sobolev spaces are Hilbert spaces and therefore easy to deal with in terms of functional analysis. The Sobolev theorem helps us to relate results obtained in this way back to $\Gamma(E)$.

Before discussing the Dirac operator in terms of Sobolev theory, we will briefly come back to the negative Sobolev spaces.
Recall that $\langle \cdot, \cdot \rangle_{L^2}$ extends to a non-degenerate pairing $H_{-l}(E) \otimes \overline{H_l(E)} \to \mathbb{C}$. Consequently, this defines an isomorphism $f \mapsto \langle f, \cdot \rangle_{L^2}$ between $H_{-l}(E)$ and $\overline{H_l(E)}^*$. By definition of $\|\cdot\|_{H_{-l}}$ we can see that this isomorphism is isometric. Indeed,

$$\|\langle f, \cdot \rangle_{L^2}\|_{\overline{H_l(E)}^*} = \sup\{|\langle f, \psi \rangle_{L^2}| \mid \|\psi\|_{H_l} = 1\} = \|f\|_{H_{-l}}. \tag{2.11}$$

This suggest that $H_{-l}(E)$ can be thought of as a space of distributions. For example, given a point $x \in M$ and a vector $v_x \in E_x$ we can define the delta distribution

$$\delta_{v_x}(\psi) := (v_x, \psi(x)) \qquad \psi \in \Gamma(E). \tag{2.12}$$

Lemma 2.5. *The delta distribution δ_{v_x} can be extended to a bounded antilinear map $\overline{H_{[\frac{n}{2}]}(E)} \to \mathbb{C}$. Hence, $\delta_{v_x} \in H_{-[\frac{n}{2}]}(E)$.*

Proof. Observe that for $0 \leqslant l \leqslant \frac{n}{2}$

$$|\delta_{v_x}(\psi)| = |(v_x, \psi(x))| \leqslant |v_x||\psi(x)| \leqslant C\|\psi\|_{H_l},$$

where the last inequality follows from the Sobolev embedding theorem. □

Remark 2.6. Since $H_l(E)$ is also a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{H_l}$, it follows by Riesz representation theorem that $H_l(E)$ is isometric isomorphic to $\overline{H_l(E)}^*$, which in turn is isometric isomorphic to $H_{-l}(E)$. We will exploit this isomorphism $H_{-l}(E) \to H_l(E)$ in the next section for a slight variation of the inner product $\langle \cdot, \cdot \rangle_{H_l}$.

2.1.1 Sobolev Theory for Dirac Operators

So far we have dealt with Sobolev spaces defined via the linear differential operator ∇. We will now focus our attention on Dirac operators $\not{D}_{\mathcal{V}}$ on twisted spinor bundles and restate Sobolev theory in terms of this operator.

Let M be an even dimensional compact spin manifold with spinor bundle $\not{S}$ and let $\mathcal{E} = \not{S} \otimes \mathcal{V}$ be a twisted spinor bundle with Dirac operator $\not{D}_{\mathcal{V}}$. We will denote the Sobolev space $H_l(\mathcal{E})$ by H_l.

By Proposition 2.3 the Dirac operator $\not{D}_{\mathcal{V}}$ on $\mathcal{E}$ extends to a bounded map

$$\not{D}_{\mathcal{V}} : H_l \to H_{l-1} \qquad \text{for } l \in \mathbb{Z}. \tag{2.13}$$

Since the usual Sobolev theory is built on the connection ∇, we introduce a slight alteration of the inner product to make it better suited for Dirac operators and spinors. Observe that $\left(\not{D}_{\mathcal{V}}^2 + I\right)^l$ extends to a bounded map

$$\left(\not{D}_{\mathcal{V}}^2 + I\right)^l : H_l \to H_{-l} \qquad \text{for } l \geqslant 0. \tag{2.14}$$

Definition 2.7. On H_l with $l \geqslant 0$, we define the inner product

$$\langle\langle \psi, \phi \rangle\rangle_{H_l} := \langle \left(\not{D}_{\mathcal{V}}^2 + I\right)^l \psi, \phi \rangle_{L^2}. \tag{2.15}$$

We will show that this product is equivalent to the usual $\langle \cdot, \cdot \rangle_{H_l}$ product from Definition 2.1. To do so, we need the following estimate.

Proposition 2.8 (Gårding's inequality). *For $l \geqslant 0$ and $\psi \in H_l$, there exists a constant $C_l > 0$ such that*

$$\|\psi\|_{H_{l+1}}^2 \leqslant C_l \left(\|\not{D}_{\mathcal{V}}\psi\|_{H_l}^2 + \|\psi\|_{H_l}^2 \right). \tag{2.16}$$

Proof. By continuity, it suffices to verify the inequality for $\psi \in \Gamma(\mathcal{E})$. Recall Lichnerowicz's Theorem 1.55 $\not{D}_{\mathcal{V}}^2 = \Delta + \frac{1}{4} r_M + F^{\mathcal{V}}$. For $l = 0$, we have that

$$\|\psi\|_{H_1}^2 = \langle (1+\Delta)\psi, \psi \rangle_{L^2} = \langle \left(I + \not{D}_{\mathcal{V}}^2 - \frac{1}{4} r_M - F^{\mathcal{V}} \right) \psi, \psi \rangle_{L^2} = \langle \not{D}_{\mathcal{V}}\psi, \not{D}_{\mathcal{V}}\psi \rangle_{L^2} + \langle \left(1 - \frac{1}{4} R - F^{\mathcal{V}} \right) \psi, \psi \rangle_{L^2}.$$

Since both curvatures r_M and $F^{\mathcal{V}}$ are bounded on the compact manifold M, it follows that there is a constant $C_0 > 0$ such that

$$\|\psi\|_{H_1}^2 \leqslant C_0 \left(\|\not{D}_{\mathcal{V}}\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2 \right).$$

Let now $l \geqslant 0$ and assume that (2.16) holds for all smaller values of l. Hence, for $\psi \in H_l$ the induction hypothesis yields

$$\begin{aligned}
\|\psi\|_{H_l}^2 = \|\nabla\psi\|_{H_{l-1}}^2 + \|\psi\|_{L^2}^2 &\leqslant C_{l-1} \left(\|\not{D}_{\mathcal{V}}\nabla\psi\|_{H_{l-2}}^2 + \|\nabla\psi\|_{H_{l-2}}^2 \right) + \|\psi\|_{L^2}^2 \\
&\leqslant C_{l-1} \left(\|\nabla\not{D}_{\mathcal{V}}\psi\|_{H_{l-2}}^2 + \|[\not{D}_{\mathcal{V}}, \nabla]\psi\|_{H_{l-2}}^2 + \|\nabla\psi\|_{H_{l-2}}^2 \right) + \|\psi\|_{L^2}^2 \\
&\leqslant C_{l-1} \left(\|\not{D}_{\mathcal{V}}\psi\|_{H_{l-1}}^2 + \|\psi\|_{H_{l-1}}^2 + \|[\not{D}_{\mathcal{V}}, \nabla]\psi\|_{H_{l-2}}^2 \right) + \|\psi\|_{L^2}^2 \\
&\leqslant C_l \left(\|\not{D}_{\mathcal{V}}\psi\|_{H_{l-1}}^2 + \|\psi\|_{H_{l-1}}^2 \right).
\end{aligned}$$

In the last inequality, we have used that $[\not{D}_{\mathcal{V}}, \nabla]$ is an 0-th order differential operator, which can be seen by a computation in a local frame. ☐

Using this estimate, we can prove that both products $\langle\langle \cdot, \cdot \rangle\rangle_{H_l}$ and $\langle \cdot, \cdot \rangle_{H_l}$ are equivalent.

Proposition 2.9. *The inner product $\langle\langle \cdot, \cdot \rangle\rangle_{H_l}$ is equivalent to the inner product $\langle \cdot, \cdot \rangle_{H_l}$.*

Proof. We need to show that there are constants $C, c > 0$ such that

$$c \, \langle\langle \psi, \psi \rangle\rangle_{H_l} \leqslant \|\psi\|_{H_l}^2 \leqslant C \, \langle\langle \psi, \psi \rangle\rangle_{H_l}.$$

The lower bound follows directly from the fact that $(\not{D}_{\mathcal{V}}^2 + I)^l : H_l \to H_{-l}$ is bounded.
For the upper bound we use induction on $l \geqslant 0$.
The case $l = 0$ is trivial. Using Gåding's inequality and the induction hypothesis for $l > 0$ we obtain

$$\begin{aligned}
\|\psi\|_{H_l}^2 &\leqslant C_l \left(\|\not{D}_{\mathcal{V}}\psi\|_{H_{l-1}}^2 + \|\psi\|_{H_{l-1}}^2 \right) \\
&\leqslant C \left(\langle (\not{D}_{\mathcal{V}}^2 + I)^{l-1} \not{D}_{\mathcal{V}}\psi, \not{D}_{\mathcal{V}}\psi \rangle_{L^2} + \langle (\not{D}_{\mathcal{V}}^2 + I)^{l-1} \psi, \psi \rangle_{L^2} \right) \\
&= C \left(\langle \not{D}_{\mathcal{V}}^2 + I)^l \psi, \psi \rangle_{L^2} \right) = C \, \langle\langle \psi, \psi \rangle\rangle_{H_l},
\end{aligned}$$

which proves the claim. ☐

The inner product $\langle\langle\cdot,\cdot\rangle\rangle_{H_l}$ induces a norm $|||\cdot|||_{H_l}$ on H_l and a corresponding norm

$$|||f|||_{H_{-l}} := \sup\{\, |\langle f,\phi\rangle_{L^2}| \mid \phi\in H_l,\ |||\phi|||_{H_l}=1\} \qquad l\geqslant 0 \tag{2.17}$$

on H_{-l}, which is equivalent to $\|\cdot\|_{H_{-l}}$. From now on we will only work with this new set of equivalent inner products and norms. By equivalence, all statements from Proposition 2.3 and Theorem 2.4 are still true for H_l equipped with $|||\cdot|||_{H_l}$.

We have seen in Remark 2.6, that the spaces H_l and H_{-l} are isomorphic. Using our new inner product $\langle\langle\cdot,\cdot\rangle\rangle_{H_l}$ we can exhibit this isomorphism explicitly.

Corollary 2.10. *The bounded map*

$$\left(\not{D}_{\mathcal{V}}^2 + I\right)^l : H_l \to H_{-l} \tag{2.18}$$

is an isometric isomorphism with respect to $\langle\langle\cdot,\cdot\rangle\rangle_{H_l}$.

Proof. Observe that $H_{-l}\to \overline{H_l}^*$, $f\mapsto\langle f,\cdot\rangle_{L^2}$ is by definition an isometric isomorphism. Since H_l is a Hilbert space with inner product $\langle\langle\cdot,\cdot\rangle\rangle_{H_l}$, it follows by Riesz representation theorem that the map $H_l\to\overline{H_l}^*$,
$g\mapsto\langle\langle g,\cdot\rangle\rangle$ is an isometric isomorphism. Composing these maps gives an isometric isomorphism $H_l\to H_{-l}$ mapping $g\in H_l$ to the unique $f\in H_{-l}$ such that $\langle f,\cdot\rangle_{L^2}=\langle\langle g,\cdot\rangle\rangle_{H_l}$.
By definition

$$\langle\langle g,\cdot\rangle\rangle_{H_l} = \langle\left(\not{D}_{\mathcal{V}}^2+I\right)^l g,\cdot\rangle_{L^2}, \tag{2.19}$$

which means that the isometric isomorphism $H_l\to H_{-l}$ is given by $\left(\not{D}_{\mathcal{V}}^2+I\right)^l$. ☐

We can now take up the task alluded to in the beginning of this section and work towards find a spectral decomposition of $\not{D}_{\mathcal{V}}^2 : L^2(E)\to H_{-2}(E)$.
As a first step we define the operator

$$T : H_0 \hookrightarrow H_{-1} \overset{\left(\not{D}_{\mathcal{V}}^2+I\right)^{-1}}{\longrightarrow} H_1 \hookrightarrow H_0. \tag{2.20}$$

This is clearly a self adjoint and positive operator. By Theorem 2.4 and the fact that the composition of a compact and a bounded operators is again compact, it follows that T is compact. Hence, we can invoke the spectral theorem for compact self adjoint operators, a proof of which can be found in [8].

Lemma 2.11. *If T is a compact, self adjoint operator on a Hilbert space H, then H admits an orthonormal basis $(\psi_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of T to eigenvalues μ_n such that*

$$\mu_n\in\mathbb{R} \qquad \lim_{n\to\infty}\mu_n = 0. \tag{2.21}$$

Since our T is positive (and injective) it follows that $\mu_n>0$ for all $n\in\mathbb{N}$. Defining $\lambda_n := \frac{1}{\mu_n}-1$ we conclude that

$$\text{(i)}\qquad \not{D}_{\mathcal{V}}^2\psi_n = \lambda_n\psi_n \tag{2.22}$$

$$\text{(ii)}\qquad \lim_{n\to\infty}\lambda_n = \infty \tag{2.23}$$

and noting that $\lambda_n\langle\psi,\psi\rangle_{L^2} = \langle\not{D}_{\mathcal{V}}^2\psi,\psi\rangle_{L^2} = \langle\not{D}_{\mathcal{V}}\psi,\not{D}_{\mathcal{V}}\psi\rangle_{L^2}\geqslant 0$ it follows that

$$\text{(iii)}\qquad \lambda_n\geqslant 0. \tag{2.24}$$

We still need to establish that the eigenfunctions ψ_n are smooth.
To do so we need the following regularity lemma.

Lemma 2.12. *If* $\psi \in L^2$ *with* $\left(\not{D}_{\mathcal{V}}^2 + I\right)^l \psi \in L^2$ *for some* $l \geqslant 0$*, then* $\psi \in H_l$*.*

Proof. Different proofs for this lemma can be found in [7], [3] and [8]. The main idea is again to use a partition of unity to divide M into chart domains taken to lie on torus and then use standard techniques from Sobolev theory such as mollifiers. □

Theorem 2.13 (Elliptic Regularity)**.** *Let* M *be an even dimensional compact spin manifold with twisted Dirac bundle* $\mathcal{E} = \not{S} \otimes \mathcal{V}$*. Then there is an orthonormal basis* $(\psi_n)_{n\in\mathbb{N}}$ *of* $L^2(\mathcal{E})$ *consisting of eigenfunctions of* $\not{D}_{\mathcal{V}}^2$ *such that all* $\psi_n \in \Gamma(\mathcal{E})$ *(i.e. such that all* ψ_n *are smooth).*

Proof. We have already established that there is an orthonormal basis (ψ_n) of L^2 of eigenfunctions of $\not{D}_{\mathcal{V}}^2$. Since $\left(\not{D}_{\mathcal{V}}^2 + I\right)^l \psi_n = (\lambda_n + 1)^l \psi_n \in L^2$ for all $l \geqslant 0$, it follows from Lemma 2.12 that $\psi_n \in H_l$ for all $l \geqslant 0$. Thus, by the Sobolev embedding theorem, $\psi_n \in \Gamma(\mathcal{E})$. □

2.2 Fredholm Operators and Index

So far, we have defined the index on a rather informal basis. In this section, we give a formal definition of the index of the chiral Dirac operator in terms of Fredholm operators.

Definition 2.14. A bounded operator $T : H \to H'$ between Hilbert spaces H and H' is *Fredholm*, if both $\ker(T)$ and $\operatorname{coker}(T)$ are finite dimensional.

Definition 2.15. The *index* of a Fredholm operator T is defined to be

$$\operatorname{ind}(T) := \dim(\ker(T)) - \dim(\operatorname{coker}(T)). \tag{2.25}$$

By the rank-nullity theorem, the index of any operator between finite dimensional vector spaces is zero.
A less trivial example is given by the operator $R : l^2 \to l^2,\ (a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)$ which has index -1.
We also observe that any isomorphism is Fredholm with index zero.

If $A : H \to W$, $B : H' \to W'$ are bounded operators between Hilbert spaces, then $A \oplus B : H \oplus H' \to W \oplus W'$ is Fredholm if and only if A and B are Fredholm with index

$$\operatorname{ind}(A \oplus B) = \operatorname{ind}(A) + \operatorname{ind}(B). \tag{2.26}$$

This is a consequence of $\ker(A \oplus B) = \ker(A) \oplus \ker(B)$ and $\operatorname{im}(A \oplus B) = \operatorname{im}(A) \oplus \operatorname{im}(B)$.

To get a handle on Fredholm operators we state the following lemma. A proof can be found in [8].

Lemma 2.16. *Let* H*,* H' *and* H'' *be Hilbert spaces.*

(1) *If* $T : H \to H'$*,* $S : H' \to H''$ *are Fredholm, then* $TS : H \to H''$ *is Fredholm with index*

$$\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S). \tag{2.27}$$

(2) *If* $K : H \to H$ *is compact, then* $I + K : H \to H$ *is Fredholm with index zero.*

(3) *If* $T : H \to H'$ *is Fredholm with (Hilbert space) adjoint* $T^* : H' \to H$*, then* T^* *is Fredholm with*

$$\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\operatorname{coker}(T^*)) = -\operatorname{ind}(T^*). \tag{2.28}$$

It turns out that the Dirac Laplacian $\not{D}_{\mathcal{V}}^2$, the Dirac operator $\not{D}_{\mathcal{V}}$ and the chiral Dirac operators $\not{D}_{\mathcal{V}}^{\pm}$ are all Fredholm. To prove this, we need the following lemma.

Lemma 2.17. *For $l \in \mathbb{Z}$,*

$$I + \not{D}_{\mathcal{V}}^2 : H_l \to H_{l-2} \tag{2.29}$$

is an isomorphism.

Proof. We've already proven the case $l = 1$ in Corollary 2.10. A proof for the general case can be found in [8]. □

We are now ready to show that the Dirac operator and its square are both Fredholm operators.

Proposition 2.18.

