Frobenius algebras, Hopf algebras and 3-categories

David Reutter

University of Oxford

Hopf algebras in Kitaev’s quantum double models
Perimeter Institute, Canada

August 3, 2017
The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models
The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models
The plan

- **Part 6.** Lattice models
- **Part 5.** Higher linear algebra
- **Part 4.** Hopf algebras
- **Part 3.** 3-categories
- **Part 2.** 2-categories
- **Part 1.** Motivation
Part 1
Motivation
What is higher algebra?

- Ordinary algebra lets us compose along a line:
  
  \[ xy^2 zyx^3 \]
What is higher algebra?

- Ordinary algebra lets us compose along a line:
  \[ xy^2 zyx^3 \]

- *Higher algebra* lets us compose in higher dimensions:
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

Next hour: Hopf algebras as a ‘shadow’ of a three dimensional theory.

David Reutter
Hopf algebras and 3-categories
August 3, 2017 5 / 34
A tradeoff between algebra and topology

Frobenius algebras
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram1.png}}
\end{array}
\end{array}
& = & \begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram2.png}}
\end{array}
\end{array}
& = & \begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram3.png}}
\end{array}
\end{array}
\\
\begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram4.png}}
\end{array}
\end{array}
& = & \begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram5.png}}
\end{array}
\end{array}
& = & \begin{array}{c}
\begin{array}{c}
\parbox{1cm}{\includegraphics{diagram6.png}}
\end{array}
\end{array}
\end{array}
\]
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

\[ \text{simpler topology} + \text{harder algebra} \quad \text{or} \quad \text{harder topology} + \text{simpler algebra} \]

Next hour: Hopf algebras as a ‘shadow’ of a three dimensional theory.
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

\[
\begin{align*}
\text{lower dimensional topology} & \quad + \quad \text{harder algebra} & \quad = & \quad \text{higher dimensional topology} & \quad + \quad \text{simpler algebra}
\end{align*}
\]
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

Study topology in terms of algebra

lower dimensional topology + harder algebra

higher dimensional topology + simpler algebra
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

study topology in terms of algebra

lower dimensional topology + harder algebra

higher dimensional topology + simpler algebra

‘outsource’ algebra to topology

Next hour: Hopf algebras as a ‘shadow’ of a three dimensional theory.
A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.

Next hour: Hopf algebras as a ‘shadow’ of a three dimensional theory.
Part 2
2-categories
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow \eta & & \uparrow \epsilon \\
A & \rightarrow & B
\end{array}
\]

object 1-morphism 2-morphism

We can compose 2-morphisms like this:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow \eta & & \uparrow \epsilon \\
A & \rightarrow & B
\end{array}
\]

vertical composition horizontal composition

These are pasting diagrams. The dual diagrams are the graphical calculus.

A 2-category with one object (the 'empty region') is a monoidal category.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow \eta & & \uparrow \epsilon \\
A & \xrightarrow{g} & B
\end{array}
\]

object \quad 1\text{-morphism} \quad 2\text{-morphism}

We can compose 2-morphisms like this:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow \eta & & \uparrow \epsilon \\
A & \xrightarrow{g} & B
\end{array}
\]

vertical composition \quad horizontal composition

These are *pasting diagrams*. 
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\eta & \downarrow & \epsilon \\
A & \xrightarrow{f} & B \\
\end{array} \]

We can compose 2-morphisms like this:

\[ \begin{array}{ccc}
A & \xrightarrow{\epsilon} & B \\
\eta & \downarrow & \epsilon \\
A & \xrightarrow{\eta} & B \\
\end{array} \]

vertical composition

\[ \begin{array}{ccc}
A & \xrightarrow{\eta} & B \\
\epsilon & \downarrow & \epsilon \\
A & \xrightarrow{\epsilon} & C \\
\end{array} \]

horizontal composition

These are pasting diagrams. The dual diagrams are the graphical calculus.
Algebra in the plane = 2-category theory

The language describing algebra in the plane is 2-category theory:

\[
\begin{array}{ccc}
\text{object} & \downarrow f & \text{2-morphism} \\
\eta & \epsilon & \eta \\
\end{array}
\]

We can compose 2-morphisms like this:

vertical composition     horizontal composition
\[
\begin{array}{ccc}
\text{vertical composition} & \downarrow f & \text{horizontal composition} \\
\eta & \epsilon & \eta \\
\end{array}
\]

These are pasting diagrams. The dual diagrams are the graphical calculus. A 2-category with one object (the 'empty region') is a monoidal category.
A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

\[\begin{array}{c}
f^* \quad f \\
\end{array}\quad \begin{array}{c}
f \quad f^* \\
\end{array}\quad \begin{array}{c}
f \quad f^* \\
\end{array}\quad \begin{array}{c}
f^* \quad f \\
\end{array}\]
A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

\[
\begin{align*}
\begin{array}{c}
\scalebox{0.7}{
\includegraphics{diagram1.png}} \\
\scalebox{0.7}{
\includegraphics{diagram2.png}} \\
\scalebox{0.7}{
\includegraphics{diagram3.png}} \\
\scalebox{0.7}{
\includegraphics{diagram4.png}}
\end{array}
\end{align*}
\]

such that the following hold:

\[
\begin{align*}
\begin{array}{c}
\scalebox{0.7}{
\includegraphics{diagram5.png}} \\
\scalebox{0.7}{
\includegraphics{diagram6.png}} \\
\scalebox{0.7}{
\includegraphics{diagram7.png}} \\
\scalebox{0.7}{
\includegraphics{diagram8.png}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\scalebox{0.7}{
\includegraphics{diagram9.png}} \\
\scalebox{0.7}{
\includegraphics{diagram10.png}} \\
\scalebox{0.7}{
\includegraphics{diagram11.png}} \\
\scalebox{0.7}{
\includegraphics{diagram12.png}}
\end{array}
\end{align*}
\]
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane
A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

\[
\begin{align*}
& \begin{array}{ccc}
  & f^* & f \\
  f & & \\
  & f & f^*
\end{array} \\
& \begin{array}{ccc}
  & f & f^* \\
  f & & \\
  & f^* & f
\end{array}
\end{align*}
\]

such that the following hold:

\[
\begin{align*}
& \begin{array}{ccc}
  & f^* & f \\
  f & & \\
  & f & f^*
\end{array} \\
= & \begin{array}{ccc}
  & f & f^* \\
  f & & \\
  & f^* & f
\end{array} \\
= & \begin{array}{ccc}
  & f^* & f \\
  f & & \\
  & f & f^*
\end{array} \\
= & \begin{array}{ccc}
  & f & f^* \\
  f & & \\
  & f^* & f
\end{array}
\end{align*}
\]

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

**Tangle hypothesis.**

$\text{Bord}_{1,0}^{2D} \cong$ free monoidal category on a dualizable object
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

\[ f^* \xrightarrow{f} B \xleftarrow{f} A \]

such that the following hold:

\[ \begin{array}{cccc}
\text{Diagram 1} & \overset{=}{=} & \text{Diagram 2} & \overset{=}{=} \\
\text{Diagram 3} & \overset{=}{=} & \text{Diagram 4} & \overset{=}{=} \\
\end{array} \]

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G$. 

David Reutter
Hopf algebras and 3-categories
August 3, 2017 8 / 34
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals ↔ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G$.

