Frobenius algebras, Hopf algebras and 3-categories

David Reutter

University of Oxford

Hopf algebras in Kitaev's quantum double models
Perimeter Institute, Canada

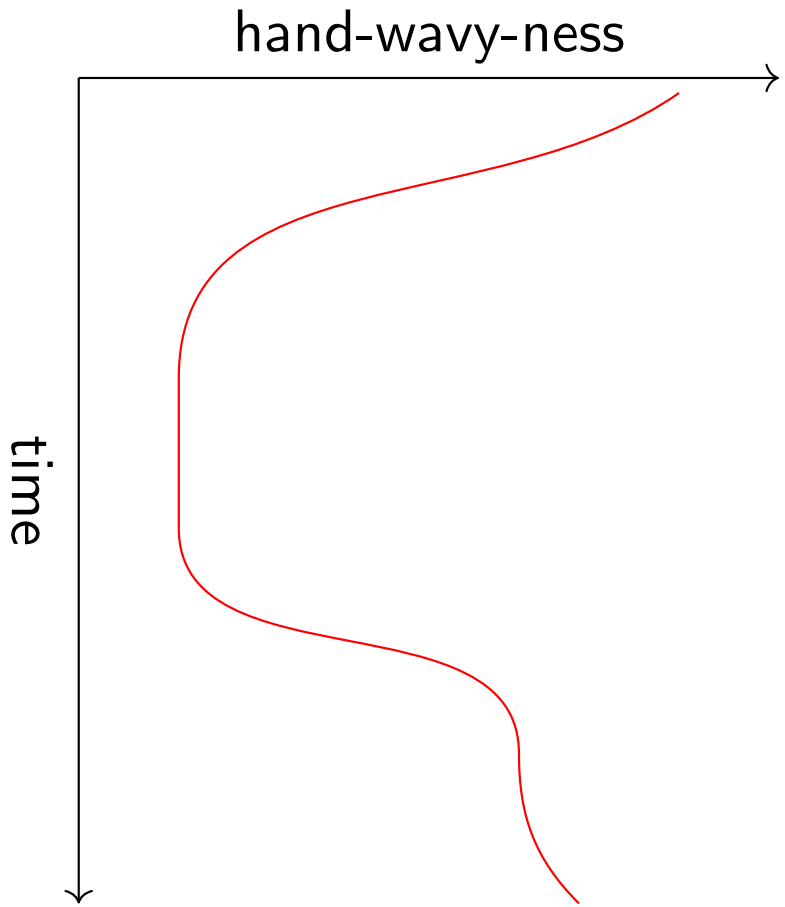
August 3, 2017

The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models

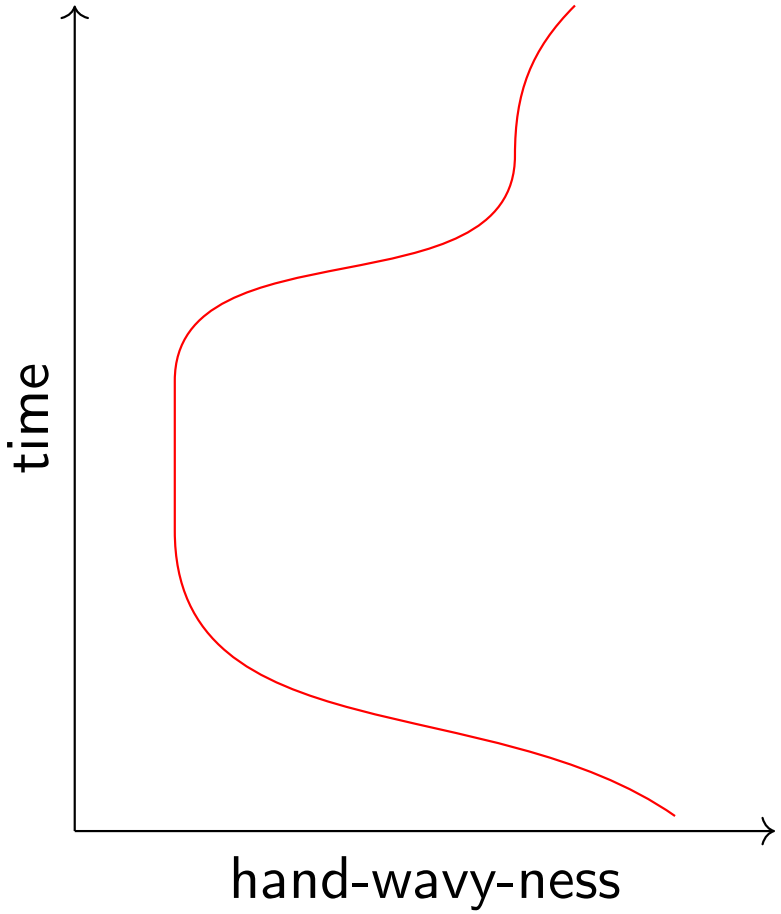
The plan

- **Part 1.** Motivation
- **Part 2.** 2-categories
- **Part 3.** 3-categories
- **Part 4.** Hopf algebras
- **Part 5.** Higher linear algebra
- **Part 6.** Lattice models



The plan

- **Part 6.** Lattice models
- **Part 5.** Higher linear algebra
- **Part 4.** Hopf algebras
- **Part 3.** 3-categories
- **Part 2.** 2-categories
- **Part 1.** Motivation



Part 1
Motivation

What is higher algebra?

- Ordinary algebra lets us compose along a line:

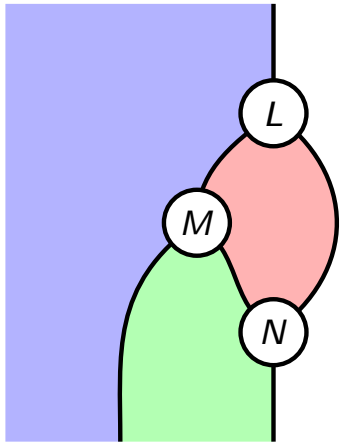
$$xy^2zyx^3$$

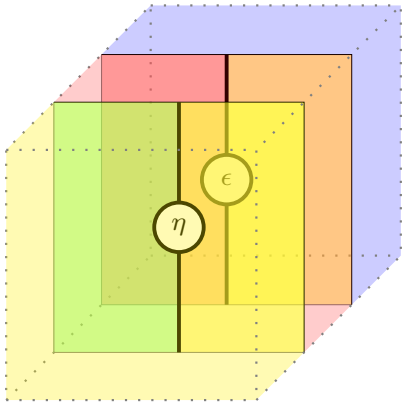
What is higher algebra?

- Ordinary algebra lets us compose along a line:

$$xy^2zyx^3$$

- *Higher algebra* lets us compose in higher dimensions:

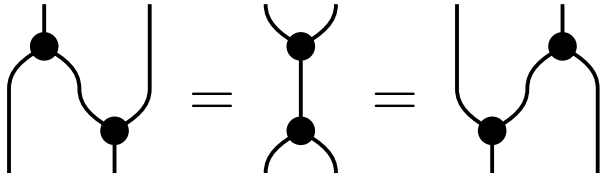




A tradeoff between algebra and topology

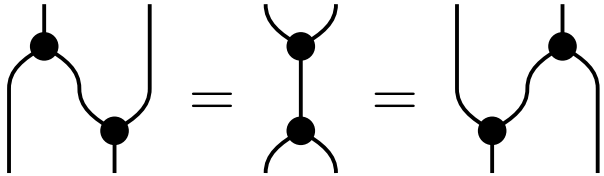
A tradeoff between algebra and topology

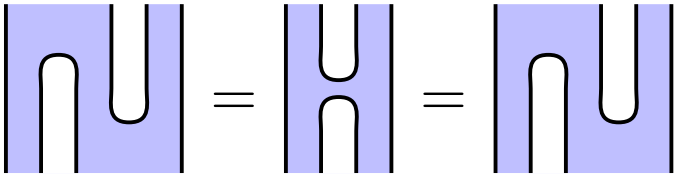
Frobenius algebras



A tradeoff between algebra and topology

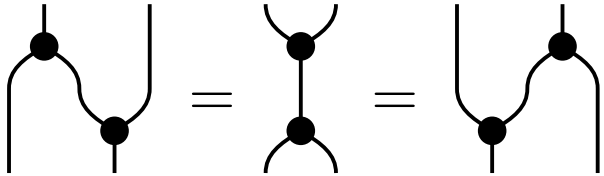
Frobenius algebras as a ‘shadow’ of a two dimensional theory.



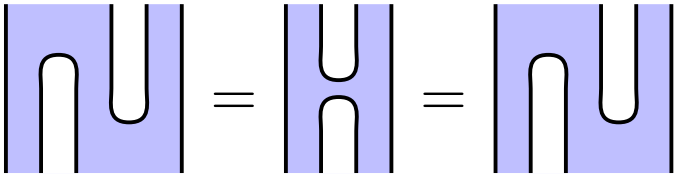


A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.



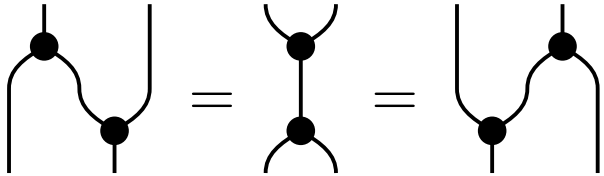
simpler topology	+	harder algebra



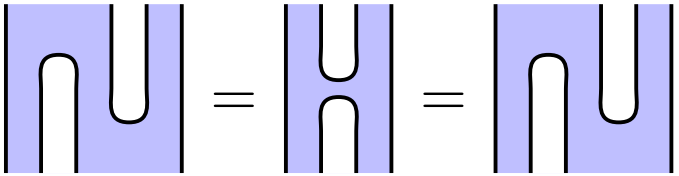
harder topology	+	simpler algebra

A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.



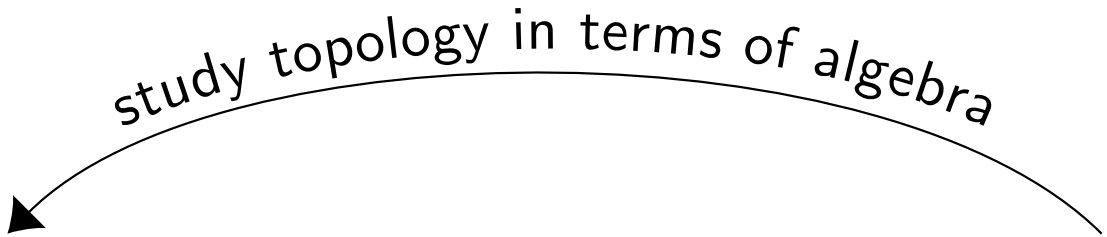
lower dimensional topology + harder algebra

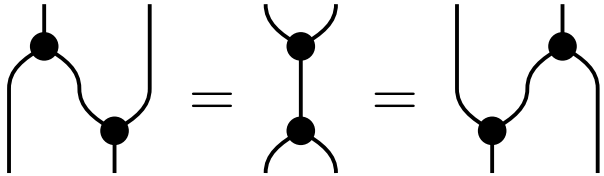


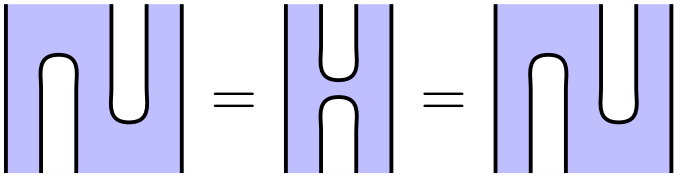
higher dimensional topology + simpler algebra

A tradeoff between algebra and topology

Frobenius algebras as a ‘shadow’ of a two dimensional theory.



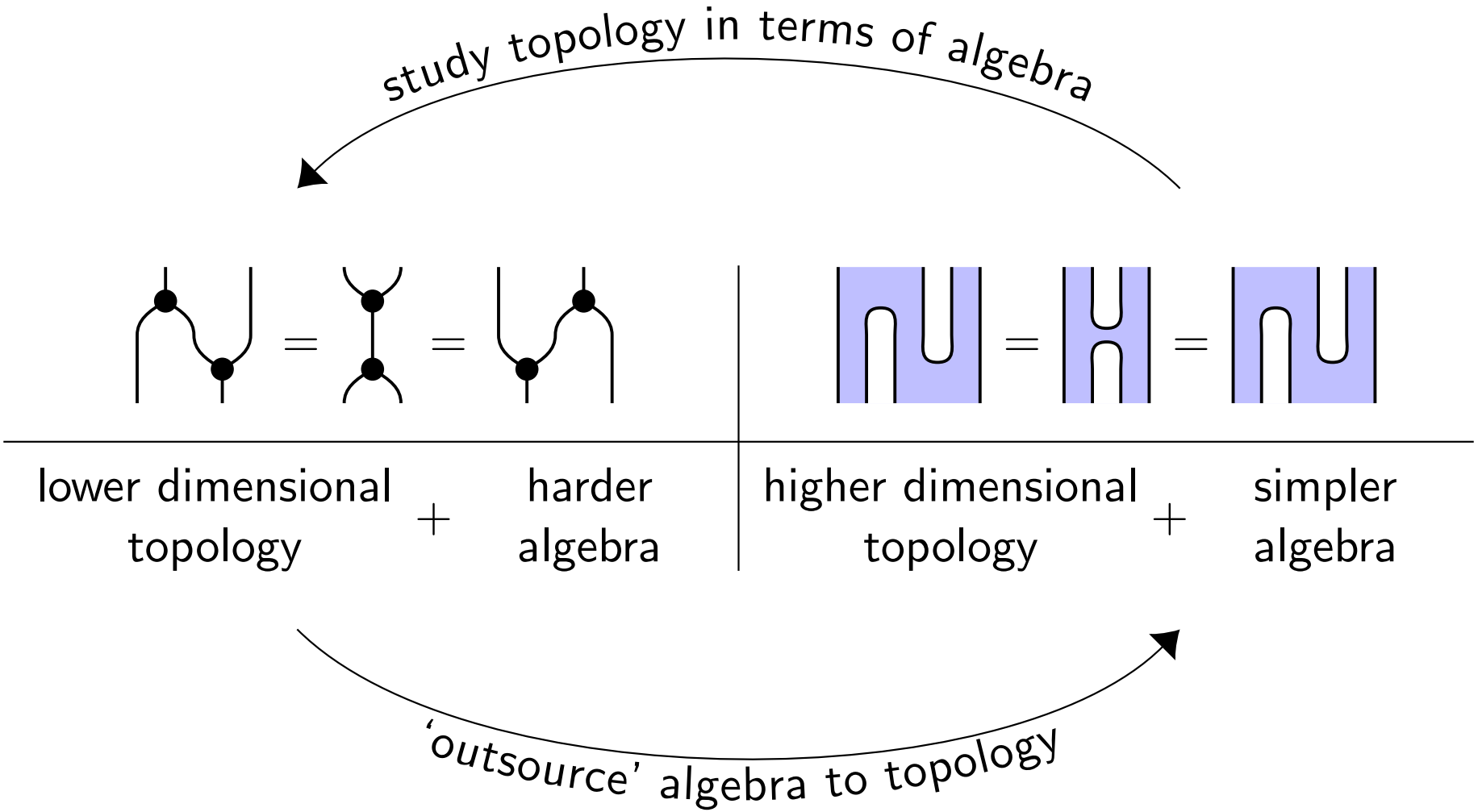




lower dimensional topology + harder algebra	higher dimensional topology + simpler algebra

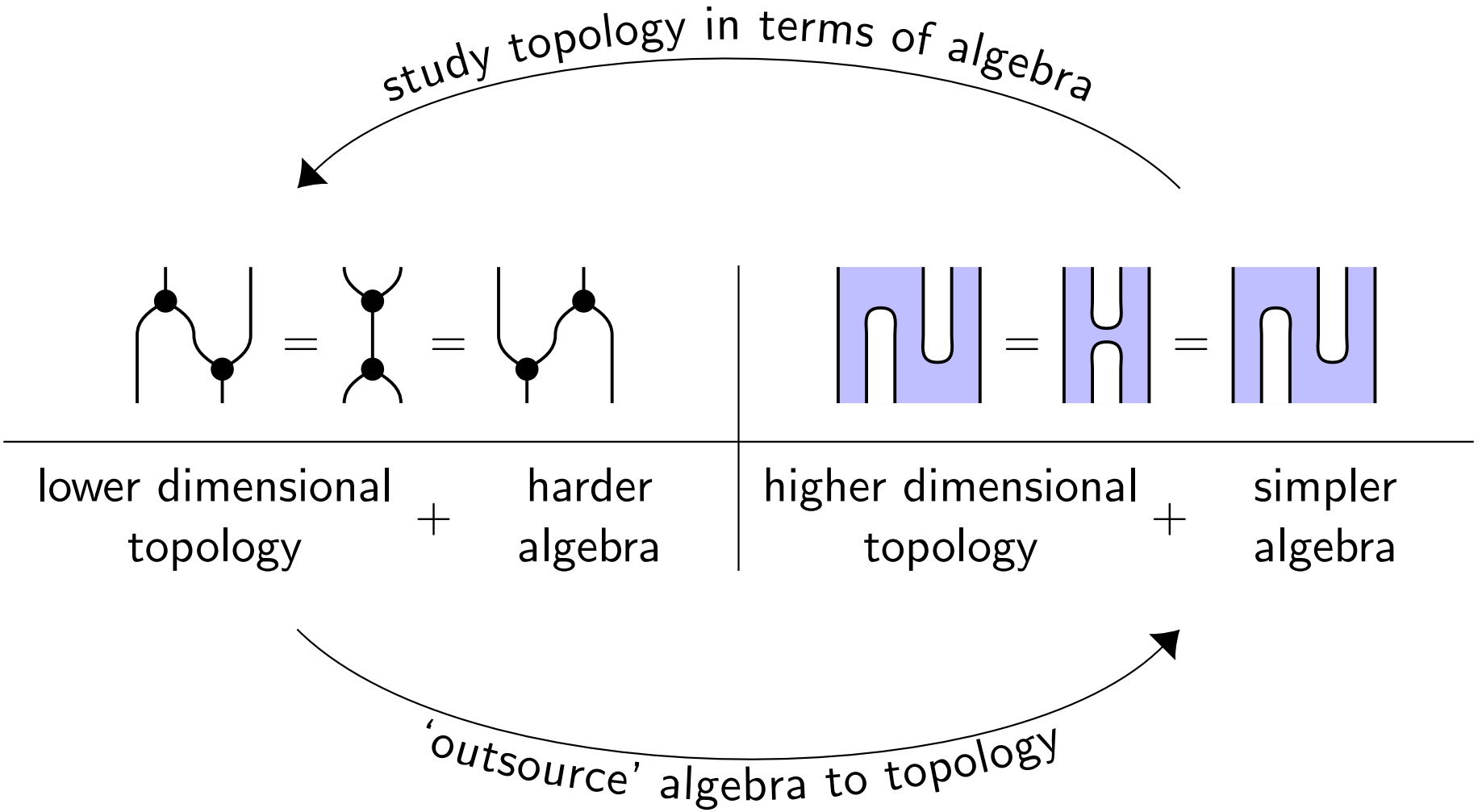
A tradeoff between algebra and topology

Frobenius algebras as a 'shadow' of a two dimensional theory.



A tradeoff between algebra and topology

Frobenius algebras as a 'shadow' of a two dimensional theory.



Next hour: Hopf algebras as a 'shadow' of a three dimensional theory.

Part 2
2-categories

Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

A	$A \xrightarrow{f} B$	$\begin{tikzcd} A \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\eta"] & B \end{tikzcd}$
object	1-morphism	2-morphism

Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

$$A \qquad\qquad A \xrightarrow{f} B \qquad\qquad \begin{tikzcd} A \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\Uparrow\eta"] & B \end{tikzcd}$$

object 1-morphism 2-morphism

We can compose 2-morphisms like this:

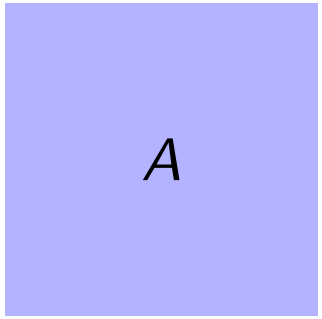
$$\begin{tikzcd} A \arrow[r, bend left=60] \arrow[r, phantom, "\Uparrow\epsilon"] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, "\Uparrow\eta"] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, bend left=60] \arrow[r, phantom, "\Uparrow\eta"] \arrow[r, bend right=60] & B \arrow[r, bend left=60] \arrow[r, phantom, "\Uparrow\epsilon"] \arrow[r, bend right=60] & C \end{tikzcd}$$

vertical composition horizontal composition

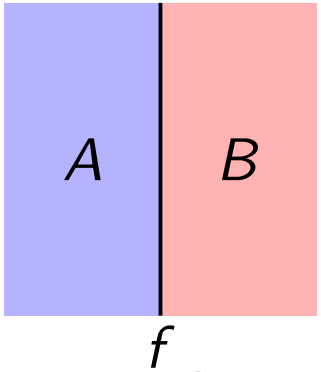
These are *pasting diagrams*.

Algebra in the plane = 2-category theory

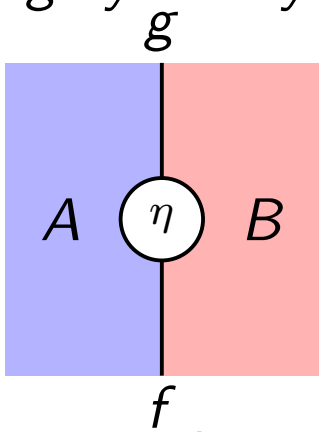
The language describing algebra in the plane is *2-category theory*:



object

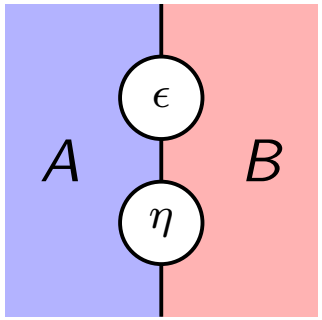


1-morphism

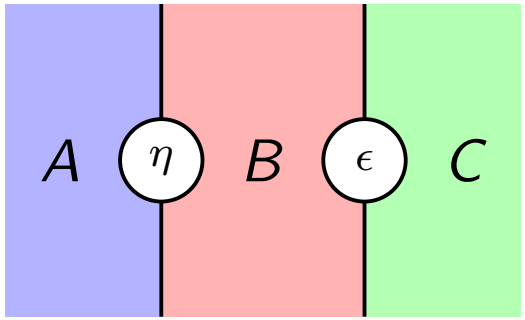


2-morphism

We can compose 2-morphisms like this:



vertical composition



horizontal composition

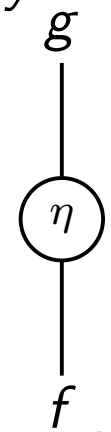
These are *pasting diagrams*.The *dual* diagrams are the graphical calculus.

Algebra in the plane = 2-category theory

The language describing algebra in the plane is *2-category theory*:

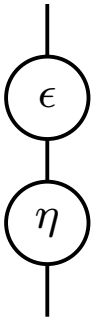
object

1-morphism

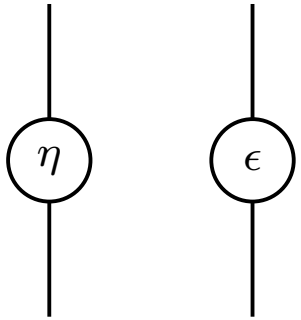


2-morphism

We can compose 2-morphisms like this:



vertical composition

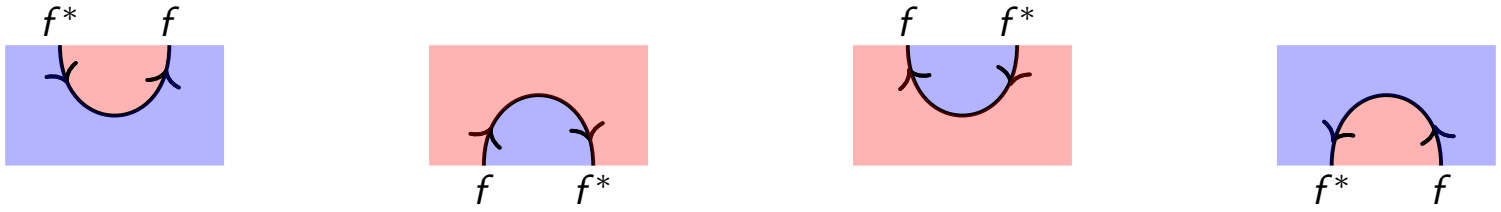


horizontal composition

These are *pasting diagrams*.The *dual* diagrams are the graphical calculus. A 2-category with one object (the 'empty region') is a *monoidal category.*

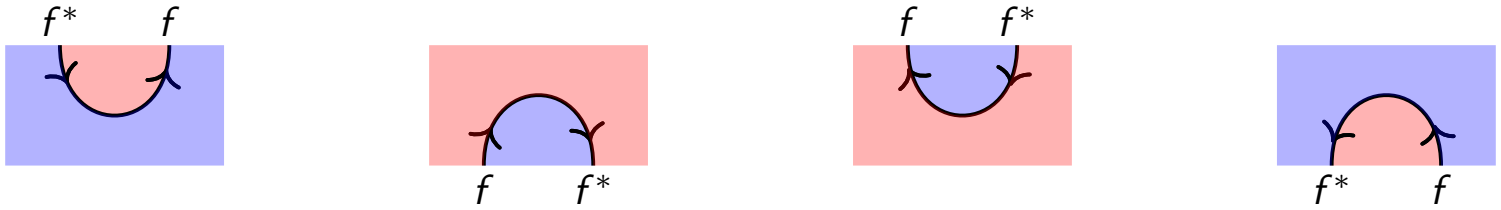
Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:

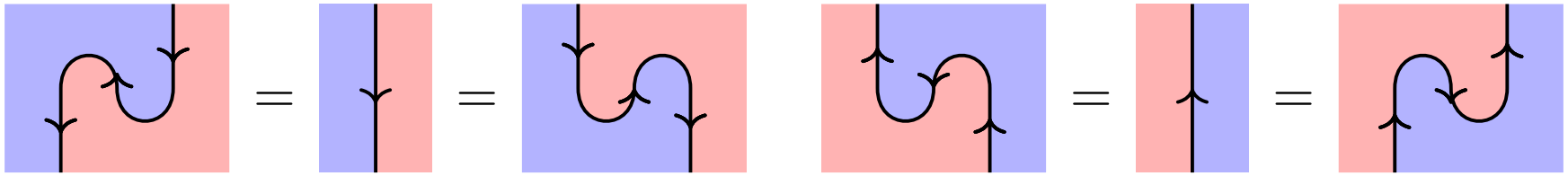


Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:

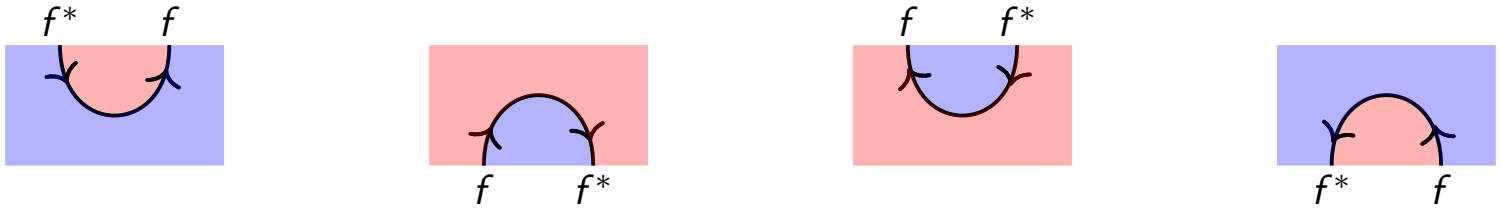


such that the following hold:

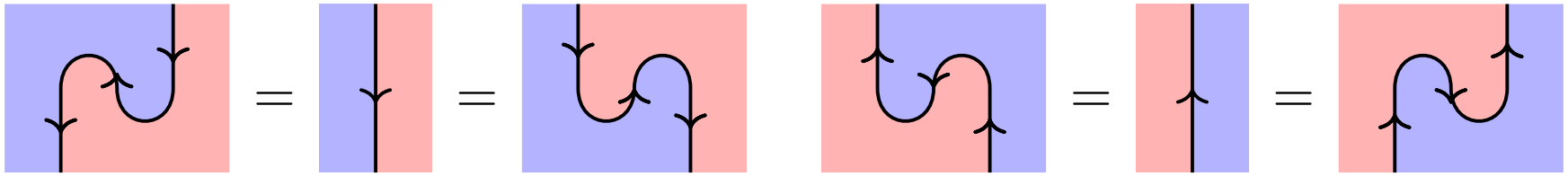


Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:



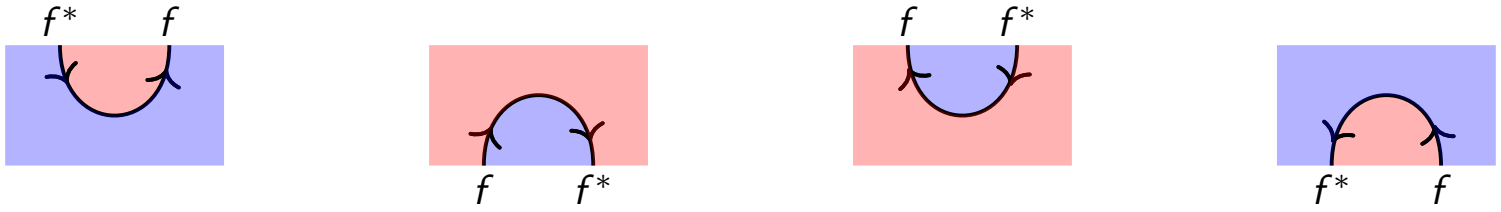
such that the following hold:



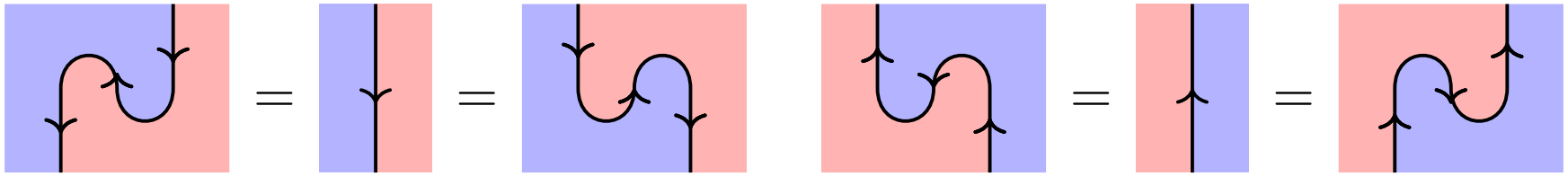
Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:



such that the following hold:

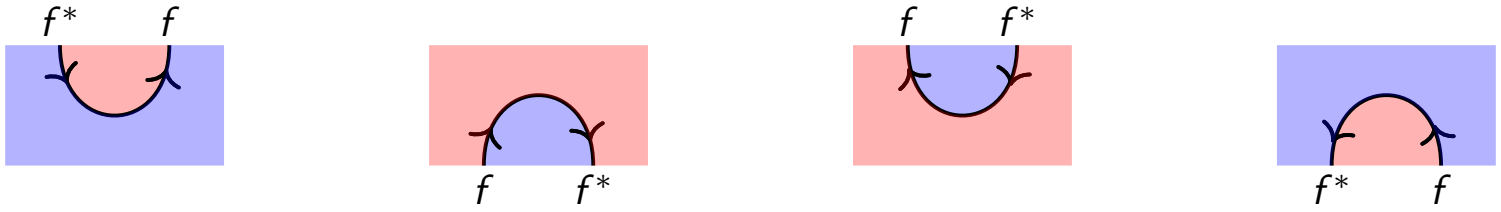


Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

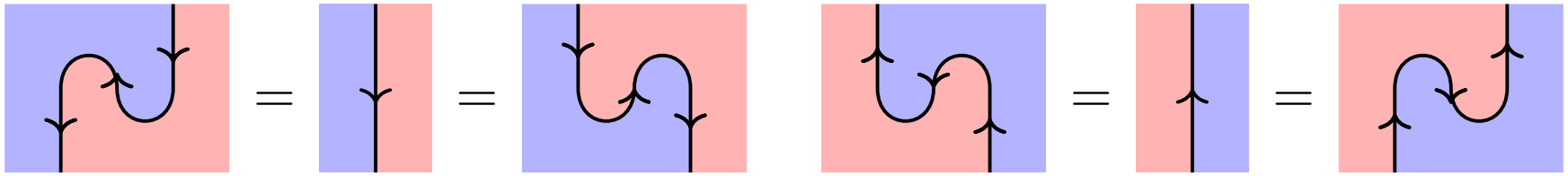
Tangle hypothesis.
$\mathbf{Bord}^{2D}_{1,0} \cong$ free monoidal category on a dualizable object

Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:



such that the following hold:

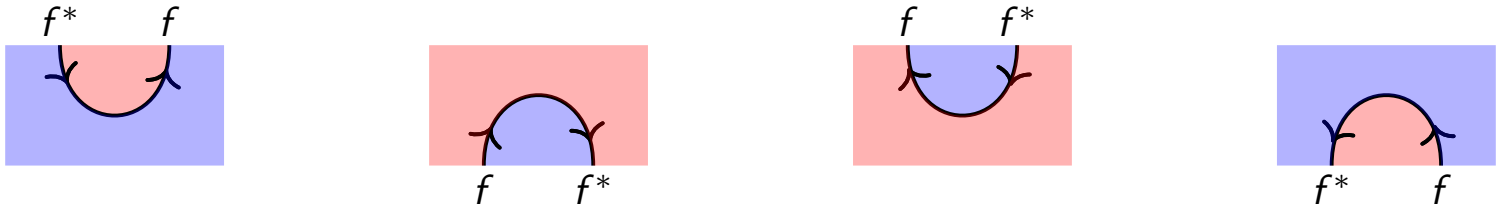


Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

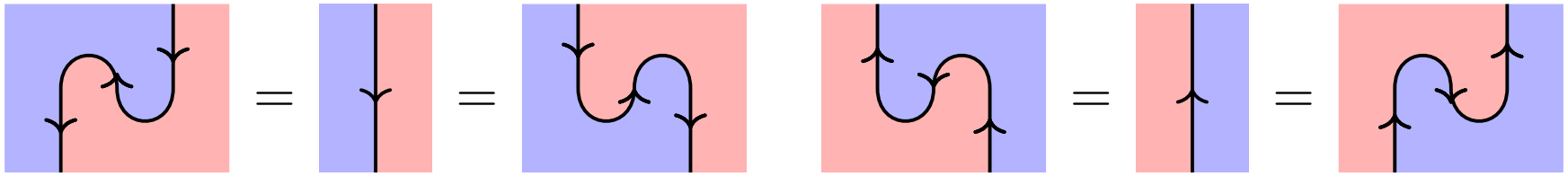
Definition. G directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on G.

Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:

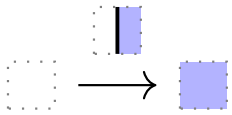
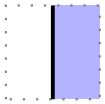


such that the following hold:



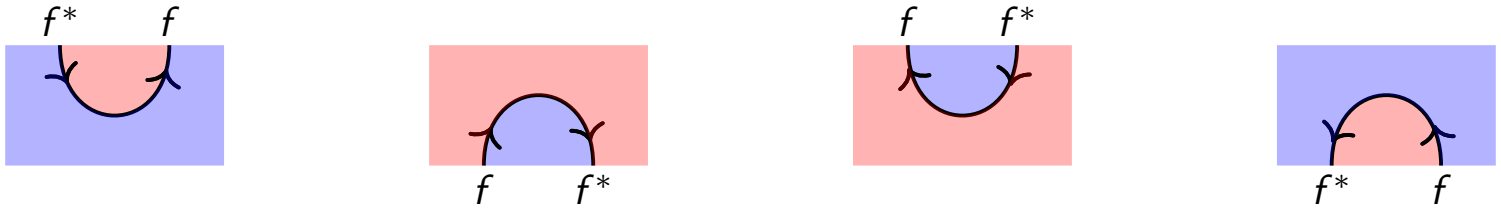
Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

Definition. G directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on G.

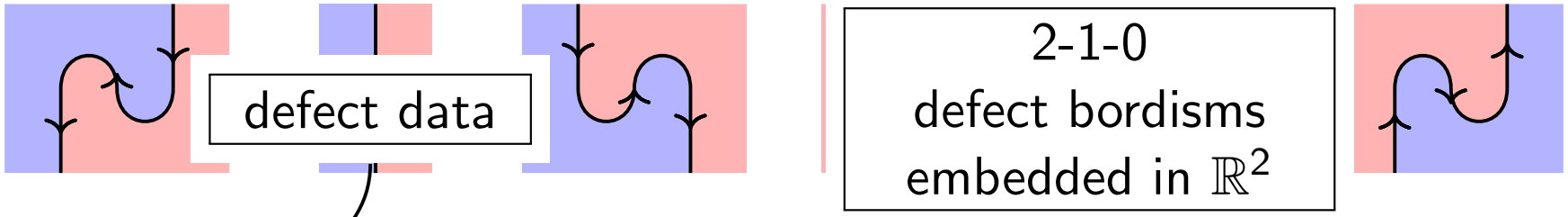
Example. $\mathcal{F}_2\Bigg($  $\Bigg)$: free 2-category on dualizable 1-morphism 

Dualizable 1-morphisms

A 1-morphism $A \xrightarrow{f} B$ has a *dual* $B \xrightarrow{f^*} A$ if there are 2-morphisms:

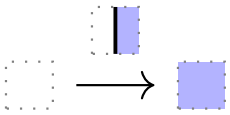
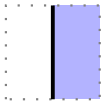


such that the following hold:



Theorem. Graphical calculus for duals $\leftrightarrow$ oriented wires in the plane

Definition. G directed graph $\Rightarrow \mathcal{F}_2(G) :=$ free 2-category with duals on G.

Example. $\mathcal{F}_2\Big($  $\Big)$: free 2-category on dualizable 1-morphism 

Frobenius algebras and dualizable 1-morphisms

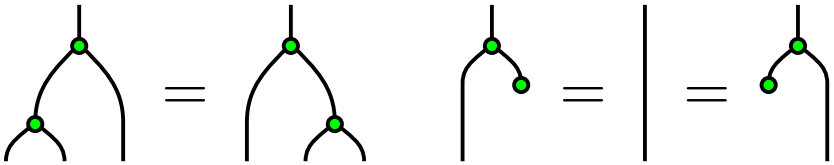
A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

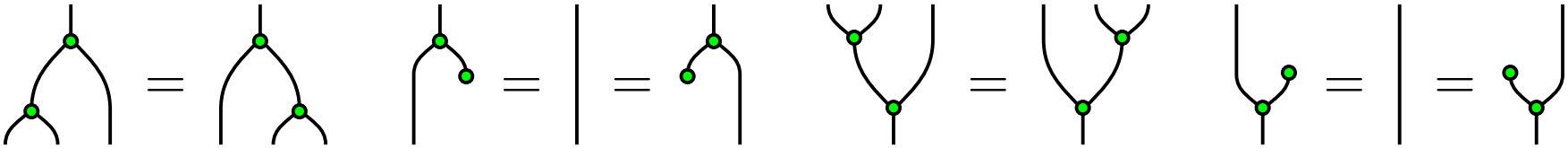
such that:



Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

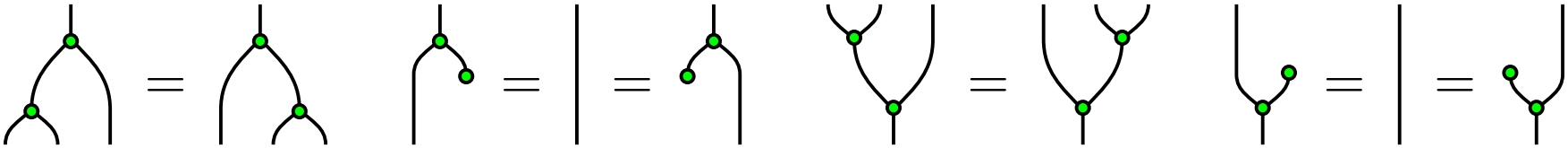
such that:

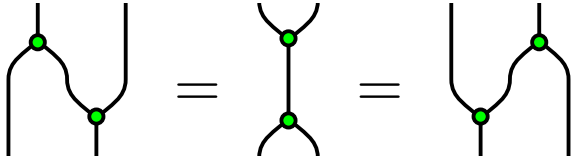


Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

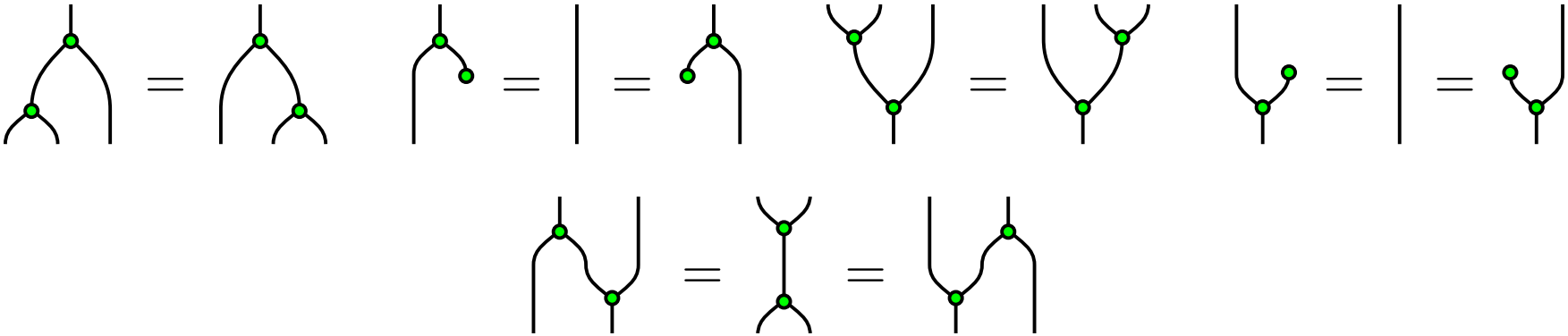




Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:

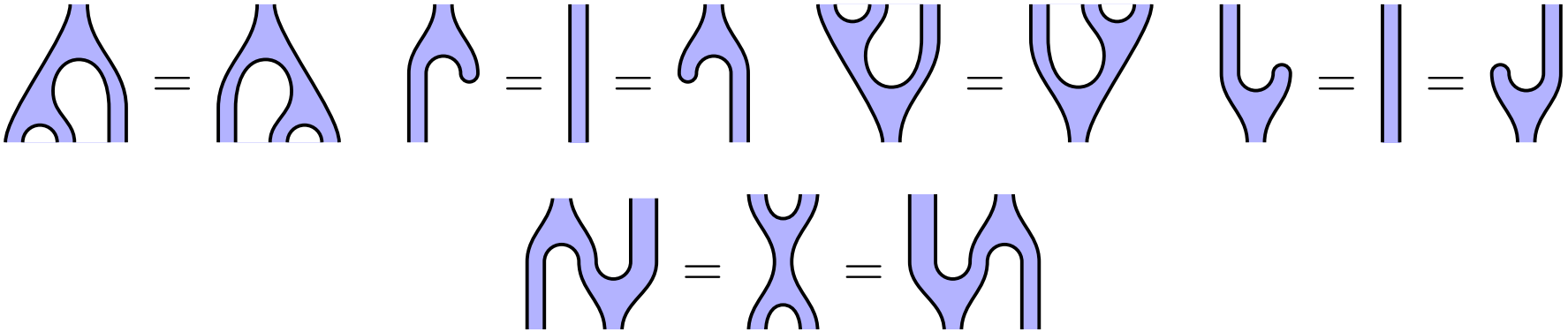


Frob: free monoidal category on a Frobenius algebra

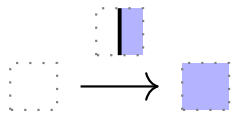
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:



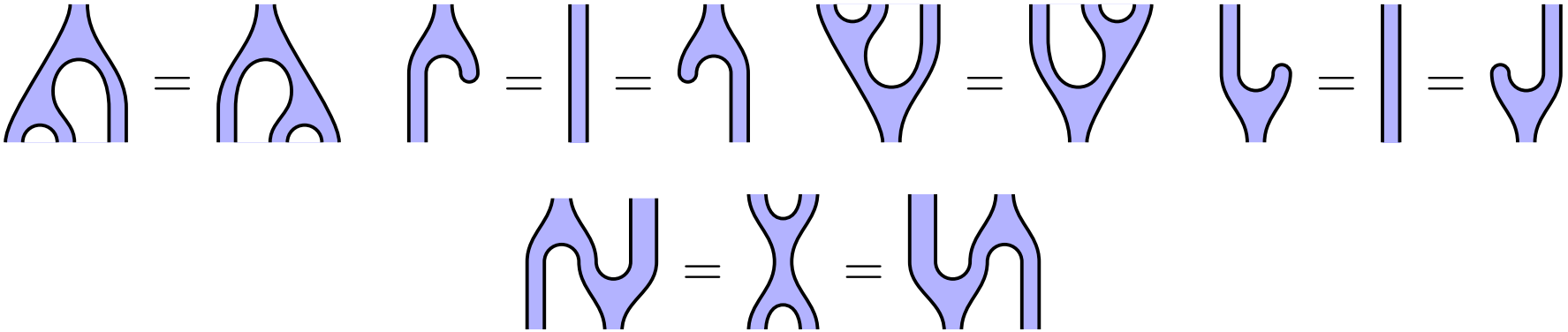
Frob: free monoidal category on a Frobenius algebra

There is a 2-functor **Frob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}\Big($  $\Big)$.

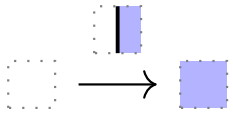
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:



Frob: free monoidal category on a Frobenius algebra

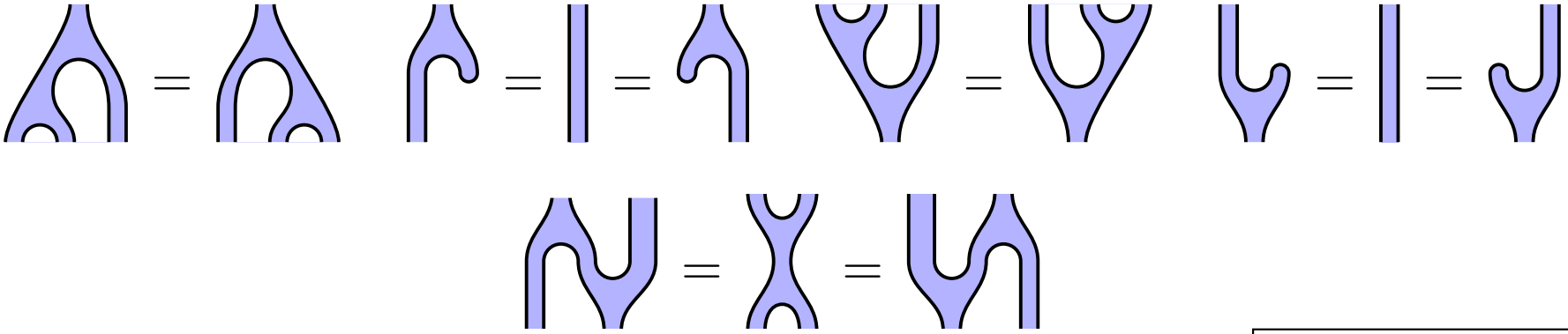
There is a 2-functor **Frob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}\Big($  $\Big)$.

Theorem. This induces a monoidal equivalence **Frob** $\cong \mathrm{F}_2\big($  ,  $\big)$.

Frobenius algebras and dualizable 1-morphisms

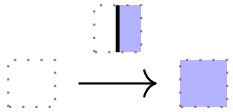
A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:



Frob: free monoidal category on a Frobenius a

open strings in the plane

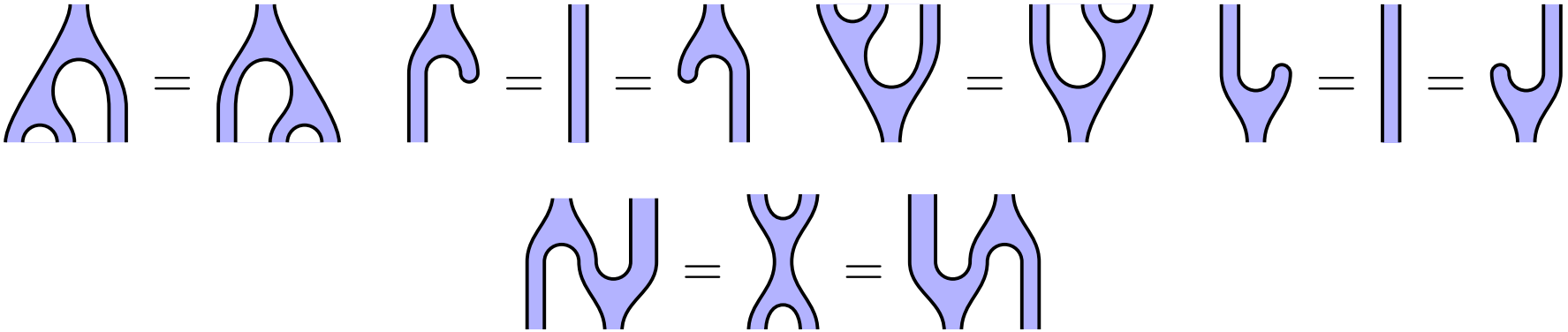
There is a 2-functor **Frob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}\Big($  $\Big)$.

Theorem. This induces a monoidal equivalence **Frob** $\cong \mathrm{F}_2\big($  $,$  $\big)$.

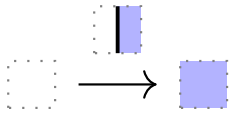
Frobenius algebras and dualizable 1-morphisms

A *Frobenius algebra* in a monoidal category is an object with morphisms:

such that:



Frob: free monoidal category on a Frobenius algebra

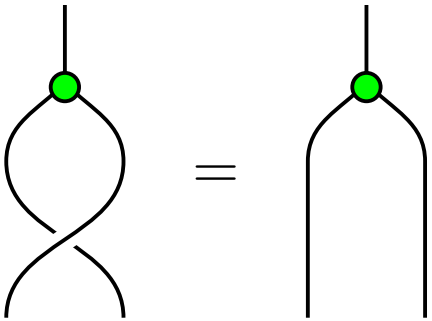
There is a 2-functor **Frob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_2 := \mathcal{F}_{\mathbf{2}}\Big($  $\Big)$.

Theorem. This induces a monoidal equivalence **Frob** $\cong \mathrm{F}_2\big($  $\big)$.

Frob as a 'shadow' of the theory of dualizable 1-morphisms in 2-categories.

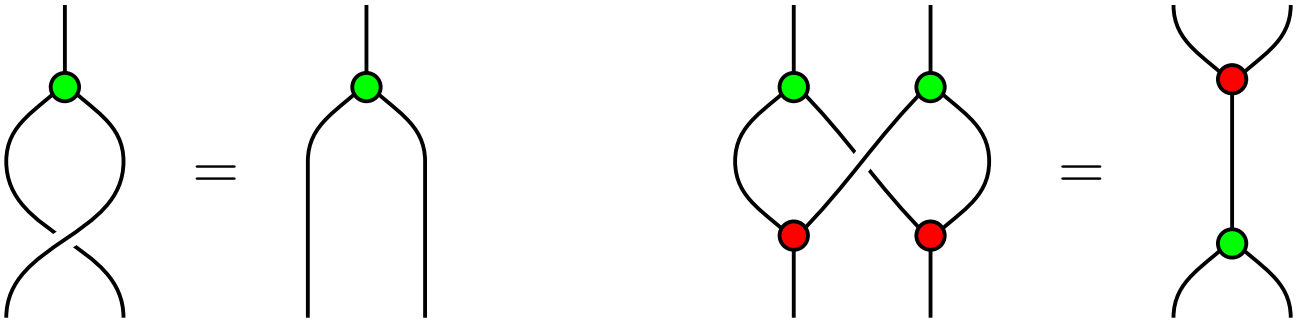
Other algebraic theories?

What about *commutative* Frobenius algebras



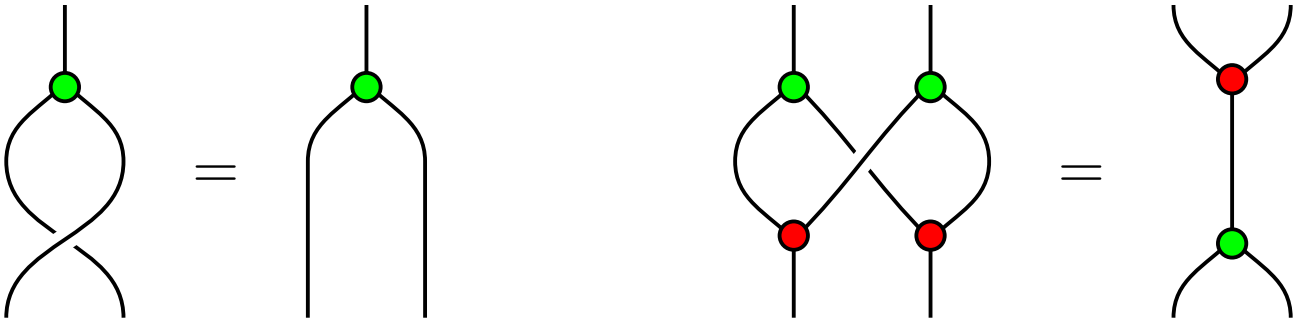
Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?



Other algebraic theories?

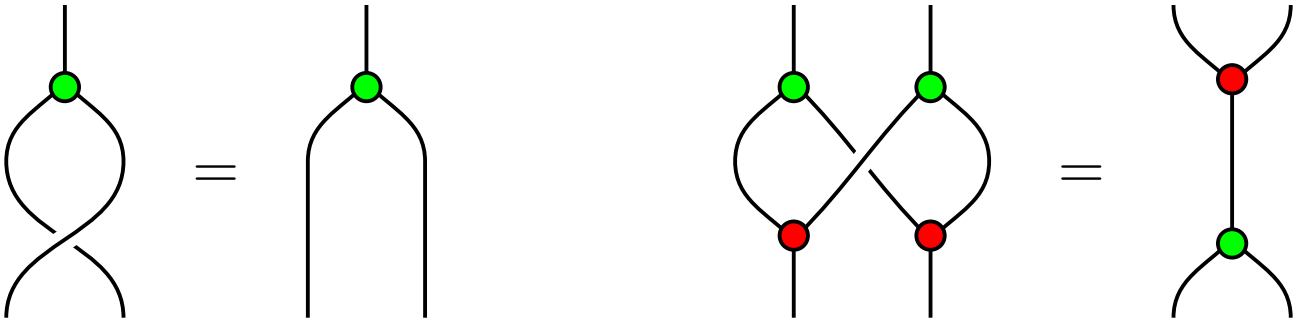
What about *commutative* Frobenius algebras or *bialgebras*?



Only make sense in (at least) three dimensional space.

Other algebraic theories?

What about *commutative* Frobenius algebras or *bialgebras*?



Only make sense in (at least) three dimensional space.

$$\Downarrow$$

Shadows of 3D structures?