(1) *The square of the Dirac operator $\not{D}_{\mathcal{V}}^2 : H_l \to H_{l-2}$ is Fredholm with index zero.*

(2) *The Dirac operator $\not{D}_{\mathcal{V}} : H_l \to H_{l-1}$ is Fredholm with index zero.*

Proof. (1) By Lemma 2.17, $I + \not{D}_{\mathcal{V}}^2 : H_{l+2} \to H_l$ is an isomorphism with inverse $(I + \not{D}_{\mathcal{V}}^2)^{-1} : H_l \to H_{l+2}$. Since $\not{D}_{\mathcal{V}}^2(I + \not{D}_{\mathcal{V}}^2)^{-1} : H_l \to H_l$ is of the form $I - K$, where $K = (I + \not{D}_{\mathcal{V}}^2)^{-1} : H_l \to H_{l+2} \hookrightarrow H_l$. By Rellich's Theorem 2.4, the operator K is compact. It follows from Lemma 2.16 that $\not{D}_{\mathcal{V}}^2(I + \not{D}_{\mathcal{V}}^2)^{-1} : H_l \to H_{-l}$ is Fredholm and has index zero. Since any isomorphism is Fredholm with index zero, it follows that $\not{D}_{\mathcal{V}}^2 = \not{D}_{\mathcal{V}}^2(I + \not{D}_{\mathcal{V}}^2)^{-1}(I + \not{D}_{\mathcal{V}}^2)$ is Fredholm with index

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^2) = \operatorname{ind}(\not{D}_{\mathcal{V}}^2(I + \not{D}_{\mathcal{V}}^2)^{-1}) + \operatorname{ind}(I + \not{D}_{\mathcal{V}}^2) = 0.$$

For (2), consider the operator $I + i\not{D}_{\mathcal{V}} : H_l \to H_{l-1}$ and observe that

$$(I - i\not{D}_{\mathcal{V}})(I + i\not{D}_{\mathcal{V}}) = I + \not{D}_{\mathcal{V}}^2 : H_l \to H_{l-2}.$$

Since this is an isomorphism, it follows that $I + i\not{D}_{\mathcal{V}} : H_l \to H_{l-1}$ is injective.
Since also $(I + i\not{D}_{\mathcal{V}})(I - i\not{D}_{\mathcal{V}}) = I + \not{D}_{\mathcal{V}}^2 : H_{l+1} \to H_{l-1}$ is an isomorphism, it follows that $I + i\not{D}_{\mathcal{V}}$ is surjective and thus that $I + i\not{D}_{\mathcal{V}} : H_l \to H_{l-1}$ is an isomorphism. Now consider $i\not{D}_{\mathcal{V}}(I + i\not{D}_{\mathcal{V}})^{-1} : H_l \to H_l$, which is again of the form $I - K$, where $K = (I + i\not{D}_{\mathcal{V}})^{-1} : H_l \to H_{l+1} \hookrightarrow H_l$ is compact.
Thus $\not{D}_{\mathcal{V}} i(I + \not{D}_{\mathcal{V}})^{-1}$ is Fredholm of index zero. By the same reasoning as for $\not{D}_{\mathcal{V}}^2$, it follows that $\not{D}_{\mathcal{V}}$ is Fredholm of index zero. □

In particular, this means that the index of $\not{D}_{\mathcal{V}}$ and $\not{D}_{\mathcal{V}}^2$ is independent of $l \in \mathbb{Z}$.

We've seen that the index of both the Dirac operator and its square vanishes. This is due to the fact that both operators are formally self-adjoint. We'll thus turn our attention to the non self-adjoint chiral Dirac operators with non-trivial index.

Corollary 2.19. *The chiral Dirac operators $\not{D}_{\mathcal{V}}^{\pm} : H_l(\not{S}^{\pm} \otimes \mathcal{V}) \to H_{l-1}(\not{S}^{\mp} \otimes \mathcal{V})$ are Fredholm with*

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^+) = \dim(\ker(\not{D}_{\mathcal{V}}^+)) - \dim(\ker(\not{D}_{\mathcal{V}}^-)) = -\operatorname{ind}(\not{D}_{\mathcal{V}}^-). \tag{2.30}$$

Proof. Since $\not{D}_{\mathcal{V}} = \not{D}_{\mathcal{V}}^+ \oplus \not{D}_{\mathcal{V}}^-$, the statement follows from equations (2.26) and (2.28). □

Corollary 2.20. *The index of $\not{D}_{\mathcal{V}}^{\pm} : H_l \to H_{l-1}$ is independent of $l \in \mathbb{Z}$.*

Proof. We remark that by the elliptic regularity theorem 2.13 $\ker(\not{D}_{\mathcal{V}}^+) \oplus \ker(\not{D}_{\mathcal{V}}^-) = \ker(\not{D}_{\mathcal{V}}) \subseteq \Gamma(\mathcal{E})$. Thus, $\dim(\ker(\not{D}_{\mathcal{V}}^{\pm}))$ and therefore also $\operatorname{ind}(\not{D}_{\mathcal{V}}^+)$ are independent of l. □

We can thus speak about *the* index of $\not{D}_{\mathcal{V}}^{\pm} : \Gamma(\not{S}^{\pm} \otimes \mathcal{V}) \to \Gamma(\not{S}^{\mp} \otimes \mathcal{V})$ and don't have to care about the Sobolev spaces between which we consider the operator.
This statement is only based on elliptic regularity and is therefore true for all elliptic operators.

2.3 The Proof of the Index Theorem

We now come to the actual proof of the index theorem, which will be divided in four steps.
Firstly, we express the index of the chiral dirac operator in terms of the supertrace of the heat operator associated to $\not{D}_{\mathcal{V}}^2$. This result is known as the McKean-Singer formula (Theorem 2.29).
In the second step, we study the heat operator in greater detail and show that it can be written as an integral over a heat kernel and that its supertrace is the integral over the pointwise supertraces of this kernel (Mercer's Theorem 2.35).
Thirdly, we show that the heat kernel has an asymptotic expansion and express the index in terms of a specific coefficient of this expansion (equation (2.63)).
Finally, we employ a scaling argument to reduce the calculation of this coefficient to the calculation of the heat kernel of a generalized harmonic oscillator.

Before we start proving the McKean-Singer formula, we give a brief introduction to superspaces.

2.3.1 Superspaces and Supertraces

Many properties we've discussed so far are related to $\mathbb{Z}_2$-gradings of certain spaces, such as the $\mathbb{Z}_2$-grading of the Clifford algebra $\mathrm{Cl}(n) = \mathrm{Cl}^0(n) \oplus \mathrm{Cl}^1(n)$ or the $\mathbb{Z}_2$- grading of the spin representation $S = S^+ \oplus S^-$.
Since especially the grading of the spin representation will play a prominent role in the following proof of the index theorem, we will give a short introduction in $\mathbb{Z}_2$-graded vector spaces (or superspaces) and the supertrace defined on their operators.

Definition 2.21. A *superspace* is a $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$.
The *grading operator* ϵ of a superspace V is the element $\epsilon \in \mathrm{End}(V)$ defined by

$$\epsilon v = \begin{cases} +1 & v \in V^+ \\ -1 & v \in V^- \end{cases} . \tag{2.31}$$

If V is a finite dimensional superspace, we define its *superdimension* to be

$$\dim_S(V) = \dim(V^+) - \dim(V^-) \tag{2.32}$$

and for $A \in \mathrm{End}(V)$, we define the *supertrace*

$$\mathrm{tr}_S(A) := \mathrm{tr}(\epsilon \circ A) \qquad A \in \mathrm{End}(V). \tag{2.33}$$

Writing an operator $A \in \mathrm{End}(V)$ in terms of a block matrix with respect to the splitting $V = V^+ \oplus V^-$, its supertrace is

$$\mathrm{tr}_S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathrm{tr}(a) - \mathrm{tr}(d). \tag{2.34}$$

We've already encountered the spin representation $S = S^+ \oplus S^-$, a superspace with grading operator $c(\omega^{\mathbb{C}})$.

Lemma 2.22. *Let $e_1 \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ with corresponding basis $\{e_I = e_{i_1} \cdots e_{i_r}\}$ of $\mathrm{Cl}_{\mathbb{C}}(n)$. If $A \in \mathrm{End}_{\mathbb{C}}(S) \cong \mathrm{Cl}_{\mathbb{C}}(n)$ is given as $\sum_I A^I e_I$, its supertrace is*

$$\mathrm{tr}_S(A) = (-2i)^{\frac{n}{2}} A^{1,\dots,n} \tag{2.35}$$

Proof. Note that $\mathrm{tr}_S(A) = \mathrm{tr}(\omega^{\mathbb{C}} A)$ and $\omega^{\mathbb{C}} = i^{\frac{n}{2}} e_1 \cdots e_n$. Therefore,

$$\mathrm{tr}_S(e_1 \cdots e_n) = \mathrm{tr}(i^{-\frac{n}{2}} (\omega^{\mathbb{C}})^2) = i^{-\frac{n}{2}} \dim(S) = i^{-\frac{n}{2}} 2^{\frac{n}{2}} = (-2i)^{\frac{n}{2}}.$$

Observe that for $A \in \mathrm{Cl}_{\mathbb{C}}(n)$ and e_i some basis element,

$$\mathrm{tr}_S(e_i A) = \mathrm{tr}(\omega^{\mathbb{C}} e_i A) = -\mathrm{tr}(e_i \omega^{\mathbb{C}} A) = -\mathrm{tr}(\omega^{\mathbb{C}} A e_i) = -\mathrm{tr}_S(A e_i).$$

For e_I a basis element of $\mathrm{Cl}_{\mathbb{C}}(n)$ with some $i \notin I$ this means that

$$\mathrm{tr}_S(e_i e_I e_i) = -\,\mathrm{tr}_S(e_I e_i^2) = \mathrm{tr}_S(e_I).$$

Since $e_i e_I e_i = (-1)^{|I|+1} e_I$, it follows that $\mathrm{tr}_S(e_I) = (-1)^{|I|+1}\,\mathrm{tr}_S(e_I)$, implying that $\mathrm{tr}_S(e_I) = 0$ if $|I|$ is even. Now assume that e_I is a basis element of $\mathrm{Cl}_{\mathbb{C}}(n)$ with $|I|$ odd. Then, $e_I\omega^{\mathbb{C}} = -\omega^{\mathbb{C}} e_I$ and thus

$$\mathrm{tr}_S(e_I) = \mathrm{tr}(\omega^{\mathbb{C}} e_I) = -\,\mathrm{tr}(e_I \omega^{\mathbb{C}}) = -\,\mathrm{tr}(\omega^{\mathbb{C}} e_I) = -\,\mathrm{tr}_S(e_I),$$

which shows that $\mathrm{tr}_S(e_I) = 0$ for all $I \neq \{1, \ldots, n\}$. □

We have to be more careful if V is an infinite-dimensional superspace.

Definition 2.23. A compact operator T on a seperable Hilbert space H is *trace-class*, if the eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ of T^*T satisfy $\sum_{n=1}^{\infty} \sqrt{\mu_n} < \infty$. For a trace class operator T, we can then define its *trace* as $\mathrm{Tr}(T) = \sum_{n=1}^{\infty} (Te_n, e_n)$, where $(e_i)_{i \in \mathbb{N}}$ is any orthonormal basis of H.

Since the composition of a bounded and a trace-class operator is still trace-class, it follows that the supertrace of a trace class operator is also well defined. We will denote supertraces on infinite dimensional vector spaces by Tr_S and supertraces on finite dimensional vector spaces by tr_S.

2.3.2 The McKean-Singer formula - Step One

We are now ready for our first step in the proof of the Atiyah-Singer index theorem.
Since $\left(\not{D}_{\mathcal{V}}^+\right)^* = \not{D}_{\mathcal{V}}^-$ it follows that

$$\ker(\not{D}_{\mathcal{V}}^- \not{D}_{\mathcal{V}}^+) = \ker(\not{D}_{\mathcal{V}}^+) \qquad \text{and} \qquad \ker(\not{D}_{\mathcal{V}}^+ \not{D}_{\mathcal{V}}^-) = \ker(\not{D}_{\mathcal{V}}^-). \tag{2.36}$$

Thus,

$$\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \dim\left(\ker(\not{D}_{\mathcal{V}}^- \not{D}_{\mathcal{V}}^+)\right) - \dim\left(\ker(\not{D}_{\mathcal{V}}^+ \not{D}_{\mathcal{V}}^-)\right). \tag{2.37}$$

In the language of superspaces, this means that

$$\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \dim_S(\ker(\not{D}_{\mathcal{V}}^2)). \tag{2.38}$$

We make the following observation: By carefully balancing the gradings, we can replace $\ker(\not{D}_{\mathcal{V}}^2)$ by some higher dimensional but easier calculable space without changing its superdimension.
It is easier to work with supertraces instead of superdimensions. Let $\mathcal{P}_{\ker(\not{D}_{\mathcal{V}}^2)}$ be the projector on the kernel of $\not{D}_{\mathcal{V}}^2$. Since $\dim_S(\ker(\not{D}_{\mathcal{V}}^2)) = \mathrm{Tr}_S\left(\mathcal{P}_{\ker(\not{D}_{\mathcal{V}}^2)}\right)$, our plan is to replace this projector with another operator of larger support but same supertrace.
We will show that such an operator is given by the heat operator $e^{-t\not{D}_{\mathcal{V}}^2}$.

Let $V(\lambda)$ be the $\lambda-$eigenspace of $\not{D}_{\mathcal{V}}^2$ in $L^2(\mathcal{E})$. By Theorem 2.13, it follows that $V(\lambda) \subseteq \Gamma(\mathcal{E})$.
Let $V_{\pm}(\lambda) := V(\lambda) \cap \Gamma(\mathcal{E}^{\pm})$.

Lemma 2.24. *For $\lambda \neq 0$, the operator $\not{D}_{\mathcal{V}}^+ : \Gamma(\mathcal{E}^+) \to \Gamma(\mathcal{E}^-)$ restricts to an isomorphism $V_+(\lambda) \to V_-(\lambda)$.*

Proof. For $\psi \in V_+(\lambda)$, it follows that $\lambda\psi = \not{D}_{\mathcal{V}}^2\psi = \not{D}_{\mathcal{V}}^- \not{D}_{\mathcal{V}}^+\psi$. Thus,

$$\lambda \not{D}_{\mathcal{V}}^+\psi = \not{D}_{\mathcal{V}}^+ \not{D}_{\mathcal{V}}^- \not{D}_{\mathcal{V}}^+\psi = \not{D}_{\mathcal{V}}^2 \not{D}_{\mathcal{V}}^+\psi.$$

Therefore, $\not{D}_{\mathcal{V}}^+(V_+(\lambda)) \subseteq V(\lambda) \cap \Gamma(\mathcal{E}^-) = V_-(\lambda)$. Since $\not{D}_{\mathcal{V}}^- \not{D}_{\mathcal{V}}^+|_{V_+(\lambda)} = \lambda\ \mathrm{Id}$, it follows that $\not{D}_{\mathcal{V}}^+$ is injective. On the other hand since also $\left.\not{D}_{\mathcal{V}}^+ \not{D}_{\mathcal{V}}^-\right|_{V_-(\lambda)} = \lambda\,\mathrm{Id}$, it follows that $\not{D}_{\mathcal{V}}^+$ is also surjective and thus an isomorphism. □

For $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ with $\sup_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} |f(\lambda)| < \infty$, we define the operator

$$f(\not{D}_{\mathcal{V}}^2) : L^2(\mathcal{E}) \to L^2(\mathcal{E}), \qquad \psi_n \mapsto f(\lambda_n)\psi_n. \tag{2.39}$$

This is a well-defined bounded operator.

Lemma 2.25. *The operator $f(\not{D}_{\mathcal{V}}^2)$ is trace class if and only if $\sum_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} |f(\lambda)| < \infty$.*

Proof. Since $\lim_{n\to\infty} |f(\lambda_n)| = 0$, the operator $f(\not{D}_{\mathcal{V}}^2)$ is a limit of finite rank operators and therefore compact. Because it is self adjoint, it is trace class if and only if $\sum_\lambda |f(\lambda)| < \infty$. □

Proposition 2.26. *Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be such that $f(0) = 1$ and $\sum_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} |f(\lambda)| < \infty$. Then*

$$\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \mathrm{Tr}(f(\not{D}_{\mathcal{V}}^-\not{D}_{\mathcal{V}}^+)) - \mathrm{Tr}(f(\not{D}_{\mathcal{V}}^+\not{D}_{\mathcal{V}}^-)) = \mathrm{Tr}_S(f(\not{D}_{\mathcal{V}}^2)). \tag{2.40}$$

Proof. For $\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)$, let $V(\lambda) \subseteq \Gamma(\mathcal{E})$ the eigenspace of $\not{D}_{\mathcal{V}}^2$ to eigenvalue λ and let $V_\pm(\lambda) = V(\lambda) \cap \Gamma(\mathcal{E}^\pm)$. Observe that

$$\mathrm{Tr}_S(f(\not{D}_{\mathcal{V}}^2)) = \sum_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} \Big(\dim(V_+(\lambda)) - \dim(V_-(\lambda))\Big) f(\lambda).$$

By Lemma 2.24, for $\lambda \neq 0$, $V_+(\lambda) \cong V_-(\lambda)$, and thus $\dim(V_+(\lambda)) = \dim(V_-(\lambda))$. Therefore,

$$\mathrm{Tr}_S(f(\not{D}_{\mathcal{V}}^2)) = \dim(V_+(0)) - \dim(V_-(0)),$$

which equals $\mathrm{ind}(\not{D}_{\mathcal{V}}^+)$ by equation (2.37). □

A specific example of a function as in Proposition 2.26 is $f_t(\lambda) := e^{-t\lambda}$ for $t > 0$. Since $\sup_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} |e^{-t\lambda}| \leqslant 1$, the map $e^{-t\not{D}_{\mathcal{V}}^2}$ defines a bounded operator on $L^2(\mathcal{E})$.

Definition 2.27. The bounded map

$$e^{-t\not{D}_{\mathcal{V}}^2} : L^2(\mathcal{E}) \to L^2(\mathcal{E}) \qquad \psi_n \mapsto e^{-t\lambda_n}\psi_n \tag{2.41}$$

is called the *heat operator*.

Lemma 2.28. *It holds that*

$$\sum_{\lambda \in \mathrm{spec}(\not{D}_{\mathcal{V}}^2)} e^{-t\lambda} < \infty \tag{2.42}$$

for all $t > 0$. In other words, the heat operator is trace class.

Proof. This fact is part of Mercer's theorem 2.35. □

Because of its importance we will restate Proposition 2.26 in terms of the heat operator.

Theorem 2.29 (McKean-Singer formula)**.** *For any $t > 0$, we have that*

$$\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \mathrm{Tr}(e^{-t\not{D}_{\mathcal{V}}^-\not{D}_{\mathcal{V}}^+}) - \mathrm{Tr}(e^{-t\not{D}_{\mathcal{V}}^+\not{D}_{\mathcal{V}}^-}) = \mathrm{Tr}_S(e^{-t\not{D}_{\mathcal{V}}^2}). \tag{2.43}$$

2.3.3 The Heat Equation - Step Two

In the first step of the proof we realized that we can replace $\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \dim_S\left(\ker(\not{D}_{\mathcal{V}}^2)\right)$ by the supertrace of the heat operator $e^{-t\not{D}_{\mathcal{V}}^2}$. In this second step we will explain why this operator is easier to deal with than the projector on the kernel and why it is called heat operator.
Consider the (Dirac-) heat equation

$$\left(\partial_t + \not{D}_{\mathcal{V}}^2\right)\psi_t(x) = 0. \tag{2.44}$$

Plugging in, we see that the operator $e^{-t\not{D}_{\mathcal{V}}^2}$ maps initial conditions $\psi_0 \in L^2(\mathcal{E})$ to solutions $\psi_t = e^{-t\not{D}_{\mathcal{V}}^2}\psi_0$. Thus, $e^{-t\not{D}_{\mathcal{V}}^2}$ is the evolution operator for the heat equation or simply the heat operator.
We will see that the full heat operator is a smoothing operator mapping distributional initial conditions to smooth solutions. This allows us to express it in terms of an integral kernel $e^{-t\not{D}_{\mathcal{V}}^2}\psi(x) = \int_M p_t(x,y)\psi(y)\,\mathrm{d}y$. Mercer's theorem 2.35 then shows that the supertrace of the heat operator can be calculated as the integral of pointwise supertraces of the heat kernel. This reduces the infinite-dimensional supertrace $\mathrm{Tr}_S(e^{-t\not{D}_{\mathcal{V}}^2})$ to an integral over finite-dimensional supertraces.

The fact that $e^{-t\not{D}_{\mathcal{V}}^2}$ is smoothing is proven in the following proposition.