**Example.** $\mathcal{F}_2 \left( \begin{array}{c} \text{Example} \\ \end{array} \right)$: free 2-category on dualizable 1-morphism
A 1-morphism $A \xrightarrow{f} B$ has a dual $B \xrightarrow{f^*} A$ if there are 2-morphisms:

such that the following hold:

**Theorem.** Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

**Definition.** $G$ directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on $G$.

**Example.** $\mathcal{F}_2 \left( \begin{array}{c} f^* \rightarrow f \\ f \rightarrow f^* \end{array} \right) :$ free 2-category on dualizable 1-morphism
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[ \text{such that:} \]

\[
\begin{array}{c}
\text{Frob} \\
\text{Frob} \\
\text{Frob}
\end{array}
\]

\[
\begin{array}{c}
\text{Frob} \\
\text{Frob} \\
\text{Frob}
\end{array}
\]

\[
\begin{array}{c}
\text{Frob} \\
\text{Frob} \\
\text{Frob}
\end{array}
\]

\[
\begin{array}{c}
\text{Frob} \\
\text{Frob} \\
\text{Frob}
\end{array}
\]

Theorem.
This induces a monoidal equivalence
\( \text{Frob} \cong \mathcal{F} \)

open strings in the plane

Frob as a 'shadow' of the theory of dualizable 1-morphisms in 2-categories.
A Frobenius algebra in a monoidal category is an object with morphisms:

\[
\begin{align*}
\text{such that:} & \\
\end{align*}
\]

\[
\begin{align*}
\text{Theorem.} & \\
\end{align*}
\]
A Frobenius algebra in a monoidal category is an object with morphisms:

\[ \text{such that:} \]

\[ \begin{align*}
\text{Frob} & = \text{Frob} \\
= & = \\
= & = \\
= & = \\
= & = \\
= & = \\
\end{align*} \]
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

\[
\begin{align*}
\text{such that:} & \\
\text{Frob} & = \text{Frob} \\
\end{align*}
\]

\[
\begin{align*}
\text{Frob} : & \text{free monoidal category on a Frobenius algebra} \\
\text{Frob} & \rightarrow \text{thickening} \\
\end{align*}
\]

There is a 2-functor \( Frob : \mathcal{C} \rightarrow \mathcal{D} \) for any category \( \mathcal{C} \).

**Theorem.** This induces a monoidal equivalence \( \text{Frob} \sim \mathcal{F} \).

**open strings** in the plane

\( \text{Frob} \) as a 'shadow' of the theory of dualizable 1-morphisms in 2-categories.

David Reutter

Hopf algebras and 3-categories

August 3, 2017 9 / 34
A Frobenius algebra in a monoidal category is an object with morphisms:

\[ \mathsf{Frob} : \text{free monoidal category on a Frobenius algebra} \]

such that:

\[ \mathsf{Frob} \]
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[ \begin{align*}
\text{Frob} : & \text{free monoidal category on a Frobenius algebra} \\
\text{Thickening} : & F_2 := \mathcal{F}_2 \left( \begin{array}{c}
\hspace{1cm} \\
\end{array} \right).
\end{align*} \]
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[
\begin{align*}
\text{Frob}: \text{free monoidal category on a Frobenius algebra}
\end{align*}
\]

There is a 2-functor \( \text{Frob} \xrightarrow{\text{thickening}} F_2 := \mathcal{F}_2 \left( \begin{array}{c}
\includegraphics{frob.png}
\end{array} \right) \).

**Theorem.** This induces a monoidal equivalence \( \text{Frob} \cong F_2 \left( \begin{array}{c}
\includegraphics{frob.png}
\end{array} \right) \).
Frobenius algebras and dualizable 1-morphisms

A Frobenius algebra in a monoidal category is an object with morphisms:

\[ Frob : \text{free monoidal category on a Frobenius algebra} \]

such that:

\[ = = = = = = = = = \]

There is a 2-functor \( Frob \) 'thickening' \( F_2 := \mathcal{F}_2 \left( \begin{array}{c} \square \rightarrow \square \end{array} \right) \).

**Theorem.** This induces a monoidal equivalence \( Frob \cong F_2 (\square, \square) \).
A *Frobenius algebra* in a monoidal category is an object with morphisms:

\[
\begin{align*}
\mathcal{Frob} : \text{free monoidal category on a Frobenius algebra} \\
\end{align*}
\]

such that:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Frob: free monoidal category on a Frobenius algebra}
\end{array}
\end{array}
\end{align*}
\]

There is a 2-functor \( \mathcal{Frob} \to \mathcal{F}_2 \) such that:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Theorem. This induces a monoidal equivalence } \mathcal{Frob} \cong \mathcal{F}_2 (\square, \square).
\end{array}
\end{array}
\end{align*}
\]

\( \mathcal{Frob} \) as a ’shadow’ of the theory of dualizable 1-morphisms in 2-categories.
Other algebraic theories?

What about *commutative* Frobenius algebras

\[ \begin{array}{c}
\includegraphics[width=0.3\textwidth]{commutative_frobenius_algebra_diagram}
\end{array} \]
What about *commutative* Frobenius algebras or *bialgebras*?

\[
\begin{align*}
\text{Diagram 1} &= \text{Diagram 2} \\
\text{Diagram 3} &= \text{Diagram 4}
\end{align*}
\]
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?

Only make sense in (at least) three dimensional space.
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?

Only make sense in (at least) three dimensional space.

⇓

Shadows of 3D structures?
Part 3
3-categories
Algebra in three dimensions $= 3$-category theory

The language describing algebra in three dimensions is 3-category theory:

\[
\begin{array}{c}
\text{object} \\
\text{1-morphism} \\
\text{2-morphism} \\
\text{3-morphism}
\end{array}
\]

We can compose 3-morphisms like this:

\[
\begin{array}{c}
\eta \\
\epsilon \\
\epsilon \\
\eta
\end{array}
\]

vertical composition

horizontal composition

layered composition

A one object (the 'empty region') 3-category is a monoidal 2-category.

A one object and one 1-morphism 3-category is a braided monoidal category.
Algebra in three dimensions $=$ 3-category theory

The language describing algebra in three dimensions is 3-category theory:

\[
\begin{array}{c}
A \\
\end{array}
\]

object
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

- **object**
- **1-morphism**

- **vertical composition**
- **horizontal composition**
- **layered composition**
Algebra in three dimensions $= 3$-category theory

The language describing algebra in three dimensions is *3-category theory*:

```
A
```

```
B
```

```
F
```

```
g
```

We can compose 3-morphisms like this:

- Vertical composition:
  
- Horizontal composition:
  
- Layered composition:

A one object (the ‘empty region’) 3-category is a *monoidal 2-category*.

A one object and one 1-morphism 3-category is a *braided monoidal category*.
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

- **Object**
- **1-morphism**
- **2-morphism**
- **3-morphism**

We can compose 3-morphisms like this:

- Vertical composition
- Horizontal composition
- Layered composition

A one object (the ‘empty region’) 3-category is a monoidal 2-category.

A one object and one 1-morphism 3-category is a braided monoidal category.
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

We can compose 3-morphisms like this:
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

We can compose 3-morphisms like this:
The language describing algebra in three dimensions is 3-category theory:

We can compose 3-morphisms like this:

vertical composition

horizontal composition
The language describing algebra in three dimensions is **3-category theory**:

- **Object**: Single entity.
- **1-Morphism**: Connecting objects.
- **2-Morphism**: Connections between 1-morphisms.
- **3-Morphism**: Connections between 2-morphisms.