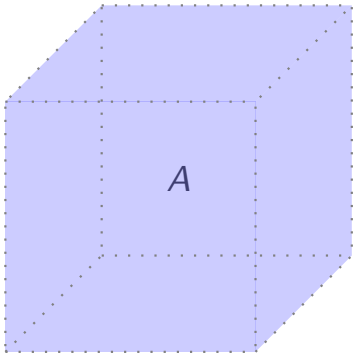
Part 3
3-categories

Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:

Algebra in three dimensions = 3-category theory

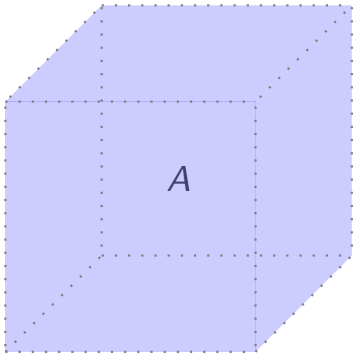
The language describing algebra in three dimensions is *3-category theory*:



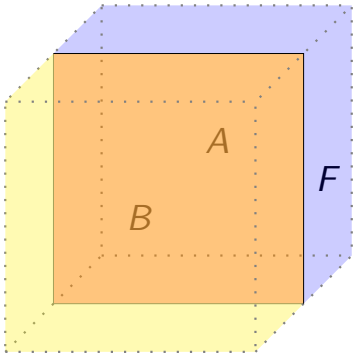
object

Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



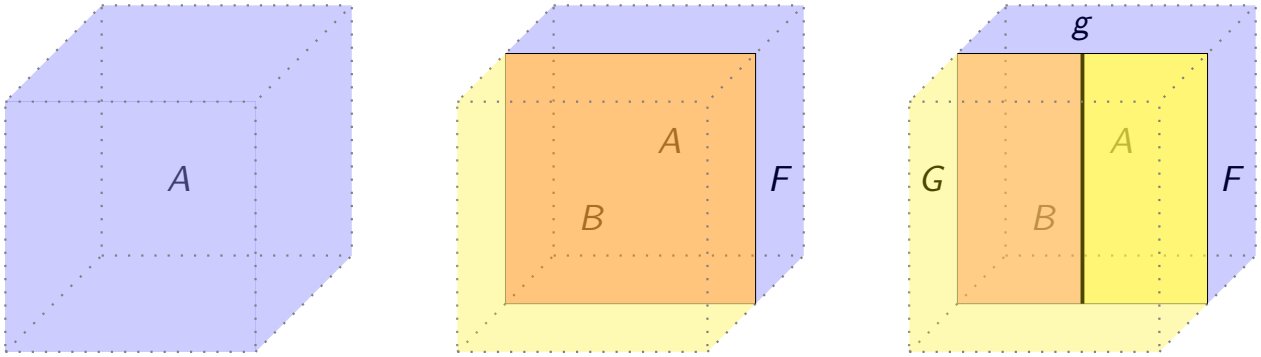
object



1-morphism

Algebra in three dimensions = 3-category theory

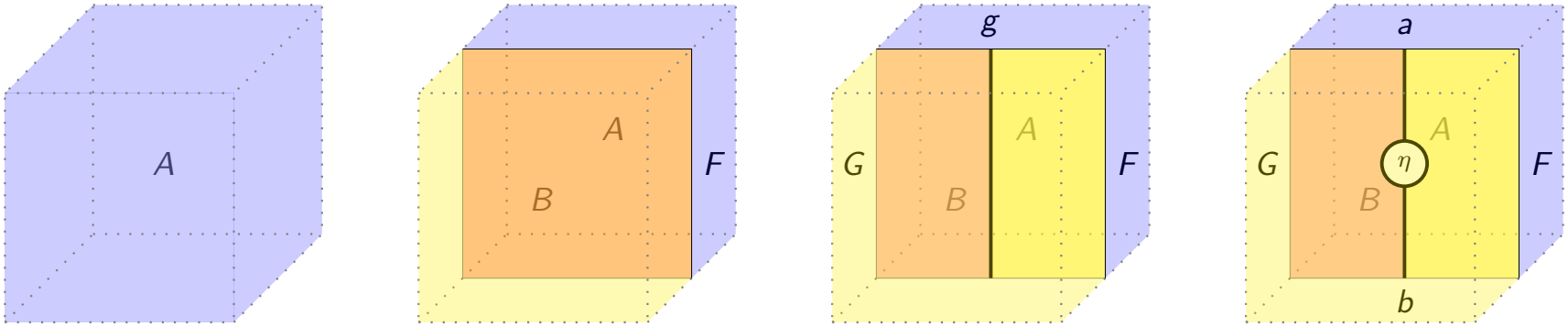
The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism

Algebra in three dimensions = 3-category theory

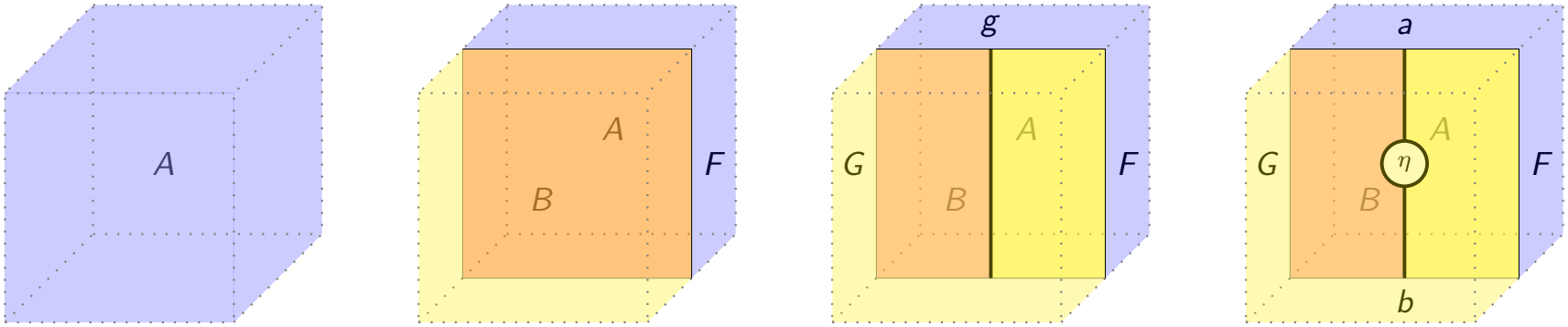
The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:

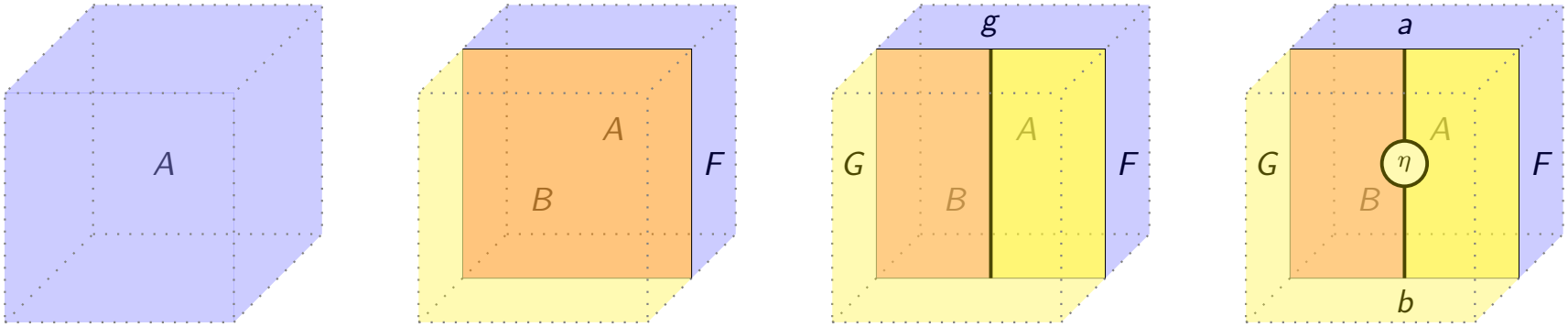


object 1-morphism 2-morphism 3-morphism

We can compose 3-morphisms like this:

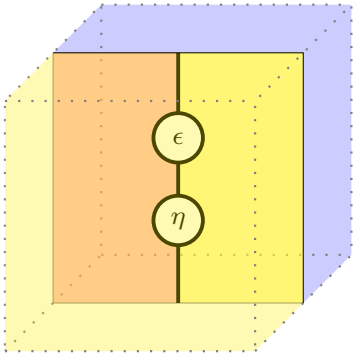
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

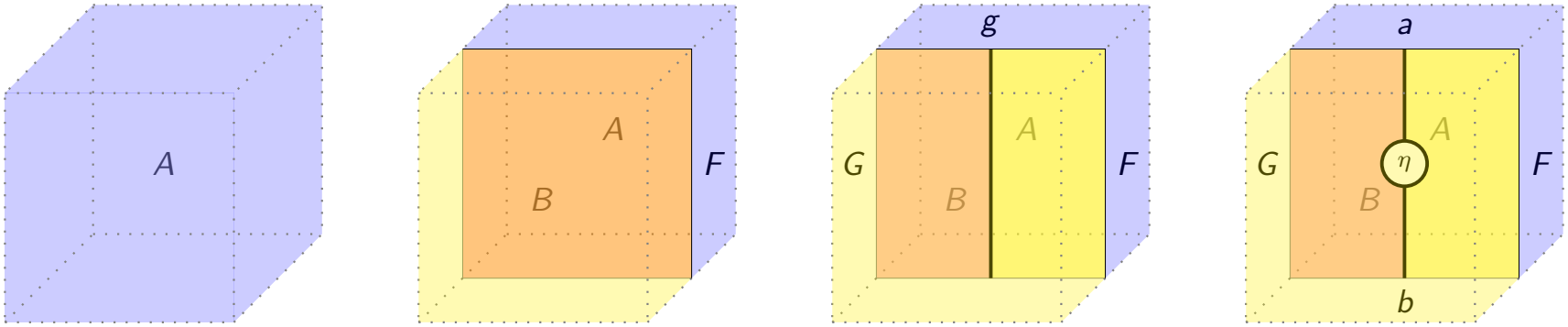
We can compose 3-morphisms like this:



vertical composition

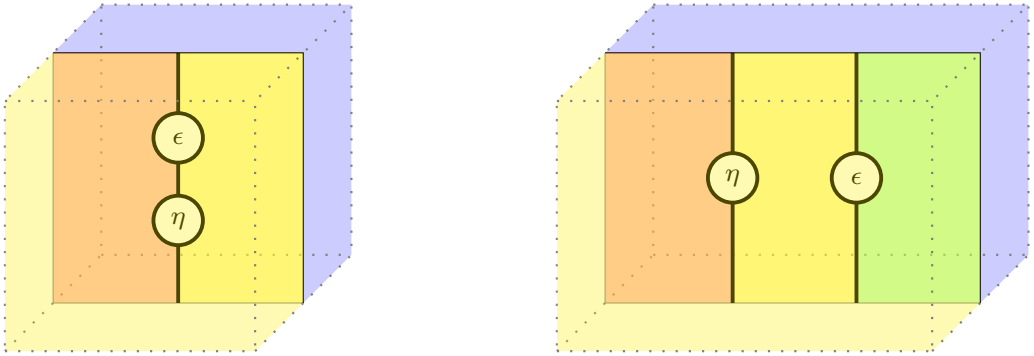
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

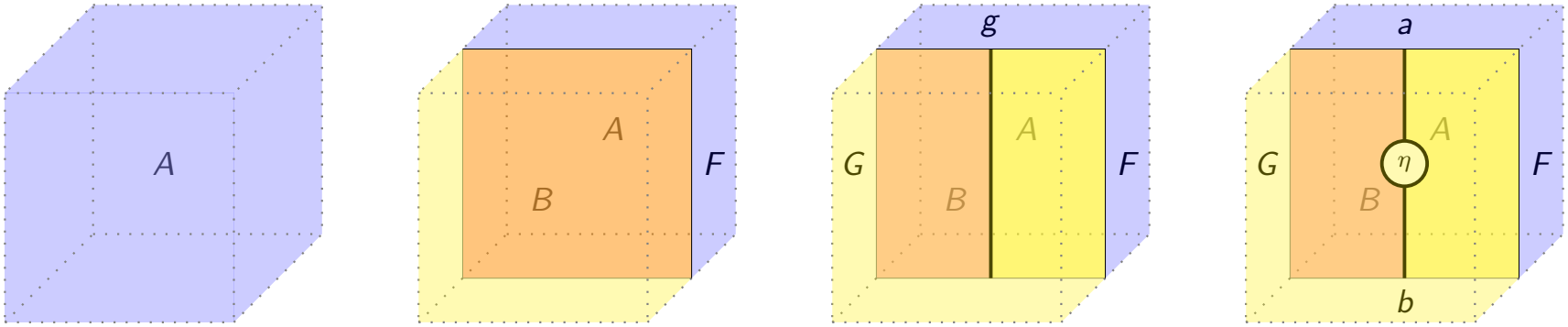
We can compose 3-morphisms like this:



vertical composition horizontal composition

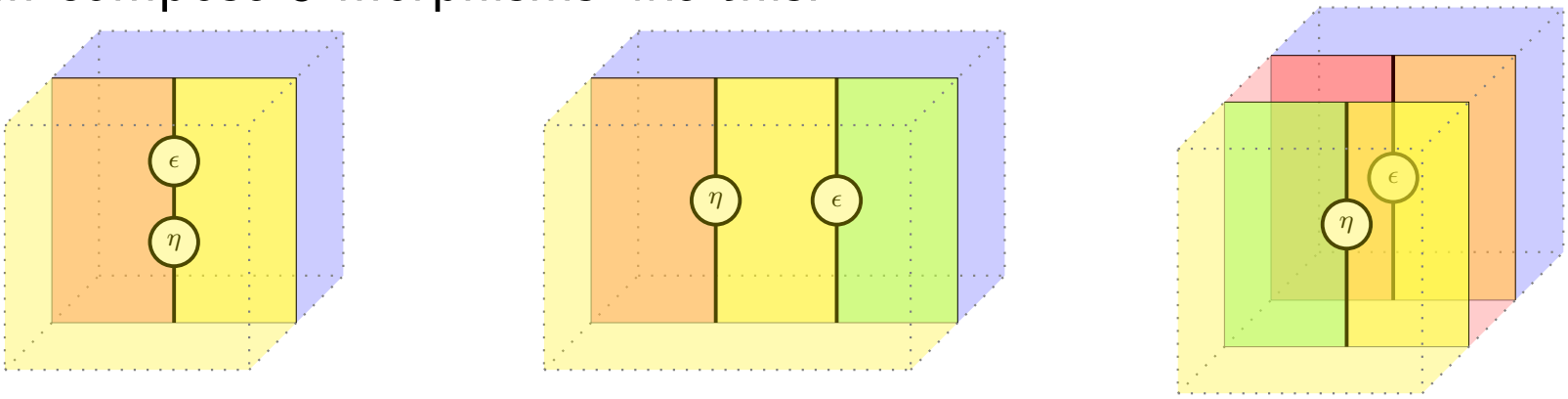
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

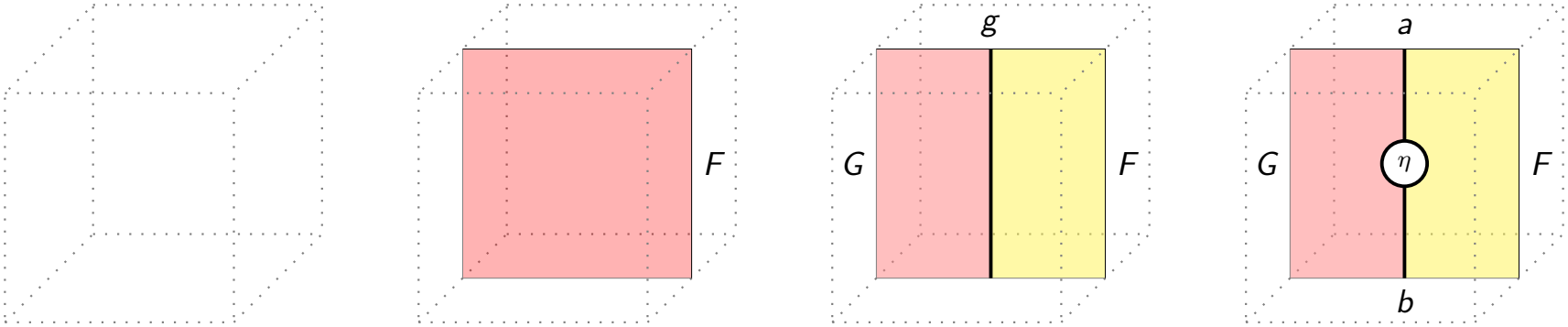
We can compose 3-morphisms like this:



vertical composition horizontal composition layered composition

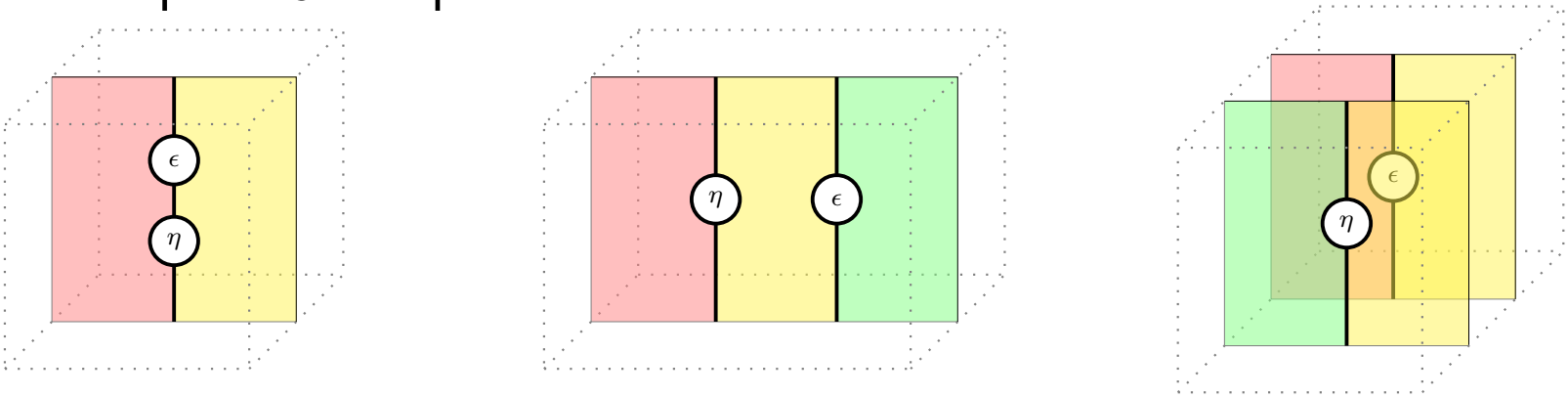
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

We can compose 3-morphisms like this:

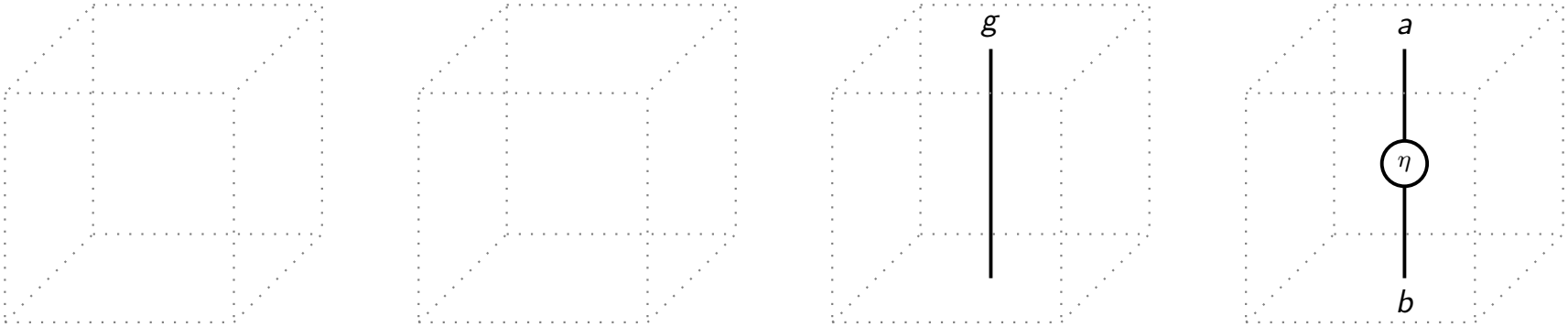


vertical composition horizontal composition layered composition

A one object (the 'empty region') 3-category is a *monoidal 2-category*.

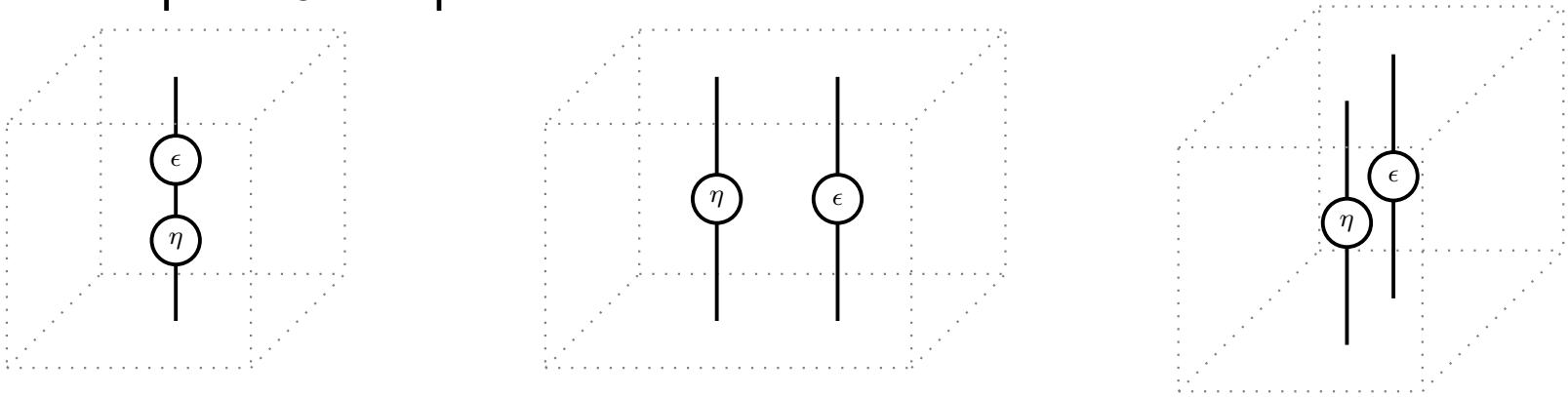
Algebra in three dimensions = 3-category theory

The language describing algebra in three dimensions is *3-category theory*:



object 1-morphism 2-morphism 3-morphism

We can compose 3-morphisms like this:

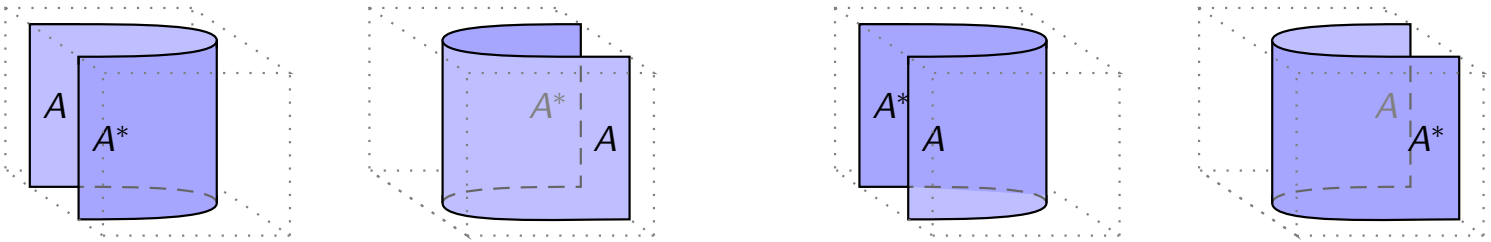


vertical composition horizontal composition layered composition

A one object (the 'empty region') 3-category is a *monoidal 2-category*.
A one object and one 1-morphism 3-category is a *braided monoidal category*.

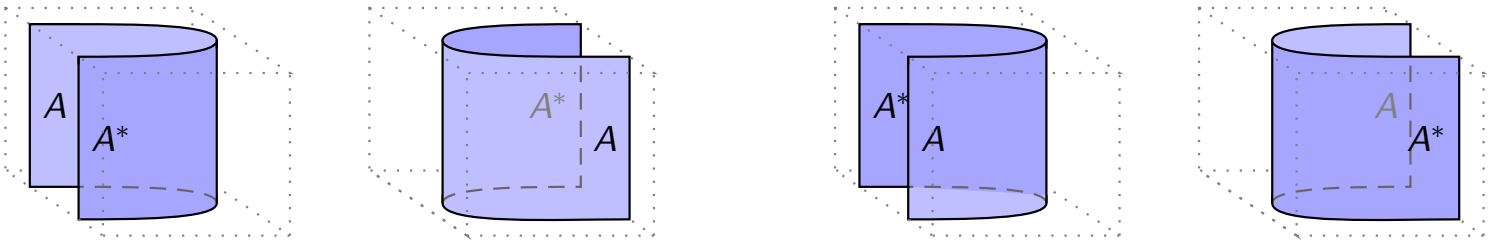
Duals in 3-categories

A 1-morphism A has an *oriented dual* A^* if there are 2-morphisms (*folds*):

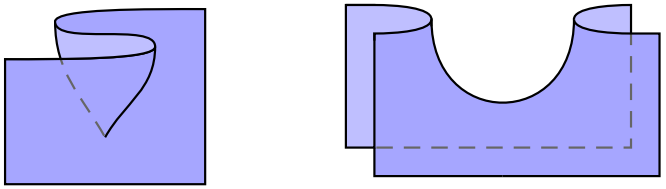


Duals in 3-categories

A 1-morphism A has an *oriented dual* A^* if there are 2-morphisms (*folds*):



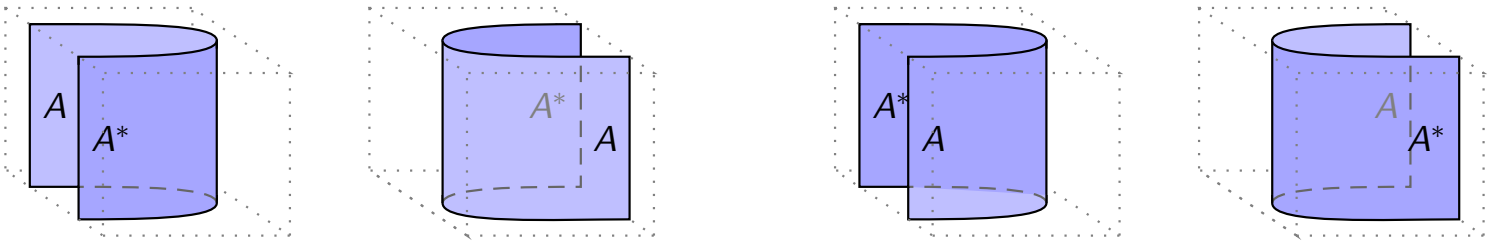
and 3-morphisms (*cusps, saddles and births/deaths of the circle*):



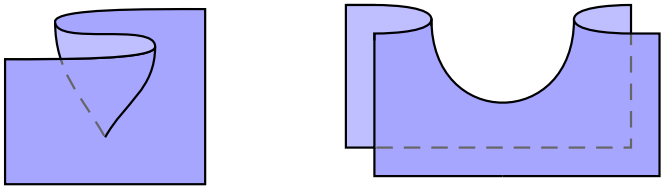
+ horizontal and vertical reflections and opposite orientation

Duals in 3-categories

A 1-morphism A has an *oriented dual* A^* if there are 2-morphisms (*folds*):

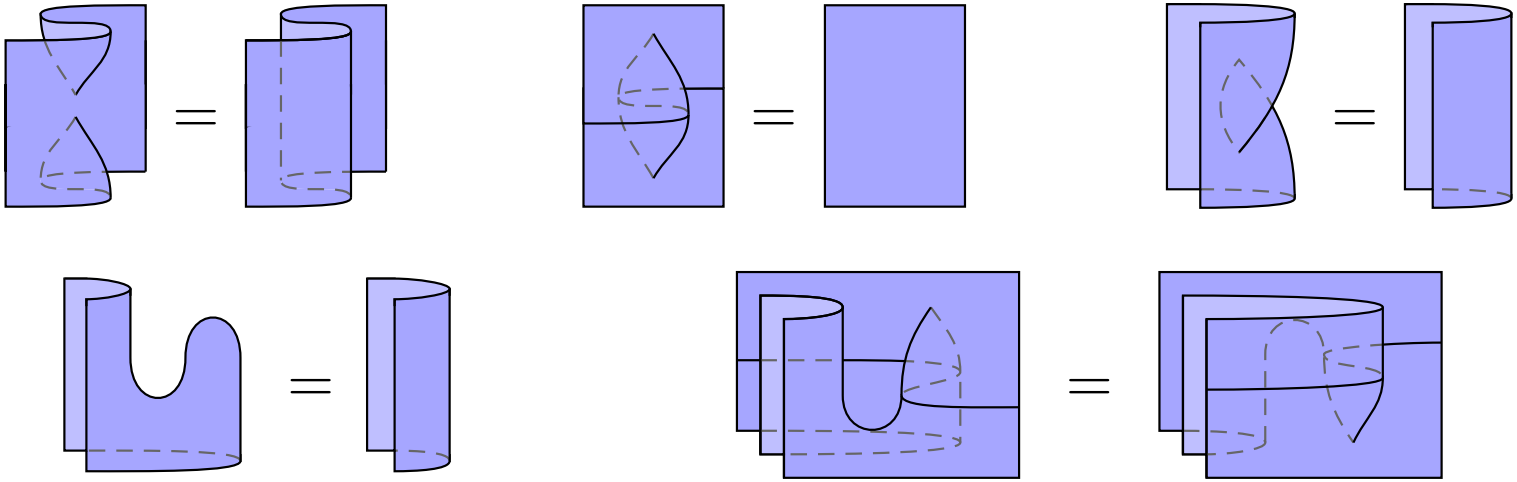


and 3-morphisms (*cusps, saddles and births/deaths of the circle*):



+ horizontal and vertical reflections and opposite orientation

such that the following hold (& reflections and opposite orientation):



Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Tangle hypothesis.
$\mathbf{Bord}^{3D}_{2,1,0} \cong$ free monoidal 2-category on a dualizable object

Duals in 3-categories

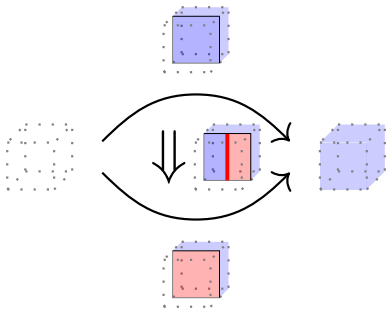
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

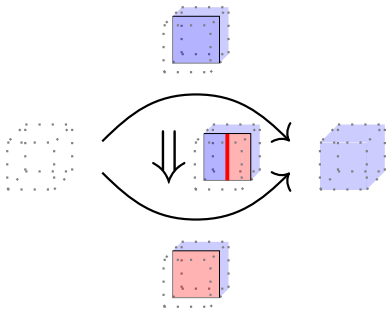


Duals in 3-categories

Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

Def. $\mathcal{F}_3(\mathbb{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathbb{G}$.