Proposition 2.30. *The image of the heat operator is contained in* $\Gamma(\mathcal{E})$

$$e^{-t\not{D}_{\mathcal{V}}^2} : L^2(\mathcal{E}) \to \Gamma(\mathcal{E}). \tag{2.45}$$

A map with this property is called a smoothing operator.

Proof. Let $\psi \in L^2(\mathcal{E})$. We will show that $e^{-t\not{D}_{\mathcal{V}}^2}\psi \in H_l(\mathcal{E})$ for all $l \geqslant 0$. By Proposition 2.12, it suffices to show that $(\not{D}_{\mathcal{V}}^2 + I)^l e^{-t\not{D}_{\mathcal{V}}^2}\psi \in L^2(\mathcal{E})$ for all $l \geqslant 0$. We write $\psi = \sum_n a_n\psi_n$, where ψ_n are smooth eigenfunctions of $\not{D}_{\mathcal{V}}^2$. Then $e^{-t\not{D}_{\mathcal{V}}^2}\psi = \sum_n a_n e^{-t\lambda_n}\psi_n$. Let $C_l = \sup_{\lambda>0}(\lambda+1)^{2l}e^{-2t\lambda}$. Then,

$$\|(\not{D}_{\mathcal{V}}^2 + I)^l e^{-t\not{D}_{\mathcal{V}}^2}\psi\|_{L^2}^2 = \sum_n |a_n|^2(\lambda_n+1)^{2l}e^{-2t\lambda_n} \leqslant C_l\sum_n |a_n|^2 = C_l\|\psi\|_{L^2}^2,$$

which proves that $e^{-t\not{D}_{\mathcal{V}}^2}\psi \in H_l$ for all $l \geqslant 0$.
By the Sobolev embedding theorem it follows that $e^{-t\not{D}_{\mathcal{V}}^2}\psi \in \Gamma(\mathcal{E})$. □

Indeed, the heat operator is smoothing for all kind of distributional initial conditions.
By duality, any operator $A_l : H_l \to H_l$ for $l \geqslant 0$ defines an operator $A_{-l} : H_{-l} \to H_{-l}$ via

$$\langle A_{-l}\psi, \phi\rangle_{L^2} := \langle \psi, A_l^*\phi\rangle_{L^2} \qquad \text{for } \psi \in H_{-l}, \phi \in H_l. \tag{2.46}$$

If the image of A_l is contained in $\Gamma(\mathcal{E})$, then this is also true for the image of $A_{-l} : H_{-l} \to H_{-l}$.

We can summarise these findings in the following corollary.

Corollary 2.31. *For all* $l \in \mathbb{Z}$, *the heat operator is a smoothing operator*

$$e^{-t\not{D}_{\mathcal{V}}^2} : H_l(\mathcal{E}) \to \Gamma(\mathcal{E}) \qquad \textit{for all } l \in \mathbb{Z}. \tag{2.47}$$

Since the heat operator is smoothing, it can be expressed in terms of a kernel. We introduce this kernel as the solution of the heat equation originating from a delta function.

Definition 2.32. For $x, y \in M$ and $t > 0$, we define the *heat kernel* $p_t(x,y)$ as the map

$$p_t(x,y) : \mathcal{E}_y \to \mathcal{E}_x \qquad \sigma_y \mapsto \left(e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\sigma_y}\right)(x). \tag{2.48}$$

Proposition 2.33. *The heat kernel* $p_t(x,y)$ *is smooth in* x, y *and* $t > 0$ *and* $\sigma_y \mapsto p_t(x,y)\sigma_y$ *is linear.*
More formally

$$p_t : (x,y) \in M \mapsto p_t(x,y) \tag{2.49}$$

is a smooth section of the external tensor product $E^* \boxtimes E$ *and* $t \mapsto p_t$ *is a smooth family of such sections.*

Proof. Linearity in σ_y is a direct consequence of linearity of $\sigma_y \mapsto \delta_{\sigma_y}$.
It follows from Corollary 2.31 that $p_t(x,y)$ is smooth in x. Since $e^{-t\not{D}_{\mathcal{V}}^2}$ is formally self-adjoint, we have for $\tau_y \in \mathcal{E}_y, \sigma_x \in \mathcal{E}_x$ that

$$\langle \delta_{\sigma_x}, e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\tau_y}\rangle_{L^2} = \langle e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\sigma_x}, \delta_{\tau_y}\rangle_{L^2},$$

i.e. that

$$(\sigma_x, p_t(x,y)\tau_y) = \langle \delta_{\sigma_x}, e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\tau_y}\rangle_{L^2} = \langle e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\sigma_x}, \delta_{\tau_y}\rangle_{L^2} = (p_t(y,x)\sigma_x, \tau_y).$$

This means that

$$p_t(y,x) = \overline{p_t(x,y)}, \tag{2.50}$$

where $\bar{\cdot}$ denotes the adjoint of a map $\mathcal{E}_y \to \mathcal{E}_x$ with respect to the inner product $(\cdot,\cdot)$. Therefore, $p_t(x,y)$ is also smooth in y. Smoothness in $t > 0$ follows from the fact that for fixed $y \in M$ and $\sigma_y \in \mathcal{E}_y$, $p_t(x,y)\sigma_y$ fulfills the heat equation

$$\left(\partial_t + \not{D}_{\mathcal{V},x}^2\right) p_t(x,y)\sigma_y = 0. \qquad \square$$

Using the eigenfunctions ψ_n of $\not{D}_{\mathcal{V}}^2$ to eigenvalues λ_n we can also write

$$p_t(x,y)\sigma_y = \sum_n \psi_n(x)\,(p_t(x,y)\sigma_y, \psi_n(x)) = \sum_n e^{-t\lambda_n}\psi_n(x)\,(\sigma_y, \psi_n(y)), \tag{2.51}$$

or

$$p_t(x,y) = \sum_n e^{-t\lambda_n}\psi_n(x) \otimes \psi_n^*(y), \tag{2.52}$$

where $\psi_n^*(y) \in \mathcal{E}_y^*$ is such that for $f_y \in \mathcal{E}_y$, $\psi_n^*(y)(f_y) := (f_y, \psi_n(y))$.

The heat kernel is indeed an integral kernel of $e^{-t\not{D}_{\mathcal{V}}^2}$.

Proposition 2.34. *For any smooth section $\psi \in \Gamma(\mathcal{E})$,*

$$\left(e^{-t\not{D}_{\mathcal{V}}^2}\psi\right)(x) = \int_M p_t(x,y)\psi(y)\,\mathrm{d}y. \tag{2.53}$$

Proof. Any section $\psi \in \Gamma(\mathcal{E})$ can be seen as the distribution

$$f \mapsto \langle \psi, f\rangle_{L^2} = \int_M (\psi(x), f(x))\,\mathrm{d}x = \int_M \delta_{\psi(x)}(f)\,\mathrm{d}x.$$

Thus, as a distribution $\psi = \int_M \delta_{\psi(x)}\,\mathrm{d}x$ and consequently,

$$\left(e^{-t\not{D}_{\mathcal{V}}^2}\psi\right)(x) = \int_M \left(e^{-t\not{D}_{\mathcal{V}}^2}\delta_{\psi(y)}\right)(x)\,\mathrm{d}y = \int_M p_t(x,y)\psi(y)\,\mathrm{d}y. \qquad \square$$

The advantage of the heat kernel is, that we can calculate the trace of the heat operator as an integral over the trace of the heat kernel on the diagonal. This result is known as Mercer's theorem.

Theorem 2.35 (Mercer)**.** *For $t > 0$, the heat operator $e^{-t\not{D}_{\mathcal{V}}^2} : L^2(\mathcal{E}) \to \Gamma(\mathcal{E})$ is trace class.*
Its supertrace is related to the fibrewise supertrace of the operator $p_t(x,x) \in \mathrm{End}(\mathcal{E}_x)$ via

$$\mathrm{Tr}_S(e^{-t\not{D}_{\mathcal{V}}^2}) = \int_M \mathrm{tr}_S(p_t(x,x))\,\mathrm{d}x. \tag{2.54}$$

Proof. To show that $e^{-t\not{D}_{\mathcal{V}}^2}$ is trace class, it suffices to prove that $\sum_n e^{-t\lambda_n} < \infty$ (see Lemma 2.25).
For $\sigma_y \in \mathcal{E}_y$, we have that

$$p_t(x,y)\sigma_y = \sum_n e^{-t\lambda_n}\psi_n(x)\,(\sigma_y, \psi_n(y)).$$

Setting $x = y$ and tracing over $\sigma_x \in \mathcal{E}_x$, we obtain

$$\mathrm{tr}(p_t(x,x)) = \sum_n e^{-t\lambda_n}|\psi_n(x)|^2.$$

Integrating over M we obtain

$$\int_M \mathrm{tr}(p_t(x,x))\,\mathrm{d}x = \int_M \sum_n e^{-t\lambda_n}|\psi_n(x)|^2\,\mathrm{d}x = \sum_n e^{-t\lambda_n}\int_M |\psi_n(x)|^2\,\mathrm{d}x = \sum_n e^{-t\lambda_n}.$$

Since $\sum_{n=1}^{N} e^{-t\lambda_n} |\psi_n(x)|^2$ is monotone in N, the monotone convergence theorem allowed us to interchange summation and integration. Now observe that $\mathrm{tr}(p_t(x,x))$ is a smooth function on a compact manifold. Therefore,

$$\infty > \int_M \mathrm{tr}(p_t(x,x))\ \mathrm{d}x = \sum_n e^{-t\lambda_n},$$

proving that $e^{-t\not{D}_{\mathcal{V}}^2}$ is trace-class.
To obtain the formula for the supertrace, we just have to insert the grading automorphism

$$\epsilon : \mathcal{E}_x \to \mathcal{E}_x, \qquad \epsilon(v_x) = \begin{cases} v & v \in \mathcal{E}_x^+ \\ -v & v \in \mathcal{E}_x^- \end{cases}$$

to obtain

$$\mathrm{Tr}_S(e^{-t\not{D}_{\mathcal{V}}^2}) = \int_M \mathrm{tr}_s(p_t(x,x))\ \mathrm{d}x. \qquad \square$$

2.3.4 The Asymptotic Expansion of the Heat Kernel - Step Three

We have seen that the index of the chiral Dirac operator $\mathrm{ind}(\not{D}_{\mathcal{V}}^+) = \mathrm{Tr}_S(\mathcal{P}_{\ker(\not{D}_{\mathcal{V}}^2)})$ can be expressed as the supertrace of the heat operator $\mathrm{Tr}_S\left(e^{-t\not{D}_{\mathcal{V}}^2}\right)$. Since $e^{-t\not{D}_{\mathcal{V}}^2} \overset{t\to\infty}{\to} \mathcal{P}_{\ker(\not{D}_{\mathcal{V}}^2)}$, we can interpret this result as stating that the supertrace of the heat operator is preserved for all times $t > 0$.
At late times $t \to \infty$, any solution of the heat equation will be in the kernel of $\not{D}_{\mathcal{V}}^2$ and thus depending on the global geometry and topology of M (which is how the analytical index $\dim_S\left(\ker(\not{D}_{\mathcal{V}}^2)\right)$ was initially defined). However, for small times $t > 0$, we expect that the solution is supported in a small neighboorhood of its initial distribution and therefore only depends on the local geometry of M.

This heuristic leads to the idea of calculating the index

$$\mathrm{ind}\left(\not{D}_{\mathcal{V}}^+\right) = \int_M \mathrm{tr}_S(p_t(x,x))\,\mathrm{d}x \qquad \forall t > 0 \tag{2.55}$$

in terms of the $t \to 0$ limit

$$\mathrm{ind}\left(\not{D}_{\mathcal{V}}^+\right) = \lim_{t\to 0} \int_M \mathrm{tr}_S(p_t(x,x))\,\mathrm{d}x. \tag{2.56}$$

To find this limit, we don't need the full heat kernel $\mathrm{tr}_S(p_t(x,x))$ but only its leading terms in an expansion in powers of t.

Definition 2.36. Let f be a continuous function on $(0,\infty)$. If there exist constants $a_n \in \mathbb{C}$ such that

$$f(t) - \sum_{n=0}^{N-1} a_n t^n = O(t^N) \quad (t \to 0) \qquad \forall N \in \mathbb{N}, \tag{2.57}$$

then we say that the (possibly non-convergent) formal power series $\sum_{n=0}^{\infty} a_n t^n$ is an *asymptotic expansion* of f. We write

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^n \quad (t \to 0). \tag{2.58}$$

An asymptotic expansion is unique: Given a continuous $f : (0,\infty) \to \mathbb{C}$, with an asymptotic expansion, we can reconstruct its expansion coefficients as $a_0 = \lim_{t\to 0} f(t),\ a_1 = \lim_{t\to 0} \frac{f(t)-a_0}{t}$ and so forth.

For our purposes we need an asymptotic expansion of $t-$families of sections of vector bundles. To be able to freely interchange derivatives and integrals with the expansion, we make the following refined definition.

Definition 2.37. Let M be a compact manifold and f_t a smooth family of sections of a hermitian vector bundle $\mathcal{V}$ with connection ∇. If there are sections $a_n \in \Gamma(\mathcal{V})$ such that

$$\sup_{x\in M} \|\partial_t^k \nabla^j \left(f(t,x) - \sum_{n=0}^{N} a_n(x) t^n \right) \| = O(t^{N-k}) \qquad (t \to 0) \qquad \forall N \geqslant k, j \in \mathbb{N}_0, \tag{2.59}$$

then we say that the formal power series $\sum_{n=0}^{\infty} a_n t^n$ is an *asymptotic expansion* of f and we write

$$f(t,x) \sim \sum_{n=0}^{\infty} a_n(x) t^n. \tag{2.60}$$

With this new definition, we can interchange derivatives (in t and x) and integrals with asymptotic expansions.

We can now expand the heat kernel $p_t(x,y)$ in powers of t.

Theorem A.4. *There exist smooth sections $B_i \in \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ over $M \times M$ with $B_0(x,x) = \mathbb{1}_{\mathcal{E}_x}$ such that*

$$p_t(x,y) \sim q_t(x,y) \sum_{j=0}^{\infty} t^j B_j(x,y), \tag{2.61}$$

where $q_t(x,y) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{4t}}$ is the Gaussian on $M \times M$.

Proof. The proof of this theorem can be found in the appendix. □

Therefore, we can expand

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^{+}) = \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j \left(\int_M \operatorname{tr}_S(B_j(x,x))\,\mathrm{d}x \right). \tag{2.62}$$

Since the left hand side is independent of $t > 0$, it follows that for $j \neq \frac{n}{2}$, $\int_M \operatorname{tr}_S(B_j(x,x))\,\mathrm{d}x = 0$ and

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^{+}) = \lim_{t\to 0} \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (4\pi)^{-\frac{n}{2}} \int_M \operatorname{tr}_S(B_{\frac{n}{2}}(x,x))\,\mathrm{d}x. \tag{2.63}$$

In principle, we can calculate $\operatorname{tr}_S(B_{\frac{n}{2}}(x,x))$ from the recursion relations found in the proof of Theorem A.4. However, for large n this method becomes highly impractical.

2.3.5 Getzler Scaling - Step Four

We've already shown that

$$\operatorname{ind}(\not{D}_{\mathcal{V}}^{2}) = \lim_{t\to 0} \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (4\pi)^{-\frac{n}{2}} \int_M \operatorname{tr}_S(B_{\frac{n}{2}}(x,x))\,\mathrm{d}x. \tag{2.64}$$

In this fourth and last step of the proof of the index theorem, we will show that not only the limit of the integrals $\lim_{t\to 0} \int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x$ exists but also the limit of its integrands $\lim_{t\to 0} \operatorname{tr}_S(p_t(x,x))$. This is far from obvious, since the asymptotic expansion of $p_t(x,x)$ has leading term $t^{-\frac{n}{2}}$. The 'fantastic cancellations' (as they were dubbed by McKean and Singer) leading to the vanishing of the terms $\operatorname{tr}_S(B_j(x,x))$ for $j < \frac{n}{2}$ are due to symmetries of the Clifford algebra.

To exploit these symmetries, in his 1985 paper [4] Ezra Getzler introduced a simultaneous rescaling of both spacetime $\mathbb{R}_{>0} \times M$ and Clifford algebra. This is inspired by supersymmetry, scaling both Clifford multplication (fermions) and differential operators (bosons) simultaneously. The scaling is chosen such that in the limit of small scaling parameter u, the rescaled heat kernel approaches $\lim_{t\to 0} \operatorname{tr}_S(p_t(x,x))$. On the other hand, the rescaled kernel is uniquely determined as the solution of a rescaled heat equation. In the $u \to 0$ limit this rescaled heat equation tends to the equation of a generalized hamonic oscillator, which can be solved exactly. This yields an explicit expression for $\lim_{t\to 0} \mathrm{tr}_S(p_t(x,x))$ in terms of the local geometry of M.

Let $\mathcal{E} = \$\!\!/ \otimes \mathcal{V}$ be a twisted spinor bundle on a compact even dimensional spin manifold M^n with Dirac operator $D\!\!\!/_{\mathcal{V}}$ and heat kernel $p_t(x,y) \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^*)$.

For a form $\alpha \in \Lambda T^*M$, we denote its projection to $\Lambda^k T^*M$ by $\alpha_{[k]}$.
We make the following observation.

Proposition 2.38. *If $\alpha \in \Gamma(\mathrm{Cl}_{\mathbb{C}}(M))$, then*

$$\mathrm{tr}_S(\alpha)\,\mathrm{d}x = (-2i)^{\frac{n}{2}}\,\sigma(\alpha)_{[n]}\,, \tag{2.65}$$

where $\sigma : \mathrm{Cl}_{\mathbb{C}}(M) \to \Lambda T^*M$ denotes the symbol map.
In particular,

$$\mathrm{tr}_S\left(p_t(x,x)\right)\mathrm{d}x = (-2i)^{\frac{n}{2}}\,\mathrm{tr}\left(\sigma\left(p_t(x,x)\right)_{[n]}\right), \tag{2.66}$$

where tr denotes the trace over $\mathcal{V}_x$.

Proof. This is a direct consequence of Lemma 2.22 rewritten in terms of the volume form $\mathrm{d}x = e^1 \wedge \cdots \wedge e^n$ for a local orthonormal basis $e_1, \ldots, e_n$ of TM. □

To define the rescaling, we will work in geodesic coordinates around a point $x_0 \in M$.
Let $W = T_{x_0}M$ and let $U = \{\underline{x} \in T_{x_0}M \mid \|\underline{x}\| < \epsilon\}$, where $\epsilon > 0$ is such that $\exp_{x_0} : U \to M$ is a diffeomorphism. Therefore, U is a chart of M. We call the corresponding coordinates *geodesic coordinates*.
Since $D\exp_{x_0}\big|_{\underline{0}} = \mathbb{1}$, it follows that the coordinate vectors $\partial_1, \ldots, \partial_n$ are orthonormal at $\underline{0} \in W$. So far we have defined a coordinate system on M. Lets trivialize our bundles.
Let $E := \mathcal{E}_{x_0}$, $V = \mathcal{V}_{x_0}$ and let $\tau(x_0,x) : \mathcal{E}_x \to E$ denote the parallel transport map along the unique geodesic in $\exp_{x_0} U$ connecting x_0 and x. Explicilty, this geodesic is given by $\gamma_x(s) := \exp_{x_0}(s\exp_{x_0}^{-1}(x))$.
We use this trivialisation to identify $\Gamma(\exp_{x_0} U, \mathcal{E})$ with $C^\infty(U, E)$.
Even though we already have a (non-orthogonal) coordinate frame ∂_i of TM, we can apply the same construction of trivialisation by parallel transport to TM. We let $e_1, \ldots, e_n$ be the orthonormal local frame of TM on $\exp_{x_0} U$ obtained by parallely transporting the vectors $\partial_i|_{x_0}$ along geodesic starting at x_0.