We can compose 3-morphisms like this:

- **Vertical composition**
- **Horizontal composition**
- **Layered composition**
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is \textit{3-category theory}:

\begin{align*}
\begin{array}{c}
\text{object} \\
1\text{-morphism} \\
2\text{-morphism} \\
3\text{-morphism}
\end{array}
\end{align*}

We can compose 3-morphisms like this:

\begin{align*}
\begin{array}{c}
\text{vertical composition} \\
\text{horizontal composition} \\
\text{layered composition}
\end{array}
\end{align*}

A one object (the ‘empty region’) 3-category is a \textit{monoidal 2-category}.
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is 3-category theory:

We can compose 3-morphisms like this:

A one object (the ‘empty region’) 3-category is a monoidal 2-category. A one object and one 1-morphism 3-category is a braided monoidal category.
Duals in 3-categories

A 1-morphism $A$ has an *oriented dual* $A^*$ if there are 2-morphisms (*folds*):

\[
\begin{array}{cccc}
A^* & A & A^* & A \\
A^* & A & A^* & A \\
\end{array}
\]
A 1-morphism $A$ has an *oriented dual* $A^*$ if there are 2-morphisms (folds):

and 3-morphisms (*cusps, saddles and births/deaths of the circle*):

+ horizontal and vertical reflections and opposite orientation
A 1-morphism $A$ has an oriented dual $A^*$ if there are 2-morphisms (folds):

and 3-morphisms (cusps, saddles and births/deaths of the circle):

such that the following hold (& reflections and opposite orientation):

David Reutter
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space
**Theorem.** graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

**Tangle hypothesis.**

$\text{Bord}_{2,1,0}^{3D} \cong \text{free monoidal 2-category on a dualizable object}$
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( \begin{array}{c} 2\text{-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \end{array} \right)$
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( \begin{array}{c} 2\text{-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \end{array} \right)$
**Theorem.** graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( \text{2-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \right)$

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$. 

Summary. The graphical calculus of $\mathcal{F}_3(\mathcal{G})$ is given by regions, surfaces and wires in three dimensional space.
**Theorem.** graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left( \begin{array}{c} 2\text{-Edges} \quad \leftrightarrow \quad \text{Edges} \quad \leftrightarrow \quad \text{Vertices} \end{array} \right)$

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$.

**Examples.**

$\mathcal{F}_3 \left( \begin{array}{c} \text{Defect data} \\ 3-2-1-0 \end{array} \right)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3 \left( \begin{array}{c} \text{Defect bordisms} \\ \text{embedded in } \mathbb{R}^3 \end{array} \right)$: free 3-category on \{two dualizable 1-morphisms, one dualizable 2-morphism\}
Duals in 3-categories

**Theorem.** graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $G$ be a 2-globular set $G = \left(\begin{array}{c} 2\text{-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices} \end{array}\right)$

**Def.** $\mathcal{F}_3(G)$: free 3-category with duals for 2- and 1-morphisms given in $G$.

**Examples.**

$\mathcal{F}_3\left(\begin{array}{c} \end{array}\right)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3\left(\begin{array}{c} \end{array}\right)$: free 3-category on \{two dualizable 1-morphisms, one dualizable 2-morphism\}

**Summary.** The graphical calculus of $\mathcal{F}_3(G)$ is given by regions, surfaces and wires in three dimensional space.
Theorem. Graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space.

Let $\mathcal{G}$ be a 2-globular set $\mathcal{G} = \left(\text{2-Edges} \leftrightarrow \text{Edges} \leftrightarrow \text{Vertices}\right)$.

**Def.** $\mathcal{F}_3(\mathcal{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathcal{G}$.

**Examples.**

$\mathcal{F}_3(\mathcal{G})$: free 3-category on a dualizable 1-morphism.

$\mathcal{F}_3(\mathcal{G})$: free 3-category on two dualizable 1-morphisms, one dualizable 2-morphism.

Summary. The graphical calculus of $\mathcal{F}_3(\mathcal{G})$ is given by regions, surfaces and wires in three dimensional space.
A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[
\begin{align*}
\text{Free} \quad & \quad \text{Braided} \quad & \quad \text{Monoidal} \\
\rightarrow & \quad \rightarrow & \quad \rightarrow \\
\end{align*}
\]
A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[
\begin{align*}
\text{cFrob} & = \text{free braided monoidal category on a commutative Frobenius algebra} \\
\end{align*}
\]

**cFrob**: free *braided* monoidal category on a commutative Frobenius algebra
A commutative Frobenius algebra is a Frobenius algebra such that:

\[ \text{cFrob} : \text{free braided monoidal category on a commutative Frobenius algebra} \]

There is a 3-functor \( \text{cFrob} \to \mathcal{F}_3 \) such that:

\[ \text{cFrob} \sim \mathcal{F}_3 \]

\( \text{cFrob} \) as a ‘shadow’ of the theory of dualizable 1-morphisms in 3-categories.
A commutative Frobenius algebra is a Frobenius algebra such that:

\[ \text{cFrob} : \text{free braided monoidal category on a commutative Frobenius algebra} \]

There is a 3-functor \( \text{cFrob} \) 'thickening' \( \rightarrow \) \( \text{F}_3 := \mathcal{F}_3 \left( \begin{array}{cc} \text{ } & \text{ } \\ \text{ } & \text{ } \end{array} \right) \).

**Theorem.** This induces a braided monoidal equivalence \( \text{cFrob} \cong \text{F}_3 \left( \begin{array}{cc} \text{ } & \text{ } \\ \text{ } & \text{ } \end{array} \right) \).
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

\[ \text{cFrob} \]: free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor \( \text{cFrob} \) 'thickening' \( 
\rightarrow 
F_3 := \mathcal{F}_3 \left( \begin{array}{c} \includegraphics[scale=0.5]{example1} \\ \includegraphics[scale=0.5]{example2} \end{array} \right) \).

**Theorem.** This induces a braided monoidal equivalence \( \text{cFrob} \cong F_3(\begin{array}{c} \includegraphics[scale=0.5]{example1} \\ \includegraphics[scale=0.5]{example2} \end{array}) \).

\( \text{cFrob} \) as a 'shadow' of the theory of dualizable 1-morphisms in 3-categories.
Part 4
Hopf algebras
A Hopf algebra in a braided monoidal category is a pair of

an algebra \( \left( \begin{array}{c}
\text{green triangle} \\
, \\
\text{green dot}
\end{array} \right) \)

and

a coalgebra \( \left( \begin{array}{c}
\text{red triangle} \\
, \\
\text{red dot}
\end{array} \right) \)
Hopf algebras

A *Hopf algebra* in a braided monoidal category is a pair of

an algebra \( (\ , \ ) \)  

a coalgebra \( (\ , \ ) \)

that form a *bialgebra*

\[
\begin{align*}
\begin{array}{c}
\text{Hopf algebras}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{and have an antipode } S \text{ fulfilling } S S = S,
\end{align*}
\]

Here, we consider more restrictive algebras.
A Hopf algebra in a braided monoidal category is a pair of

an algebra \( (\begin{array}{c}
\text{\includegraphics{algebra.png}}
\end{array}, \begin{array}{c}
\text{\includegraphics{algebra.png}}
\end{array}) \)

a coalgebra \( (\begin{array}{c}
\text{\includegraphics{coalgebra.png}}
\end{array}, \begin{array}{c}
\text{\includegraphics{coalgebra.png}}
\end{array}) \)

that form a bialgebra

and have an antipode; an endomorphism \( S \) fulfilling

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]

\[ \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} = \begin{array}{c}
\text{\includegraphics{antipode.png}}
\end{array} \]
Hopf algebras