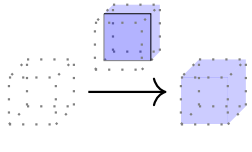
Duals in 3-categories

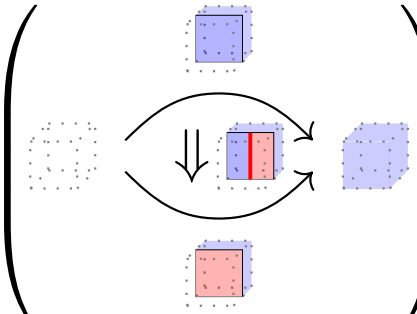
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

Def. $\mathcal{F}_3(\mathbb{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathbb{G}$.

Examples.

$\mathcal{F}_3\Big($  $\Big)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3\Bigg($  $\Bigg)$: free 3-category on $\begin{cases} \text{two dualizable 1-morphisms} \\ \text{one dualizable 2-morphism} \end{cases}$

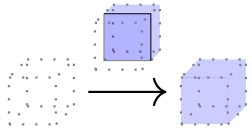
Duals in 3-categories

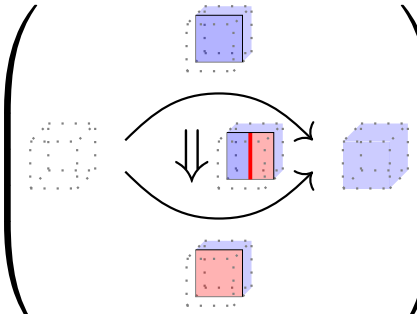
Theorem. graphical calculus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices} \right)$

Def. $\mathcal{F}_3(\mathbb{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathbb{G}$.

Examples.

$\mathcal{F}_3\Big($  $\Big)$: free 3-category on a dualizable 1-morphism

$\mathcal{F}_3\Bigg($  $\Bigg)$: free 3-category on $\begin{cases} \text{two dualizable 1-morphisms} \\ \text{one dualizable 2-morphism} \end{cases}$

Summary. The graphical calculus of $\mathcal{F}_3(\mathbb{G})$ is given by regions, surfaces and wires in three dimensional space.

Duals in 3-categories

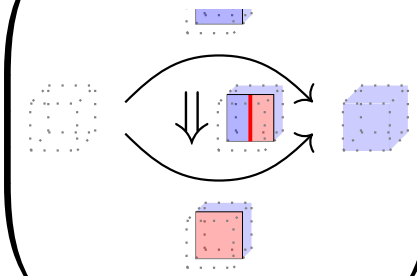
Theorem $\boxed{\text{defect data}}$ culus for duals $\leftrightarrow$ oriented surfaces in 3D space

Let $\mathbb{G}$ be a 2-globular set $\mathbb{G} = \left(\text{2-Edges} \rightrightarrows \text{Edges} \rightrightarrows \text{Vertices}\right)$

Def. $\mathcal{F}_3(\mathbb{G})$: free 3-category with duals for 2- and 1-morphisms given in $\mathbb{G}$.

Examples.

$\mathcal{F}_3\Big($  $\boxed{\begin{array}{c}\text{3-2-1-0}\\ \text{defect bordisms}\\ \text{embedded in } \mathbb{R}^3\end{array}}$ gory on a dualizable 1-morphism

$\mathcal{F}_3\Bigg($  $\Bigg)$: free 3-category on $\begin{cases}\text{two dualizable 1-morphisms}\\ \text{one dualizable 2-morphism}\end{cases}$

Summary. The graphical calculus of $\mathcal{F}_3(\mathbb{G})$ is given by regions, surfaces and wires in three dimensional space.

Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

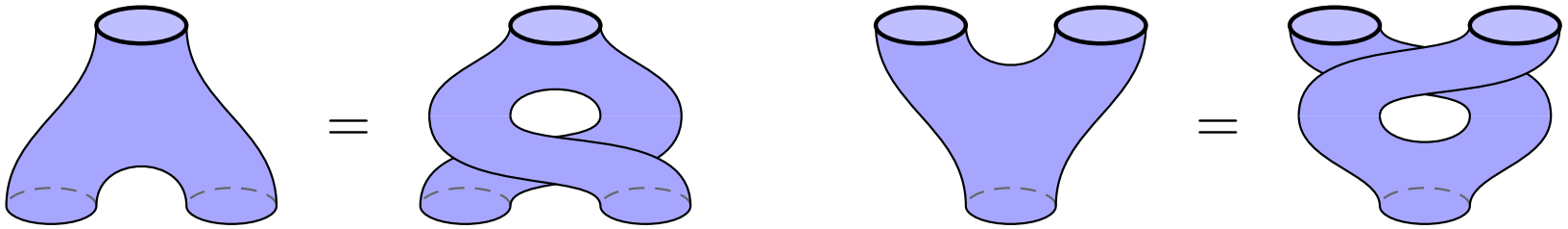
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

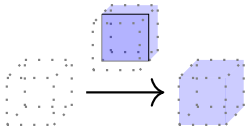
cFrob: free *braided* monoidal category on a commutative Frobenius algebra

Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:

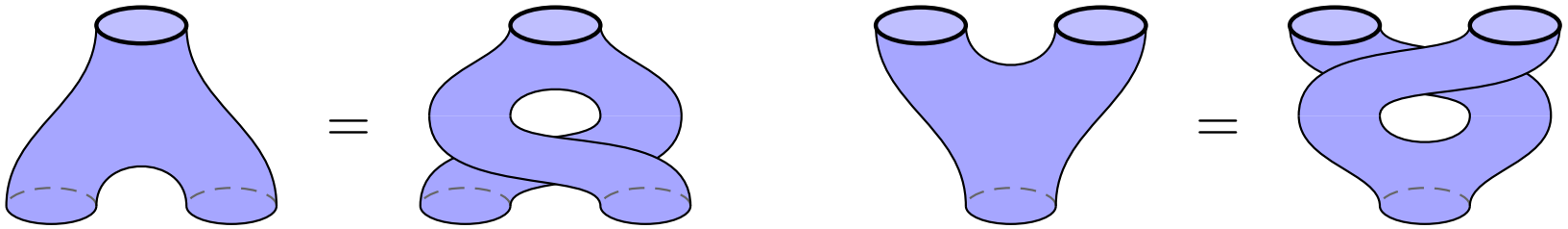


cFrob: free *braided* monoidal category on a commutative Frobenius algebra

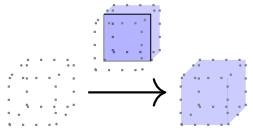
There is a 3-functor **cFrob** $\xrightarrow{\text{'thickening'}}$ $\mathrm{F}_3 := \mathcal{F}_3\Big($  $\Big)$.

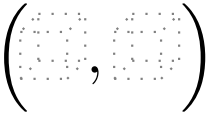
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:



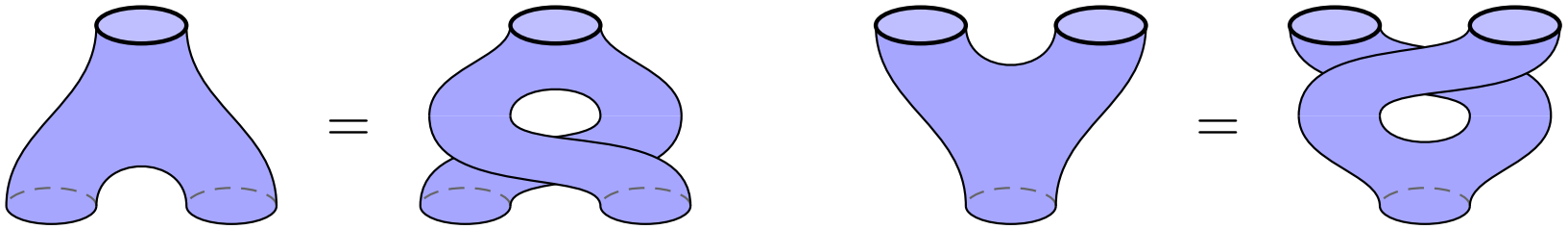
cFrob: free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor **cFrob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_3 := \mathcal{F}_3\Big($  $\Big)$.

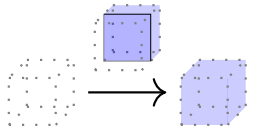
Theorem. This induces a braided monoidal equivalence **cFrob** $\cong \mathrm{F}_3\Big($  $\Big)$.

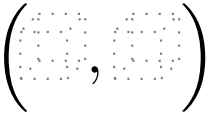
Commutative Frobenius algebras

A *commutative* Frobenius algebra is a Frobenius algebra such that:



cFrob: free *braided* monoidal category on a commutative Frobenius algebra

There is a 3-functor **cFrob** $\overset{\text{'thickening'}}{\longrightarrow}$ $\mathrm{F}_3 := \mathcal{F}_3\Big($  $\Big)$.

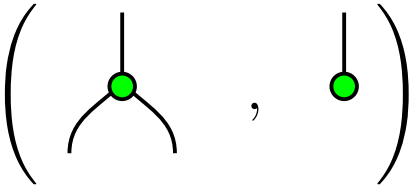
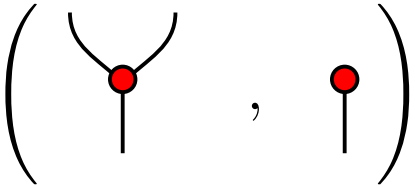
Theorem. This induces a braided monoidal equivalence **cFrob** $\cong \mathrm{F}_3\Big($  $\Big)$.

cFrob as a 'shadow' of the theory of dualizable 1-morphisms in 3-categories.

Part 4
Hopf algebras

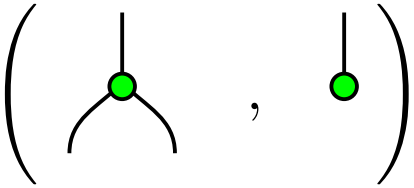
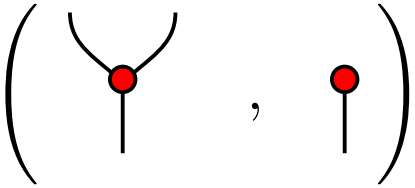
Hopf algebras

A *Hopf algebra* in a braided monoidal category is a pair of

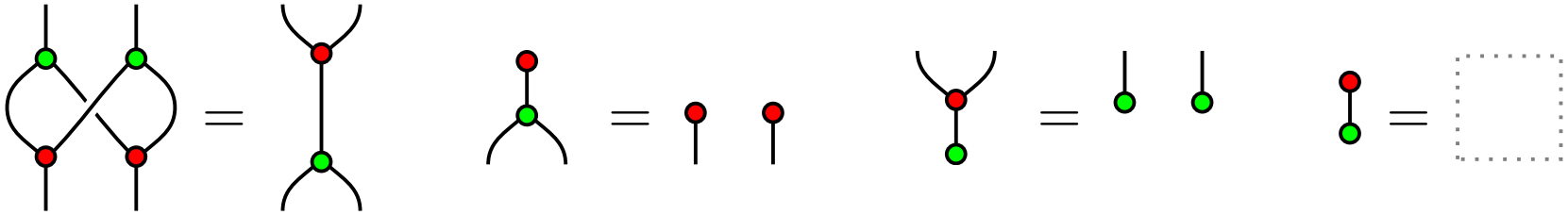
an algebra  a coalgebra 

Hopf algebras

A *Hopf algebra* in a braided monoidal category is a pair of

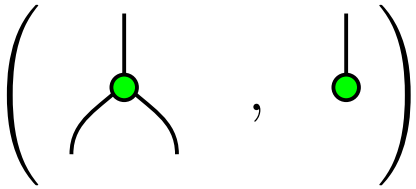
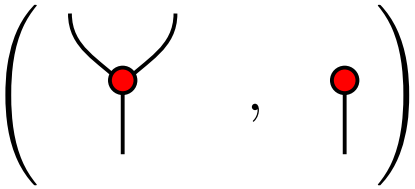
an algebra  a coalgebra 

that form a *bialgebra*

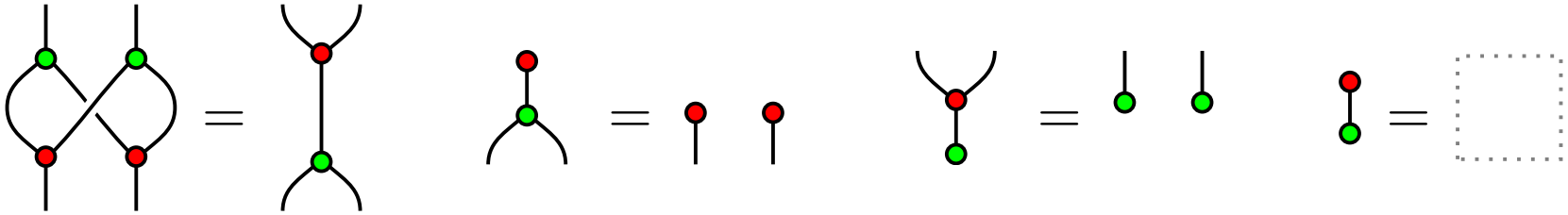


Hopf algebras

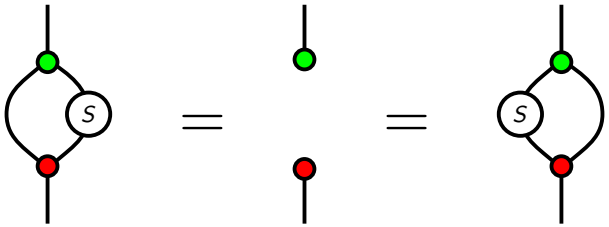
A *Hopf algebra* in a braided monoidal category is a pair of

an algebra  a coalgebra 

that form a *bialgebra*

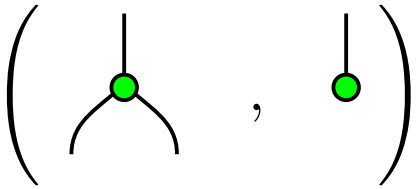
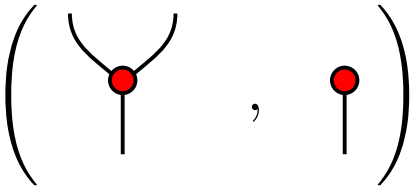


and have an *antipode*; an endomorphism S fulfilling

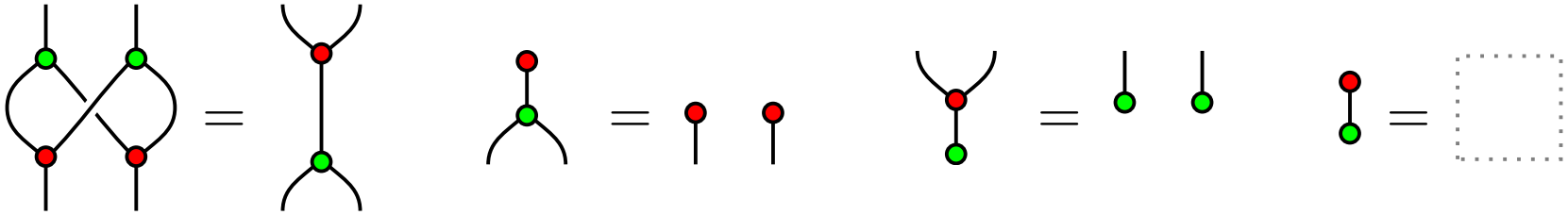


Hopf algebras

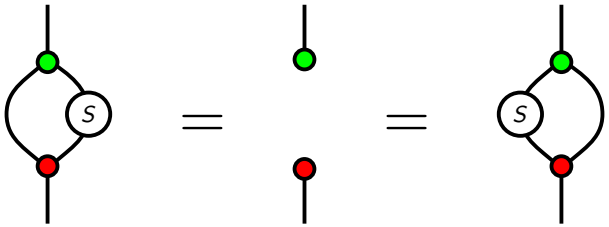
A *Hopf algebra* in a braided monoidal category is a pair of

an algebra  a coalgebra 

that form a *bialgebra*



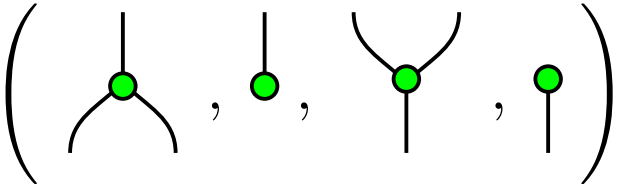
and have an *antipode*; an endomorphism S fulfilling

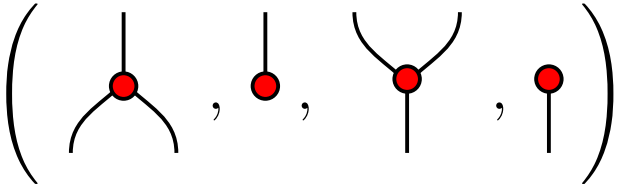


Here, we consider more restrictive algebras.

Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

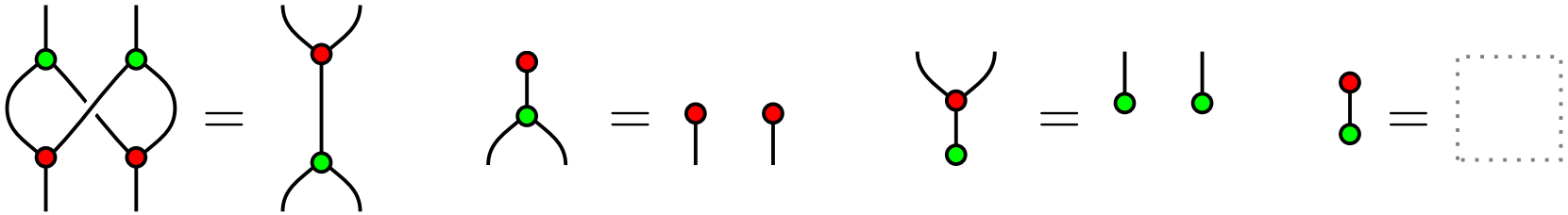




Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

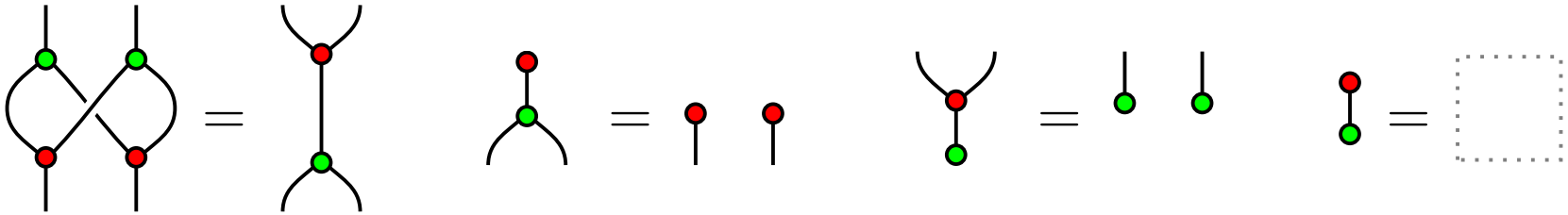
that form a *bialgebra*



Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*

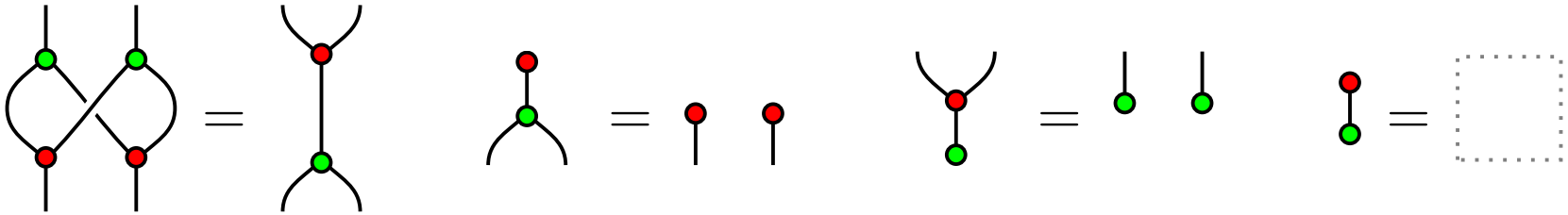


and such that

Unimodular Hopf algebras

A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*



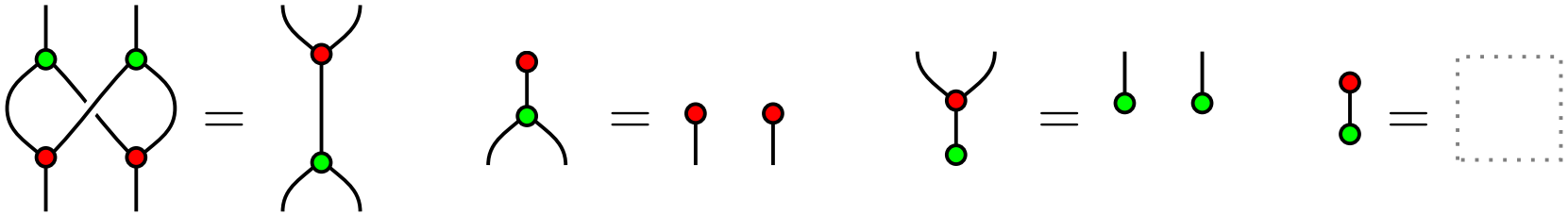
and such that

uHopf: free *braided* monoidal category on a unimodular Hopf algebra

Unimodular Hopf algebras

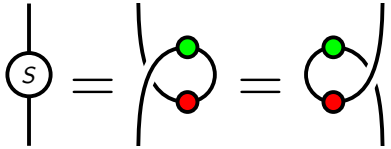
A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*



and such that

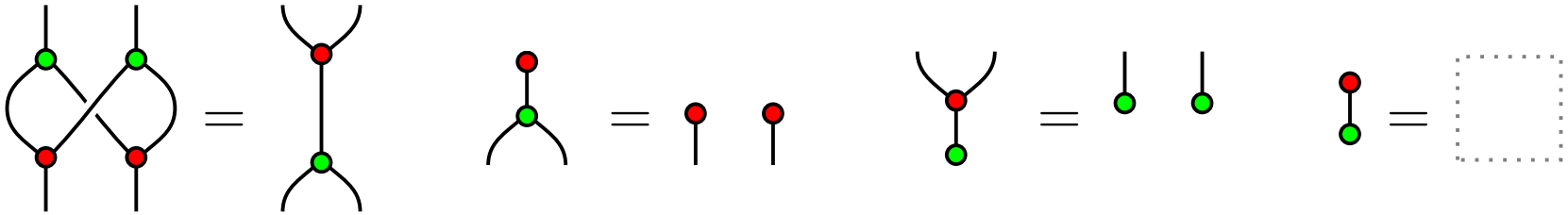
uHopf: free *braided* monoidal category on a unimodular Hopf algebra

Theorem. The antipode of a unimodular Hopf algebra is  .

Unimodular Hopf algebras

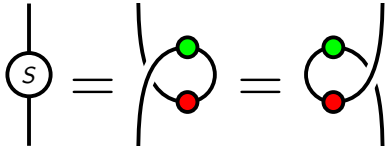
A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*



and such that

uHopf: free *braided* monoidal category on a unimodular Hopf algebra

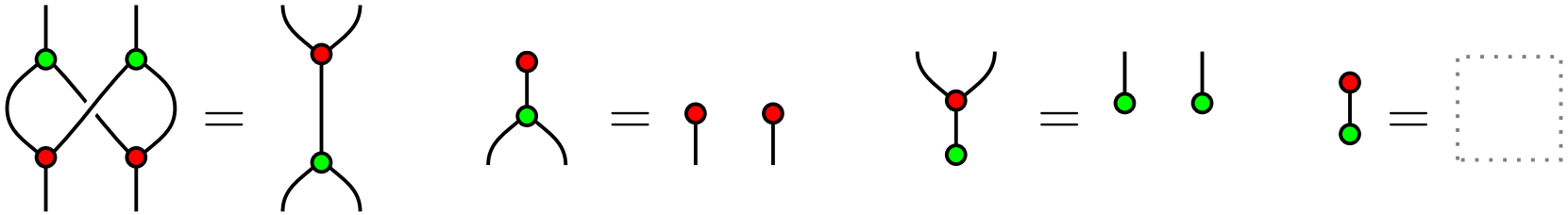
Theorem. The antipode of a unimodular Hopf algebra is  .

U. Hopf algebras in $\mathbf{Vect}_k$ are *finite dimensional unimodular Hopf algebras*.

Unimodular Hopf algebras

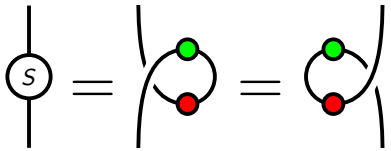
A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*



and such that

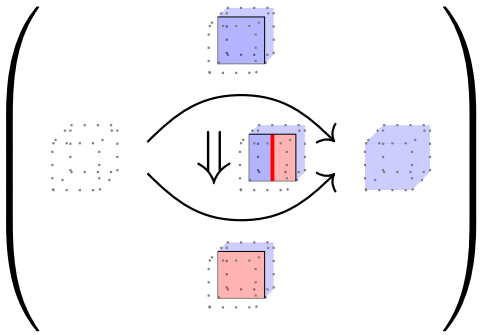
uHopf: free *braided* monoidal category on a unimodular Hopf algebra

Theorem. The antipode of a unimodular Hopf algebra is  .

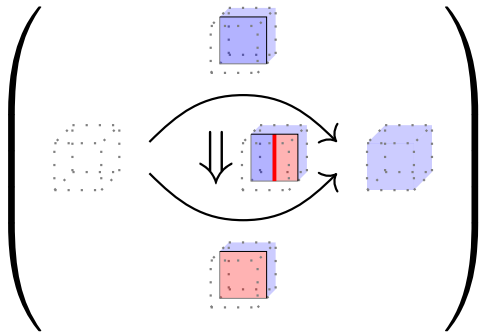
U. Hopf algebras in $\mathbf{Vect}_k$ are *finite dimensional unimodular Hopf algebras*.

Example. Any finite dimensional semisimple and cosemisimple Hopf algebra.