Definition 2.39. We define the *local heat kernel at* x_0, $k^{x_0} : \mathbb{R}_{>0} \times U \to \Lambda W^* \otimes \mathrm{End}(V)$ by

$$k^{x_0}(t,\underline{x}) := \sigma\left(\tau(x_0, \exp_{x_0}\underline{x})p_t(\exp_{x_0}\underline{x}, x_0)\right), \tag{2.67}$$

where $\sigma : \mathrm{End}(S) \cong \mathrm{Cl}_{\mathbb{C}}(n) \to \Lambda W^*$ is the symbol map.

We are now ready to introduce the rescaling.

Definition 2.40. We define the *Getzler scaling* by $0 < u \leqslant 1$ of an element $\alpha \in C^\infty(\mathbb{R}_{>0} \times U, \Lambda W^* \otimes \mathrm{End}(V))$ by

$$(\delta_u\alpha)(t,\underline{x}) := \sum_{i=0}^{n} u^{-i}\alpha(u^2t, u\underline{x})_{[i]} \qquad \alpha \in C^\infty(\mathbb{R}_{>0} \times U, \Lambda W^* \otimes \mathrm{End}(V)). \tag{2.68}$$

Under Getzler scaling operators on $C^\infty(\mathbb{R}_{>0} \times U, \Lambda W^* \otimes \mathrm{End}(V))$ transform as

$$\begin{aligned} \delta_u\phi(\underline{x})\delta_u^{-1} &= \phi(u\underline{x}) \qquad && \phi \in C^\infty(U) \\ \delta_u\partial_t\delta_u^{-1} &= u^{-2}\partial_t && \\ \delta_u\partial_i\delta_u^{-1} &= u^{-1}\partial_i. && \end{aligned}$$

On the other hand, the algebraic structure transforms into

$$\begin{aligned} \delta_u\epsilon(w)\delta_u^{-1} &= u^{-1}\epsilon(w) \qquad && w \in W^* \\ \delta_u\iota(w)\delta_u^{-1} &= u \quad \iota(w) && w \in W^*, \end{aligned}$$

where the maps $\epsilon : W^* \to \mathrm{End}(\Lambda W^*)$ and $\iota : W^* \to \mathrm{End}(\Lambda W^*)$ are the exterior and interior product from Definition 1.10. Therefore, Clifford multplication by $w \in W^*$, which acts on ΛW^* as $c(w) = \epsilon(w) - \iota(w)$ (see Proposition 1.11) changes to

$$\delta_u c(w)\delta_u^{-1} = u^{-1}\left(\epsilon(w) + u^2\iota(w)\right) \tag{2.69}$$

We can interpret the rescaled Clifford multiplication as coming from a rescaled Clifford algebra $\mathrm{Cl}\left(W^*, u^2(\cdot,\cdot)\right)$. From the point of view of physics, the limit $u \to 0$ can thus be understood as the classical $\hbar \to 0$ limit turning the quantized Clifford algbera $e_i \cdot e_j + e_j \cdot e_j = \hbar\delta_{ij}$ back into the classical exterior algebra $e_i \cdot e_j + e_j \cdot e_i = 0$.

Under rescaling, the heat kernel behaves as follows.

Definition 2.41. We define the rescaled heat kernel as

$$r^{x_0}(u,t,\underline{x}) = u^n(\delta_u k^{x_0})(t,\underline{x}). \tag{2.70}$$

The factor u^n is added such that r^{x_0} is still a heat kernel with the right initial condition $\lim_{t\to 0} r^{x_0}(u,t,\underline{x})\sigma_{x_0} = \delta_{\sigma_{x_0}}$.

Getzler scaling is chosen such that

$$r^{x_0}(u,1,\underline{0}) = \sum_{i=0}^{n} u^{n-i}k^{x_0}(u^2,\underline{0})_{[i]}, \tag{2.71}$$

placing the top-level component $k^{x_0}(u^2,\underline{0})_{[n]}$ as leading order term in u.
This means that if the $u \to 0$ limit exists, then the top-level part $r^{x_0}(u,1,\underline{0})_{[n]}$ converges to $\lim_{t\to 0} k^{x_0}(t,\underline{0})_{[n]}$. It then follows from Proposition 2.38 that if the limit exists, we have that

$$\lim_{u\to 0} \mathrm{tr}_{\mathcal{V}_{x_0}}\left(r^{x_0}(u,1,\underline{0})_{[n]}\right) = \lim_{t\to 0}(-2i)^{-\frac{n}{2}}\,\mathrm{tr}_S(p_t(x_0,x_0))\,\mathrm{d}x, \tag{2.72}$$

where $\mathrm{tr}_{\mathcal{V}_{x_0}}$ denotes the trace over the vector space $\mathcal{V}_{x_0}$.

To finish the proof of the index theorem, we therefore have to show that $\lim_{u\to 0} r^{x_0}(u,1,\underline{0})$ exists and calculate its value explicitly.

Using Lichnerowicz Theorem 1.55 for $\Delta_{\mathcal{E}}$, we find that in our trivialisation by parallel transport the square of the Dirac operator is the $\mathrm{End}(E) \cong \mathrm{Cl}_{\mathbb{C}}(n) \otimes \mathrm{End}(V)$-valued differential operator

$$L = -\sum_{i,j=1}^{n} g^{ij}\left(\nabla^{\mathcal{E}}_{\partial_i}\nabla^{\mathcal{E}}_{\partial_j} - \Gamma^k_{ij}\nabla^{\mathcal{E}}_{\partial_k}\right) + \frac{r_M}{4} + F^{\mathcal{V}} : C^\infty(U,E) \to C^\infty(U,E), \tag{2.73}$$

where $\nabla^{\mathcal{E}}_{\partial_i}$ is the trivialized connection on E (see equation (2.81)), $r_M \in C^\infty(U)$ the scalar curvature of M and $F^{\mathcal{V}} \in C^\infty(U, \mathrm{End}(E))$ is the trivialized Clifford curvature of $\nabla^{\mathcal{V}}$ (see Definition 1.53).

The local heat kernel k^{x_0} fulfills the heat equation

$$(\partial_t + L)k^{x_0}(t,\underline{x}) = 0, \tag{2.74}$$

from which it follows that the rescaled heat kernel fulfills

$$\left(\partial_t + u^2\delta_u L\delta_u^{-1}\right) r^{x_0}(u,t,\underline{x}) = 0. \tag{2.75}$$

To work out the small u limit of $r(u,t,\underline{x})$ we set out to find an asymptotic expansion of r in powers of u.

Proposition 2.42. *There exists $\Lambda W^* \otimes \mathrm{End}(V)$-valued polynomials γ_i on $\mathbb{R} \times W$ such that*

$$r^{x_0}(u,t,\underline{x}) \sim q_t(\underline{x}) \sum_{i=-n}^{\infty} u^i\gamma_i(t,\underline{x}) \qquad (u \to 0), \tag{2.76}$$

with $\gamma_i(0,\underline{0}) = 0$ for $i \neq 0$ and $\gamma_0(0,\underline{0}) = 1$.

Proof. It follows from Theorem A.4 that $p_t(x,y)$ has an asymptotic expasion in terms of sections $B_i \in \Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)$ with $B_0(x_0,x_0) = \mathbb{1}_E$. We localize these sections to functions $A_j^{x_0} \in C^\infty(U, \Lambda W^* \otimes \mathrm{End}(V))$, explicitly given by $A_j^{x_0}(\underline{x}) = \sigma\left(\tau(x_0, \exp_{x_0}\underline{x})B_j(\exp_{x_0}\underline{x}, x_0)\right)$. Thus, we have the local asymptotic expansion

$$k^{x_0}(t,\underline{x}) \sim q_t(\underline{x})\sum_{j=0}^{\infty} t^j A_j^{x_0}(\underline{x}) \qquad (t \to 0).$$

Expanding $A_j^{x_0}(\underline{x}) = \sum_{i=0}^n A_{j,[i]}^{x_0}(\underline{x}) \in \bigoplus_{i=0}^n \Lambda^i W^* \otimes \mathrm{End}(V)$, we obtain

$$k^{x_0}(t,\underline{x}) \sim q_t(\underline{x})\sum_{j=0}^{\infty}\sum_{i=0}^{n} t^j A_{j,[i]}^{x_0}(\underline{x})$$

and therfore

$$r^{x_0}(u,t,\underline{x}) \sim q_t(\underline{x})\sum_{j=0}^{\infty}\sum_{i=0}^{n} t^j u^{2j-i} A_{j,[i]}^{x_0}(u\underline{x}).$$

Taylor expanding $A_{j,[i]}^{x_0}(u\underline{x})$ in powers of u, we obtain an asymptotic series

$$r^{x_0}(u,t,\underline{x}) \sim q_t(\underline{x})\sum_{j=-n}^{\infty} u^j \gamma_j(t,\underline{x}) \qquad (u \to 0),$$

where γ_j are polynomials in t and in $\underline{x}$.
Explicitly, expanding $A_{j,[i]}^{x_0}(u\underline{x}) = \sum_{k\in\mathbb{N}_0^n} A_{j,[i],k}^{x_0} u^{|k|}\underline{x}^k$ we obtain

$$\gamma_j(t,\underline{x}) = \sum_{\substack{a\in\mathbb{N}; i\leqslant n; k\in\mathbb{N}_0^n\\ 2a-i+|k|=j}} A_{j,[i],k}^{x_0} t^a \underline{x}^k.$$

In particular,

$$\sum_{j=-n}^{\infty} u^j\gamma_j(0,\underline{0}) = \sum_{i=0}^{n} u^{-i} A_{0,[i]}^{x_0}(\underline{0}).$$

Since $A_0^{x_0}(\underline{0}) = \sigma\left(B_0(x_0,x_0)\right) = \sigma\left(\mathbb{1}_E\right) = 1$, it follows that $A_{0,[i]}^{x_0}(\underline{0}) = \delta_{i,0}1$.
This means that $\sum_{j=-n}^{\infty} u^j\gamma_j(0,\underline{0}) = 1$, which proves that $\gamma_i(0,\underline{0}) = 0$ for $i \neq 0$ and $\gamma_0(0,\underline{0}) = 1$. □

So far, the asymptotic expansion of $r^{x_0}(u,t,\underline{x})$ has leading term u^{-n} and shows no sign of convergence as $u \to 0$. However, we will prove in Proposition 2.46 that there exists an operator K such that

$$L(u) = u^2\delta_u L\delta_u^{-1} = K + O(u) \qquad (u \to 0). \tag{2.77}$$

Using this, we can show that the first non-vanishing leading term of $r^{x_0}(u,t,\underline{x})$ is indeed u^0 and that r therefore converges for $u \to 0$.

Proposition 2.43. *The polynomials γ_j vanish identically for $j < 0$.*
Therefore, $\lim_{u\to 0} r^{x_0}(u,t,\underline{x})$ exists and equals $r^{x_0}(0,t,\underline{x}) := q_t(\underline{x})\gamma_0(t,\underline{x})$. Furthermore, $r^{x_0}(0,t,\underline{x})$ is uniquely determined by the differential equation

$$(\partial_t + K)r^{x_0}(0,t,\underline{x}) = 0 \qquad \gamma_0(0,\underline{0}) = 1, \tag{2.78}$$

where K is the operator from Proposition 2.46.

Proof. We expand the equation

$$(\partial_t + L(u))q_t(\underline{x})\sum_{j=-n}^{\infty} u^j\gamma_j = 0$$

in powers of u. In leading order u^{-n} we obtain the equation

$$(\partial_t + K)q_t(\underline{x})\gamma_{-n}(t,\underline{x}) = 0.$$

Since γ_j are polynomials and thus power series in t, the solution to the heat equation is uniquely determined by the value $\gamma_j(0,\underline{0})$ (we can obtain a recurrence relation similar as in the proof of Lemma A.3 which shows that there is a unique formal power series solution to $(\partial_t + K)q_t(\underline{x})F = 0$). Since $\gamma_{-n}(0,\underline{0}) = 0$, it follows that $\gamma_{-n} = 0$. We can now proceed inductively to the next higher term in the expansion and prove that $\gamma_{-j} = 0$ for all $j > 0$. For $j = 0$ we have the first non-trivial initial condition

$$(\partial_t + K)q_t(\underline{x})\gamma_0(t,\underline{x}) = 0 \qquad \gamma_0(0,\underline{0}) = 1.$$

This implies that $r(u,t,\underline{x}) \sim q_t(\underline{x})\sum_{j=0}^n u^j\gamma_j(t,\underline{x})$and thus that $\lim_{u\to 0} r(u,t,\underline{x})$ exists and equals $q_t(\underline{x})\gamma_0$. Since $\gamma_0(t,\underline{x})$ is a polynomial and therefore a power series in t, it follows that $r(0,t,\underline{x})$ is uniquely determined by the above equation. □

We are now left with calculating the leading order part K of $L(u)$. To do this, we need two auxiliary lemma. The following technical lemma summarises the most important properties of geodesic coordinates and parallel trivialisations.

Lemma 2.44. *Let $\mathcal{E}$ be a vector bundle with connection $\nabla^{\mathcal{E}}$ considered in geodesic coordinates with trivialisation by parallel transport.*

(1) *Let $\mathcal{R} = \sum_{i=1}^n \underline{x}^i\partial_i$ be the radial vector field. For $X \in C^\infty(U,E)$ we have that*

$$\nabla^{\mathcal{E}}_{\mathcal{R}}X = \sum_i \underline{x}^i\partial_i X = \mathcal{R}X. \tag{2.79}$$

In particular, if σ^{x_0} is a vector in $E = \mathcal{E}_{x_0}$ and $\sigma \in \Gamma(\exp_{x_0} U, \mathcal{E})$ is obtained by parallel transporting σ^{x_0} along geodesics we obtain that

$$\nabla_{\mathcal{R}}\sigma = 0 \text{ and thus that } (\nabla\sigma)_{x_0} = 0. \tag{2.80}$$

(2) *Let $K^{\mathcal{E}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ be the curvature of $\nabla^{\mathcal{E}}$. Then,*

$$\nabla^{\mathcal{E}}_{\partial_i} = \partial_i - \frac{1}{2}\sum_{j=1}^n K^{\mathcal{E}}(\partial_i,\partial_j)_{x_0}\underline{x}^j + O(\|\underline{x}\|^2) : C^\infty(U,E) \to C^\infty(U,E). \tag{2.81}$$

Proof. A proof can be found in [2]. □

Using this lemma, we can find the local appearance of the (twisted) spin connection $\nabla^{\mathcal{E}}$.

Lemma 2.45. *We work in geodesic coordinates with trivialisation by parallel transport.*

(1) *The function $c(e^i) \in C^\infty(U, \mathrm{End}(E))$ is constant and equals $c^i := c\left(dx^i\big|_{x_0}\right) \in \mathrm{End}(E)$.*

(2) *The covariant derivative $\nabla^{\mathcal{E}}_{\partial_i}$ on $C^\infty(U,E)$ is given by*

$$\nabla^{\mathcal{E}}_{\partial_i} = \partial_i + \frac{1}{4}\sum_{j;k<l} R_{klij}(\underline{0})\underline{x}^j c^k c^l + \sum_{k<l} f_{ikl}(\underline{x})c^k c^l + g(\underline{x}), \tag{2.82}$$

where $f_{ikl}(\underline{x}) = O(\|\underline{x}\|^2) \in C^\infty(U)$ and $g(\underline{x}) = O(\|\underline{x}\|) \in C^\infty(U, \mathrm{End}(V))$.

Proof. (1) Let $\mathcal{R} = \sum_{i=1}^n \underline{x}^i\partial_i$. By Lemma 2.44, $\nabla^{\mathcal{E}}_{\mathcal{R}}e^i = 0$ and therefore

$$\mathcal{R}c(e^i) = \nabla^{\mathcal{E}}_{\mathcal{R}}c(e^i) = c(\nabla_{\mathcal{R}}e^i) = 0.$$

This implies that $c(e^i)$ is constant as a function $C^\infty(U,E)$ and equal to $c(e^i)_{x_0} = c(\,dx^i\big|_{x_0}) =: c^i$.
(2) It follows from Lemma 2.44, that

$$\nabla^{\mathcal{E}}_{\partial_i} = \partial_i - \frac{1}{2}\sum_{j=1}^n K^{\mathcal{E}}(\partial_i,\partial_j)_{x_0}\underline{x}^j + O(\|\underline{x}\|^2).$$

Observe that $K^{\mathcal{E}}(\partial_i,\partial_j) = R^{\$}(\partial_i,\partial_j) + K^{\mathcal{V}}(\partial_i,\partial_j)$ and that by Proposition 1.49

$$R^{\$}(\partial_i,\partial_j)_{x_0} = R^{\$}(e_i,e_j)_{x_0} = -\frac{1}{2}\sum_{k<l} R_{ijkl}(\underline{0})c^k c^l$$

This proves the claim. □

Finally, we can work out how the rescaled heat equation looks in the $u \to 0$ limit.

Proposition 2.46. *The operator $L(u) := u^2 \delta_u L \delta_u^{-1}$ is of the form $L(u) = K + O(u)$, where*

$$K = -\sum_{i=1}^{n} \left(\partial_i - \frac{1}{4} \sum_{j=1}^{n} [R]_{ij}^{x_0} \underline{x}_j \right)^2 + K^{\mathcal{V}}(x_0), \tag{2.83}$$

where $[R]^{x_0}$ denotes the antisymmetric $n \times n$ matrix with coefficients in $\Lambda W^* = \Lambda T_{x_0}^* M$ given by

$$[R]_{ij}^{x_0} = g(\partial_j, R(\cdot,\cdot)\partial_i)_{x_0} = \sum_{k<l} R_{klji}(x_0)\, dx^k \wedge dx^l\big|_{x_0} \in \Lambda^2 W^* \tag{2.84}$$

and where $K^{\mathcal{V}}(x_0) \in \Lambda^2 W^* \otimes \mathrm{End}(V)$ is the curvature of $\nabla^{\mathcal{V}}$ at x_0.