A Hopf algebra in a braided monoidal category is a pair of

an algebra \( (\text{\textbullet}, \text{\textbullet}) \)  
a coalgebra \( (\text{\textbullet}, \text{\textbullet}) \)

that form a bialgebra

\[
\begin{align*}
\begin{tikzpicture}[scale=0.8] 
\draw[thick, green, fill=green] (0,0) circle (0.1); 
\draw[thick, red, fill=red] (1,0) circle (0.1); 
\draw[thick, green, fill=green] (2,0) circle (0.1); 
\draw[thick, red, fill=red] (3,0) circle (0.1); 
\draw[thick, green, fill=green] (-1,0) circle (0.1); 
\draw[thick, red, fill=red] (4,0) circle (0.1); 
\end{tikzpicture}
\end{align*}
\]

and have an antipode; an endomorphism \( S \) fulfilling

\[
\begin{align*}
\begin{tikzpicture}[scale=0.8] 
\draw[thick, green, fill=green] (0,0) circle (0.1); 
\draw[thick, red, fill=red] (1,0) circle (0.1); 
\draw[thick, green, fill=green] (2,0) circle (0.1); 
\draw[thick, red, fill=red] (3,0) circle (0.1); 
\draw[thick, red, fill=red] (0,1) circle (0.1); 
\draw[thick, green, fill=green] (1,1) circle (0.1); 
\draw[thick, red, fill=red] (2,1) circle (0.1); 
\draw[thick, green, fill=green] (3,1) circle (0.1); 
\draw[thick, green, fill=green] (-1,0) circle (0.1); 
\draw[thick, green, fill=green] (4,0) circle (0.1); 
\draw[thick, green, fill=green] (-1,1) circle (0.1); 
\draw[thick, green, fill=green] (4,1) circle (0.1); 
\end{tikzpicture}
\end{align*}
\]

Here, we consider more restrictive algebras.
A unimodular Hopf algebra is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right) \quad \left( \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \right)
\]
Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} \right) \quad \left( \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} \right)
\]

that form a *bialgebra*

\[
\begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} = \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} \quad \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} = \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} \quad \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} = \begin{array}{c}
  \text{,} \\
  \text{,} \\
  \text{,} \\
  \text{,} \\
\end{array} = \begin{array}{c}
\end{array}
\]

The antipode of a unimodular Hopf algebra is

\[ S \]

U. Hopf algebras in \( \text{Vect}_k \) are finite dimensional unimodular Hopf algebras.

Example. Any finite dimensional semisimple and cosemisimple Hopf algebra.

David Reutter

Hopf algebras and 3-categories

August 3, 2017 18 / 34
A *unimodular Hopf algebra* is a pair of Frobenius algebras that form a bialgebra and such that

\[
\begin{align*}
\text{unimodular Hopf algebra} & \text{ is a pair of Frobenius algebras} \\
\begin{pmatrix}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array}
\end{array}
\end{pmatrix}
& \begin{pmatrix}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array}
\end{array}
\end{pmatrix}
\end{align*}
\]

and such that

\[
\begin{align*}
\begin{pmatrix}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array}
\end{array}
\end{pmatrix}
& \begin{pmatrix}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4.png}
\end{array}
\end{array}
\end{pmatrix}
\end{align*}
\]
A *unimodular Hopf algebra* is a pair of Frobenius algebras
\[
\left( \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array} \right), \quad \left( \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array} \right)
\]
that form a *bialgebra* and such that
\[
\text{uHopf}: \text{free braided monoidal category on a unimodular Hopf algebra}
\]
A *unimodular Hopf algebra* is a pair of Frobenius algebras that form a *bialgebra*

\[
\begin{array}{c}
\text{that form a bialgebra}
\end{array}
\]

\[
\begin{array}{c}
\text{and such that}
\end{array}
\]

\[
\begin{array}{c}
\text{uHopf: free braided monoidal category on a unimodular Hopf algebra}
\end{array}
\]

**Theorem.** The antipode of a unimodular Hopf algebra is

\[
\begin{array}{c}
\text{The antipode of a unimodular Hopf algebra is}
\end{array}
\]
Unimodular Hopf algebras

A unimodular Hopf algebra is a pair of Frobenius algebras

\[
\left( \begin{array}{cccc}
\alpha & \beta & \gamma & \delta
\end{array} \right)
\quad \left( \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} \right)
\]

that form a bialgebra

\[
\begin{align*}
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta
\end{array} &= \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array}
\end{align*}
\]

and such that

\[
\begin{align*}
\begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} &= \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array}
\end{align*}
\]

\textbf{uHopf}: free braided monoidal category on a unimodular Hopf algebra

\textbf{Theorem}. The antipode of a unimodular Hopf algebra is \( S = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} = \begin{array}{cccc}
\wp & \eta & \xi & \iota
\end{array} \). 

U. Hopf algebras in \textbf{Vect}_k are finite dimensional unimodular Hopf algebras.
Unimodular Hopf algebras

A **unimodular Hopf algebra** is a pair of Frobenius algebras

\[
\left( \begin{array}{c}
\text{left side}
\end{array} \right) \quad \left( \begin{array}{c}
\text{right side}
\end{array} \right)
\]

that form a **bialgebra**

\[
\begin{array}{c}
\text{diagram 1}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 2}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 3}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 4}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 5}
\end{array}
\]

and such that

\[
\begin{array}{c}
\text{diagram 6}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 7}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 8}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 9}
\end{array}
\quad = \quad \begin{array}{c}
\text{diagram 10}
\end{array}
\]

**uHopf**: free braided monoidal category on a unimodular Hopf algebra

**Theorem.** The antipode of a unimodular Hopf algebra is

\[
S = \begin{array}{c}
\text{diagram 11}
\end{array}
\]

U. Hopf algebras in \textbf{Vect}_k are **finite dimensional unimodular Hopf algebras**.

**Example.** Any finite dimensional semisimple and cosemisimple Hopf algebra.
A topological 3-category

Start with $\mathcal{F}_3$:

Definition. $\mathcal{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

Explicitly, invertibility of the saddles means:

This 3-category is new.
A topological 3-category

Start with $\mathcal{F}_3$:

free 'topological' 3-category on
- a blue surface
- a red surface
- a blue-red boundary wire
A topological 3-category

Start with $\mathcal{F}_3$:

free 'topological' 3-category on
- a blue surface
- a red surface
- a blue-red boundary wire

**Definition.** $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

is inverse to

and

$=$
A topological 3-category

Start with $\mathcal{F}_3$:

free ‘topological’ 3-category on
- a blue surface
- a red surface
- a blue-red boundary wire

**Definition.** $\mathcal{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

is inverse to and

Explicitly, invertibility of the saddles means:
A topological 3-category

Start with $\mathcal{F}_3$:

free ‘topological’ 3-category on
- a blue surface
- a red surface
- a blue-red boundary wire

**Definition.** $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

- is inverse to
- and

Explicitly, invertibility of the saddles means:

- $=$
- $=$
- $=$
- $=$
A Hopf algebra in $\mathcal{H}$

There is a Hopf algebra in $\mathcal{H}$. 

\[
\left(\begin{array}{c}

\end{array}\right)
\left(\begin{array}{c}

\end{array}\right)
\]
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:
There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:
A Hopf algebra in $\mathcal{H}$