A topological 3-category

Start with $\mathcal{F}_3$ 

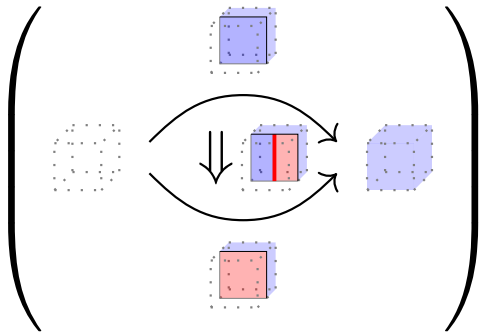
A topological 3-category

Start with $\mathcal{F}_3$  :

free 'topological' 3-category on

- a blue surface
- a red surface
- a blue-red boundary wire

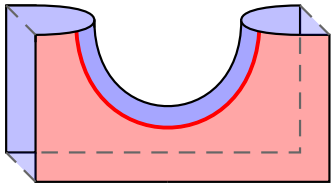
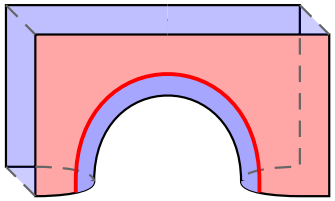
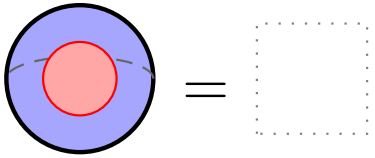
A topological 3-category

Start with $\mathcal{F}_3$  :

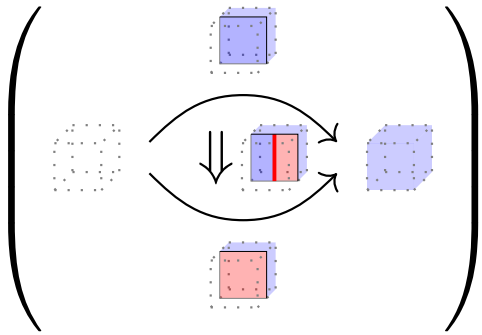
free 'topological' 3-category on

- a blue surface
- a red surface
- a blue-red boundary wire

Definition. $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

 is inverse to  and 

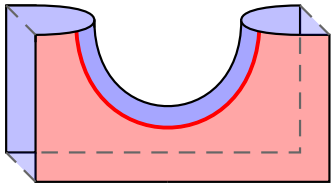
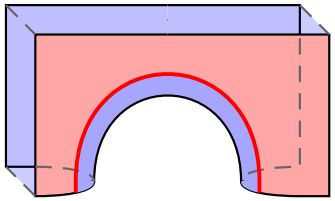
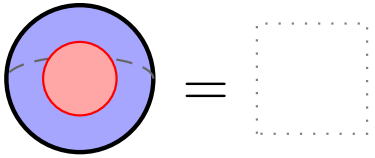
A topological 3-category

Start with $\mathcal{F}_3$  :

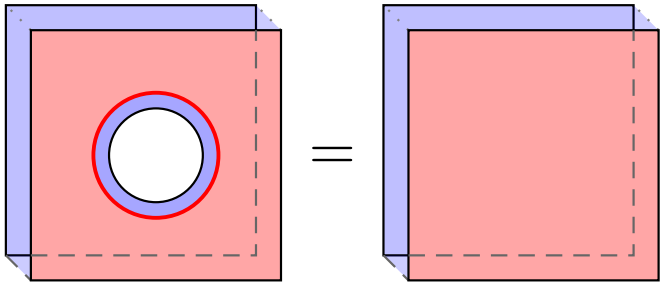
free 'topological' 3-category on

- a blue surface
- a red surface
- a blue-red boundary wire

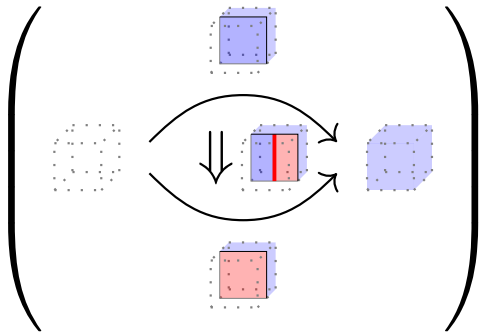
Definition. $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

 is inverse to  and 

Explicitly, invertibility of the saddles means:



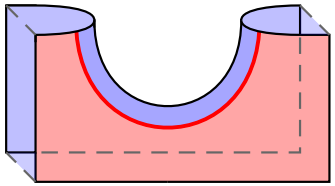
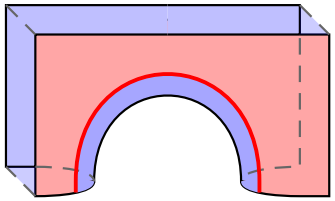
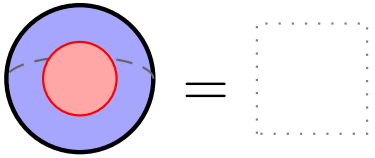
A topological 3-category

Start with $\mathcal{F}_3$  :

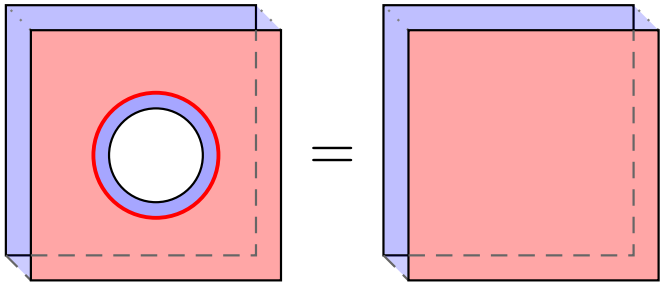
free 'topological' 3-category on

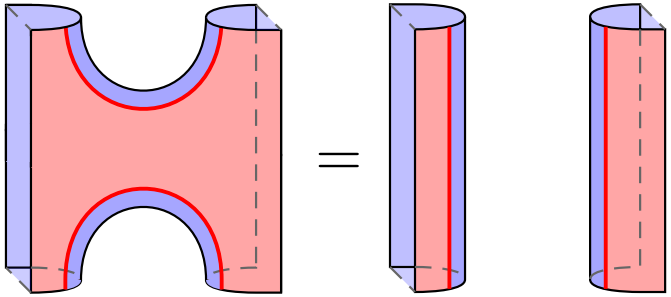
- a blue surface
- a red surface
- a blue-red boundary wire

Definition. $\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:

 is inverse to  and 

Explicitly, invertibility of the saddles means:



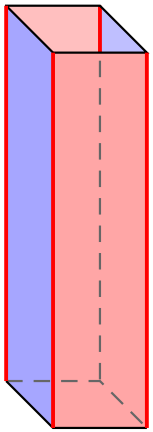


A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.

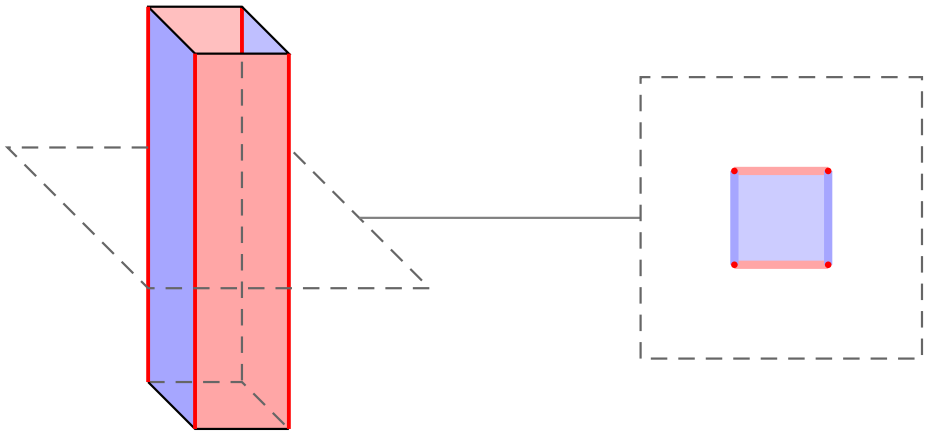
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:



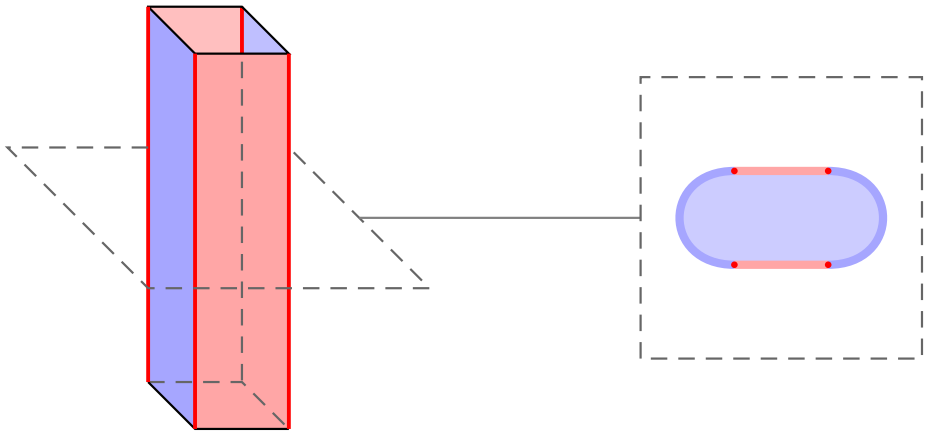
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:



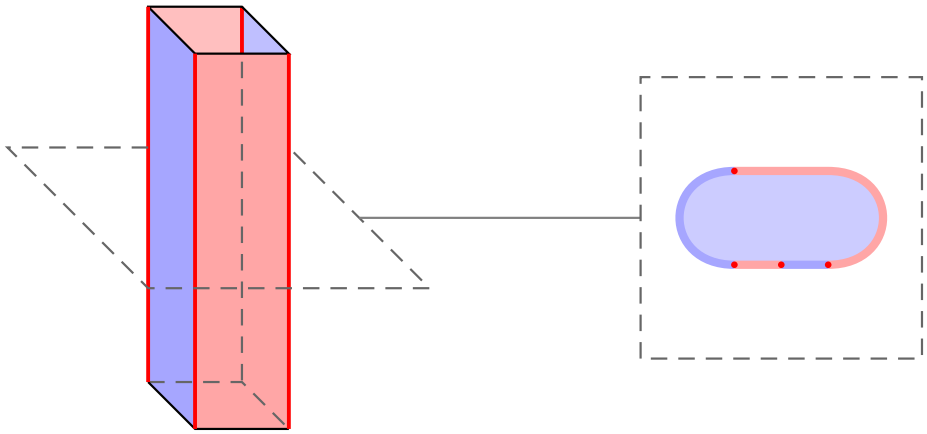
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:



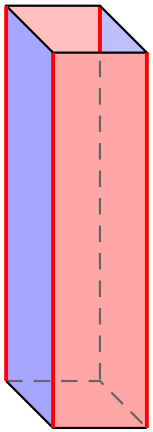
A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:

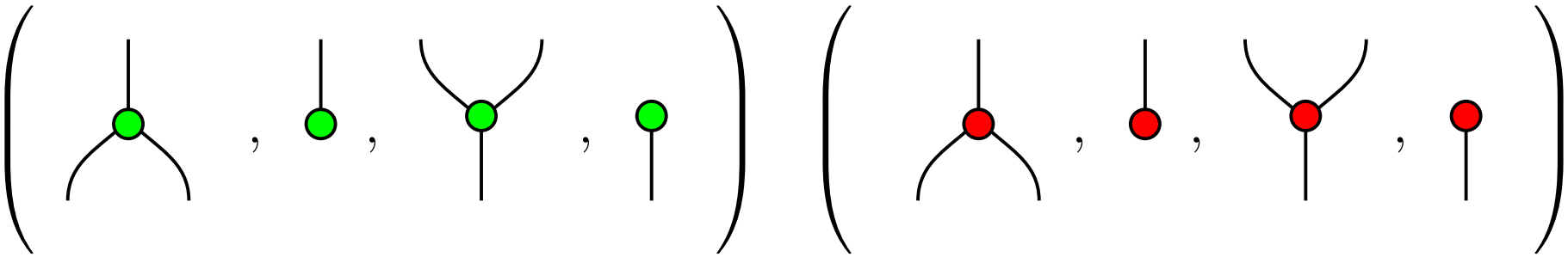


A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:

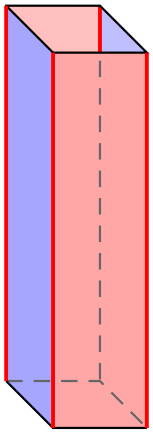


The two interacting Frobenius structures are:

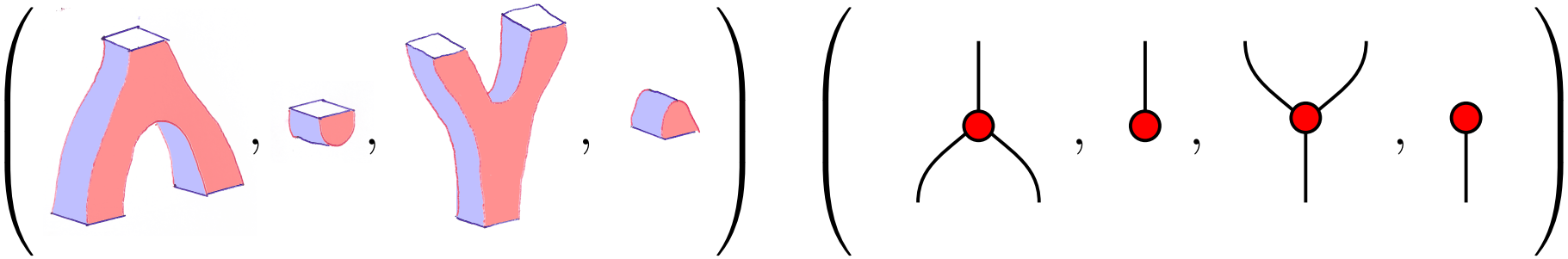


A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:

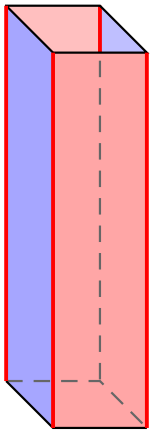


The two interacting Frobenius structures are:

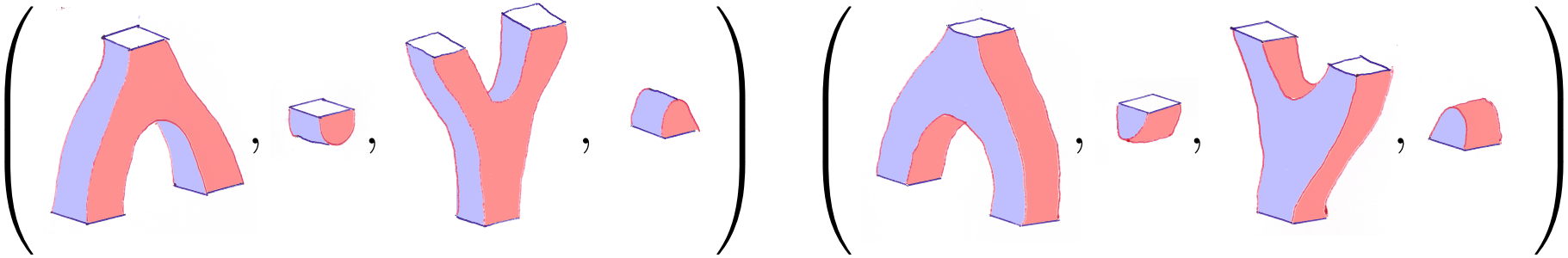


A Hopf algebra in $\mathbb{H}$

There is a Hopf algebra in $\mathbb{H}$.
It lives on the following 'thickened' wire:



The two interacting Frobenius structures are:

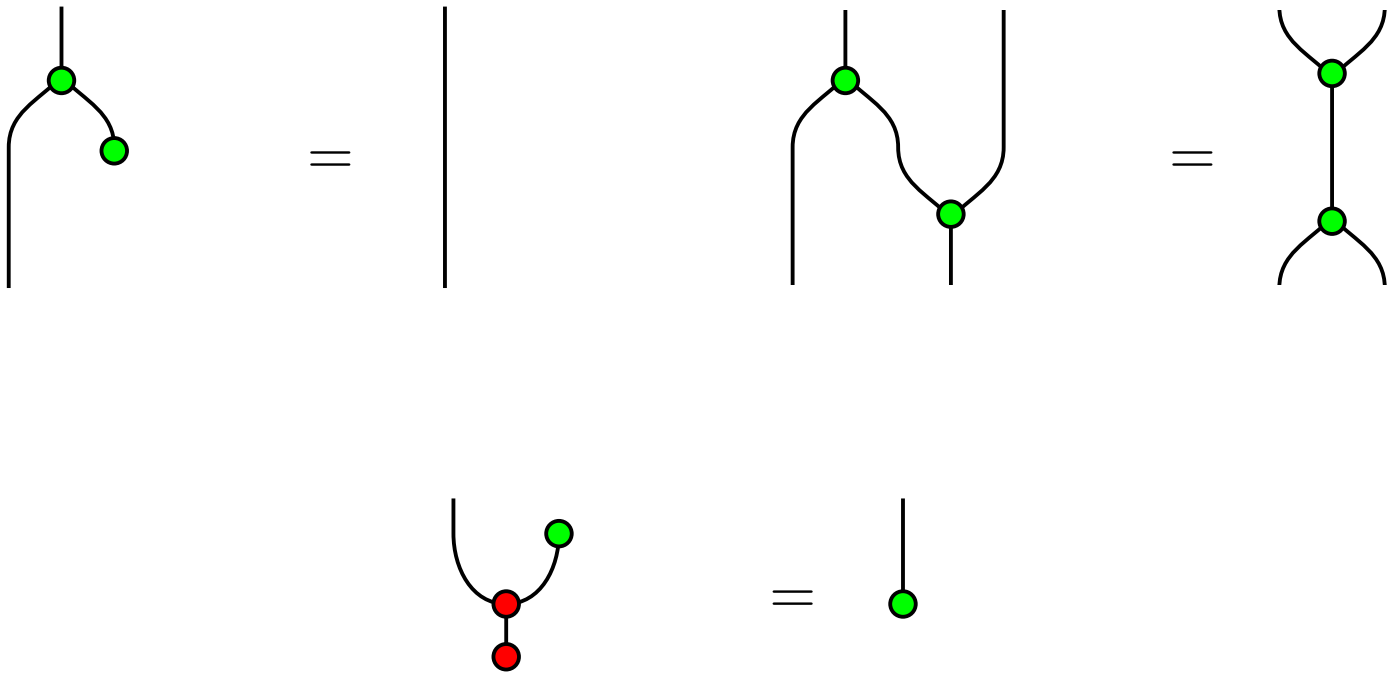


A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:

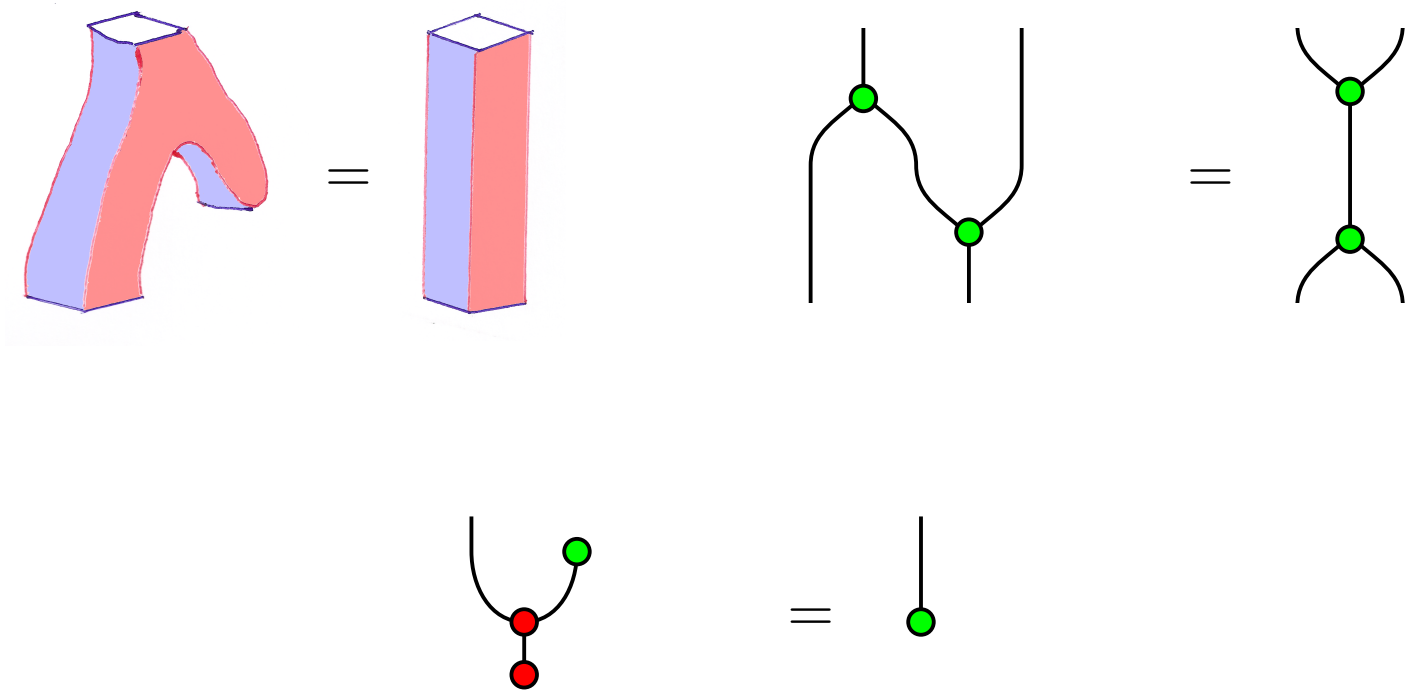
A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:



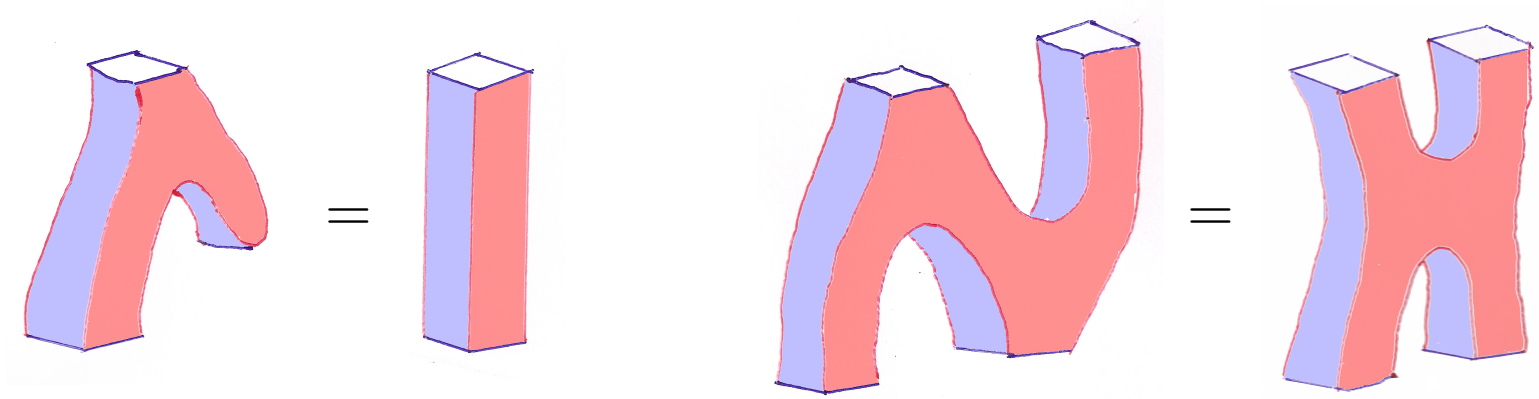
A Hopf algebra in $\mathbb{H}$

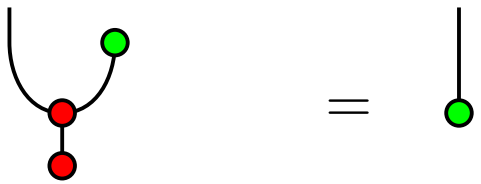
Let's check (some of) the axioms of unimodular Hopf algebras:



A Hopf algebra in $\mathbb{H}$

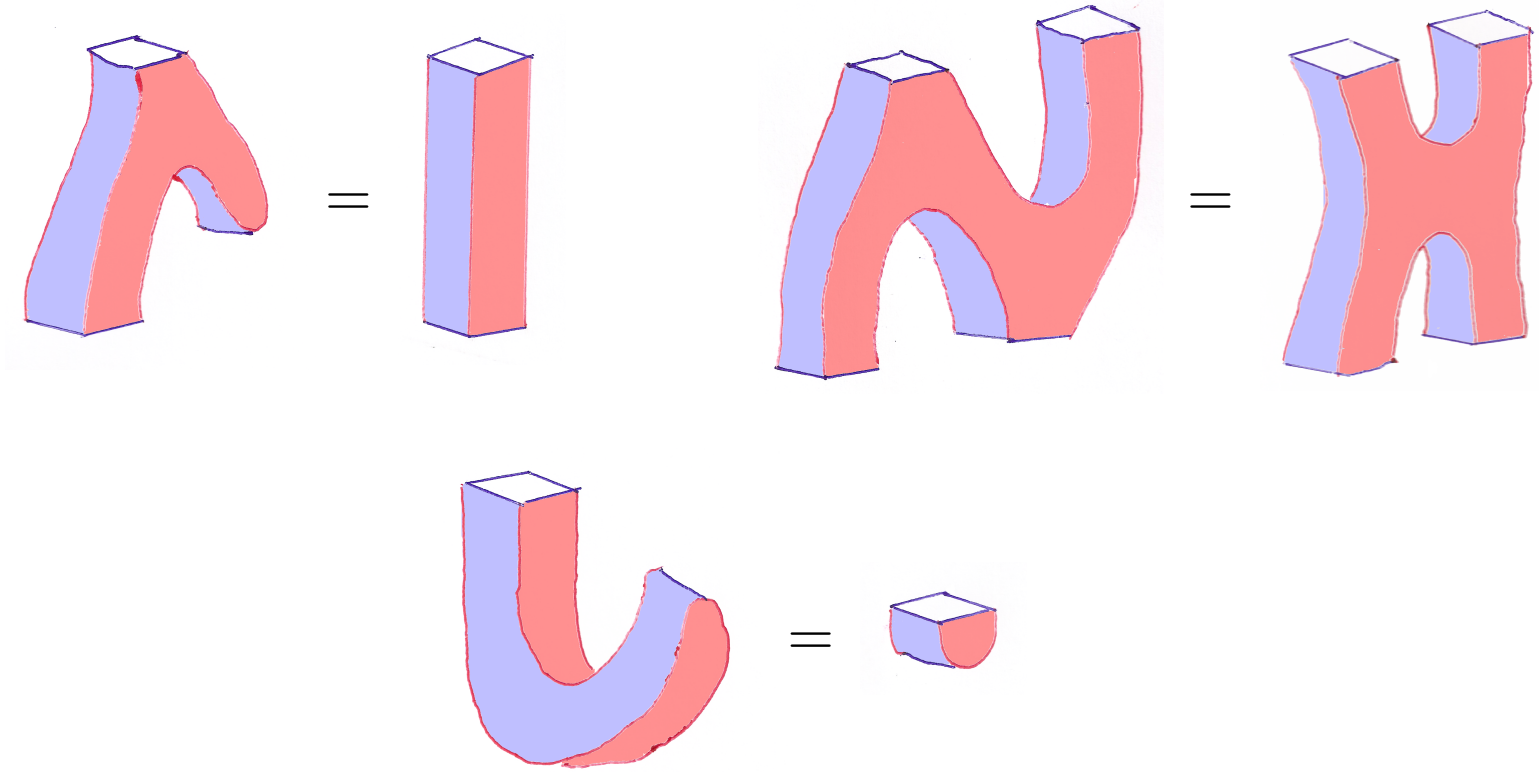
Let's check (some of) the axioms of unimodular Hopf algebras:





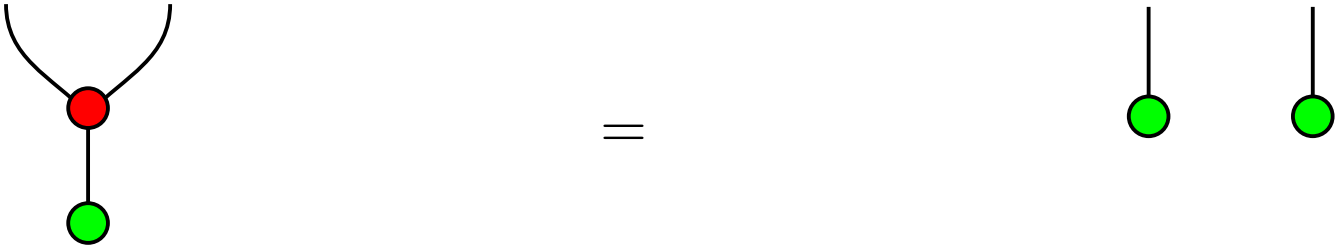
A Hopf algebra in $\mathbb{H}$

Let's check (some of) the axioms of unimodular Hopf algebras:



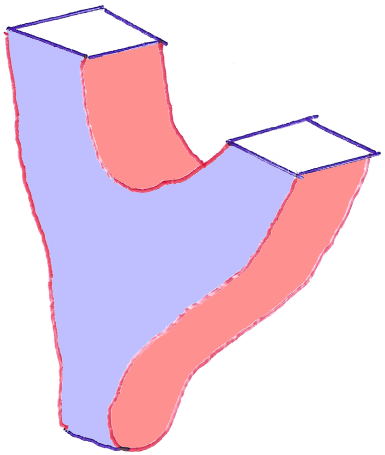
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:

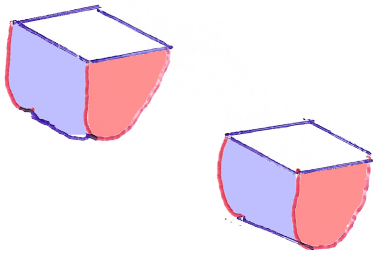


A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:

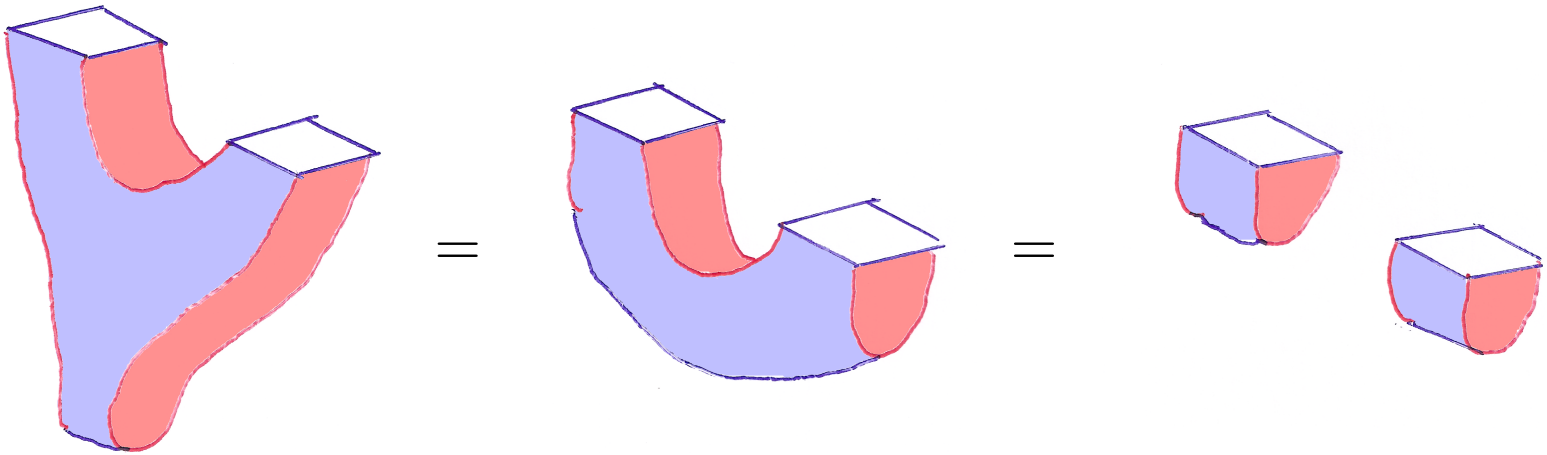


$=$



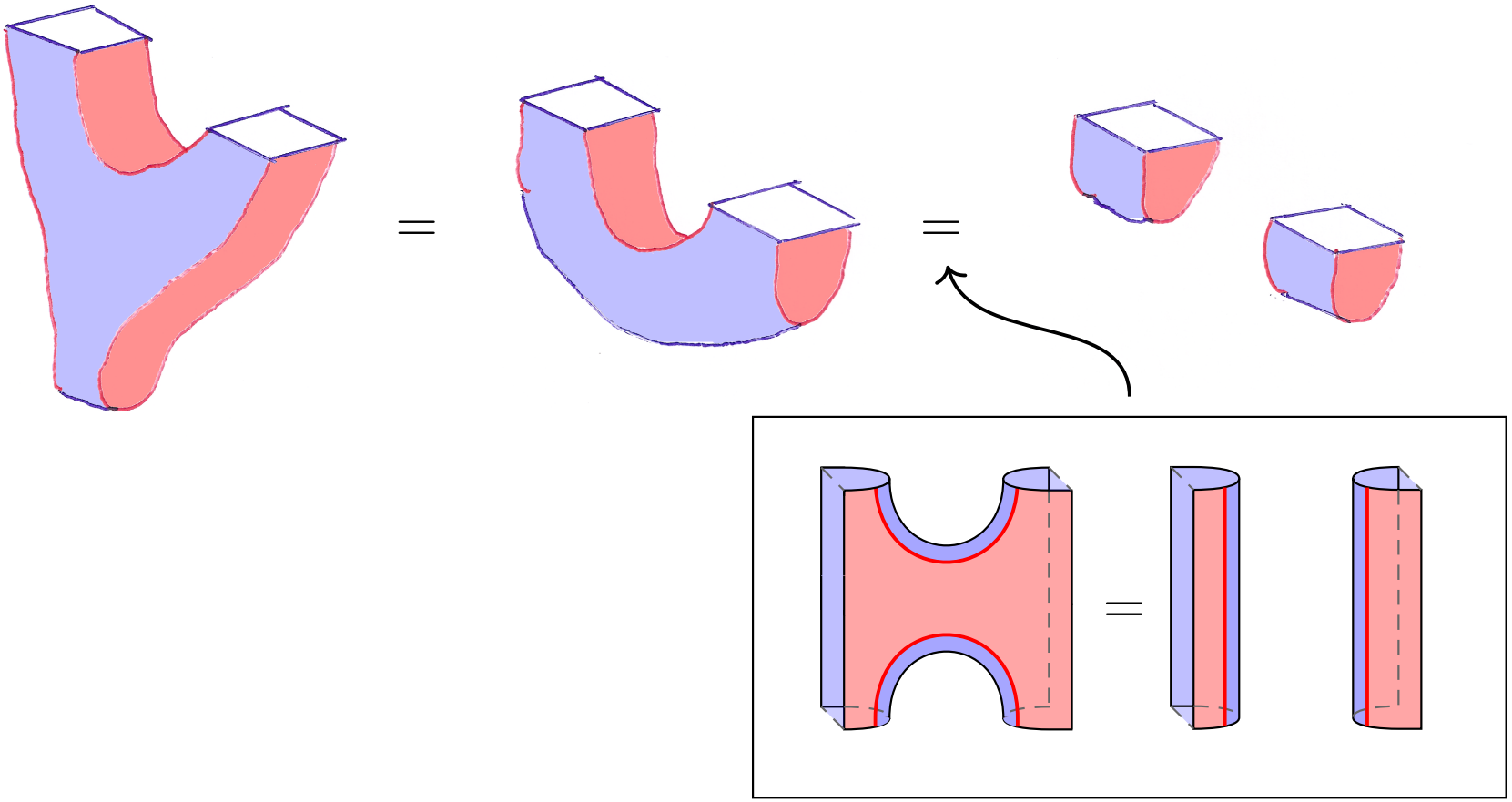
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



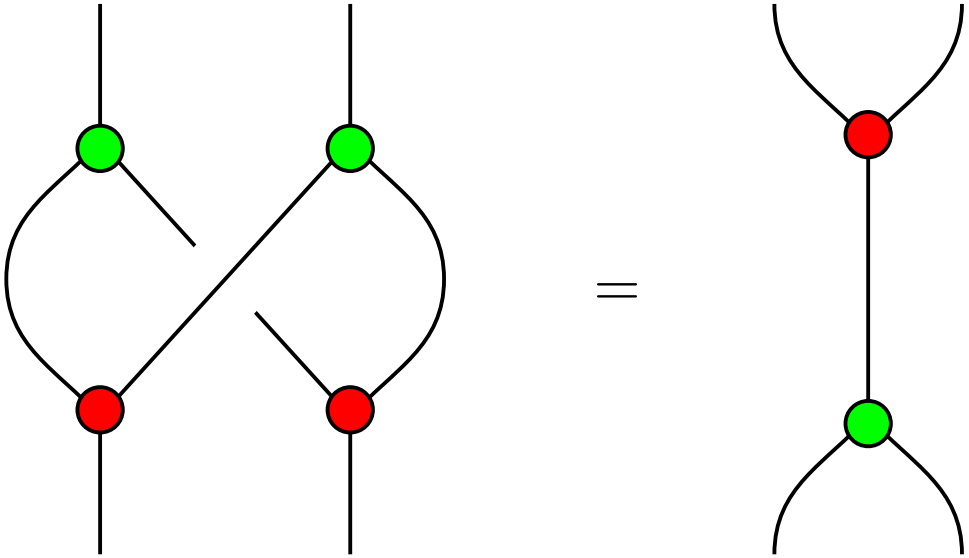
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



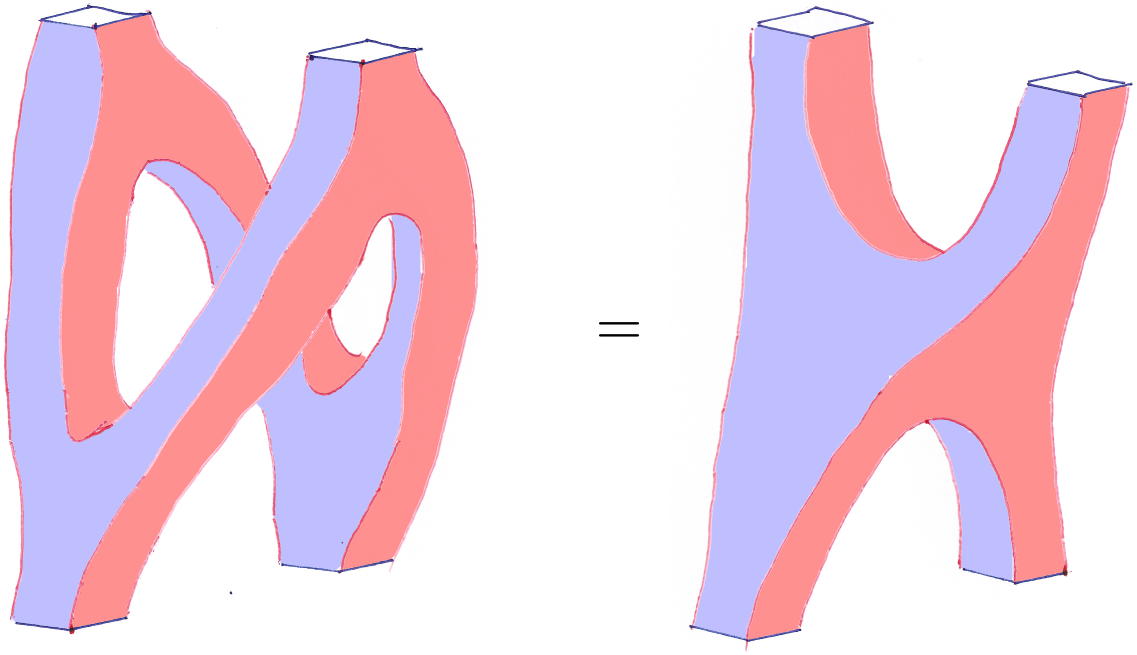
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



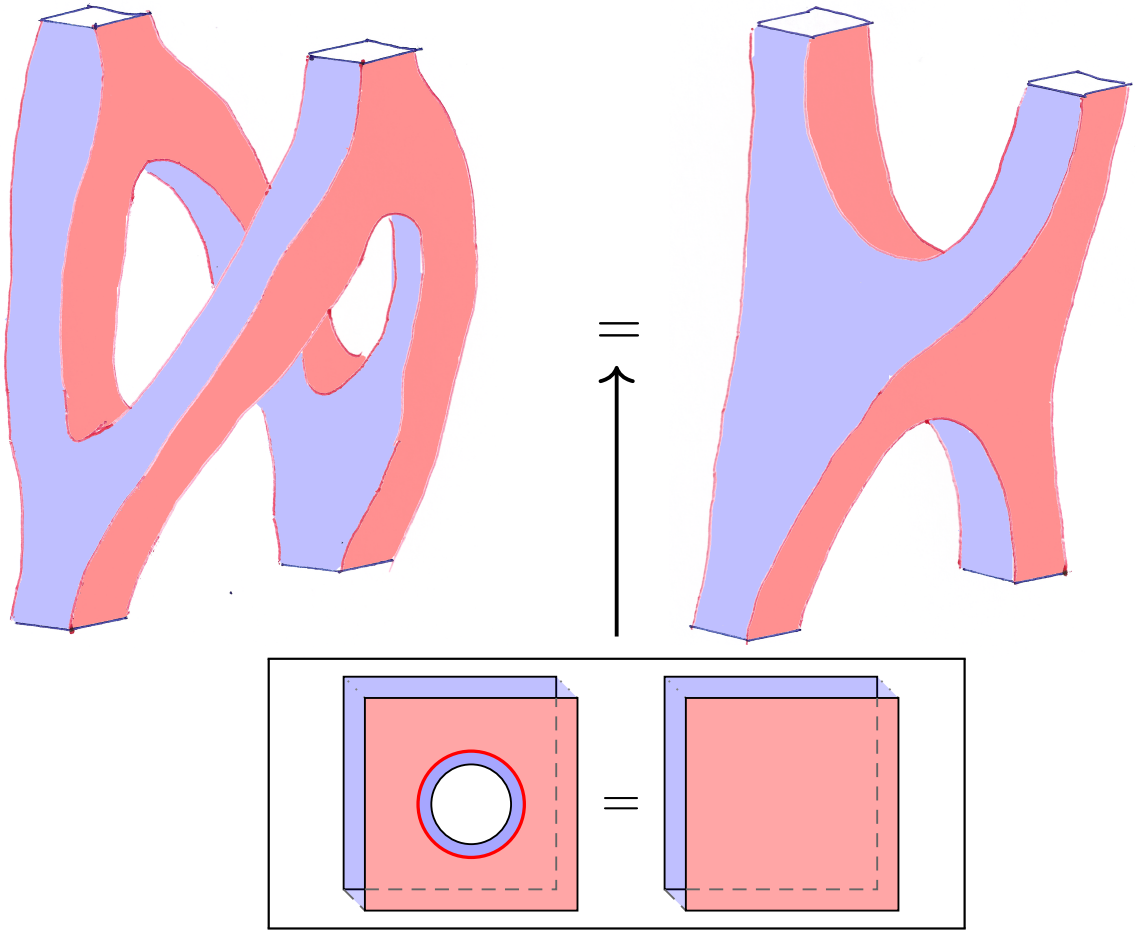
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



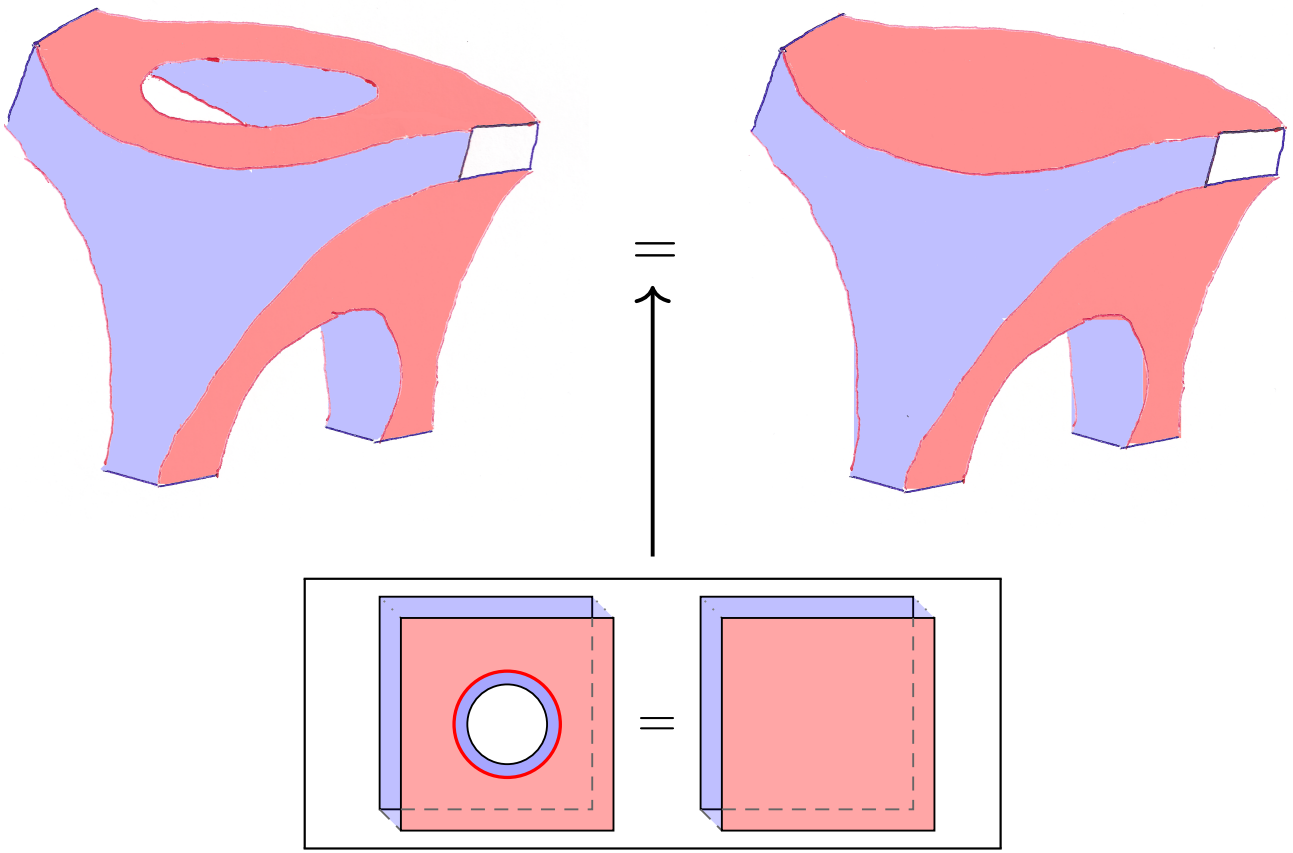
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



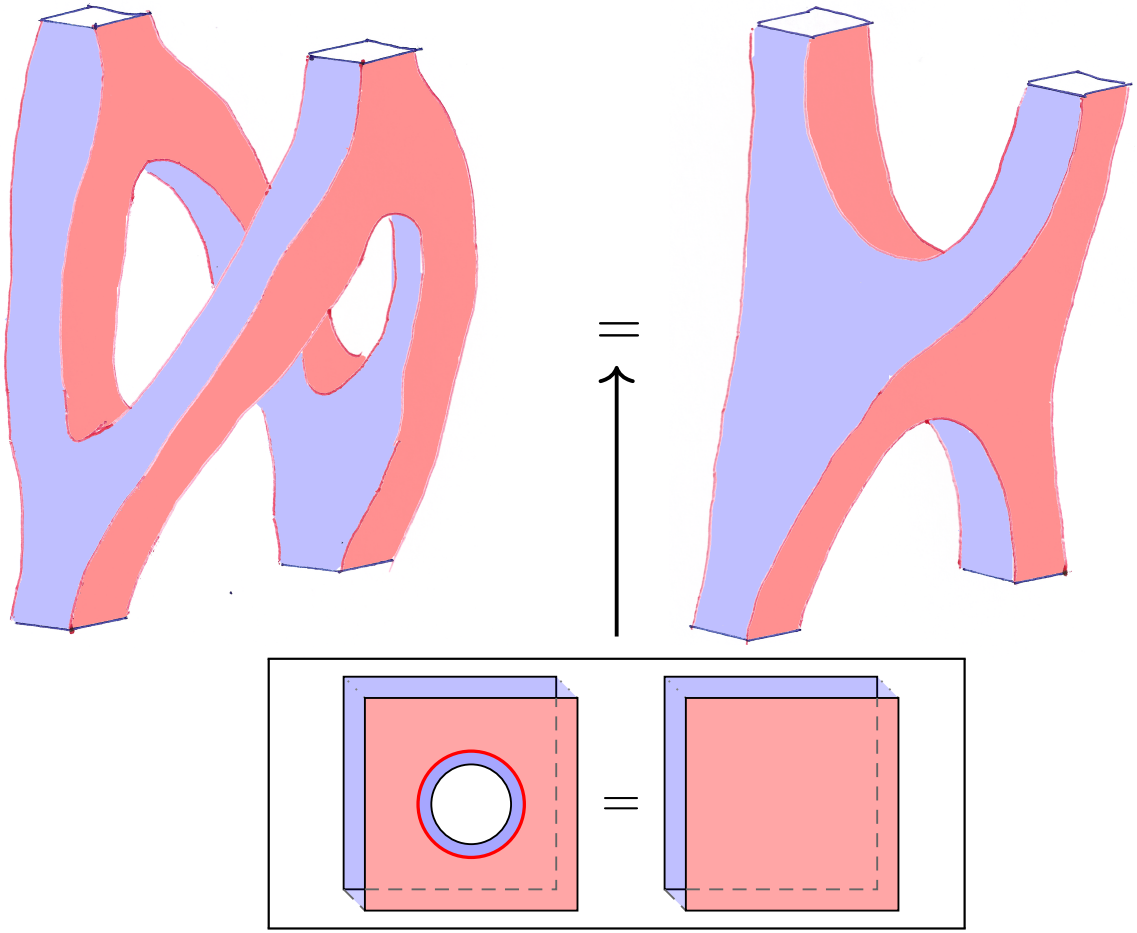
A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



A Hopf algebra in $\mathbb{H}$

The bialgebra laws correspond to the invertiblity of the saddle:



A Hopf algebra in $\mathbb{H}$

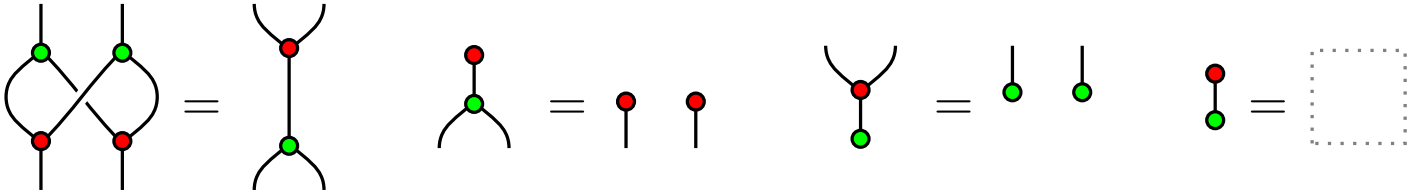
Summary. uHopf is a shadow of a simpler 3-category.

A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

A *unimodular Hopf algebra* is a pair of Frobenius algebras

that form a *bialgebra*

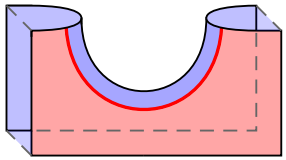


and such that

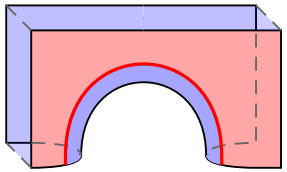
A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

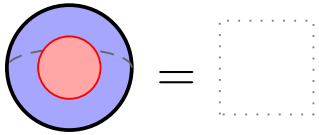
$\mathbb{H}$ is the free 3-category with duals on two surfaces and a boundary wire, such that the following hold:



is inverse to



and



A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

A Hopf algebra in $\mathbb{H}$

Summary. **uHopf** is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

Conjecture. This induces a braided equivalence **uHopf** $\cong \mathbb{H}\left(\quad,\quad\right)$.

A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

Conjecture. This induces a braided equivalence **uHopf** $\cong \mathbb{H}\Big($  $\Big)$.

Several Hopf algebraic calculations simplify in this 3D model.

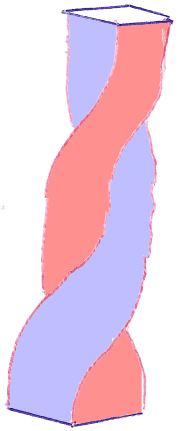
A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

Conjecture. This induces a braided equivalence **uHopf** $\cong \mathbb{H}\left(\quad,\quad\right)$.

Several Hopf algebraic calculations simplify in this 3D model.
For example, the antipode is the half twist:



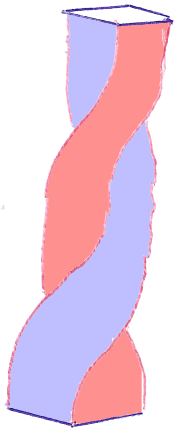
A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

Conjecture. This induces a braided equivalence **uHopf** $\cong \mathbb{H}\left(\quad,\quad\right)$.

Several Hopf algebraic calculations simplify in this 3D model.
For example, the antipode is the half twist:



$\Rightarrow$ The antipode is an algebra antihomomorphism.

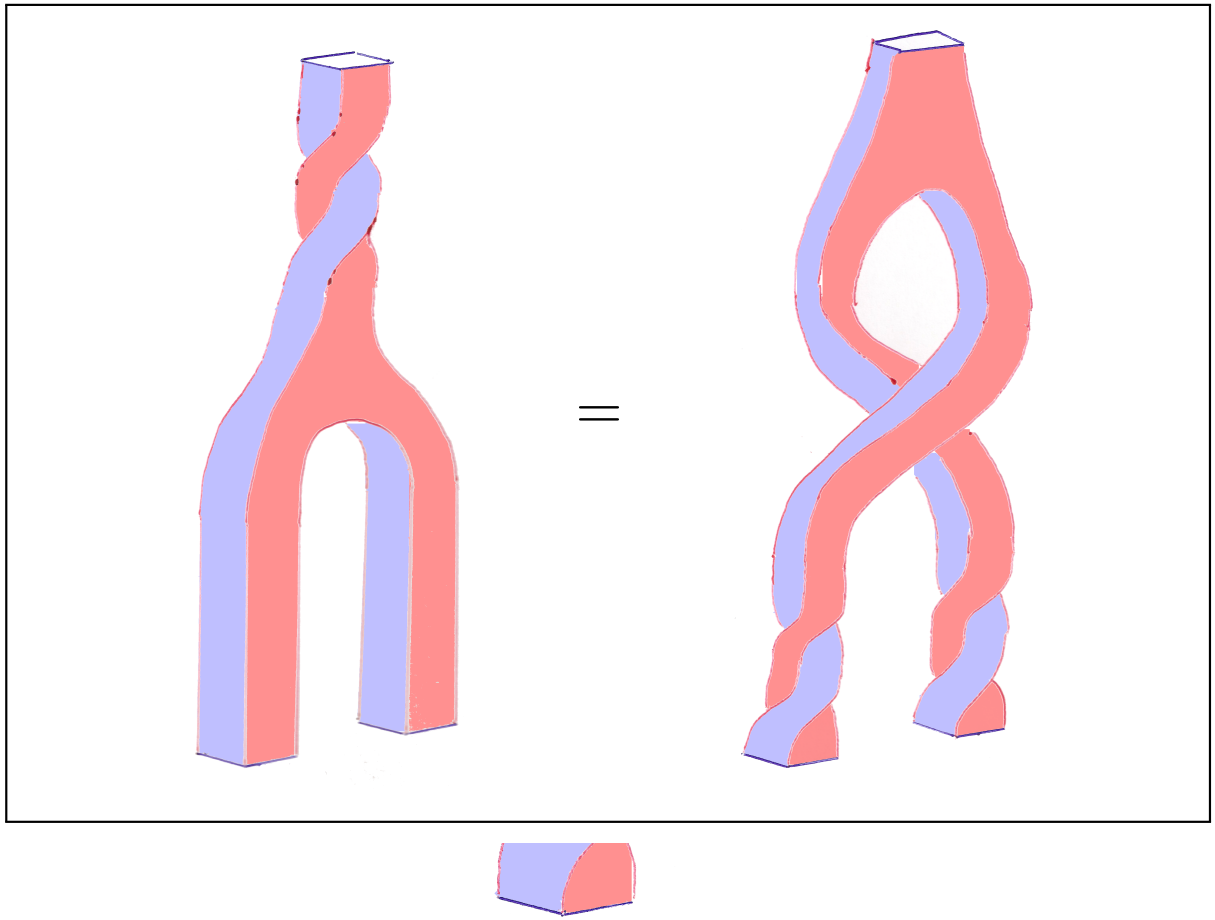
A Hopf algebra in $\mathbb{H}$

Summary. ul

Formally, we h

Conjecture.

Several Hopf
For example,



$\Rightarrow$ The antipode is an algebra antihomomorphism.

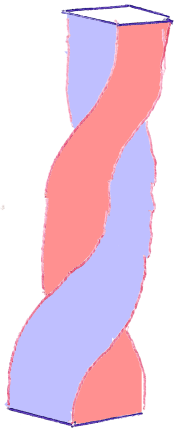
A Hopf algebra in $\mathbb{H}$

Summary. uHopf is a shadow of a simpler 3-category.

Formally, we have defined a 3-functor **uHopf** $\to \mathbb{H}$.

Conjecture. This induces a braided equivalence **uHopf** $\cong \mathbb{H}\left(\quad,\quad\right)$.

Several Hopf algebraic calculations simplify in this 3D model.
For example, the antipode is the half twist:



$\Rightarrow$ The antipode is an algebra antihomomorphism.
$\Rightarrow$ In a unimodular Hopf algebra, the antipode squares to the twist.
In particular, in a symmetric monoidal category, its 4th power is trivial.

Part 5
Higher linear algebra

Representations

So far: algebraic structures in terms of generators & relations

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $\mathbf{uHopf} \to \mathcal{C}$

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors **uHopf** $\to \mathcal{C}$

Linear representations - representation functors with target **Vect**

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors **uHopf** $\to \mathcal{C}$

Linear representations - representation functors with target **Vect**

What are the appropriate *linear* targets for higher categorical theories?

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors **uHopf** $\to \mathcal{C}$

Linear representations - representation functors with target **Vect**

What are the appropriate *linear* targets for higher categorical theories?

Expectations:

- symmetric monoidal *n*-categories **nVect** categorifying **Vect**

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $\mathbf{uHopf} \to \mathcal{C}$

Linear representations - representation functors with target **Vect**

What are the appropriate *linear* targets for higher categorical theories?