Proof. Using Lemma 2.45, and writing $\epsilon^i = \epsilon(dx^i|_{x_0}) \in \mathrm{End}(\Lambda W^*)$ and $\iota^i := \iota(dx^i|_{x_0}) \in \mathrm{End}(\Lambda W^*)$ we have that

$$\begin{aligned}\nabla_{\partial_i}^{\mathcal{E},u} := u\delta_u \nabla_{\partial_i}^{\mathcal{E}} \delta_u^{-1} &= \partial_i + u\frac{1}{4}\sum_{j,k<l} R_{klij}(\underline{0}) u x^j u^{-2}\left(\epsilon^k + u^2\iota^k\right)\left(\epsilon^l + u^2\iota^l\right) + u f_{ikl}(u\underline{x}) u^{-1}\left(\epsilon^k + u^2\iota^k\right) + u g(u\underline{x}) \\ &= \partial_i + \frac{1}{4}\sum_{j;k<l} R_{klij}(\underline{0}) x^j \epsilon^k \epsilon^l + O(u)\end{aligned}$$

Writing $[R]_{ij}^{x_0} = \sum_{k<l} R_{klji}(\underline{0})\, dx^k \wedge dx^l\big|_{x_0} = -\sum_{k<l} R_{klij}(\underline{0})\epsilon^k\epsilon^l$, we find that

$$\nabla_{\partial_i}^{\mathcal{E},u} = \partial_i - \frac{1}{4}\sum_{j=1}^{n} [R]_{ij}^{x_0} x^j + O(u).$$

Since in geodesic coordinates $g^{ij} = \delta^{ij} + O(\|\underline{x}\|)$ and $\Gamma_{ij}^k = O(\|\underline{x}\|)$, it follows that $g^{ij}(u\underline{x}) = \delta^{ij} + O(u)$ and $\Gamma_{ij}^k(u\underline{x}) = O(u)$ and thus that

$$L(u) = -\sum_{i,j=1}^{n} \left(\nabla_{\partial_i}^{\mathcal{E},0}\right)^2 + u^2 \delta_u \left(\frac{r_M}{4} + F^{\mathcal{V}}\right)\delta_u^{-1} + O(u).$$

Since $r_M \in C^\infty(U)$, it follows that $u^2\delta_u r_M \delta_u^{-1} = u^2 r_M(u\underline{x}) = O(u^2)$ and using

$$F^{\mathcal{V}} = \sum_{i<j} K^{\mathcal{V}}(e_i, e_j) c(e^i) c(e^j) = \sum_{i<j} K^{\mathcal{V}}(e_i, e_j) c^i c^j$$

we find that

$$\begin{aligned} u^2 \delta_u F^{\mathcal{V}} \delta_u^{-1} &= \sum_{i<j} u^2 K^{\mathcal{V}}(e_i, e_j)(u\underline{x}) u^{-2}\left(\epsilon^i + u^2\iota^i\right)\left(\epsilon^j + u^2\iota^j\right) = \sum_{i<j} K^{\mathcal{V}}(e_i, e_j)_{x_0}\epsilon^i\epsilon^j + O(u) \\ &= \sum_{i<j} K^{\mathcal{V}}(\partial_i, \partial_j)_{x_0}\, dx^i \wedge dx^j\big|_{x_0} + O(u) = K^{\mathcal{V}}(\cdot,\cdot)_{x_0} + O(u) \in \Lambda^2 W^* \otimes \mathrm{End}(V). \end{aligned}$$

This proves the claim. □

This operator K is a generalized harmonic oscillator, whose heat kernel can be calculated explicitly (Theorem B.3). Both from the point of view of physics and mathematics, the appearance of this operator shouldn't come as too much of a surprise. In fact, we have seen that the bosonic analogon of the $\mathfrak{spin}_{2n}$ Lie algebra is the metaplectic algebra $\mathfrak{mp}_{2n}$, the subalgebra of the Weyl algebra generated by quadratic elements. Up to the constant $K^{\mathcal{V}}$, the harmonic oscillator is precisely such a quadratic element.

Since $r^{x_0}(0, t, \underline{x})$ is uniquely determined by the harmonic oscillator heat equation, we conclude this section with the following corollary.

Corollary 2.47. *The function $r^{x_0}(0, t, \underline{x}) \in \Lambda W^* \otimes \mathrm{End}(V)$ is given by*

$$r(0,t,\underline{x}) = q_t(\underline{x}) \det^{\frac{1}{2}}\left(\frac{t[R]^{x_0}/2}{\sinh(t[R]^{x_0}/2)}\right) \exp\left(-\frac{1}{4t}\underline{x}^t\Big(t[R]^{x_0}/2\coth(t[R]^{x_0}/2) - 1\Big)\underline{x}\right)\exp\Big(-tK^{\mathcal{V}}(x_0)\Big). \tag{2.85}$$

By nilpotency of $[R]^{x_0}$ and $K^{\mathcal{V}}$, this is the product of the Gaussian with a polynomial in t and $\underline{x}$.

Proof. By Proposition 2.43, the function $r^{x_0}(0,t,\underline{x}) = q_t(\underline{x})\gamma_0(t,\underline{x})$ is uniquely determined by

$$(\partial_t + K)r^{x_0}(0,t,\underline{x}) \qquad \gamma_0(0,\underline{0}) = 1.$$

It follows from Theorem B.3 that γ_0 is uniquely determined by the equation and has the form given in (2.85). $\square$

Summarising, we have shown in Proposition 2.43 that the limit $\lim_{u\to 0} r^{x_0}(u,t,\underline{x})$ exists and we have calculated this limit in Corollary 2.47 explicitly in terms of the curvatures R and K. By equation (2.72), we have therefore calculated $\lim_{t\to 0} \mathrm{tr}_S(p_t(x,x))$ and proven the index theorem.

2.4 The Index Theorem

Given a $r\times r$ matrix A valued in some finite-dimensional commutative algebra $\mathcal{A}$, we define the $\mathcal{A}-$ valued formal power series $\det^{\frac{1}{2}}\left(\frac{sA/2}{\sinh sA/2}\right)$ and the $\mathrm{Mat}_{r\times r}(\mathcal{A})$-valued formal power series $\exp(-sA)$ with formal parameter s as in equations (B.8) and (B.10). If $A \in \mathrm{Mat}_{r\times r}$ is nilpotent, these formal power series are polynomials and can thus be evaluated at $s = 1$, defining $\det^{\frac{1}{2}}\left(\frac{A/2}{\sinh(A/2)}\right)$ and $\exp(-A)$.

In the following, our commutative algebra will be $\mathcal{A} = \Omega^{\text{even}}(M) := \bigoplus_{k=0}^{[\frac{n}{2}]} \Omega^{2k}(M)$.

Definition 2.48. Let $\mathcal{V} \to M$ be a rank r vector bundle over a manifold M and let $\nabla^{\mathcal{V}}$ be a connection on $\mathcal{V}$ with curvature $K^{\mathcal{V}} \in \Omega^2(M, \mathrm{End}(\mathcal{V}))$. Let $v_1,\dots,v_r$ be a local framing of $\mathcal{V}|_U$ and let $[K^{\mathcal{V}}]$ be the $r\times r$ matrix of two-forms given by

$$[K^{\mathcal{V}}]^i{}_j := v^i\left(K^{\mathcal{V}}(\cdot,\cdot)v_j\right) \in \Omega^2(U). \tag{2.86}$$

We define

$$\widehat{A}(\nabla^{\mathcal{V}})\Big|_U := \det^{\frac{1}{2}}\left(\frac{[K^{\mathcal{V}}]/2}{\sinh\left([K^{\mathcal{V}}]/2\right)}\right) \in \Omega^{\text{even}}(U). \tag{2.87}$$

Since $[K^{\mathcal{V}}]$ is nilpotent, this is a well defined $\Omega^{\text{even}}(U)-$polynomial in the components $[K^{\mathcal{V}}]_{ij} \in \Omega^2(U)$ and by definition of the determinant, it is independent of the choice of local frame.
Therefore, this defines a global form

$$\widehat{A}(\nabla^{\mathcal{V}}) \in \Omega^{\text{even}}(M), \tag{2.88}$$

called the *$\widehat{A}$-form of* $\nabla^{\mathcal{V}}$.
We also define

$$\mathrm{ch}(\nabla^{\mathcal{V}}) := \mathrm{Tr}\left(\exp(-[K^{\mathcal{V}}])\right) \in \Omega^{\text{even}}(U, \mathrm{End}(\mathcal{V})), \tag{2.89}$$

which is also a $\Omega^{\text{even}}(U)$-polynomial independent of the local frame.
Therefore,

$$\mathrm{ch}\left(\nabla^{\mathcal{V}}\right) \in \Omega^{\text{even}}(M), \tag{2.90}$$

is a global form, called the *Chern character form of* $\nabla^{\mathcal{V}}$.

Since $\frac{z/2}{\sinh(z/2)}$ is an even function, its Taylor series has only even terms and the $\widehat{A}-$form has only components in degrees divisible by four,

$$\widehat{A}(\nabla^{\mathcal{V}}) \in \bigoplus_{i=0}^{[\frac{n}{4}]} \Omega^{4i}(M). \tag{2.91}$$

Recall that we denote the projection of a differential form $\alpha \in \Omega(M)$ to $\Omega^k(M)$ by $\alpha_{[k]}$. Having introduced this notation, we can finally state the index theorem.

Theorem 2.49 (Local Index Theorem)**.** *Let $p_t(x,y)$ be the heat kernel of the Dirac operator $\not{D}_{\mathcal{V}}$ of a twisted spinor bundle $\mathcal{E} = \not{S}\otimes\mathcal{V}$ on an even dimensional compact spin manifold M^n.*
Then for $x \in M$, the limit $\lim_{t\to 0}\mathrm{tr}_S(p_t(x,x))$ exists, is uniform in $x \in M$ and equals

$$\lim_{t\to 0} \mathrm{tr}_S(p_t(x,x))\ \mathrm{d}x = (2\pi i)^{-\frac{n}{2}}\left(\widehat{A}(\nabla)\wedge \mathrm{ch}(\nabla^{\mathcal{V}})\right)_{[n]}(x), \tag{2.92}$$

where ∇ is the Levi-Civita connection on TM and $\nabla^{\mathcal{V}}$ is the connection on $\mathcal{V}$.
Given a local frame of TM and $\mathcal{V}$ around $x \in M$, this can be written as

$$\lim_{t\to 0} \operatorname{tr}_S(p_t(x,x))\ \mathrm{d}x = (2\pi i)^{-\frac{n}{2}} \left(\det{}^{\frac{1}{2}}\left(\frac{[R]/2}{\sinh([R]/2)}\right) \wedge \operatorname{Tr}\left(\exp(-[K^{\mathcal{V}}])\right)\right)_{[n]} \tag{2.93}$$

where $[R]$ and $[K^{\mathcal{V}}]$ are the matrices of two-forms obtained from the Riemann curvature R of M and the curvature $K^{\mathcal{V}}$ of $\nabla^{\mathcal{V}}$ as in (2.86).

Proof. We have seen in the discussion leading to equation (2.72) that if the limit $\lim_{u\to 0} r^{x_0}(u,1,\underline{0})$ exists, then the limit $\lim_{t\to 0}\operatorname{tr}_S(p_t(x_0,x_0))$ also exists and

$$\lim_{t\to 0} \operatorname{tr}_S(p_t(x_0,x_0)) = (-2i)^{\frac{n}{2}} \operatorname{tr}_{\mathcal{V}_{x_0}}\left(r^{x_0}(0,1,\underline{0})_{[n]}\right).$$

Uniformity of the limit follows from our construction starting with the asymptotic expansion (Theorem A.4), which was uniform both in x and y. From Corollary 2.47 we have the explicit expression

$$r^x(0,1,\underline{0}) = (4\pi)^{-\frac{n}{2}} \det{}^{\frac{1}{2}}\left(\frac{[R]^x/2}{\sinh([R]^x/2)}\right) \wedge \exp(-K^{\mathcal{V}}),$$

such that

$$\lim_{t\to 0} \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (2\pi i)^{-\frac{n}{2}} \left(\det{}^{\frac{1}{2}}\left(\frac{[R]^x/2}{\sinh([R]^x/2)}\right)\operatorname{tr}_V(\exp(-K^{\mathcal{V}}))\right)_{[n]} = (2\pi i)^{-\frac{n}{2}}\left(\widehat{A}(\nabla)\wedge \operatorname{ch}(\nabla^{\mathcal{V}})\right)_{[n]}.$$

□

Observe that for a vector bundle $\mathcal{V}$, both $\widehat{A}(\nabla^{\mathcal{V}})$ and $\operatorname{ch}(\nabla^{\mathcal{V}})$ are closed forms and therefore define equivalence classes in de-Rham cohomology. It turns out that these classes do only depend on the bundle $\mathcal{V}$ and not on the choice of connection $\nabla^{\mathcal{V}}$.

Definition 2.50. Let $\mathcal{V} \to M$ be a vector bundle over a manifold M. We define the *$\widehat{A}$–genus of $\mathcal{V}$* as the cohomology class

$$\widehat{A}(\mathcal{V}) = \left[\widehat{A}(\nabla^{\mathcal{V}})\right] \in \bigoplus_{i=0}^{[\frac{n}{4}]} \mathrm{H}^{4i}_{\mathrm{dR}}(M), \tag{2.94}$$

where $\nabla^{\mathcal{V}}$ is any connection on $\mathcal{V}$ with $\widehat{A}$–form $\widehat{A}(\nabla^{\mathcal{V}})$.
Similarly, we define the *Chern character of $\mathcal{V}$* as the cohomology class

$$\operatorname{ch}(\mathcal{V}) = \left[\operatorname{ch}(\nabla^{\mathcal{V}})\right] \in \mathrm{H}^{\mathrm{even}}_{\mathrm{dR}}(M). \tag{2.95}$$

Finally, this leads to the Atiyah-Singer index theorem.

Theorem 2.51 (Atiyah-Singer Index Theorem)**.** *Let $p_t(x,y)$ be the heat kernel of the Dirac operator $\not{D}_{\mathcal{V}}$ of a twisted Dirac bundle $\mathcal{E} = \not{S} \otimes \mathcal{V}$ on an even dimensional compact spin manifold M^n. Then*

$$\operatorname{ind}(\not{D}^+_{\mathcal{V}}) = (2\pi i)^{-\frac{n}{2}} \int_M \left(\hat{A}(TM)\wedge \operatorname{ch}(\mathcal{V})\right)_{[n]}. \tag{2.96}$$

Proof. Since $\lim_{t\to 0}\operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (2\pi i)^{-\frac{n}{2}}\left(\widehat{A}(\nabla)\wedge \operatorname{ch}(\nabla^{\mathcal{V}})\right)_{[n]}$ is uniform in x, it follows that

$$\operatorname{ind}(\not{D}^+_{\mathcal{V}}) = \lim_{t\to 0}\int_M \operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = \int_M \lim_{t\to 0}\operatorname{tr}_S(p_t(x,x))\,\mathrm{d}x = (2\pi i)^{-\frac{n}{2}}\int_M \left(\widehat{A}(TM)\wedge \operatorname{ch}(\mathcal{V})\right)_{[n]}.$$ □

Chapter 3

Applications and Outlook

In this chapter, we discuss several applications of the Atiyah-Singer index theorem. We start with a discussion of its direct consequences, such as the integrality of the $\widehat{A}$-genus of a spin manifold. We will then prove the Riemann-Roch theorem as an example of how a 'classical' index theorem can be deduced from the Atiyah-Singer index theorem. We will also give a very brief introduction on how the index theorem is used in the study of solutions to non-linear PDEs such as the Seiberg-Witten equations.
Finally, we give an outlook on how the index theorem for Dirac operators on spin manifolds can be used as the starting point of a proof of the index theorem for general elliptic operators.

3.1 First Examples

For any smooth $n-$dimensional manifold M we can define the $\widehat{A}-$genus

$$\widehat{A}(M) := \left[\det{}^{\frac{1}{2}}\left(\frac{[R]/2}{\sinh([R]/2)}\right)\right] \in \bigoplus_{i=0}^{\left[\frac{n}{4}\right]} H^{4i}_{\mathrm{dR}}(M), \tag{3.1}$$

where $[R]$ is the curvature of some metric on M. From this definition we can infer that the value $(2\pi i)^{-\left[\frac{n}{2}\right]}\int_M \widehat{A}(M)_{[n]}$ (which is sometimes also called the $\widehat{A}$-genus of M) is a real number. Using Chern-Weil theory, one can improve this statement and show that $(2\pi i)^{-\left[\frac{n}{2}\right]}\int_M \widehat{A}(M)_{[n]} \in \mathbb{Q}$.
For a general manifold M, this statement can't be improved any further.
However, the $\widehat{A}-$genus of an even dimensional spin manifold equals the index of the Dirac operator, which is by definition an integer. Therefore, a first non-trivial application of the index theorem is the following.

Corollary 3.1. *On an even dimensional spin manifold M^n, the $\widehat{A}-$genus $(2\pi i)^{-\frac{n}{2}}\int_M \widehat{A}(M)_{[n]}$ is an integer.*

Without knowing the index theorem, this is a highly non-trivial statement. Indeed, the relation between having a spin structure and an integral $\widehat{A}-$genus was one of the observations that led to the discovery of the Atiyah-Singer index theorem.

Let M be a smooth manifold, which we equip with a metric g with corresponding curvature R. Using the Taylor expansion $\frac{z/2}{\sinh(z/2)} = 1 - \frac{z^2}{24} + O(z^4)$, we see that

$$\det\left(\frac{R/2}{\sinh(R/2)}\right) = 1 - \frac{\mathrm{Tr}(R^2)}{24} + \text{terms in } \bigoplus_{i\geqslant 4}\Omega^i(M) \tag{3.2}$$

and therefore that

$$\widehat{A}(\nabla) = \det{}^{\frac{1}{2}}\left(\frac{R/2}{\sinh(R/2)}\right) = 1 - \frac{\mathrm{Tr}(R^2)}{48} + \text{terms in } \bigoplus_{i\geqslant 4}\Omega^i(M). \tag{3.3}$$

Since $\widehat{A}(\nabla)_{[2]} = 0$, the $\widehat{A}$-genus of a two dimensional manifold vanishes and the first non-trivial instance of Corollary 3.1 can be found in dimension 4.
For a four dimensional manifold M^4, the $\widehat{A}-$genus is

$$\widehat{A}(\nabla) = 1 - \frac{\mathrm{Tr}(R^2)}{48}. \tag{3.4}$$

As an example, we will calculate the $\widehat{A}-$ genus of the four dimensional manifold $\mathbb{CP}^2$. We can use the Fubini-Study metric to calculate $\mathrm{tr}(R^2) = 24\,\mathrm{vol}_{\mathbb{CP}^2}$, which means that $\widehat{A}(\nabla) = 1 - \frac{1}{2}\,\mathrm{vol}_{\mathbb{CP}^2}$. Therefore,

$$(2\pi i)^{-2}\int_{\mathbb{CP}^2}\widehat{A}(\nabla)_{[4]} = \frac{-1}{4\pi^2}\int_{\mathbb{CP}^2}\left(-\frac{1}{2}\right)\mathrm{vol}_{\mathbb{CP}^2} = \frac{1}{8\pi^2}\,\mathrm{vol}(\mathbb{CP}^2). \tag{3.5}$$

Since $\mathbb{CP}^2 = S^5/S^1$ and since the Fubini study metric ω_{FS} is the quotient metric of the round metrics on S^5 and S^1, it follows that $\mathrm{vol}(\mathbb{CP}^2) = \frac{\mathrm{vol}(S^5)}{\mathrm{vol}(S^1)} = \frac{\pi^2}{2}$. Therefore,

$$(2\pi i)^{-2}\int_{\mathbb{CP}^2}\widehat{A}(\nabla)_{[4]} = \frac{1}{16}. \tag{3.6}$$

This shows that $\mathbb{CP}^2$ cannot be a spin manifold. Of course, this result could have been much easier obtained by calculating the second Stieffel-Whitney class of $\mathbb{CP}^2$ and noticing that it is not zero. Indeed, Corollary 3.1 is of little use as an obstruction theorem for spin structures (since the second Stieffel-Whitney class is usually much easier to compute than the $\widehat{A}-$genus). However, this calculation shows that it is by no means obvious that the $\widehat{A}-$genus of a spin manifold is an integer.