There is a Hopf algebra in $\mathcal{H}$. It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:

$$\left(\begin{array}{cccc} \text{green}, & \text{green}, & \text{green}, & \text{green} \\ \text{red}, & \text{red}, & \text{red}, & \text{red} \end{array}\right)$$
There is a Hopf algebra in $H$. It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$. It lives on the following ‘thickened’ wire:

The two interacting Frobenius structures are:

\[
\left(\begin{array}{c}
\left(\begin{array}{c}
A, \\
Y
\end{array}\right), \\
\end{array}\right)
\]
Let’s check (some of) the axioms of unimodular Hopf algebras:
Let’s check (some of) the axioms of unimodular Hopf algebras:
A Hopf algebra in $H$

Let’s check (some of) the axioms of unimodular Hopf algebras:
A Hopf algebra in $\mathbb{H}$

Let’s check (some of) the axioms of unimodular Hopf algebras:
A Hopf algebra in $H$

Let’s check (some of) the axioms of unimodular Hopf algebras:
The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $H$

The bialgebra laws correspond to the invertibility of the saddle:

\[ = \]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:

\[ = \quad = \]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:

![Diagram](attachment:hopf_diagram.png)
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:

\[
\begin{align*}
\text{Definition.} & \quad H \text{ is this 3-category with the additional restriction that the} \\
& \text{following two saddles are inverse to each other:} \\
& \text{Explicitly, this means:} \\
& \text{This 3-category is new.}
\end{align*}
\]
A Hopf algebra in \( \mathbb{H} \)

The bialgebra laws correspond to the invertibility of the saddle:

\[
\begin{align*}
\text{Definition.} & \quad \mathbb{H} \text{ is this 3-category with the additional restriction that the following two saddles are inverse to each other:} \\
& \quad \text{Explicitly, this means:} \\
& \quad \text{This 3-category is new.}
\end{align*}
\]
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertibility of the saddle:
Summary. \texttt{uHopf} is a shadow of a simpler 3-category.
A Hopf algebra in $\mathbb{H}$

Summary. $\text{uHopf}$ is a shadow of a simpler 3-category.

A unimodular Hopf algebra is a pair of Frobenius algebras

\[
\left( \begin{array}{cc}
\begin{array}{c}
\begin{array}{c}
\text{bialgebra}
\end{array}
\end{array}
\end{array} \right)
\]

that form a bialgebra

\[
\begin{array}{c}
\begin{array}{c}
\text{and such that}
\end{array}
\end{array}
\]

$\Rightarrow$ The antipode is an algebra antihomomorphism.

$\Rightarrow$ In a unimodular Hopf algebra, the antipode squares to the twist.

In particular, in a symmetric monoidal category, its 4th power is trivial.
A Hopf algebra in \( \mathbb{H} \)

**Summary.** \( u\text{Hopf} \) is a shadow of a simpler 3-category.

\( \mathbb{H} \) is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

- is inverse to
- and

\[ \begin{array}{c}
\text{is inverse to} \\
\text{and} \\
= \\
\end{array} \]

In a unimodular Hopf algebra, the antipode squares to the twist. In particular, in a symmetric monoidal category, its 4th power is trivial.
Summary. uHopf is a shadow of a simpler 3-category.
Formally, we have defined a 3-functor $u\text{Hopf} \to \mathcal{H}$.
Summary. $\mathfrak{u}\text{Hopf}$ is a shadow of a simpler 3-category. Formally, we have defined a 3-functor $\mathfrak{u}\text{Hopf} \rightarrow \mathbb{H}$.

Conjecture. This induces a braided equivalence $\mathfrak{u}\text{Hopf} \cong \mathbb{H}\left(\begin{array}{c} \bullet \\ \circ \\ \circ \end{array}\right)$.
A Hopf algebra in $\mathbb{H}$

**Summary.** $u\text{Hopf}$ is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor $u\text{Hopf} \rightarrow \mathbb{H}$.

**Conjecture.** This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}(\text{,})$.

Several Hopf algebraic calculations simplify in this 3D model.
Summary. \textbf{uHopf} is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor \textbf{uHopf} \rightarrow \mathbb{H}.

Conjecture. This induces a braided equivalence \textbf{uHopf} \cong \mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}(\mathbb{H}else:\text{In a unimodular Hopf algebra, the antipode squares to the twist.}

In particular, in a symmetric monoidal category, its 4th power is trivial.

Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:
A Hopf algebra in $\mathbb{H}$

Summary. $u\text{Hopf}$ is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor $u\text{Hopf} \rightarrow \mathbb{H}$.

Conjecture. This induces a braided equivalence $u\text{Hopf} \cong \mathbb{H}(\cdot, \cdot)$.

Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:

⇒ The antipode is an algebra antihomomorphism.

$\Rightarrow$
A Hopf algebra in $\mathbb{H}$

**Summary.** $uHopf$ is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor $uHopf : H \to H$.

**Conjecture.**

This induces a braided equivalence $uHopf \sim H$.

Several Hopf algebraic calculations simplify in this 3D model.

For example, the antipode is the half twist:

$\Rightarrow$ The antipode is an algebra antihomomorphism.
**Summary.** \(u\text{Hopf}\) is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor \(u\text{Hopf} \rightarrow \mathbb{H}\).

**Conjecture.** This induces a braided equivalence \(u\text{Hopf} \simeq \mathbb{H}(\overrightarrow{\bullet}, \overrightarrow{\bullet})\).

Several Hopf algebraic calculations simplify in this 3D model. For example, the antipode is the half twist:

\[\Rightarrow\text{ The antipode is an algebra antihomomorphism.}\]
\[\Rightarrow\text{ In a unimodular Hopf algebra, the antipode squares to the twist.}\]

In particular, in a symmetric monoidal category, its 4th power is trivial.
Part 5
Higher linear algebra
Representations

So far: algebraic structures in terms of generators & relations
Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories
Representations

So far: algebraic structures in terms of generators & relations
Now: \textit{representations} - instances of these structures in concrete categories
U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \rightarrow \mathcal{C}$
Representations

So far: algebraic structures in terms of generators & relations
Now: \textit{representations} - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors \textit{uHopf} $\to \mathcal{C}$

\textit{Linear representations} - representation functors with target \textit{Vect}
Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \to \mathcal{C}$

*Linear representations* - representation functors with target $\text{Vect}$

What are the appropriate *linear* targets for higher categorical theories?
Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $C$: braided monoidal functors $\text{uHopf} \to C$

*Linear representations* - representation functors with target $\text{Vect}$

What are the appropriate *linear* targets for higher categorical theories?

*Expectations:*

- symmetric monoidal $n$-categories $n\text{Vect}$ categorifying $\text{Vect}$
Representations

So far: algebraic structures in terms of generators & relations
Now: **representations** - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \to \mathcal{C}$

**Linear representations** - representation functors with target $\text{Vect}$

What are the appropriate *linear* targets for higher categorical theories?

**Expectations:**
- symmetric monoidal $n$-categories $\mathbf{nVect}$ categorifying $\text{Vect}$
- recover $\mathbf{nVect}$ from $(n+1)\text{Vect}$: $(n+1)\text{Vect}(I, I) \cong \mathbf{nVect}$
Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $u\text{Hopf} \to \mathcal{C}$

*Linear representations* - representation functors with target $\text{Vect}$

What are the appropriate *linear* targets for higher categorical theories?