Expectations:

- symmetric monoidal *n*-categories **nVect** categorifying **Vect**
- recover **nVect** from $(\mathbf{n}+\mathbf{1})\mathbf{Vect}$: $(\mathbf{n}+\mathbf{1})\mathbf{Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{nVect}$

Representations

So far: algebraic structures in terms of generators & relations

Now: *representations* - instances of these structures in concrete categories

U. Hopf algebras in a BMC $\mathcal{C}$: braided monoidal functors $\mathbf{uHopf} \to \mathcal{C}$

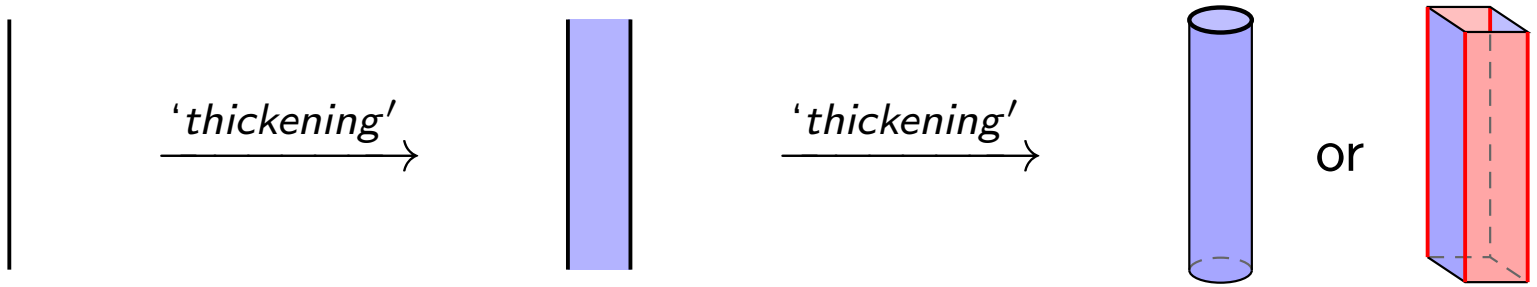
Linear representations - representation functors with target **Vect**

What are the appropriate *linear* targets for higher categorical theories?

Expectations:

- symmetric monoidal *n*-categories **nVect** categorifying **Vect**
- recover **nVect** from $(\mathbf{n}+\mathbf{1})\mathbf{Vect}$: $(\mathbf{n}+\mathbf{1})\mathbf{Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{nVect}$

nVect a 'shadow' of $(\mathbf{n}+\mathbf{1})\mathbf{Vect}$ $\leftrightsquigarrow$ $(\mathbf{n}+\mathbf{1})\mathbf{Vect}$ a 'thickening' of **nVect**



Higher linear algebra

				objects	morphisms
Vect				f.d. vector spaces	linear maps

Higher linear algebra

			objects	1-morphisms	2-morphisms
Vect				f.d. vector spaces	linear maps
2Vect			finite semisimple categories	linear functors	natural transformations

Higher linear algebra

			objects	1-morphisms	2-morphisms
Vect				f.d. vector spaces	linear maps
2Vect			finite semisimple categories	linear functors	natural transformations
			f.d. semisimple algebras	f.d. bimodules	intertwiners

Higher linear algebra

			objects	1-morphisms	2-morphisms
Vect				f.d. vector spaces	linear maps
2Vect		$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
			f.d. semisimple algebras	f.d. bimodules	intertwiners

Higher linear algebra

		objects	1-morphisms	2-morphisms	3-morphisms
Vect				f.d. vector spaces	linear maps
2Vect		$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
			f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect		fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$	$\cong$	f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$	$\cong$	f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$	$\cong$	f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

They are *symmetric monoidal* 1-,2- and 3-categories with duals.

Higher linear algebra

	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$		f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

They are *symmetric monoidal* 1-,2- and 3-categories with duals.

$$\mathbf{3Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{2Vect} \qquad \mathbf{2Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{Vect} \qquad \mathbf{Vect}(\mathrm{I},\mathrm{I}) = \mathbb{C}$$

Higher linear algebra

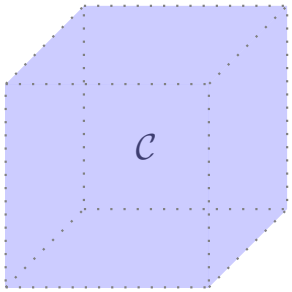
	tensor unit	objects	1-morphisms	2-morphisms	3-morphisms
$\mathbb{C}$	1				complex numbers
Vect	$\mathbb{C}$			f.d. vector spaces	linear maps
2Vect	**Vect**	$\mathrm{Rep}(-)$ $\cong$	finite semisimple categories	linear functors	natural transformations
	$\mathbb{C}$		f.d. semisimple algebras	f.d. bimodules	intertwiners
3Vect	**Vect**	fusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

They are *symmetric monoidal* 1-,2- and 3-categories with duals.

$$\mathbf{3Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{2Vect} \qquad \mathbf{2Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{Vect} \qquad \mathbf{Vect}(\mathrm{I},\mathrm{I}) = \mathbb{C}$$

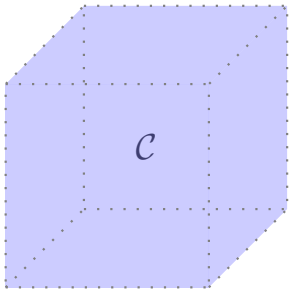
Various generalizations are possible.

The 3-category **3Vect**

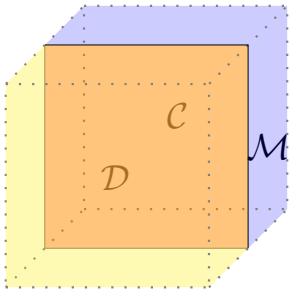


fusion category

The 3-category **3Vect**

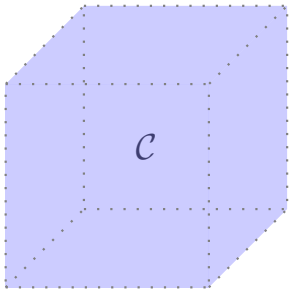


fusion category

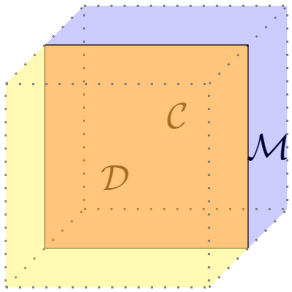


bimodule category

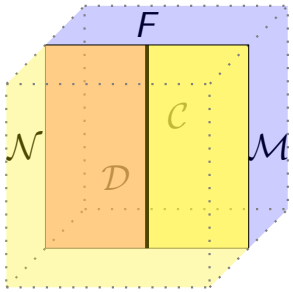
The 3-category **3Vect**



fusion category

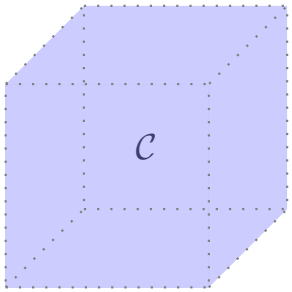


bimodule category

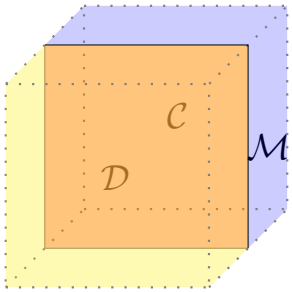


intertwining functor

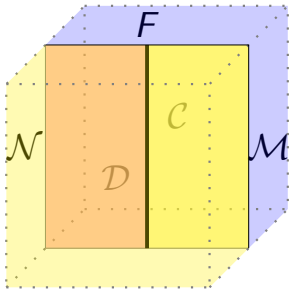
The 3-category **3Vect**



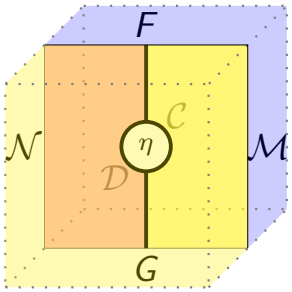
fusion category



bimodule category

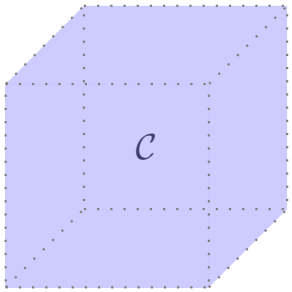


intertwining functor

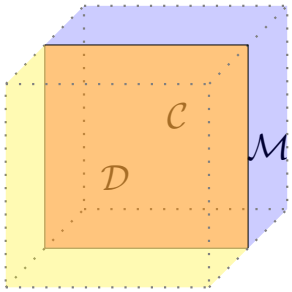


natural transformation

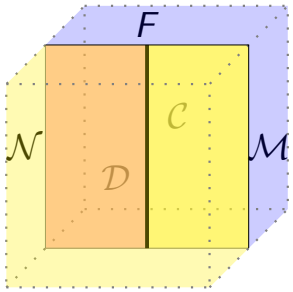
The 3-category **3Vect**



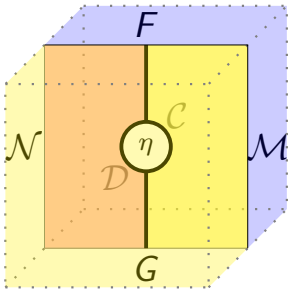
fusion category



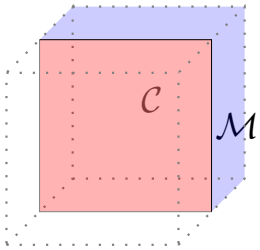
bimodule category



intertwining functor

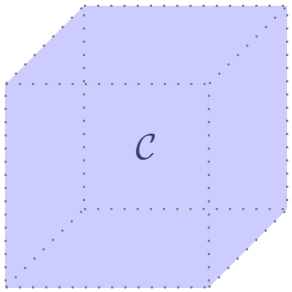


natural transformation

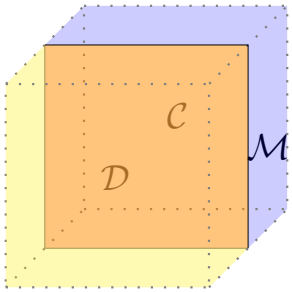


right $\mathcal{C}$-module $\mathcal{M}$

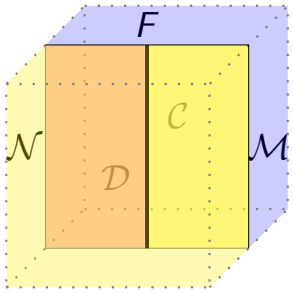
The 3-category **3Vect**



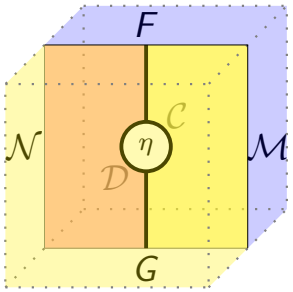
fusion category



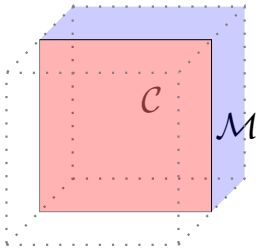
bimodule category



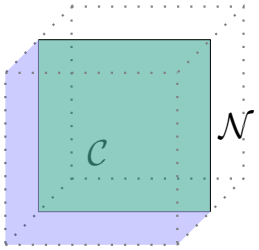
intertwining functor



natural transformation

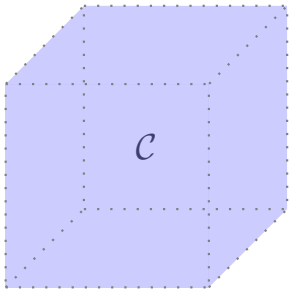


right $\mathcal{C}$-module $\mathcal{M}$

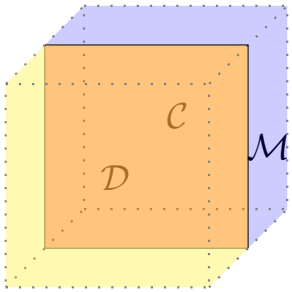


left $\mathcal{C}$-module $\mathcal{N}$

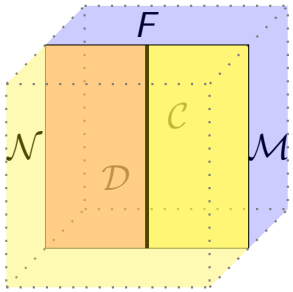
The 3-category **3Vect**



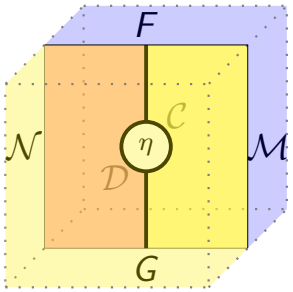
fusion category



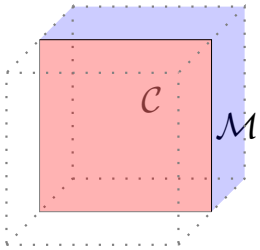
bimodule category



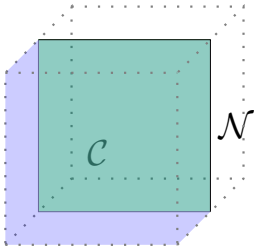
intertwining functor



natural transformation

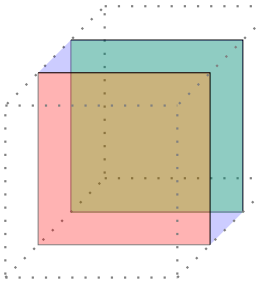


right $\mathcal{C}$-module $\mathcal{M}$



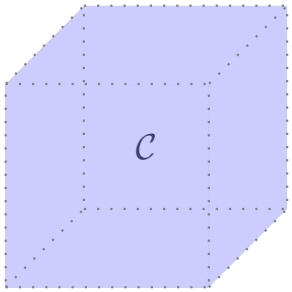
left $\mathcal{C}$-module $\mathcal{N}$

$\Rightarrow$

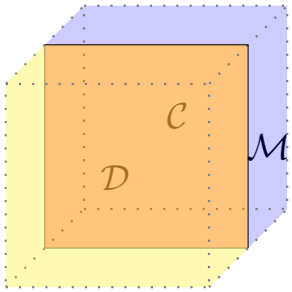


relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

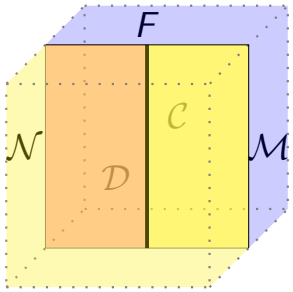
The 3-category **3Vect**



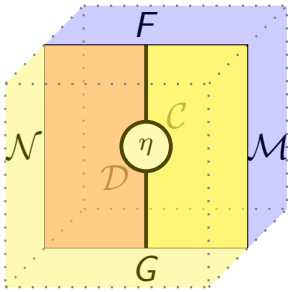
fusion category



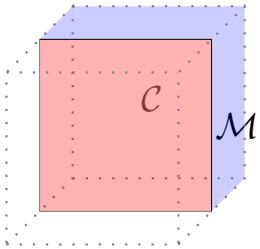
bimodule category



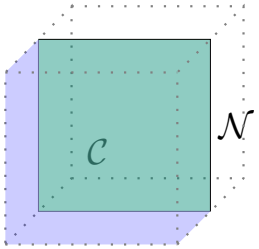
intertwining functor



natural transformation

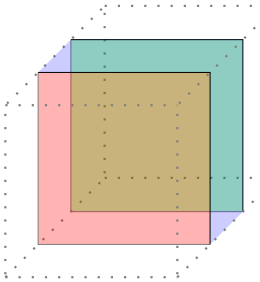


right $\mathcal{C}$-module $\mathcal{M}$



left $\mathcal{C}$-module $\mathcal{N}$

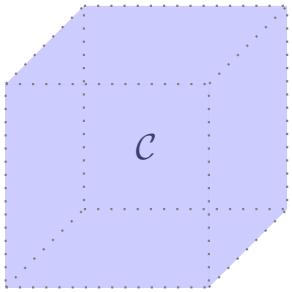
$\Rightarrow$



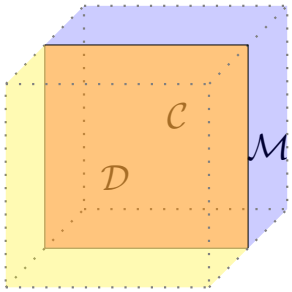
relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

relative Deligne product: universal for $\mathcal{C}$-bilinear functors out of $\mathcal{M} \times \mathcal{N}$

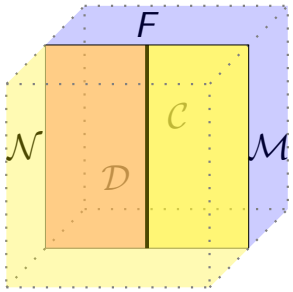
The 3-category **3Vect**



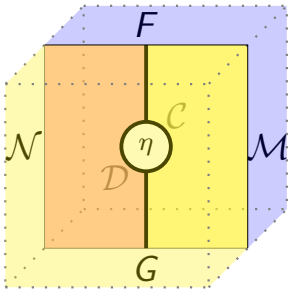
fusion category



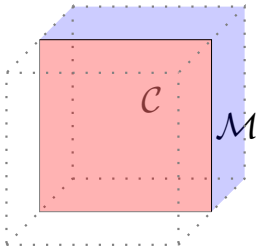
bimodule category



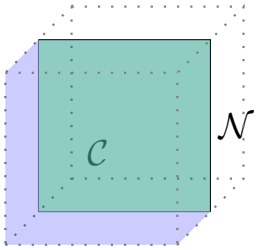
intertwining functor



natural transformation

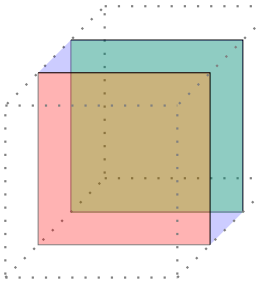


right $\mathcal{C}$-module $\mathcal{M}$

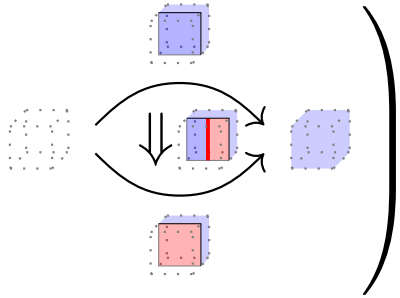


left $\mathcal{C}$-module $\mathcal{N}$

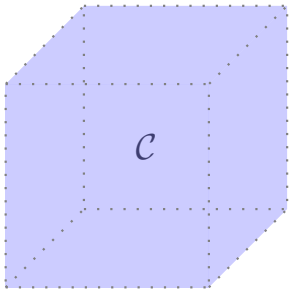
$\Rightarrow$



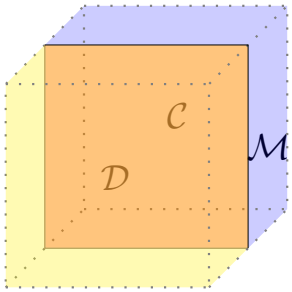
relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

$\mathcal{F}_3$  $\to$ **3Vect**

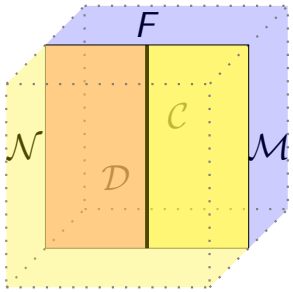
The 3-category **3Vect**



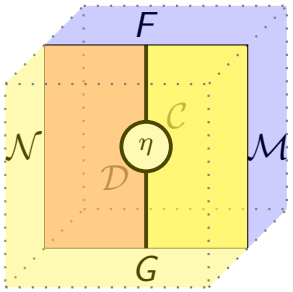
fusion category



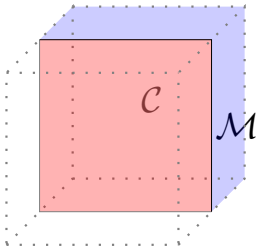
bimodule category



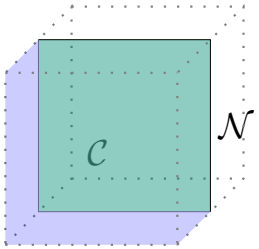
intertwining functor



natural transformation

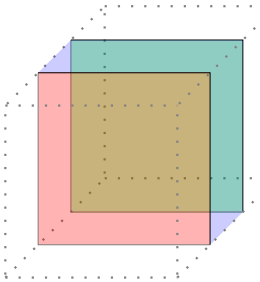


right $\mathcal{C}$-module $\mathcal{M}$

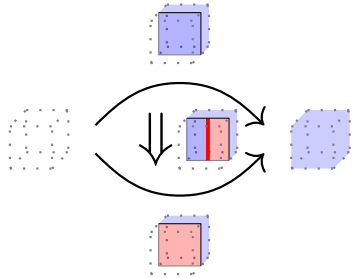


left $\mathcal{C}$-module $\mathcal{N}$

$\Rightarrow$

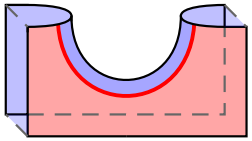
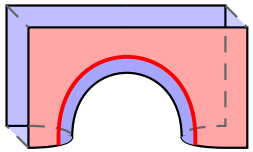
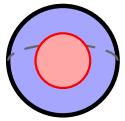


relative Deligne product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$

$\mathcal{F}_3$  $\to$ **3Vect** : $\begin{cases} \text{a fusion category } \mathcal{C} \\ \text{two right } \mathcal{C}\text{-module categories } \mathcal{M}, \mathcal{N} \\ \text{an intertwining functor } \mathcal{M} \to \mathcal{N} \end{cases}$

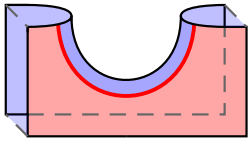
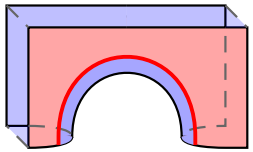
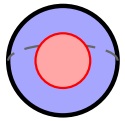
Hopf algebras and fusion categories - a sketch

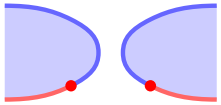
This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

Hopf algebras and fusion categories - a sketch

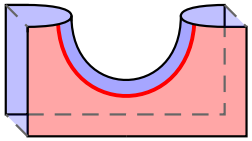
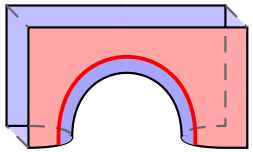
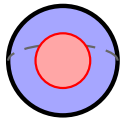
This 3-functor factors through $\mathbb{H}$ if the following hold:

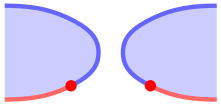
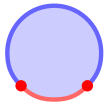
 is inverse to  and  $=$ 

$\Rightarrow$  is *isomorphic* to 

Hopf algebras and fusion categories - a sketch

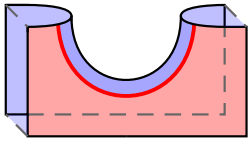
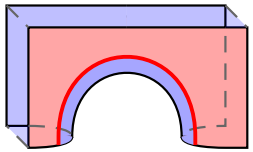
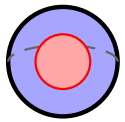
This 3-functor factors through $\mathbb{H}$ if the following hold:

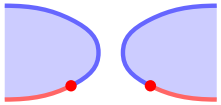
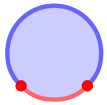
 is inverse to  and  $=$ 

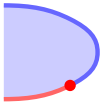
$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

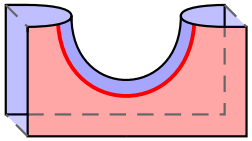
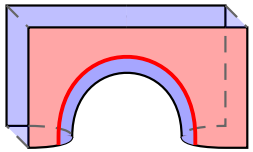
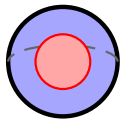
 is inverse to  and  $=$ 

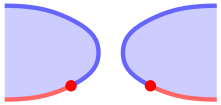
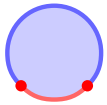
$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

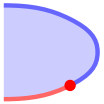
In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

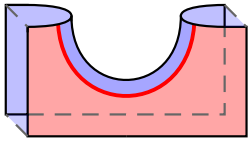
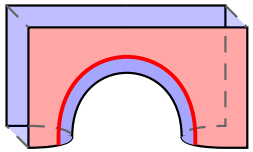
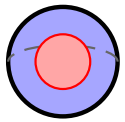
$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

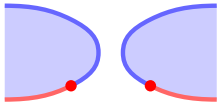
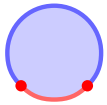
In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

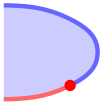
$$\text{Data of a 3-functor } \mathbb{H} \to \mathbf{3Vect} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\ \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect} \end{cases}$$

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

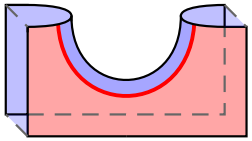
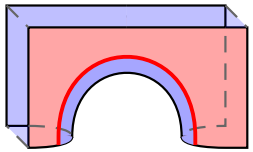
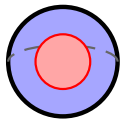
In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

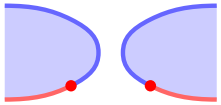
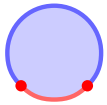
$$\text{Data of a 3-functor } \mathbb{H} \to \mathbf{3Vect} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\ \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect} \end{cases}$$

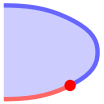
If $\mathcal{M}$ is the *regular* module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N}$

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

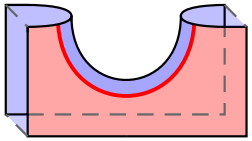
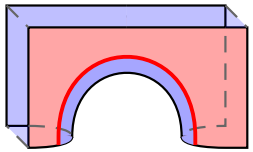
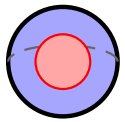
In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

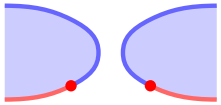
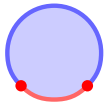
$$\begin{array}{c}\text{Data of a 3-functor}\\ \mathbb{H} \to \mathbf{3Vect}\end{array} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\ \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect} \end{cases}$$

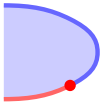
If $\mathcal{M}$ is the *regular* module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \mathbf{Vect}$.

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

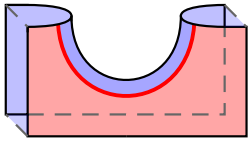
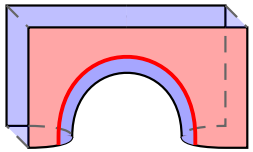
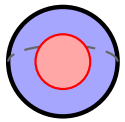
$$\text{Data of a 3-functor } \mathbb{H} \to \mathbf{3Vect} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\ \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect} \end{cases}$$

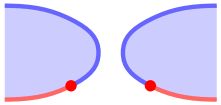
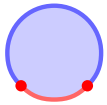
If $\mathcal{M}$ is the *regular* module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \mathbf{Vect}$.

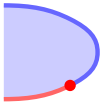
A $\mathcal{C}$-module structure on **Vect** is the same as a monoidal functor $\mathcal{C} \to \mathbf{Vect}$.

Hopf algebras and fusion categories - a sketch

This 3-functor factors through $\mathbb{H}$ if the following hold:

 is inverse to  and  $=$ 

$\Rightarrow$  is *isomorphic* to  and  is *isomorphic* to 

In other words,  $: \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect}$ is an *adjoint equivalence*!

$$\begin{array}{c}\text{Data of a 3-functor}\\ \mathbb{H} \to \mathbf{3Vect}\end{array} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a left and a right module category } \mathcal{M}, \mathcal{N} \\ \text{an adjoint equivalence } \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathbf{Vect} \end{cases}$$

If $\mathcal{M}$ is the *regular* module $\mathcal{C}$, then $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{N} \cong \mathbf{Vect}$.

A $\mathcal{C}$-module structure on **Vect** is the same as a monoidal functor $\mathcal{C} \to \mathbf{Vect}$.