We can use the Atiyah-Singer index theorem in combination with Lichnerowicz Theorem 1.55 to find an obstruction for a spin manifold to have positive scalar curvature.

Corollary 3.2 (Lichnerowicz)**.** *Let M^n be a compact manifold which admits a spin structure (i.e. for which there exists a Riemann metric g such that (M, g) is a spin manifold) and such that the $\widehat{A}-$genus $(2\pi i)^{-\frac{n}{2}}\int_M \widehat{A}(TM)$ is non-zero. Then M admits no metric of strictly positive scalar curvature.*

Proof. The condition that M admits a spin structure is equivalent to the vanishing of the second Stieffel-Whitney class $w_2(TM)$ and thus independent of the choice of specific metric on M. Now suppose there exists a metric g such that $r_M > 0$. It follows by Lichnerowicz's formula that

$$\not{D}^2 = \Delta_{\not{S}} + \frac{r_M}{4}.$$

Since $r_M > 0$, this means that $\ker(\not{D}^2) = 0$ and consequently that $\ker(\not{D}) \cong \mathrm{coker}(\not{D}) = 0$. Therefore, the same is true for the chiral Dirac operator $\not{D}^+$, which implies that $\mathrm{ind}(\not{D}^+) = 0$. Since we assumed that M is spin, the Atiyah-Singer index theorem applies and the non-vanishing of the $\widehat{A}-$genus contradicts

$$0 = \mathrm{ind}(\not{D}^+) = \int_M \widehat{A}(TM)_{[n]} \neq 0.$$

□

3.2 The Riemann Roch Theorem

We've already alluded to the fact that on spin manifolds many 'classical' index theorems (such as the Signature theorem or the Hirzebruch-Riemann-Roch theorem) can be expressed in terms of the index theorm for a twisted Dirac operator. To give an example on how this can be done, we will deduce the classical two-dimensional Riemann-Roch theorem from the index theorem and prove that the space of holomorphic one-forms on a compact Riemann surface of genus g has dimension g.

Proposition 3.3. *Let X be a compact Riemann surface. Then X is a spin manifold.*

Proof. A proof using characteristic classes can, for example, be found in [6]. □

We recall from Examples 1.6, 1.22 and 1.32 that under the identification $\mathrm{Spin}_2 \cong S^1$, the adjoint representation is given by

$$\mathrm{Ad} : S^1 \to \mathrm{Gl}(\mathbb{C}) \qquad z \mapsto z^2 \tag{3.7}$$

and the spinor representations are

$$\pi_{S^+} : S^1 \to \mathrm{Gl}(\mathbb{C}) \qquad z \mapsto z^{-1} \tag{3.8}$$
$$\pi_{S^-} : S^1 \to \mathrm{Gl}(\mathbb{C}) \qquad z \mapsto z. \tag{3.9}$$

Let $P^*_{\mathrm{Spin}_2}$ be a spin structure on X.
Since X is a complex manifold, both its tangent bundle TX and its cotangent bundle T^*X are complex vector bundles. Because we want to use a hermitian metric to identify tangent and cotangent vectors we need to equip the cotangent bundle with the conjugate complex structure and consider $\overline{T^*X}$ instead of T^*X. As real vector bundles they are isomorphic.

We denote the (4-real dimensional) spinor bundle on X by $\not{S} = \not{S}^+ \oplus \not{S}^-$, where $\not{S}^+$ and $\not{S}^-$ are the one-complex dimensional vector bundles associated to $P^*_{\mathrm{Spin}_2}$ via the representations π_{S^+} and π_{S^-}.
Since the cotangent bundle $\overline{T^*X}$ is associated to $P^*_{\mathrm{Spin}_2}$ via the map $\mathrm{Ad} : S^1 \to \mathrm{Gl}(\mathbb{C})$, it follows from the above explicit expressions (3.7) -(3.9) that

$$\not{S}^+ \otimes \not{S}^- \cong \mathbb{C}, \quad \not{S}^- \otimes \not{S}^- \cong \overline{T^*X}, \quad \not{S}^+ \otimes \overline{T^*X} \cong \not{S}^-, \tag{3.10}$$

where $\mathbb{C}$ denotes the trivial bundle over X.
We let $\mathcal{V} = \not{S}^-$ and consider the twisted Dirac operator $\not{D}_{\mathcal{V}} : \Gamma(\not{S} \otimes \mathcal{V}) \to \Gamma(\not{S} \otimes \mathcal{V})$.
Using the isomorphisms (3.10), the twisted bundle $\mathcal{E} = \not{S} \otimes \mathcal{V}$ decomposes as $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, where

$$\mathcal{E}^+ = \not{S}^+ \otimes \mathcal{V} \cong \mathbb{C}, \quad \mathcal{E}^- = \not{S}^- \otimes \mathcal{V} \cong \overline{T^*X}. \tag{3.11}$$

Therefore, the Dirac operator acts between

$$\not{D}_{\not{S}^-} : \Gamma(\mathbb{C} \oplus \overline{T^*X}) \to \Gamma(\mathbb{C} \oplus \overline{T^*X}).$$

To find its explicit form, we have to determine how the Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(\overline{T^*X})$ acts on $\mathrm{End}_{\mathbb{C}}(\not{S})$.

Since X is a complex manifold, it has a global complex structure $J : TM \to TM$. Thus, we can mirror the fibrewise construction of the spin representation (1.57) to obtain an action of the Clifford algebra bundle.
In this global construction, the vector space V is replaced by $\overline{T^*X}$ with its complex structure $\overline{J} = -J$ induced from the complex manifold X. Its complexification decomposes as

$$\overline{T^*_{\mathbb{C}}X} = (T^*X)^{0,1} \oplus (T^*X)^{1,0}, \tag{3.12}$$

where $(T^*X)^{0,1}$ and $(T^*X)^{1,0}$ are the $+i$ and $-i$ eigenspace of $\overline{J}$. In terms of real coordinates x, y and corresponding complex coordinates $z = x + iy$ on X, the bundle $(T^*X)^{1,0}$ has local basis $dz = dx + idy$ and the bundle $(T^*X)^{0,1}$ has local basis $d\overline{z} = dx - idy$. Thus, our construction of the spin representation defines a map

$$\widetilde{c} : \mathrm{Cl}_{\mathbb{C}}(\overline{T^*X}) \to \mathrm{End}\left(\Lambda(T^*X)^{0,1}\right) = \mathrm{End}\left(\mathbb{C} \oplus (T^*X)^{0,1}\right), \tag{3.13}$$

which maps

$$dx = \frac{1}{2}(d\overline{z} + dz) \in \Lambda(T^*X)^{0,1} \oplus \Lambda(T^*X)^{1,0} \quad \text{and} \quad dy = \frac{i}{2}(d\overline{z} - dz) \in \Lambda(T^*X)^{0,1} \oplus \Lambda(T^*X)^{1,0} \tag{3.14}$$

to

$$\widetilde{c}(dx) = \frac{1}{\sqrt{2}}\left(\epsilon(d\overline{z}) - \iota(dz)\right) \qquad \widetilde{c}(dy) = \frac{i}{\sqrt{2}}\left(\epsilon(d\overline{z}) + \iota(dz)\right). \tag{3.15}$$

We will now investigate how this map $\widetilde{c}$ is related to Clifford multiplication $c : \mathrm{Cl}_{\mathbb{C}}(T^*X) \to \mathrm{End}(\$)$ on the spinor bundle $\$$. Restricting to a trivialising subset $U \subseteq X$, $\$ \otimes \mathcal{V}|_U \cong \$ \otimes \mathbb{C}|_U \cong \$|_U$. Thus, by construction both maps c and $\widetilde{c}$ are locally equivalent. Globally this can't be true since $\mathbb{C} \oplus \overline{T^*X}$ is not the spinor bundle $\$$.

Horever, observe that $\overline{T^*X}$ and $(T^*X)^{0,1}$ are isomorphic as complex vector bundles and define

$$c \otimes \mathbb{1}_{\mathcal{V}} : \mathrm{Cl}_{\mathbb{C}}(\overline{T^*X}) \to \mathrm{End}(\$) \otimes \mathrm{End}(\mathcal{V}) \cong \mathrm{End}(\$ \otimes \mathcal{V}) \cong \mathrm{End}(\mathbb{C} \oplus \overline{T^*X}) \cong \mathrm{End}(\mathbb{C} \oplus (T^*X)^{0,1}). \quad (3.16)$$

This yields a global equivalence of $c \otimes \mathbb{1}_{\mathcal{V}}$ and $\widetilde{c} : \mathrm{Cl}_{\mathbb{C}}(\overline{T^*X}) \to \mathrm{End}(\mathbb{C} \oplus (T^*X)^{0,1})$ and we see that $\widetilde{c}$ defines the twisted Clifford action on $\$ \otimes \mathcal{V} \cong \mathbb{C} \oplus (T^*X)^{0,1}$.

We are now ready to compute the Dirac operator (where we identify $\overline{T^*X}$ with $(T^*X)^{0,1}$)

$$D\!\!\!/_{\mathcal{V}} : \Gamma\left(\mathbb{C} \oplus (T^*X)^{0,1}\right) \to \Gamma\left(\mathbb{C} \oplus (T^*X)^{0,1}\right). \quad (3.17)$$

Let ∂_x and ∂_y be a local frame of TX and let dx and dy be the corresponding dual frame of $\overline{T^*X}$.
Then, using (3.15)

$$D\!\!\!/_{\mathcal{V}} = \sqrt{2}\left(\Big(\epsilon(d\overline{z}) - \iota(dz)\Big)\nabla_{\partial_x} + i\Big(\epsilon(d\overline{z}) + \iota(dz)\Big)\nabla_{\partial_y}\right). \quad (3.18)$$

Restricting to the subspace $\mathcal{E}^+ = \$^+ \otimes \mathcal{V} = \mathbb{C}$, the chiral Dirac operator

$$D\!\!\!/_{\mathcal{V}}^+ : \Gamma(\mathcal{E}^+) \cong C^\infty(X) \to \Gamma(\mathcal{E}^-) \cong \Gamma((T^*X)^{0,1}) \quad (3.19)$$

becomes

$$D\!\!\!/_{\mathcal{V}}^+ f = \sqrt{2}\,(\partial_x f + i\partial_y f)\, d\overline{z} = \sqrt{2}\partial_{\overline{z}} f d\overline{z}. \quad (3.20)$$

This is exactly $\sqrt{2}$ times the Dolbeault operator

$$\overline{\partial} : C^\infty(X) \to \Gamma((T^*X)^{0,1}). \quad (3.21)$$

The Atiyah-Singer index theorem then states that

$$\mathrm{ind}(\overline{\partial}) = \mathrm{ind}(D\!\!\!/_{\mathcal{V}}^+) = (2\pi i)^{-1}\int_X \widehat{A}(X) \wedge \mathrm{ch}(\mathcal{V})_{[2]}. \quad (3.22)$$

From the expansion (3.3) it follows that $\widehat{A}(X) = 1$. Since $\mathcal{V}^2 = \$^- \otimes \$^- = \overline{T^*X}$ and $\overline{T^*X}$ is complex isomorphic to the tangent bundle TX, we have that $\mathrm{ch}(\mathcal{V}) = \frac{1}{2}\mathrm{ch}(\overline{T^*X}) = \frac{1}{2}\mathrm{ch}(TX)$. Therefore,

$$\mathrm{ind}(\overline{\partial}) = \frac{1}{2}\int_X \frac{1}{2\pi i}\mathrm{ch}(TX)_{[2]} \quad (3.23)$$

For a complex m-dimensional manifold, the top chern class $\frac{1}{(2\pi i)^m}\mathrm{ch}(TX)_{[2m]}$ equals the Euler class $e(TX)$, whose integral over X yields the euler characteristic. Therefore,

$$\mathrm{ind}(\overline{\partial}) = \frac{1}{2}\int_X e(TX) = \frac{1}{2}\chi(X) = 1 - g. \quad (3.24)$$

This is the Hirzebruch-Riemann-Roch theorem for a Riemann surface. It can easily be generalised to higher dimensional complex manifolds, giving a formula for $\mathrm{ind}(\overline{\partial} + \overline{\partial}^*)$ in terms of certain characteristic classes (see equation (22) in the introduction).

Since $\ker(\overline{\partial})$ is the space of holomorphic functions on the compact manifold X, it follows that $\ker(\overline{\partial}) = \mathbb{C}$. This means that

$$\dim\Big(\mathrm{coker}\left(\overline{\partial} : C^\infty(X) \to \Gamma((T^*X)^{0,1})\right)\Big) = g. \quad (3.25)$$

Consider the Dolbeault cohomology associated to the operator $\overline{\partial}$,

$$H^{p,q}_{\overline{\partial}}(X) := \frac{\ker\overline{\partial} : \Gamma((T^*X)^{p,q}) \to \Gamma((T^*X)^{p,q+1})}{\mathrm{im}\,\overline{\partial} : \Gamma((T^*X)^{p,q-1}) \to \Gamma((T^*X)^{p,q}}). \quad (3.26)$$

It follows from Hodge theory that $H^{p,q}_{\overline{\partial}}(X) \cong \overline{H^{q,p}_{\overline{\partial}}}(X)$. Therefore, the space of holomorphic one-forms on X, $H^{1,0}_{\overline{\partial}}(X)$ is (antilinear) isomorphic to the space

$$H^{0,1}_{\overline{\partial}}(X) = \frac{\ker \overline{\partial} : \Gamma((T^*X)^{0,1}) \to \Gamma((T^*X)^{0,2})}{\operatorname{im} \overline{\partial} : C^\infty(X) \to \Gamma((T^*X)^{0,1})}. \tag{3.27}$$

Since X is complex one-dimensional, $(T^*X)^{0,2} = 0$ and thus $H^{0,1}_{\overline{\partial}} = \operatorname{coker}(\overline{\partial} : C^\infty(X) \to \Gamma((T^*X)^{0,1}))$. Therefore, we can restate equation (3.25) as the following.

Theorem 3.4 (Riemann-Roch)**.** *Let X be a compact Riemann surface of genus g. Then the space of holomorphic one-forms $H^{1,0}_{\overline{\partial}}(X)$ has dimension g.*

3.3 The Index Theorem in Seiberg-Witten Theory

A modern application of the index theorem is the study of solution spaces of non-linear PDEs. We briefly discuss the main motivations behind studying these spaces and show how the index theorem can be used in this context. We will focus on the specific example of Seiberg-Witten theory. Our outline roughly follows [5].

Most of the techniques discussed in this section were developed to classify different smooth structures on homeomorphic manifolds. The main idea underlying these techniques is to use some geometrical partial differential equation (which evidently depends on the smooth structure of M) and consider its space of solutions $\mathcal{M}$. One can then hope that certain topological invariants of the space $\mathcal{M}$ do also depend on the smooth structure of M. These invariants could then be used to distinguish different smooth structures on homeomorphic spaces.

The solution space $\mathcal{M}$ of a *linear* PDE is a vector spaces with the dimension being its only invariant.
For the Laplace equation $\Delta\psi = 0$ this leads to Hodge theory (with $b_k = \dim\big(\ker \Delta : \Omega^p(M) \to \Omega^p(M)\big)$, the k.th Betti-number of M) and for more general elliptic operators to index theories as discussed in this essay.
However, all these quantities are *topological* invariants of M and none of them depends on the specific smooth structure of M.
To obtain a solution space with richer structure, one has to consider non-linear PDEs. A pioneer of this approach was Simon Donaldson who used the Yang-Mills equation on a four dimensional manifold M to construct an invariant out of their solution space which indeed depended on the smooth structure of M.
A technically much easier set of partial differential equations leading to similar results as Donaldson theory are the Seiberg-Witten equations, which we will briefly discuss in the following.

Both in Donaldson and Seiberg-Witten theory, the solution space $\mathcal{M}$ turns out to be (up to singular points) a smooth manifold. One can use the index theorem to calculate the local dimension of this manifold. This works for the following reason.

Heuristic. Let $N : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ be a non-linear elliptic operator between vector bundles $\mathcal{V}$ and $\mathcal{W}$ with solution space $\mathcal{M} = \{u \in \Gamma(\mathcal{V}) \mid N(u) = 0\}$. Let $u_0 \in \mathcal{M}$ such that $L = DN|_{u_0}$ is surjective (i.e. such that $\mathcal{M}$ is smooth at u_0). Then, the implicit function theorem implies that in a neighboorhood of u_0, $\mathcal{M}$ looks like $\ker(L)$. In particular, the local dimension of $\mathcal{M}$ around u_0 is given by $\dim(\ker(L)) = \operatorname{ind}(L)$ (since $\operatorname{coker}(L) = 0$, due to surjectivity of L), which can be calculated using the index theorem.

In the following, we give a very brief overview about how this heuristic is used in the case of Seiberg-Witten theory.

Definition 3.5. In analogy to the Spin group, we define the *complex Spin group* Spin^c_n by the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^c_n \to \mathrm{SO}_n \times \mathrm{U}(1) \to 1.$$

A *spin^c-structure* on a compact Riemannian manifold (M, g) is a lift of the oriented orthonormal coframe bundle $P^*_{\mathrm{SO}_n}$ to a Spin^c_n- bundle $P^*_{\mathrm{Spin}^c_n}$.

One of the main advantages of the notion of a spin^c structure is that it is much less restrictive than the notion of a spin structure. Intuitively, a spin^c structure is a spin structure up to an arbitrary phase. In particular, every spin manifold is also spin^c. Also, one can show that every smooth compact 4-manifold is spin^c.

For even n, the spin^c representation $\pi^c_S : \mathrm{Spin}^c_n \to \mathrm{End}_{\mathbb{C}}(S)$ is defined as the restriction of the spinor representation $c : \mathrm{Cl}_{\mathbb{C}}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ to Spin^c_n. To a spin^c manifold we associate the spinor bundles $\not{S}^{\pm} = P_{\mathrm{Spin}^c_n} \times_{\pi^c_{S^\pm}} S^{\pm}$ and $\not{S} = \not{S}^+ \oplus \not{S}^-$. Additionaly, on every spin^c manifold M, there is an hermitian line bundle L associated to $P^*_{\mathrm{Spin}^c_n}$ via the representation $\mathrm{Spin}^c_n \to \mathrm{SO}_n \times \mathrm{U}(1) \twoheadrightarrow \mathrm{U}(1)$.

While we were able to lift the Levi-Civita connection from $P^*_{\mathrm{SO}_n}$ to $P^*_{\mathrm{Spin}_n}$, we now need a connection on $P^*_{\mathrm{SO}_n} \times \big(P_{\mathrm{U}(1)}(L)\big)$ to lift to a connection on $P^*_{\mathrm{Spin}^{\mathbb{C}}_n}$. Thus, given a connection A on L, we get a spin^c connection $\nabla_A : \Gamma(\not{S}) \to \Gamma(T^*M \otimes \not{S})$.We can then define the Dirac operator

$$\not{D}_A : \Gamma(\not{S}) \xrightarrow{\nabla_A} \Gamma(T^*M \otimes \not{S}) \xrightarrow{c} \Gamma(\not{S}). \tag{3.28}$$

Since M is four dimensional, the Hodge-star $* : \Omega^k(M) \to \Omega^{n-k}(M)$ defines an operator $* : \Omega^2(M) \to \Omega^2(M)$ that squares to the identity. We can therefore decompose $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$ in the space of self-dual and anti self-dual forms

$$\Omega^2_+(M) = \{\omega \in \Omega^2(M) \mid * \omega = \omega\} \quad \Omega^2_-(M) = \{\omega \in \Omega^2(M) \mid * \omega = -\omega\}. \tag{3.29}$$

Since L is a line bundle, it follows that $\mathrm{End}(L) \cong \mathbb{C}$ is trivial. Therefore, the curvature is a two-form $F_A \in \Omega^2(M)$, which we decompose as $F_A = F_A^+ + F_A^-$. By extending the Clifford multiplication in a certain way, we also define a squaring map $q : \not{S}^+ \to i\Lambda^2_+ T^*M$.