*Expectations:*

- symmetric monoidal $n$-categories $\text{nVect}$ categorifying $\text{Vect}$
- recover $\text{nVect}$ from $(n + 1)\text{Vect}$: $(n + 1)\text{Vect}(I, I) \cong \text{nVect}$

$\text{nVect}$ a ‘shadow’ of $(n + 1)\text{Vect} \iff (n + 1)\text{Vect}$ a ‘thickening’ of $\text{nVect}$

\[\text{Diagram}\]

\[\text{Diagram}\]

\[\text{Diagram}\]
### Higher linear algebra

<table>
<thead>
<tr>
<th>Objects</th>
<th>Morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vect</td>
<td>f.d. vector spaces</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals.

3Vect(I, I) ∼ = 2Vect(I, I) ∼ = Vect(I, I) = C

Various generalizations are possible.
**Higher linear algebra**

<table>
<thead>
<tr>
<th></th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vect</td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
<td></td>
</tr>
<tr>
<td>2Vect</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals.

$$3Vect((I, I)) \cong 2Vect \cong Vect((I, I)) = C$$

Various generalizations are possible.
Higher linear algebra

<table>
<thead>
<tr>
<th></th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vect</strong></td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
<td></td>
</tr>
<tr>
<td><strong>2Vect</strong></td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals.

\[3\text{Vect}(\mathcal{I}, \mathcal{I}) \cong 2\text{Vect}(\mathcal{I}, \mathcal{I}) \cong \text{Vect}(\mathcal{I}, \mathcal{I}) = \mathbb{C}\]

Various generalizations are possible.
### Higher linear algebra

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vect</strong></td>
<td></td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
<td></td>
</tr>
<tr>
<td><strong>2Vect</strong></td>
<td>![Rep(-)]</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td>![Rep(-)]</td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
</tbody>
</table>
### Higher linear algebra

<table>
<thead>
<tr>
<th></th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vect</td>
<td></td>
<td></td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
</tr>
<tr>
<td>2Vect</td>
<td></td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>f.d. semisimple algebras</td>
<td></td>
<td>intertwiners</td>
</tr>
<tr>
<td>3Vect</td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
<td>natural transformations</td>
</tr>
</tbody>
</table>
### Higher linear algebra

<table>
<thead>
<tr>
<th>tensor unit</th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vect</strong></td>
<td>$\mathbb{C}$</td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
<td></td>
</tr>
<tr>
<td><strong>2Vect</strong></td>
<td>$\mathbb{C}$</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{C}$</td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
<tr>
<td><strong>3Vect</strong></td>
<td>$\mathbb{C}$</td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals. Various generalizations are possible.
### Higher linear algebra

<table>
<thead>
<tr>
<th></th>
<th>tensor unit</th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>complex numbers</td>
</tr>
<tr>
<td>Vect</td>
<td>$\mathbb{C}$</td>
<td></td>
<td>f.d. vector spaces</td>
<td></td>
<td>linear maps</td>
</tr>
<tr>
<td>2Vect</td>
<td>Vect</td>
<td>$\text{Rep}(-)$</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{C}$</td>
<td>$\cong$</td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
<tr>
<td>3Vect</td>
<td>Vect</td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
<td>natural transformations</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals.

$3\text{Vect}(\mathbb{I}, \mathbb{I}) \cong 2\text{Vect}(\mathbb{I}, \mathbb{I}) \cong \text{Vect}(\mathbb{I}, \mathbb{I}) = \mathbb{C}$

Various generalizations are possible.

---

David Reutter

Hopf algebras and 3-categories

August 3, 2017 26 / 34
### Higher linear algebra

<table>
<thead>
<tr>
<th>tensor unit</th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C</strong></td>
<td>1</td>
<td></td>
<td></td>
<td>complex numbers</td>
</tr>
<tr>
<td><strong>Vect</strong></td>
<td><strong>C</strong></td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
<td></td>
</tr>
<tr>
<td>2<strong>Vect</strong></td>
<td><strong>Vect</strong></td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td></td>
<td><strong>C</strong></td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
<tr>
<td>3<strong>Vect</strong></td>
<td><strong>Vect</strong></td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
</tr>
</tbody>
</table>

Various generalizations are possible.

They are *symmetric monoidal* 1-, 2- and 3-categories with duals.
## Higher linear algebra

<table>
<thead>
<tr>
<th>tensor unit</th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>1</td>
<td></td>
<td></td>
<td>complex numbers</td>
</tr>
<tr>
<td>$\text{Vect}$</td>
<td>$\mathbb{C}$</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td>$2\text{Vect}$</td>
<td>$\text{Rep}(-)$</td>
<td>$\mathbb{C}$</td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
</tr>
<tr>
<td>$3\text{Vect}$</td>
<td>$\text{Vect}$</td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
</tr>
</tbody>
</table>

They are *symmetric monoidal* 1-, 2- and 3-categories with duals.

$$3\text{Vect}(I, I) \cong 2\text{Vect} \quad 2\text{Vect}(I, I) \cong \text{Vect} \quad \text{Vect}(I, I) = \mathbb{C}$$
### Higher linear algebra

<table>
<thead>
<tr>
<th></th>
<th>tensor unit</th>
<th>objects</th>
<th>1-morphisms</th>
<th>2-morphisms</th>
<th>3-morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>complex numbers</td>
</tr>
<tr>
<td>$\text{Vect}$</td>
<td>$\mathbb{C}$</td>
<td></td>
<td></td>
<td>f.d. vector spaces</td>
<td>linear maps</td>
</tr>
<tr>
<td>$2\text{Vect}$</td>
<td>$\text{Vect}$</td>
<td>$\text{Rep}(-)$</td>
<td>finite semisimple categories</td>
<td>linear functors</td>
<td>natural transformations</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td></td>
<td></td>
<td>f.d. semisimple algebras</td>
<td>f.d. bimodules</td>
<td>intertwiners</td>
</tr>
<tr>
<td>$3\text{Vect}$</td>
<td>$\text{Vect}$</td>
<td>fusion categories</td>
<td>finite semisimple bimodule categories</td>
<td>intertwining functors</td>
<td>natural transformations</td>
</tr>
</tbody>
</table>

They are symmetric monoidal 1-, 2- and 3-categories with duals.

$$3\text{Vect}(I, I) \cong 2\text{Vect} \quad 2\text{Vect}(I, I) \cong \text{Vect} \quad \text{Vect}(I, I) = \mathbb{C}$$

Various generalizations are possible.
The 3-category $\mathbf{3Vect}$

<table>
<thead>
<tr>
<th>C</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>

fusion category
The 3-category $\mathbf{3Vect}$

- Fusion category $C$
- Bimodule category $M$
- Intertwining functor $C \to M$
- Relative Deligne product $M \mathbin{\bowtie} C$
- Natural transformation $\eta$

$\Rightarrow$

$\mathbf{3Vect} :$

- A fusion category $C$
- Two right $C$-module categories $M, N$
- An intertwining functor $M \to N$

David Reutter
Hopf algebras and 3-categories
August 3, 2017 27 / 34
The 3-category $\mathbf{3Vect}$

- **Fusion category**
- **Bimodule category**
- **Intertwining functor**

A fusion category $C$, two right $C$-module categories $M$, $N$, and an intertwining functor $F : M \to N$ relative Deligne product $M \bowtie_C N \to \mathbf{3Vect}$.
The 3-category $\mathbf{3Vect}$

- **Fusion category**
- **Bimodule category**
- **Intertwining functor**
- **Natural transformation**

$\mathbf{3Vect}$: a fusion category $\mathcal{C}$, two right $\mathcal{C}$-module categories $\mathcal{M}, \mathcal{N}$, an intertwining functor $\mathcal{M} \rightarrow \mathcal{N}$, and a natural transformation $\eta$. The $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ are related by the relative Deligne product $\mathcal{M} \sqcup^\mathcal{C} \mathcal{N} \rightarrow \mathbf{3Vect}$. 