Definition 3.6. Let M^4 be a smooth compact 4-manifold. Choose a spin^c structure and let $\not{S}^{\pm}$ be the associated spinor bundles and L the associated line bundle. Let μ be a fixed self-dual two form. The *Seiberg-Witten equations* are the equations

$$\not{D}_A \phi = 0 \qquad F_A^+ = q(\phi) + i\mu, \tag{3.30}$$

for (A, ϕ), where A is a connection on L and ϕ is a section of $\not{S}^+$.

If we define the operator $\mathcal{F}(A, \phi) := \big(D_A\psi, F_A^+ - q(\psi) - i\mu\big)$, then the Seiberg-Witten equations are $\mathcal{F}(A, \phi) = 0$. We denote the space of solutions by $\mathfrak{m} = \{(A, \phi) \mid \mathcal{F}(A, \phi) = 0\}$.
Since the equations are formulated in terms of a connection A, they are gauge invariant under the group of $U(1)-$gauge transformations, i.e. under the group $C^\infty(M, S^1)$ acting on $\mathfrak{m}$ as

$$h.(A, \phi) \mapsto (A - 2h^{-1}dh, h\phi) \qquad h \in C^\infty(M, S^1). \tag{3.31}$$

It is therefore natural to consider the moduli space of solutions $\mathcal{M} = \mathfrak{m}/C^\infty(M, S^1)$. We can topologize $\mathcal{M}$ in a suitable way.

Proposition 3.7. *The moduli space $\mathcal{M}$ is compact. For generic μ, $\mathcal{M}$ is a smooth manifold with dimension*

$$\dim(\mathcal{M}) = b^1 - 1 - b^2_+ + \frac{c_1^2 - \tau}{4}, \tag{3.32}$$

where $b^1 = \dim(H^1(M))$, $b^2_+ = \dim(H^2_+(M))$, $\tau = b^2_+ - b^2_-$ *and* $c_1 = c_1(L)$.

Proof sketch. We linearize the Seiberg-Witten equations at a point $(A_0, \phi_0) \in \mathfrak{m}$ and obtain an operator
$L_{(A_0,\phi_0)} = D\mathcal{F}|_{(A_0,\phi_0)}$ acting on $(A_0 + i\alpha, \phi_0 + \psi)$. To calculate the index of this operator we decompose it in several elliptic operators and calculate their indices in terms of characteristic classes on M. In particular, we have to calculate the index of the Dirac operator $\not{D}_{A_0}$. This can be done using the Atiyah-Snger index theorem

$$\mathrm{ind}(\not{D}_{A_0}) = (2\pi i)^{-2} \int_M \Big(\widehat{A}(M) \wedge \mathrm{ch}(L)\Big)_{[n]}.$$

Following our heuristic, thei gives a formula for the local dimension of $\mathcal{M}$. □

We can use the so obtained explicit expression for the dimension to assign an invariant to the solution space $\mathcal{M}$ which depends on the smooth structure of M.
For our manifold M with spin^c structure s , we let $d := b^1 - 1 - b^2_+ + \frac{c_1^2 - \tau}{4}$. If $d < 0$, then the Seiberg-Witten

equations have no solutions and we assign the invariant $\mathrm{SW}(M,s) = 0$.
If $d = 0$, it follows that $\mathcal{M}$ is a compact and zero dimensional manifold and thus a finite set of points. We define the Seiberg-Witten invariant of M as $\mathrm{SW}(M,s) = \#(\mathcal{M})$, the number of points of $\mathcal{M}$ (actually one weights the points with ± 1 according to the orientation of $\mathcal{M}$).
If $d > 0$, the situation is more complicated and one defines the Seiberg-Witten invariant $\mathrm{SW}(M,s)$ as the integral of a specific characteristic form over the smooth manifold $\mathcal{M}$.
Considering SW as a function $\mathrm{SW}(M,\cdot) : \mathcal{S}_M \to \mathbb{Z}$ over the (non-empty) set of spin^c structures on M, we have found an invariant which only depends on the manifold M and its smooth structure.

3.4 Outlook

We conclude this essay with an outline of how the Atiyah-Singer index theorem for Dirac operators can be generalised and what role the Dirac operator plays in the index theorem for general elliptic operators.
In the above discussion of the Riemann-Roch theorem we have seen that the Dolbeault operator $\overline{\partial}$ on a Riemann surface is a twisted Dirac operator. Indeed, there is a whole class of operators having similar properties than Dirac operators.

Definition 3.8. A *Clifford bundle* $\mathcal{E}$ over a compact Riemannian manifold is a hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ with a compatible connection $\nabla^{\mathcal{E}}$, and a graded action $c : \mathrm{Cl}_{\mathbb{C}}(M) \to \mathrm{End}(\mathcal{E})$ (graded means that $c(v) : \mathcal{E}^{\pm} \to \mathcal{E}^{\mp}$ for all $v \in T^*M$) such that

$$\big(c(v)e, e'\big) = -\big(e, c(v)e'\big) \quad \forall e, e' \in \mathcal{E},\ v \in T^*M \qquad \text{and} \qquad [\nabla^{\mathcal{E}}, c(\alpha)] = c(\nabla\alpha) \quad \forall \alpha \in \Gamma(\mathrm{Cl}_{\mathbb{C}}(M)). \tag{3.33}$$

Given a Clifford bundle $\mathcal{E}$, we can define its associated Dirac operator as $\not{D} = \sum_i c(e^i)\nabla^{\mathcal{E}}_{e_i}$ for some orthonormal frame $e_1, \ldots, e_n$ of TM.

The twisted Dirac bundles $\mathcal{E} = \not{S} \oplus \mathcal{V}$ discussed in this essay are examples of Clifford bundles.
Our proof of the Riemann-Roch theorem in the last section was basically about establishing that the bundle $\Lambda(T^*M)^{0,1}$ carries an action $\tilde{c}$ of $\mathrm{Cl}_{\mathbb{C}}(M)$ (equation (3.13)) which makes it into a Clifford bundle with associated Dirac operator $\sqrt{2}\,\left(\overline{\partial} + \overline{\partial}^*\right)$.
It is in fact no coincidence that this operator is equal to a twisted Dirac operator. One can prove that on a spin manifold every Clifford bundle is a twisted spinor bundle.
Since every manifold is locally a spin manifold, every Clifford bundle is locally a twisted spinor bundle.
This leads to the following extension of the Atiyah-Singer index theorem to general non-spin manifolds.

Theorem 3.9. *The index of a Dirac operator on a Clifford module $\mathcal{E}$ over a compact oriented even dimensional Riemannian manifold is given by*

$$\mathrm{ind}(\not{D}^+) = (2\pi i)^{-\frac{n}{2}} \int_M \left(\widehat{A}(M) \wedge \mathrm{ch}(\mathcal{E}/S)\right)_{[n]}. \tag{3.34}$$

The twisted chern class $\mathrm{ch}(\mathcal{E}/S)$ is defined in such a way that if $\mathcal{E} = \not{S} \otimes \mathcal{V}$ is a twisted Dirac bundle for some bundle $\mathcal{V}$, then $\mathrm{ch}(\mathcal{E}/S) = \mathrm{ch}(\mathcal{V})$.

This generalised index theorem subsumes all classical index theorems such as the Hirzebruch-Riemann-Roch theorem, the Signature theorem and the Chern-Gauss Bonnet theroem.

In fact, even more is true. Let M be an even-dimensional compact spin manifold and let $K(M)$ be the abelian group generated from the monoid of isomorphism classes of complex vector bundles on M under $\oplus$. We also introduce the group $\mathrm{Ell}(M)$ of abstract elliptic operators on M, defined in [1]. Using K-theory, one can prove that the map $K(M) \to \mathrm{Ell}(M),\ [\mathcal{V}] \mapsto \not{D}_{\mathcal{V}}$ is an isomorphism. Therefore, any elliptic operator on M is generated by twisted Dirac operators on certain twisted spinor bundles. In this sense, the index theorem for twisted Dirac operators proven in this essay is the fundamental result leading to the index theorem for general elliptic operators on even-dimensional compact spin manifolds.

Appendix

A The Asymptotic Expansion

The following section is devoted to the proof of the existence of an asymptotic expansion of the heat kernel $p_t(x,y)$. The proof consists of two parts. First, we show that there is a (local) formal power series solution to the heat equation (Lemma A.3). Then we show that this formal power series is indeed an asymptotic expansion to the actual heat kernel.

Let M be a compact even dimensional spin manifold with a Dirac bundle $\mathcal{E} = \not{S} \otimes \mathcal{V}$ and Dirac operator $\not{D}_{\mathcal{V}}$.
For the first part of this section we will fix a point $x_0 \in M$ and work in geodesic coordinates around x_0 as introduced at the beginning of the fourth step of the proof of the index theorem. We will globalize our results only for the proof of Theorem A.4.

Let $W_{x_0} = T_{x_0}M$, $U_{x_0} = \{\underline{x} \in T_{x_0}M \mid \|\underline{x}\| < \epsilon_{x_0}\}$, where $\epsilon_{x_0} > 0$ is small enough such that $\exp_{x_0} : U_{x_0} \to M$ is a diffeomorphism and let $\tau(x_0, x) : \mathcal{E}_x \to \mathcal{E}_{x_0}$ denote the parallel transport map. Let

$$L = -\sum_{i,j=1}^{n} g^{ij}\left(\nabla^{\mathcal{E}}_{\partial_i}\nabla^{\mathcal{E}}_{\partial_j} - \Gamma^k_{ij}\nabla^{\mathcal{E}}_{\partial_k}\right) + \frac{r_M}{4} + F^{\mathcal{V}} : C^\infty(U_{x_0}, \mathcal{E}_{x_0}) \to C^\infty(U_{x_0}, \mathcal{E}_{x_0}), \qquad \text{(A.1)}$$

be the triviaized Dirac operator on U_{x_0} and let $p_t(x, x_0)$ be the heat kernel of $\not{D}^2_{\mathcal{V}}$ on $M \times M$.

To prove our main theorem we need to perform some auxiliary calculations. On U_{x_0} define the *Gaussian*

$$q_t(\underline{x}) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|\underline{x}\|^2}{4t}} \qquad \underline{x} \in U_{x_0}, \qquad \text{(A.2)}$$

where $\|\underline{x}\|^2 = \sum_{i=1}^n (\underline{x}^i)^2$. Let $\Delta = \nabla^*\nabla = d^*d$ be the Laplace-Beltrami operator on $C^\infty(U_{x_0})$.

Lemma A.1. *There is a smooth function $a \in C^\infty(U_{x_0})$, such that*

$$(\partial_t + \Delta)q_t(\underline{x}) = \frac{a}{t} q_t(\underline{x}). \qquad \text{(A.3)}$$

Proof. Let $r(\underline{x}) := \|\underline{x}\|$ be the distance function. If $h \in C^\infty(U_{x_0})$ is of the form $h(\underline{x}) = f(\|\underline{x}\|^2)$, then $\nabla h = dh = 2\|\underline{x}\| f'(\|\underline{x}\|^2)dr$, where $dr = \sum_{i=1}^n \frac{\underline{x}^i}{\|\underline{x}\|} dx^i$.
Since we are working in geodesic coordinates, we have that $dr^\sharp = \sum_{i=1}^n \frac{\underline{x}^i}{\|\underline{x}\|}\partial_i = \frac{\partial}{\partial r}$.
Thus, $\Delta h = \nabla^*(2rf'(r^2)dr) = -2\operatorname{div}(rf'(r^2)\frac{\partial}{\partial r})$. To calculate the divergence, we use the local formula $\operatorname{div}(X) = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}X^a)$, where g is the determinant of the metric g_{ij}. Therefore,

$$\Delta h = -2\left(rf'(r^2)\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial r} + 2r^2 f''(r^2) + nf'(r^2)\right).$$

Applying this formula to the Gaussian $q_t(\underline{x})$ with $f(z) = e^{-\frac{z}{4t}}$ and simplifying $\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial r} = \frac{1}{2g}\frac{\partial g}{\partial r}$, we obtain

$$\Delta q_t = \left(\frac{r}{4t}\frac{1}{g}\frac{\partial g}{\partial r} - \left(\frac{r}{2t}\right)^2 + \frac{n}{2t}\right) q_t.$$

Because $\partial_t q_t = (-\frac{n}{2t} + \frac{r^2}{4t^2})q_t$, we have that

$$(\partial_t + \Delta)q_t = \frac{r}{4t}\frac{1}{g}\frac{\partial g}{\partial r}q_t. \qquad \square$$

Lemma A.2. *Let $q_t(\underline{x})$ be the Gaussian and $\sigma_t \in C^\infty(U_{x_0}, \mathcal{E}_{x_0})$. Then*

$$(\partial_t + L)q_t(\underline{x})\sigma_t(\underline{x}) = q_t(\underline{x}) \left(\partial_t + L + t^{-1}\nabla^{\mathcal{E}}_{r\partial_r} + t^{-1}a\right) \sigma_t(\underline{x}), \tag{A.4}$$

where $a \in C^\infty(U_{x_0}))$ is the function from the statement of Lemma A.1.

Proof. Since $\partial_t + L = \partial_t + \Delta^{\mathcal{E}} + \frac{r_M}{4} + F^{\mathcal{V}}$, we only need to consider the ∂_t and $\Delta^{\mathcal{E}}$ parts.
From $\Delta^{\mathcal{E}} = -\sum_{i,j=1}^n g^{ij}\left(\nabla^{\mathcal{E}}_{\partial_i}\nabla^{\mathcal{E}}_{\partial_j} - \Gamma^k_{ij}\nabla^{\mathcal{E}}_{\partial_k}\right)$, it follows that

$$\Delta^{\mathcal{E}}(q_t\sigma_t) = (\Delta q_t)\,\sigma_t - 2\sum_{i,j=1}^n g^{ij}(\nabla_{\partial_i}q_t)(\nabla^{\mathcal{E}}_{\partial_j}\sigma_t) + q_t\Delta^{\mathcal{E}}\sigma_t.$$

Since we are working in geodesic coordinates, it follows that $g^{rj} = \delta^{rj}$ and thus

$$\sum_{i,j} g^{ij}\nabla_{\partial_i}q_t\nabla^{\mathcal{E}}_{\partial_j}\sigma_t = \frac{\partial q_t}{\partial r}\nabla^{\mathcal{E}}_{\partial_r}\sigma_t = -\frac{1}{2t}q_t\nabla^{\mathcal{E}}_{r\partial_r}\sigma_t.$$

The claim follows from

$$(\partial_t + \Delta^{\mathcal{E}})q_t\sigma_t = \sigma_t(\partial_t + \Delta)q_t + q_t(\partial_t + \Delta^{\mathcal{E}})\sigma_t + q_t t^{-1}\nabla^{\mathcal{E}}_{r\partial_r}\sigma_t$$

and Lemma A.1. $\square$

We can now turn our attention to the first step of the proof and show that locally there is a formal power series solution to the heat equation.

Lemma A.3. *There are unique smooth* $\mathrm{End}(\mathcal{E}_{x_0})$*-valued functions $A_i^{x_0}(\underline{x})$ on U_{x_0} with $A_0^{x_0}(\underline{0}) = \mathbb{1}_{\mathcal{E}_{x_0}}$ such that the formal power series*

$$F(t,\underline{x}) := q_t(\underline{x})\sum_{j=0}^\infty t^j A_j^{x_0}(\underline{x}) \tag{A.5}$$

is a formal solution to the local heat equation $(\partial_t + L)F(t,\underline{x}) = 0$.
Moreover, if we let $F^N(t,\underline{x}) := q_t(\underline{x})\sum_{j=0}^N t^j A_j^{x_0}(\underline{x})$, then

$$(\partial_t + L)F^N = q_t L(A_N)t^N. \tag{A.6}$$

Proof. Since the Gaussian $q_t(\underline{x}) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4t}}$ solves the Euclidean heat equation $(\partial_t - \sum_{i=1}^n \partial_i^2)q_t(\underline{x}) = 0$, we make the Ansatz

$$F(t,\underline{x}) = q_t(\underline{x})\sum_{j=0}^\infty t^j A_j^{x_0}(\underline{x}).$$

Using Lemma A.2, we find that

$$(\partial_t + L)F(t,\underline{x}) = q_t(\partial_t + L + t^{-1}\nabla_{r\partial_r} + t^{-1}a)\sum_{j=0}^\infty t^j A_j^{x_0}.$$

Equating coefficients of powers of t, we obtain the system

$$\begin{aligned} (\nabla_{r\partial_r} + a)A_0^{x_0} &= 0 \\ (j + \nabla_{r\partial_r} + a)A_j^{x_0} &= -LA_{j-1}^{x_0} \qquad j \geqslant 1. \end{aligned}$$

Fixing $\underline{y} \in U_{x_0}$ and setting $f_0(s) = A_0^{x_0}(s\underline{y})$, the first equation can be turned into the ODE

$$\dot{f}_0(s) = -a(s\underline{y})f_0(s) \qquad f_0(0) = A_0^{x_0}(0) = \mathbb{1}_{\mathcal{E}_{x_0}},$$

which we can solve to get $A_0^{x_0}(s\underline{y})$.
By smooth dependence on initial conditions, this determines the function $A_0^{x_0} \in C^\infty(U_{x_0}, \mathrm{End}(\mathcal{E}_{x_0}))$.
Setting $f_j(t) := s^j A_0^{x_0}(t\underline{y})^{-1} A_j^{x_0}(t\underline{y})$, the other equations can be written as

$$\dot{f}_j(s) = -s^{j-1}(A_0^{x_0}(s\underline{y}))^{-1} L A_{j-1}^{x_0}(s\underline{y}) \qquad f_j(0) = 0,\ j \geqslant 1.$$

We can solve this inductively to obtain all $A_j^{x_0}$. Given the initial condition $A_0^{x_0}(\underline{0}) = \mathbb{1}_{\mathcal{E}_{x_0}}$, they are uniquely determined by this construction.
Furthermore, we observe that

$$(\partial_t + L) q_t \sum_{j=0}^N t^j A_j^{x_0} = q_t(\partial_t + L + t^{-1}\nabla_{r\partial_r} + t^{-1}a) \sum_{j=0}^N t^j A_j^{x_0} = q_t t^N L(A_N^{x_0}). \qquad \square$$

We can now return to the global setting and prove the main theorem of this section.

Theorem A.4. *There exist smooth sections $B_i \in \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ over $M \times M$ with $B_0(x,x) = \mathbb{1}_{\mathcal{E}_x}$ such that*

$$p_t(x,y) \sim q_t(x,y) \sum_{j=0}^\infty t^j B_j(x,y), \tag{A.7}$$

where $q_t(x,y) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{4t}}$ is the Gaussian on $M \times M$.