David Reutter

Hopf algebras and 3-categories
The 3-category $\mathbf{3Vect}$

- Fusion category
- Bimodule category
- Intertwining functor
- Natural transformation

Right $C$-module $\mathcal{M}$
The 3-category $3\text{Vect}$

- Fusion category
- Bimodule category
- Intertwining functor
- Natural transformation

Right $C$-module $\mathcal{M}$  
Left $C$-module $\mathcal{N}$
The 3-category $\mathbf{3Vect}$

- fusion category
- bimodule category
- intertwining functor
- natural transformation

- right $C$-module $\mathcal{M}$
- left $C$-module $\mathcal{N}$

relative Deligne product $\mathcal{M} \boxtimes_C \mathcal{N}$
The 3-category $\mathbf{3Vect}$

- **Fusion category**
- **Bimodule category**
- **Intertwining functor**
- **Natural transformation**

**Relative Deligne product:** universal for $C$-bilinear functors out of $M \times N$.
The 3-category $\mathbf{3Vect}$

- **fusion category**
- **bimodule category**
- **intertwining functor**
- **natural transformation**

- right $C$-module $\mathcal{M}$
- left $C$-module $\mathcal{N}$

$$\Rightarrow$$

**relative Deligne product** $\mathcal{M} \boxtimes_C \mathcal{N}$

$$\mathcal{F}_3 \left( \begin{array}{ccc} \mathcal{C} & \mathcal{M} & \mathcal{D} \\ \mathcal{C} & \mathcal{N} & \mathcal{D} \end{array} \right) \rightarrow \mathbf{3Vect}$$
The 3-category $\mathbf{3Vect}$

$\xymatrix{\mathcal{C} \ar[rr] & & \mathcal{D} \ar[rr] & & \mathcal{M} \ar[rr] & & \mathcal{C} \ar[rr] & & \mathcal{D} \ar[rr] & & \mathcal{M} \ar[rr] & & \mathcal{N} \ar[rr] & & \mathcal{F} \ar[rr] & & \eta \ar[rr] & & \mathcal{G}}$

- fusion category
- bimodule category
- intertwining functor
- natural transformation

right $\mathcal{C}$-module $\mathcal{M}$ left $\mathcal{C}$-module $\mathcal{N}$ relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

$\mathcal{F}_3 \left( \begin{array}{c} \mathcal{C} \\ \mathcal{M} \\ \mathcal{N} \end{array} \right) \rightarrow \mathbf{3Vect} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{two right } \mathcal{C}\text{-module categories } \mathcal{M}, \mathcal{N} \\ \text{an intertwining functor } \mathcal{M} \rightarrow \mathcal{N} \end{cases}$
This 3-functor factors through $\mathbb{H}$ if the following hold:

\[
\begin{array}{c}
\text{is inverse to} \\
& \text{and}
\end{array}
\]

In other words, $M \boxtimes C \cong N^\sim$ is an adjoint equivalence.
This 3-functor factors through $\mathbb{H}$ if the following hold:

- is inverse to

and

$\Rightarrow$ is isomorphic to
This 3-functor factors through $\mathbb{H}$ if the following hold:

1. is inverse to

2. $\Rightarrow$ is isomorphic to

and

3. $\Rightarrow$ is isomorphic to

Data of a 3-functor $\mathbb{H} \to 3\text{Vect}$:

- a fusion category $\mathbb{C}$
- a left and a right module category $\mathbb{M}$, $\mathbb{N}$
- an adjoint equivalence $\mathbb{M} \triangleright \mathbb{C} \mathbb{N}^{-} \to \text{Vect}$

If $\mathbb{M}$ is the regular module $\mathbb{C}$, then $\mathbb{C} \triangleright \mathbb{C} \mathbb{N} \cong \mathbb{N}$.

A $\mathbb{C}$-module structure on $\text{Vect}$ is the same as a monoidal functor $\mathbb{C}^{-} \to \text{Vect}$.
This 3-functor factors through $\mathbb{H}$ if the following hold:

\[
\Rightarrow \quad \text{is inverse to} \quad \quad \text{and} \quad \quad =
\]

\[
\Rightarrow \quad \text{is isomorphic to} \quad \quad \text{and} \quad \quad \text{is isomorphic to}
\]

In other words, $\mathcal{M} \boxtimes_C \mathcal{N} \to \textbf{Vect}$ is an adjoint equivalence!
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

\[ \text{is inverse to} \quad \Rightarrow \quad \text{is isomorphic to} \quad \text{and} \quad \text{is isomorphic to} \]

In other words, $\mathbb{H} \to 3\text{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathbb{H} \to 3\text{Vect}$:

- a fusion category $\mathcal{C}$
- a left and a right module category $\mathcal{M}, \mathcal{N}$
- an adjoint equivalence $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \text{Vect}$
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathcal{H}$ if the following hold:

$\Rightarrow$ is inverse to $\Rightarrow = \Rightarrow$

$\Rightarrow$ is isomorphic to $\Rightarrow$ and $\Rightarrow$ is isomorphic to $\Rightarrow$

In other words, $\Rightarrow : \mathcal{M} \boxtimes \mathcal{C} \mathcal{N} \rightarrow \text{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathcal{H} \rightarrow 3\text{Vect}$:

- a fusion category $\mathcal{C}$
- a left and a right module category $\mathcal{M}, \mathcal{N}$
- an adjoint equivalence $\mathcal{M} \boxtimes \mathcal{C} \mathcal{N} \rightarrow \text{Vect}$

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes \mathcal{C} \mathcal{N} \cong \mathcal{N}$
Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

$$\Rightarrow \quad \text{is inverse to} \quad \text{and} \quad \text{is isomorphic to}$$

In other words, $\mathbb{H} : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \text{Vect}$ is an \textit{adjoint equivalence}!

Data of a 3-functor $H : 3\text{Vect} \to \mathcal{C}$:

\begin{align*}
\{ & \text{a fusion category } \mathcal{C} \\
& \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\
& \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \text{Vect} \}
\end{align*}

If $\mathcal{M}$ is the \textit{regular} module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \text{Vect}$. 
This 3-functor factors through $\mathcal{H}$ if the following hold:

$\Rightarrow$ is inverse to $\Rightarrow$ and $\Rightarrow$ is isomorphic to $\Rightarrow$ and $\Rightarrow$ is isomorphic to $\Rightarrow$

In other words, $\Rightarrow : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \rightarrow \textbf{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathcal{H} \rightarrow 3\textbf{Vect}$:

- a fusion category $\mathcal{C}$
- a left and a right module category $\mathcal{M}, \mathcal{N}$
- an adjoint equivalence $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \rightarrow \textbf{Vect}$

If $\mathcal{M}$ is the regular module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \textbf{Vect}$.

A $\mathcal{C}$-module structure on $\textbf{Vect}$ is the same as a monoidal functor $\mathcal{C} \rightarrow \textbf{Vect}$.
This 3-functor factors through $\mathbb{H}$ if the following hold:

- $\mathbb{H}$ is inverse to $\mathbb{H}$
- $\mathbb{H} = \emptyset$

$\Rightarrow$ $\mathbb{H}$ is isomorphic to $\emptyset$ and $\mathbb{H}$ is isomorphic to $\emptyset$.

In other words, $\mathbb{H} : M \boxtimes_C N \to \text{Vect}$ is an adjoint equivalence!

Data of a 3-functor $\mathbb{H} : 3\text{Vect} \to 3\text{Vect}$:

- a fusion category $C$
- a left and a right module category $M, N$
- an adjoint equivalence $M \boxtimes_C N \to \text{Vect}$

If $M$ is the regular module $C$, then $C \boxtimes_C N \cong N \cong \text{Vect}$.

A $C$-module structure on $\text{Vect}$ is the same as a monoidal functor $C \to \text{Vect}$.

Data of a 3-functor $\mathbb{H} : 3\text{Vect}$ with $M = C$:

- a fusion category $C$
- a monoidal functor $C \to \text{Vect}$
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$
Tannaka reconstruction

Given a fusion category $C$ with a monoidal functor $C \xrightarrow{F} \text{Vect}$, the following vector space is a unimodular Hopf algebra:

![Diagram](image-url)

This is a version of Tannaka reconstruction: If $C = \text{Rep}(H)$ and $\text{forget} : C \rightarrow \text{Vect}$, this recovers the Hopf algebra $H$. Conversely, any fusion category with fibre functor $C \rightarrow \text{Vect}$ is of the form $\text{Rep}(H)$ with $H$ constructed as above.

Proof. Follows from an old result of M. M"uger.

Question. Is there a completely graphical proof, independent of the target $3\text{Vect}$?
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$
⇒ The following vector space is a unimodular Hopf algebra:

\[
\begin{array}{c}
\text{a scalar 2-morphism in } 3\text{Vect} \\
= \text{a vector space}
\end{array}
\]
Tannaka reconstruction

Given a fusion category $C$ with a monoidal functor $C \xrightarrow{F} \text{Vect}$
⇒ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $C = \text{Rep}(H) \xrightarrow{\text{forget}} \text{Vect}$, this recovers the Hopf algebra $H$. 
Tannaka reconstruction

Given a fusion category $C$ with a monoidal functor $C \xrightarrow{F} \text{Vect}$ ⇒ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $C = \text{Rep}(H) \xrightarrow{\text{forget}} \text{Vect}$, this recovers the Hopf algebra $H$.

Conversely, any fusion category with fibre functor $C \rightarrow \text{Vect}$ is of the form $\text{Rep}(H)$ with $H$ constructed as above.

Proof. Follows from an old result of M. Müger.

Question. Is there a completely graphical proof, independent of the target $3\text{Vect}$?
Tannaka reconstruction

Given a fusion category $C$ with a monoidal functor $C \xrightarrow{F} \text{Vect}$

$\Rightarrow$ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:
If $C = \text{Rep}(H) \xrightarrow{\text{forget}} \text{Vect}$, this recovers the Hopf algebra $H$.

Conversely, any fusion category with fibre functor $C \rightarrow \text{Vect}$ is of the form $\text{Rep}(H)$ with $H$ constructed as above.

Proof. Follows from an old result of M. Müger.\textsuperscript{1}

\textsuperscript{1}Theorem 6.20 in [Müger, From subfactors to categories and topology I, 2003]
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \text{Vect}$

⇒ The following vector space is a unimodular Hopf algebra:

This is a version of Tannaka reconstruction:

If $\mathcal{C} = \text{Rep}(H)$

Conversely, a $\text{Rep}(H)$ with

\[ H. \]

is of the form $\mathcal{C}$

\[ \text{Question.} \]

Is there a completely graphical proof, independent of the target $\text{3Vect}$?

\[ \text{Proof.} \] Follows from an old result of M. Müger.\(^1\)

\[ \text{1Theorem 6.20 in [Müger, From subfactors to categories and topology I, 2003]} \]
Part 6
Lattice models
Lattice models and $\textbf{3Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs
Lattice models and $\textbf{3Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$
Lattice models and $3\text{Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$
Lattice models and $\mathbf{3Vect}$

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix}
\]

= 

\[
\begin{array}{c}
\mathcal{N} \\
\mathcal{M} \\
\mathcal{M} \\
\mathcal{N}
\end{array}
\]

$\mathcal{N} = \mathbf{Vect}_{\mathbb{Z}_2}$

two possible boundaries:
smooth and rough

$\mathbf{Vect}_{\mathbb{Z}_2}$ and $\mathbf{Vect}_{\mathbb{Z}_2}$
Lattice models and \textbf{3Vect}

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space

\[
\begin{pmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

$=$

\[
\begin{pmatrix}
\mathcal{N} \\
\mathcal{M} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mathcal{M} \\
\mathcal{N} \\
\end{pmatrix}
\]

a scalar 2-morphism in \textbf{3Vect}

$=$ a vector space
Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space $\begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \mathbb{Vect}_{\mathbb{Z}_2}$

surface codes $\mathcal{C} = \mathbb{Vect}_{\mathbb{Z}_2}$
Lattice models and 3Vect

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

<table>
<thead>
<tr>
<th>Surface codes</th>
<th>$\mathcal{C} = \text{Vect}_{\mathbb{Z}_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>two possible boundaries: smooth and rough</td>
<td>two module categories: $\text{Vect}_{\mathbb{Z}_2}$ and Vect</td>
</tr>
</tbody>
</table>
Lattice surgery

topologically protected operations on surface codes via
splitting or merging of lattices along smooth or rough boundaries²

²[Horsman et al., Surface code quantum computing by lattice surgery, 2012]
Lattice surgery

topologically protected operations on surface codes via splitting or merging of lattices along smooth or rough boundaries

\[^2\text{[Horsman et al., Surface code quantum computing by lattice surgery, 2012]}\]
Lattice surgery

topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries\(^2\)

\(^2\)[Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]
Lattice surgery

topologically protected operations on surface codes via \textit{splitting} or \textit{merging} of lattices along smooth or rough boundaries\textsuperscript{2}

\textsuperscript{2}[Horsman et al., \textit{Surface code quantum computing by lattice surgery}, 2012]
Lattice surgery

topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries\(^2\)

\(^2\)[Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]
Many open questions:

- Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
- Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
- For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\hat{\mathcal{C}}$. Is something similar true for Hopf algebras?
- ...
Many open questions:

- Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
- Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
- For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\hat{\mathcal{C}}$. Is something similar true for Hopf algebras?
- ...

Maybe most interestingly:

For a defect TQFT, what is the physical meaning of the conditions:
Many open questions:

- Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
- Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
- For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \rightarrow \hat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\hat{\mathcal{C}}$. Is something similar true for Hopf algebras?
- ... 

Maybe most interestingly:

For a defect TQFT, what is the physical meaning of the conditions:

Thanks for listening!

David Reutter  Hopf algebras and 3-categories  August 3, 2017  33 / 34
Weak Hopf algebras

If we *drop* the second condition

![Diagram of weak Hopf algebra]

In fact, every fusion category induces such a functor. The corresponding Hopf algebra coincides with the Kitaev-Kong construction.
Weak Hopf algebras

If we *drop* the second condition

![Diagram]

we only obtain a *weak* Hopf algebra on

![Diagram]

but have more functors \( \mathbb{H} \to 3\text{Vect} \).
Weak Hopf algebras

If we *drop* the second condition

we only obtain a *weak* Hopf algebra on

but have more functors $\mathbb{H} \to 3\text{Vect}$. In fact, *every* fusion category induces such a functor.
Weak Hopf algebras

If we *drop* the second condition

we only obtain a *weak* Hopf algebra on

but have more functors \( \mathbb{H} \to 3\text{Vect} \).

In fact, *every* fusion category induces such a functor. The corresponding Hopf algebra coincides with the Kitaev-Kong construction.