Proof. We will use our local formal power series from Lemma A.3 to construct an asymptotic expansion of the heat kernel. Since M is compact, there is an $\epsilon > 0$ such that

$$U := \{(\underline{x}, y) \mid y \in M, \underline{x} \in T_y M,\ \|\underline{x}\| < \epsilon\}$$

is diffeomorphic to

$$\exp U := \{(x,y) \in M \times M \mid d(x,y) < \epsilon\}$$

via the diffeomorphism $(\underline{x}, y) \mapsto (\exp_y \underline{x}, y)$.

Following Lemma A.3 we can construct a formal power series

$$\widetilde{F}(t, \underline{x}, y) := q_t(\underline{x}) \sum_{j=0}^\infty t^j A_j^y(\underline{x}) \in \mathrm{End}(\mathcal{E}_y) \qquad (\underline{x}, y) \in U,$$

which is smooth in $y \in M$ by construction.
By uniqueness of the A_j^y's we observe that if $\exp_y \underline{x} = \exp_{y'} \underline{x}'$, then $A_j^y(\underline{x}) = A_j^{y'}(\underline{x}')$.
Thus, we can pull this series back to $\exp U$ and define a formal power series

$$\widetilde{F}(t,x,y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{4t}} \sum_{j=0}^\infty t^j \widetilde{B}_j(x,y) \qquad (x,y) \in \exp U$$

in $\Gamma(\exp U, \mathcal{E} \boxtimes \mathcal{E}^*)$, where $\widetilde{B}_j(x,y) = \tau(x,y) A_j^y(\underline{x})$ (and $\underline{x}$ is such that $\exp_y \underline{x} = x$).

If we introduce the bump function $\psi : \mathbb{R}_{\geqslant 0} \to [0,1]$

$$\psi(s) := \begin{cases} 1 & s \leqslant \epsilon^2/16 \\ 0 & s \geqslant \epsilon^2/4 \end{cases},$$

we can extend $\widetilde{F}$ to a power series of global sections

$$F(t,x,y) = \psi(d(x,y)^2)\, q_t(x,y) \sum_{j=0}^\infty t^j \widetilde{B}_j(x,y) \qquad (x,y) \in M.$$

We claim that this formal power series is indeed asymptotic to the heat kernel $p_t(x,y)$. To prove this let $N > \frac{n}{2}$ and set

$$F^N(t,x,y) = \psi(d(x,y)^2) q_t(x,y) \sum_{j=0}^N t^j \widetilde{B}_j(x,y),$$

which is a smooth section of $\Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$. Fix $\sigma_y \in \mathcal{E}_y$ and let

$$f^N(t,x) = \big(F^N(t,x,y) - p_t(x,y)\big)\,\sigma_y.$$

For $\eta \in \Gamma(\mathcal{E})$ we have that

$$\int_M (p_t(x,y)\sigma_y, \eta(x))_{\mathcal{E}_x}\,\mathrm{d}x \xrightarrow{t\to 0} (\sigma_y, \eta(y))_{\mathcal{E}_y}.$$

Since $F^N(t,x,y) = \psi(d(x,y)^2)q_t(x,y)\sum_{j=0}^N t^j \widetilde{B}_j(x,y)$ and since parallel transport preserves the inner product on $\mathcal{E}$, it follows that

$$\lim_{t\to 0}\int_M \big(F^N(t,x,y)\sigma_y, \eta(x)\big)_{\mathcal{E}_x}\,\mathrm{d}x = \lim_{t\to 0}\int_M \psi(d(x,y)^2)q_t(x,y)\,(B_0(x,y)\sigma_y, \eta(x))_{\mathcal{E}_x}\,\mathrm{d}x$$

$$= \lim_{t\to 0}\int_M \psi(d(x,y)^2)q_t(x,y)\,\left(A_0^y(\underline{x}), \tau(y,x)\eta(x)\right)_{\mathcal{E}_y}\,\mathrm{d}x.$$

Changing coordinates back from M to U_y (noticing that the integrand vanishes away from $\exp_y U_y$) this equals

$$\lim_{t\to 0}\int_{U_y} \psi(\|\underline{x}\|^2)q_t(\underline{x})\,\left(A_0^y(\underline{x}), \eta(\underline{x})\right)_{\mathcal{E}_y}\,\mathrm{d}\underline{x},$$

where $\eta(\underline{x}) := \tau(y, \exp_y \underline{x})\eta(\exp_y \underline{x})$.
Since for every compactly supported $h \in C_c^\infty(U)$, we have that $\lim_{t\to 0}\int_U h(\underline{x})q_t(\underline{x})\,\mathrm{d}\underline{x} = h(\underline{0})$, this implies

$$\lim_{t\to 0}\int_M (F^N(t,x,y)\sigma_y, \eta(x))_{\mathcal{E}_x}\,\mathrm{d}x = \left(A_0^y(0)\sigma_y, \eta(\underline{0})\right)_{\mathcal{E}_y} = (\sigma_y, \eta(y))_{\mathcal{E}_y}.$$

Therefore,

$$\lim_{t\to 0}\int_M \big(f^N(t,x), \eta(x)\big)_{\mathcal{E}_x}\,\mathrm{d}x = 0 \qquad \text{for all } \eta \in \Gamma(\mathcal{E}). \tag{A.8}$$

Let $r^N(t,x) := (\partial_t + \not{D}_{\mathcal{V}}^2)f^N(t,x) = (\partial_t + \not{D}_{\mathcal{V}}^2)F^N(t,x,y)\sigma_y$. Notice that $F^N(t,x,y)$ and thus $r^N(t,x)$ vanish for $d(x,y) \geqslant \frac{\epsilon}{2}$. Therefore, both sections are supported in $\exp_y U_y$. For $x = \exp_y \underline{x} \in \exp_y U_y$ we have that

$$\tau(y,x)r^N(t,x) = (\partial_t + L)\psi(\|\underline{x}\|^2)\widetilde{F}^N(t,\underline{x},y)\sigma_y = q_t(\underline{x})\psi(\underline{x})t^N L(A_N^y)\sigma_y + [L,\psi]\widetilde{F}^N(t,\underline{x},y)\sigma_y,$$

where $\widetilde{F}^N(t,\underline{x},y) = q_t(\underline{x})\sum_{j=0}^N t^j A_j^y(\underline{x}) \in \mathrm{End}(\mathcal{E}_y)$ and we have used Lemma A.3.
Since the first order operator $[L,\psi]$ vanishes for $d(x,y) \leqslant \frac{\epsilon}{4}$ and since $\widetilde{F}^N(t,\underline{x},y)$ is surpressed by $e^{-\frac{\epsilon}{32t}}$ for $d(x,y) > \frac{\epsilon}{4}$, the t-dependence of $r^N(t,x)$ is governed by the term $q_t(\underline{x})t^N L(A_N^y)$.
Thus, there exists a constant $C'_{\sigma_y,y} > 0$ such that

$$\sup_{x\in M}\|r^N(t,x)\| = \sup_{x\in \exp_y U}\|r^N(t,x)\| \leqslant C'_{\sigma_y,y}t^{N-\frac{n}{2}}.$$

Since M is constant we therefore have constants $C'_{l,\sigma_y,y} > 0$ such that in the Sobolev norms $\|\cdot\|_l$ for $H_l(\mathcal{E})$

$$\|r^N(t,\cdot)\|_{H_l} \leqslant C'_{l,\sigma_y,y}t^{N-\frac{n}{2}}.$$

To relate $r^N(t,x)$ back to the section $f^N(t,x)$, let $\{\psi_m\}_m$ be an orthonormal basis for $\Gamma(\mathcal{E})$ of eigenfunctions of the Dirac operator $\not{D}_{\mathcal{V}}^2$. Expanding the smooth section

$$f^N(t,x) = \sum_m a_m(t)\psi_m(x),$$

equation (A.8) implies that

$$\lim_{t\to 0}|a_m(t)| = 0.$$

On the other hand, if we expand $r^N(t,x) = \sum_m b_m(t)\psi_m(x)$ and notice that $r^N(t,x) = (\partial_t + \not{D}_{\mathcal{V}}^2)f^N(t,x)$ we find that $b_m = \dot{a}_m + \lambda_m a_m$. Solving this equation with $a_m(0) = 0$ yields $a_m(t) = \int_0^t e^{\lambda_m(s-t)}b_m(s)\,\mathrm{d}s$. The Cauchy-Schwarz inequality implies that

$$|a_m(t)|^2 \leqslant \int_0^t e^{2\lambda_m(s-t)}\,\mathrm{d}s\int_0^t |b_m(s)|^2\,\mathrm{d}s \leqslant t\int_0^t |b_m(s)|^2\,\mathrm{d}s.$$

In terms of the Sobolev norm $\|\cdot\|_{H_l}$ this translates into

$$\|f^N(t,\cdot)\|_{H_l}^2 \leqslant t\int_0^t \|r^N(s,\cdot)\|_{H_l}^2 \,\mathrm{d}s \leqslant t\int_0^s C'_{l,\sigma_y,y} s^{2N-n}\,\mathrm{d}s \leqslant C_{l,\sigma_y,y} t^{2N-n+2}.$$

Since this is true for all $l \geqslant 0$ and in particular for $l \geqslant \frac{n}{2}$, the Sobolev embedding theorem implies that

$$\sup_{x\in M} \|f^N(t,x)\|_{\mathcal{E}_x} \leqslant C_{\sigma_y,y} t^{N-\frac{n}{2}+1},$$

which in turn means that

$$\sup_{x\in M} \|p_t(x,y) - F^N(t,x,y)\|_{\mathcal{E}_x\boxtimes\mathcal{E}_y^*} \leqslant C_y t^{N-\frac{n}{2}+1}.$$

Since M is compact, we can pick the constant $C := \sup_{y\in M} C_y$ to obtain

$$\sup_{x,y\in M} \|p_t(x,y) - F^N(t,x,y)\|_{\mathcal{E}_x\boxtimes\mathcal{E}_y^*} \leqslant C t^{N-\frac{n}{2}+1}.$$

Similarly, we can show that

$$\sup_{x,y\in M} \|\partial_t^k \nabla^\alpha \left(p_t(x,y) - F^N(t,x,y)\right)\| \leqslant C_{k,\alpha} t^{N-\frac{n}{2}-k-\frac{\alpha}{2}+1},$$

which proves that

$$p_t(x,y) \sim F(t,x,y) = q_t(x,y)\sum_{j=0}^\infty t^j \psi(d(x,y)^2)\widetilde{B}_j(x,y) =: q_t(x,y)\sum_{j=0}^\infty t^j B_j(x,y).$$

Notice that the powers $t^{-\frac{n}{2}-\frac{\alpha}{2}}$ are necessary because of the prefactor $q_t(x,y)$.
Tracing through the proof we find $B_0(x,x) = \psi(d(x,x)^2)\widetilde{B}_0(x,x) = A_0^x(\underline{0}) = \mathbb{1}_{\mathcal{E}_x}$. ☐

B Mehler's Formula

In the following section we calculate the heat kernel of the harmonic oscillator as it appears in the rescaling limit of Getzler's proof the Atiyah-Singer index theorem.

Let $\omega \in \mathbb{R}$, $f \in \mathbb{C}$ and consider the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{16} + f \tag{B.1}$$

acting on $\mathbb{C}$-valued functions on $\mathbb{R}$.
Since we are working on the non-compact space $\mathbb{R}$, we have to require slightly more than smoothness of our solutions. Indeed, we require that our heat kernel lives in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Proposition B.1. *There exists a heat kernel $p_t \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ which is smooth in $t > 0$ and such that*

$$(\partial_t + H_x)p_t(x,y) = 0 \tag{B.2}$$

$$\lim_{t\to 0}\int_{\mathbb{R}} p_t(x,y)\phi(y)\,\mathrm{d}y = \phi(x) \qquad \forall\phi\in\mathcal{S}(\mathbb{R}). \tag{B.3}$$

At $y = 0$, it is given by

$$p_t(x,0) = (4\pi t)^{-\frac{1}{2}}\left(\frac{tr/2}{\sinh(tr/2)}\right)^{\frac{1}{2}} \exp\left(-tr/2\coth(tr/2)\frac{x^2}{4t} - tf\right). \tag{B.4}$$

Proof. Let's first consider the case $\omega = 4$ and $f = 0$, such that $H = -\frac{d^2}{dx^2} + x^2$.
We guess that the solution to the heat equation looks like a Gaussian in x, y. Since H is self-adjoint, this Gaussian has to be symmetric in x and y. Thus, we make the ansatz

$$p_t(x,y) = \exp(a_t(x^2+y^2)/2 + b_t xy + c_t).$$

Plugging this in the heat equation we find the following equations for the coefficients

$$\dot{a}_t/2 = a_t^2 - 1 = b_t^2, \qquad \dot{c}_t = a_t.$$

They have solutions

$$a_t = -\coth(2t + C), \quad b_t = \operatorname{cosech}(2t + C), \quad c_t = -\frac{1}{2}\log\sinh(2t + C) + D.$$

Substituting in the inital condition $\lim_{t\to 0} p_t(x, y) = \delta_y(x)$, we obtain

$$C = 0, \qquad D = \log\left((2\pi)^{-\frac{1}{2}}\right)$$

and therefore the solution

$$p_t(x, y) = \Big(2\pi\sinh(2t)\Big)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(\coth(2t)(x^2 + y^2) - 2\operatorname{cosech}(2t)xy\right)\right).$$

By a change of coordinate, we can recover the heat kernel of the full harmonic oscillator $H = -\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{16} + f$, proving its existence. In particular, for $y = 0$ we find that

$$p_t(x, 0) = (4\pi t)^{-\frac{1}{2}} \left(\frac{tr/2}{\sinh(tr/2)}\right)^{\frac{1}{2}} \exp\left(-tr/2\coth(tr/2)\frac{x^2}{4t} - tf\right)$$

solves $(\partial_t - \frac{d^2}{dx^2} + \frac{\omega^2 x^2}{16} + f)p_t(x, 0) = 0.$ □

We will now generalize this result to heat kernels of generalized harmonic oscillators as the operator appearing in Proposition 2.46.

Definition B.2. Let $\mathcal{A}$ be a finite dimensional commutative algebra over $\mathbb{C}$ with unit. Let R be an $n \times n$ antisymmetric matrix and let F be an $N \times N$ matrix, both with values in $\mathcal{A}$. The *generalized harmonic oscillator* is the operator

$$H = -\sum_{i=1}^{n}(\partial_i - \frac{1}{4}\sum_j R_{ij}x_j)^2 + F \tag{B.5}$$

acting on $\mathcal{A} \otimes \operatorname{End}(\mathbb{C}^N)$-valued functions on $\mathbb{R}^n$.

Consider the Taylor expansion of the holomorphic function $z \mapsto \frac{z}{\sinh(z)} = 1 + \sum_{k=1}^{\infty} a_k z^{2k}$ and define the formal power series in the parameter t with coefficients in $\operatorname{Mat}_{n\times n}(\mathcal{A})$

$$\frac{tR/2}{\sinh(tR/2)} := 1 + \sum_{k=1}^{\infty} t^{2k} a_k (R/2)^{2k}. \tag{B.6}$$

Since multiplication and addition of formal power series yields again formal power series, the determinant

$$\det\left(\frac{tR/2}{\sinh(tR/2)}\right) = 1 + \sum_{k=1}^{\infty} t^k h_k(R) \tag{B.7}$$

and its square root

$$\det{}^{\frac{1}{2}}\left(\frac{tR/2}{\sinh(tR/2)}\right) = 1 + \sum_{k=1}^{\infty} t^k f_k(R) \tag{B.8}$$

define formal power series with coefficients in $\mathcal{A}$. Here h_k and f_k are homogenous polynomials of degree k with respect to the coefficients R_{ij}. We also define the $\operatorname{Mat}_{n\times n}(\mathcal{A})$-valued formal power series

$$tR/2\coth(tR/2) = 1 + \sum_{k=1}^{\infty} t^{2k} b_k (R/2)^{2k}, \quad \text{where} \quad z\coth(z) = 1 + \sum_{k=1}^{\infty} b_k z^{2k} \tag{B.9}$$

and the $\operatorname{Mat}_{N\times N}(\mathcal{A})$-valued formal power series

$$e^{-tF} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} t^n (-F)^n. \tag{B.10}$$

We can now calculate the heat kernel of the generalized harmonic oscillator.

Theorem B.3 (Mehler). *The $\mathcal{A} \otimes \operatorname{End}(\mathbb{C}^N)$-valued formal power series in t*

$$A(t, \underline{x}) := \left(\det\left(\frac{tR/2}{\sinh(tR/2)}\right)\right)^{\frac{1}{2}} \exp\left(-\frac{1}{4t}\underline{x}^t \left(tR/2\coth(tR/2) - 1\right)\underline{x}\right)\exp(-tF), \tag{B.11}$$

is such that

$$p_t(\underline{x}) := q_t(\underline{x})A(t, \underline{x}) \tag{B.12}$$

is the unique formal power series solution to the heat equation $(\partial_t + H)p_t(\underline{x}) = 0$ with $A(0, \underline{x}) = 1$.
Here $q_t(\underline{x}) = e^{-\frac{\|\underline{x}\|^2}{4t}}$ is the Gaussian.

Proof. We have to prove that $\partial_t p_t = -Hp_t$. Since both sides of the equation are analytic with respect to R_{ij}, it suffices to prove the result for $R_{ij} \in \mathbb{R}$.
Pick an orthonormal basis for $\mathbb{R}^n$ such that the antisymmetric matrix R decomposes in a direct sum of 2×2-blocks of the form $\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}$. Thus, our equation decouples and we are left with proving that the two dimensional kernel

$$p_t(\underline{x}) = q_t(\underline{x})\frac{tr/2}{\sin(tr/2)}\exp\left(-\frac{\|\underline{x}\|^2}{4t}\left(tr/2\coth(tr/2) - 1\right)\right)exp(-tF)$$

solves the heat equation for

$$H = -(\partial_1 - \frac{1}{4}rx_2)^2 - (\partial_2 + \frac{1}{4}rx_1)^2 + F = -(\partial_1^2 + \partial_2^2) + \frac{1}{2}r(x_2\partial_1 - x_1\partial_2) - \frac{r^2}{16}(x_1^2 + x_2^2) + F.$$

Since $(x_2\partial_1 - x_1\partial_2)\|\underline{x}\|^2 = 0$, it follows that

$$Hp_t = (-(\partial_1^2 + \partial_2^2) - \frac{r^2\|\underline{x}\|^2}{16} + F)p_t.$$

Therefore, the statement follows from Proposition B.1 by replacing r by ir (which is of course only possible in the context of formal power series). □

Bibliography

[1] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators on Compact Manifolds.* Bull. Amer. Math. Soc. **69**, 1963.

[2] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators.* Springer, 1992.

[3] D.S. Freed, *Geometry of Dirac Operators.* University of Chicago lecture notes, 1987.

[4] E. Getzler, *A Short Proof of the Local Atiyah-Singer Index Theorem.* Topology, Vol. **25**, 1985.

[5] M. Hutchings and C. Taubes, *An Introduction to the Seiberg-Witten Equations on Symplectic Manifolds.* Park City-AIS summer institute lecture notes, 1997.

[6] B. Lawson and M-L. Michelsohn, *Spin Geometry.* Princeton University Press, 1989.

[7] J. Roe, *Elliptic Operators, Topology and Asymptotic Methods.* Chapman & Hall / CRC, 1988.

[8] A.J. Wassermann, *The Atiyah-Singer Index Theorem.* Part III lecture notes, 2010.