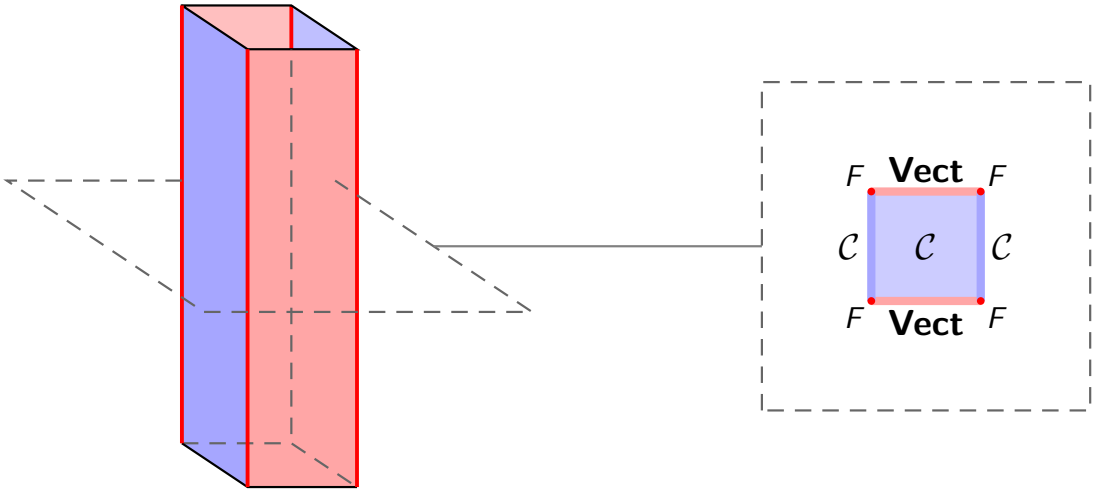
$$\begin{array}{c}\text{Data of a 3-functor}\\ \mathbb{H} \to \mathbf{3Vect} \text{ with } \mathcal{M} = \mathcal{C}\end{array} : \begin{cases} \text{a fusion category } \mathcal{C} \\ \text{a monoidal functor } \mathcal{C} \to \mathbf{Vect} \end{cases}$$

Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$

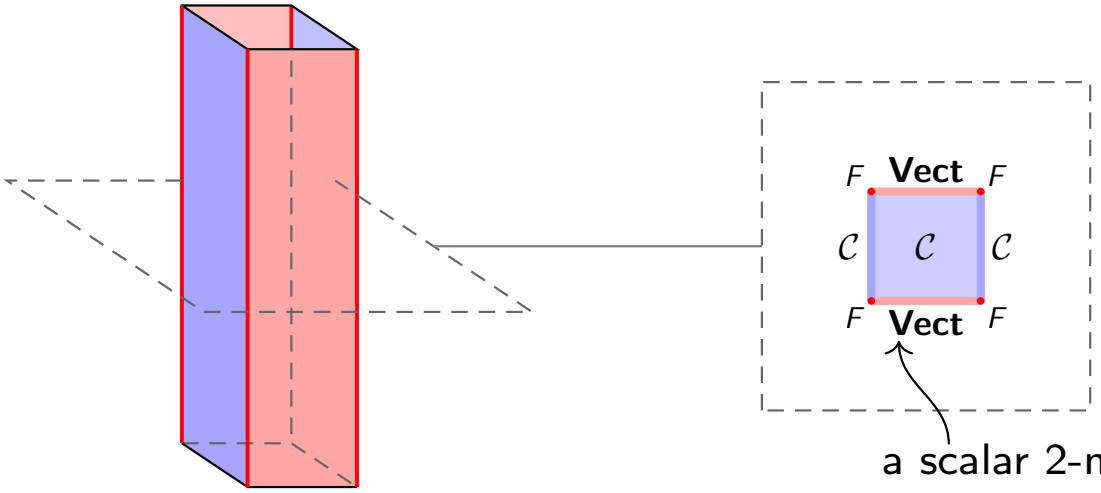
Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



Tannaka reconstruction

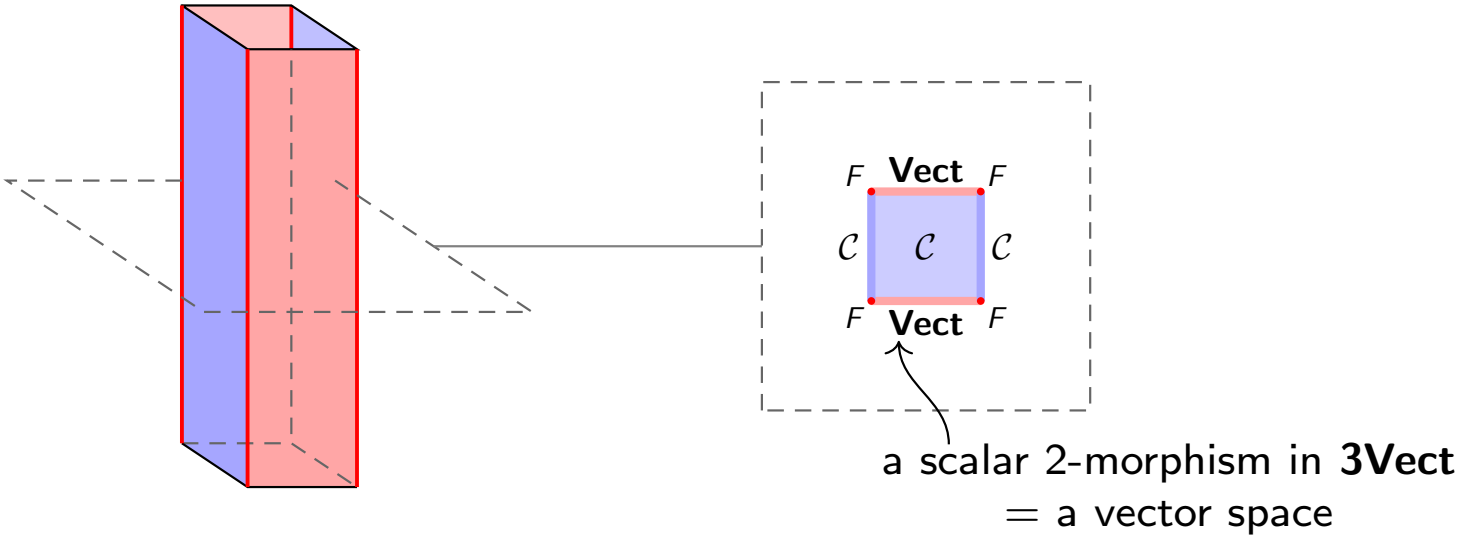
Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



a scalar 2-morphism in **3Vect**
= a vector space

Tannaka reconstruction

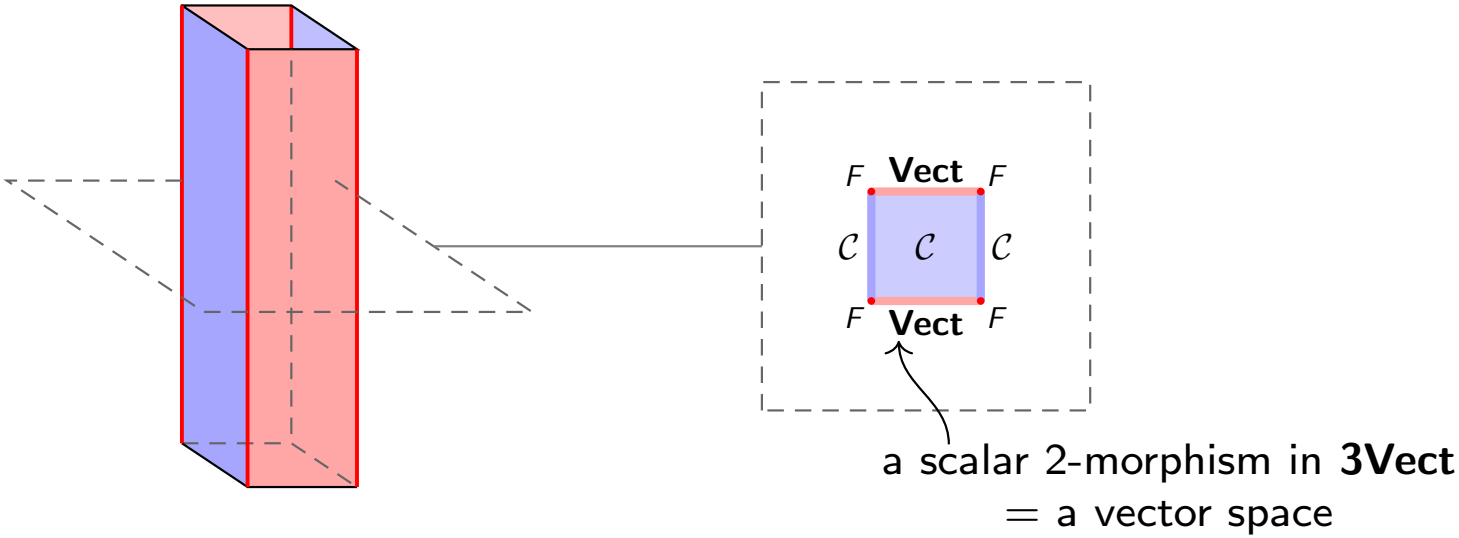
Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



This is a version of Tannaka reconstruction:
If $\mathcal{C} = \mathrm{Rep}(H) \xrightarrow{\text{forget}} \mathbf{Vect}$, this recovers the Hopf algebra H.

Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:

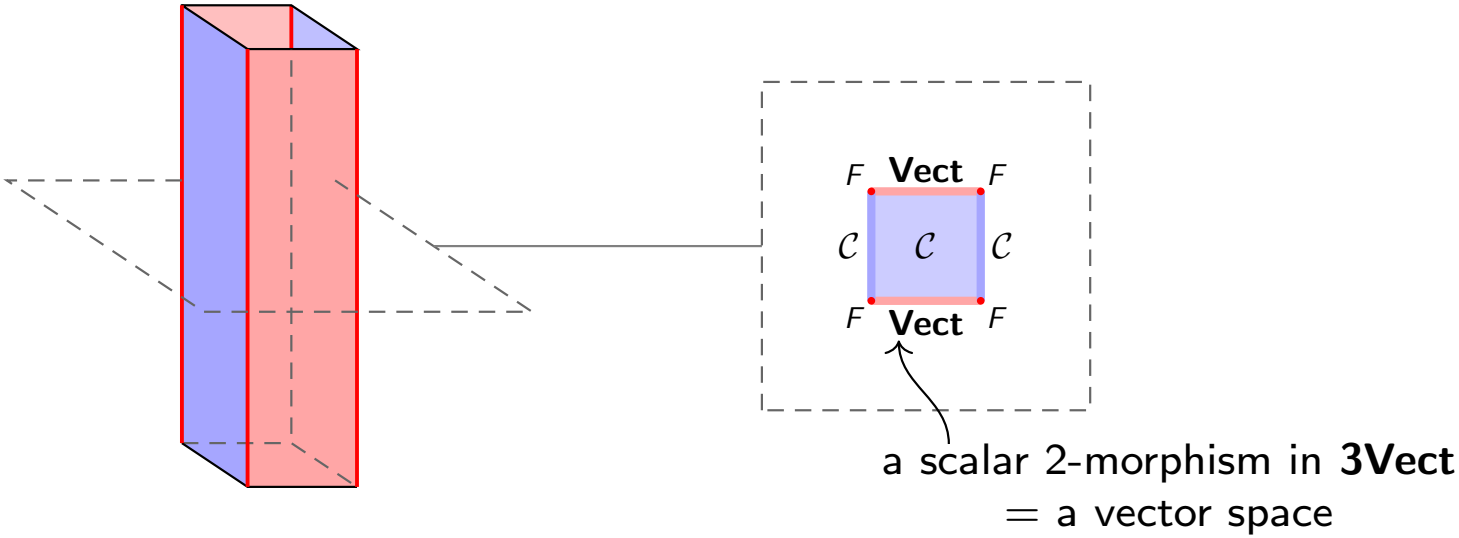


This is a version of Tannaka reconstruction:
If $\mathcal{C} = \mathrm{Rep}(H) \xrightarrow{\text{forget}} \mathbf{Vect}$, this recovers the Hopf algebra H.

Conversely, any fusion category with fibre functor $\mathcal{C} \to \mathbf{Vect}$ is of the form $\mathrm{Rep}(H)$ with H constructed as above.

Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



This is a version of Tannaka reconstruction:
If $\mathcal{C} = \mathrm{Rep}(H) \xrightarrow{\text{forget}} \mathbf{Vect}$, this recovers the Hopf algebra H.

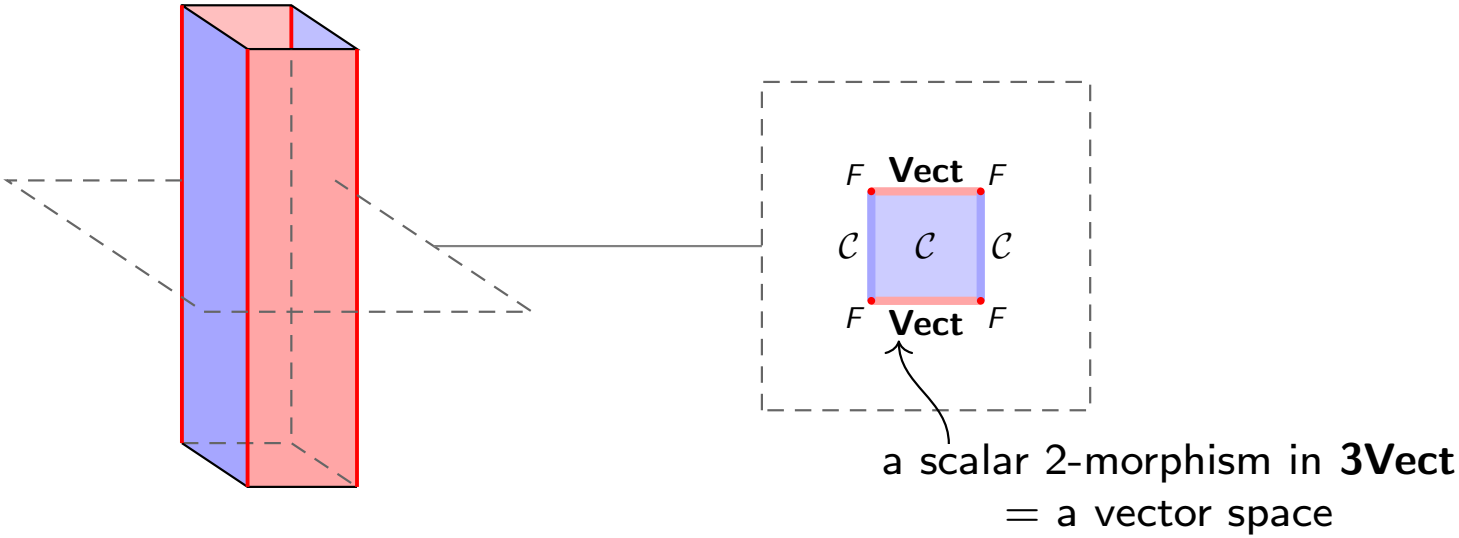
Conversely, any fusion category with fibre functor $\mathcal{C} \to \mathbf{Vect}$ is of the form $\mathrm{Rep}(H)$ with H constructed as above.

Proof. Follows from an old result of M. Müger.[1]

[1] Theorem 6.20 in [Müger, *From subfactors to categories and topology I*, 2003]

Tannaka reconstruction

Given a fusion category $\mathcal{C}$ with a monoidal functor $\mathcal{C} \xrightarrow{F} \mathbf{Vect}$
$\Rightarrow$ The following vector space is a unimodular Hopf algebra:



This is a version of Tannaka reconstruction:

If $\mathcal{C} = \mathrm{Rep}(H$ ... H.

Conversely, a ... **ct** is of the form $\mathrm{Rep}(H)$ with

Question.
Is there a completely graphical proof, independent of the target **3Vect**?

Proof. Follows from an old result of M. Müger.[1]

[1] Theorem 6.20 in [Müger, *From subfactors to categories and topology I*, 2003]

Part 6
Lattice models

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

Lattice models and **3Vect**

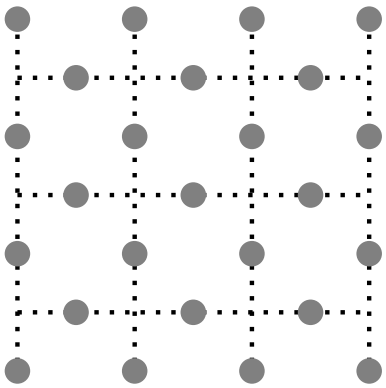
Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

$$\text{bulk of lattice} \leftrightsquigarrow \text{a fusion category } \mathcal{C}$$
$$\text{boundary of lattice} \leftrightsquigarrow \text{a } \mathcal{C}\text{-module category } \mathcal{M}$$

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

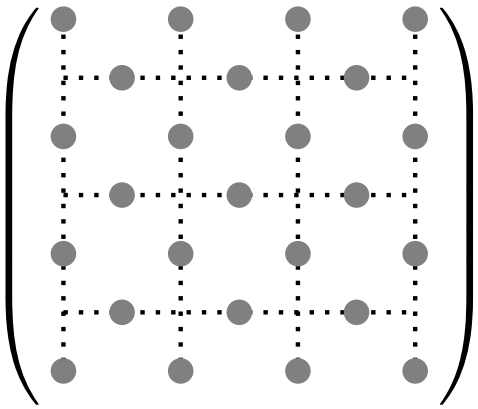
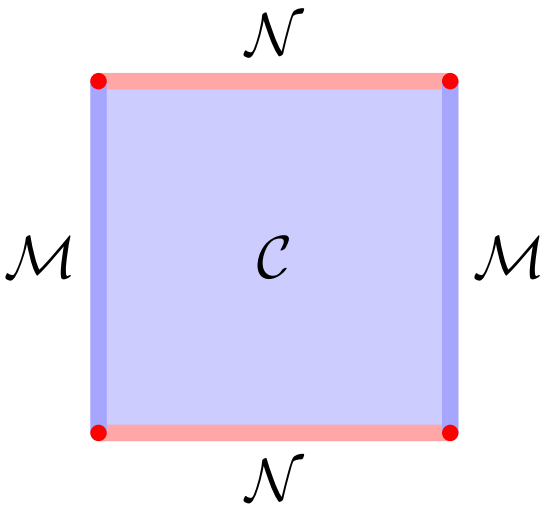
bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightsquigarrow$ a $\mathcal{C}$-module category $\mathcal{M}$



Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

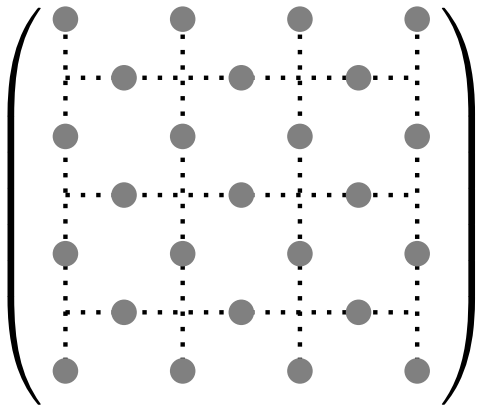
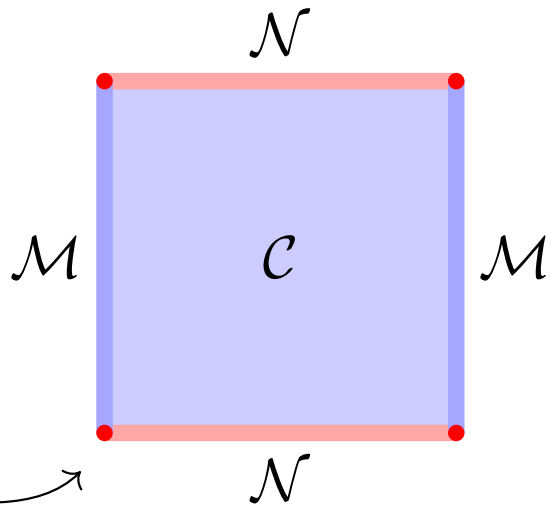
bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightsquigarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space  $=$ 

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightsquigarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

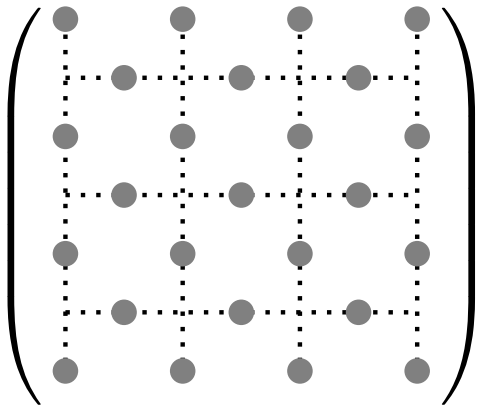
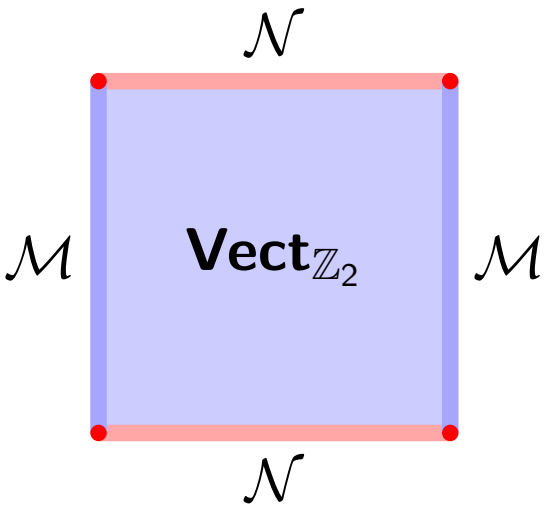
Ground space  $=$ 

a scalar 2-morphism in **3Vect**
$=$ a vector space

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightsquigarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

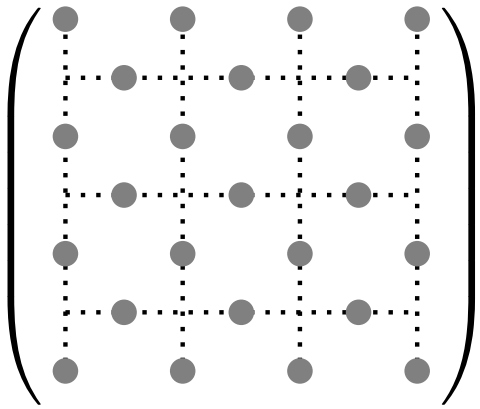
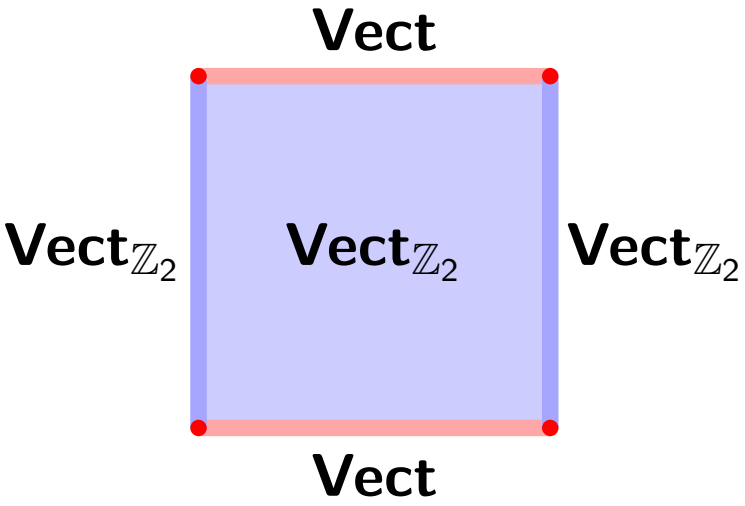
Ground space  $=$ 

surface codes	$\mathcal{C} = \mathbf{Vect}_{\mathbb{Z}_2}$

Lattice models and **3Vect**

Kitaev or Levin-Wen lattice models with boundaries $\Rightarrow$ defect TQFTs

bulk of lattice $\leftrightsquigarrow$ a fusion category $\mathcal{C}$
boundary of lattice $\leftrightsquigarrow$ a $\mathcal{C}$-module category $\mathcal{M}$

Ground space  $=$ 

surface codes	$\mathcal{C} = \mathbf{Vect}_{\mathbb{Z}_2}$
two possible boundaries: *smooth* and *rough*	two module categories: $\mathbf{Vect}_{\mathbb{Z}_2}$ and $\mathbf{Vect}$

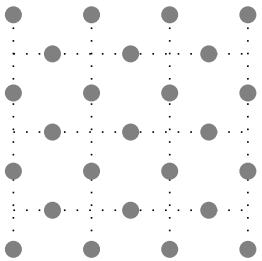
Lattice surgery

topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries[2]

[2] [Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]

Lattice surgery

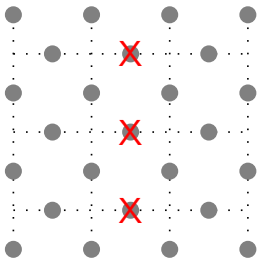
topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries[2]



[2] [Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]

Lattice surgery

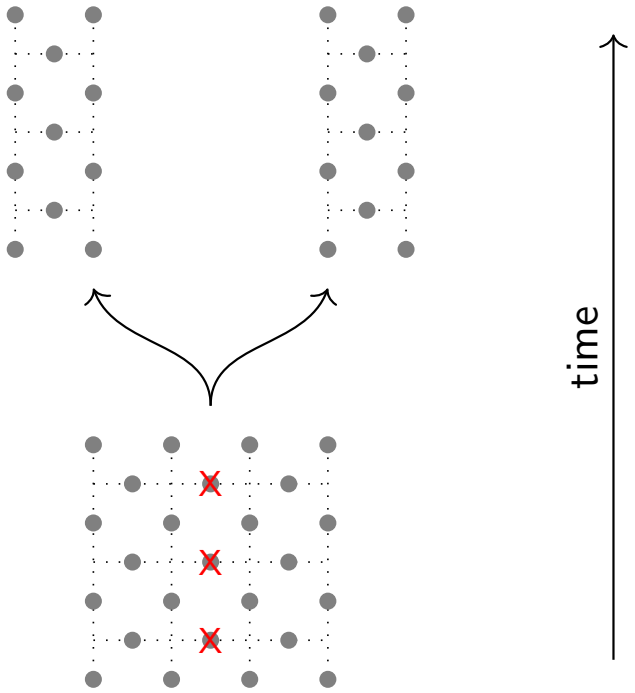
topologically protected operations on surface codes via
splitting or *merging* of lattices along smooth or rough boundaries[2]



[2] [Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]

Lattice surgery

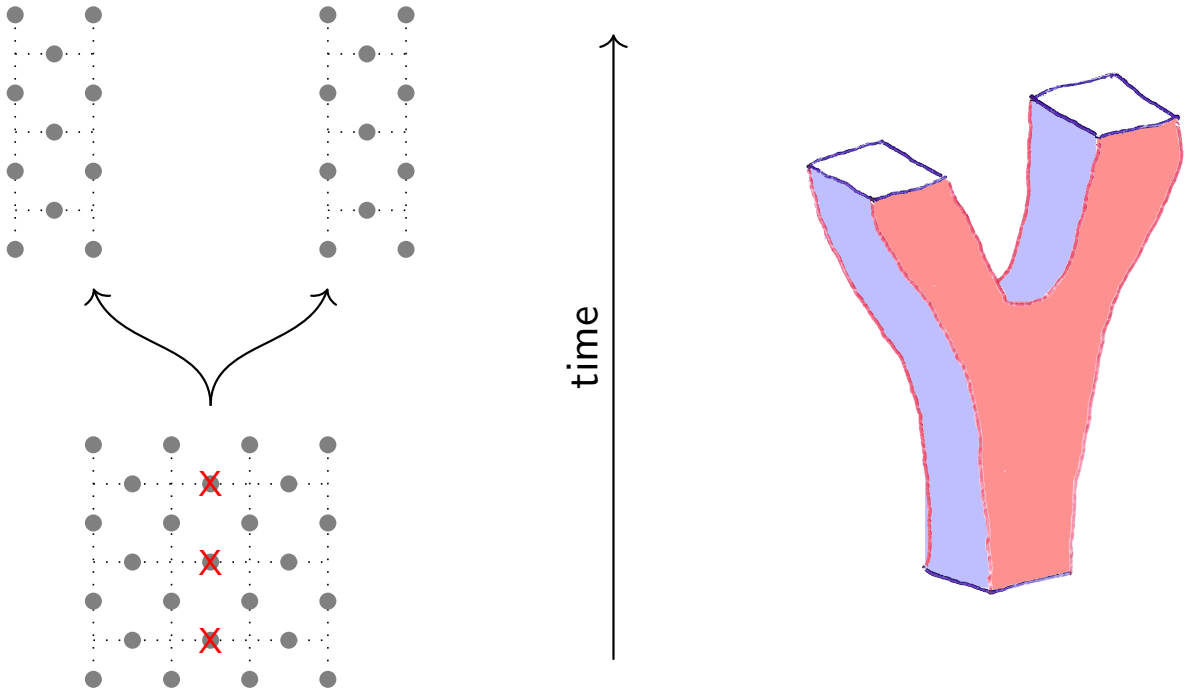
topologically protected operations on surface codes via *splitting* or *merging* of lattices along smooth or rough boundaries[2]



[2] [Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]

Lattice surgery

topologically protected operations on surface codes via
splitting or *merging* of lattices along smooth or rough boundaries[2]



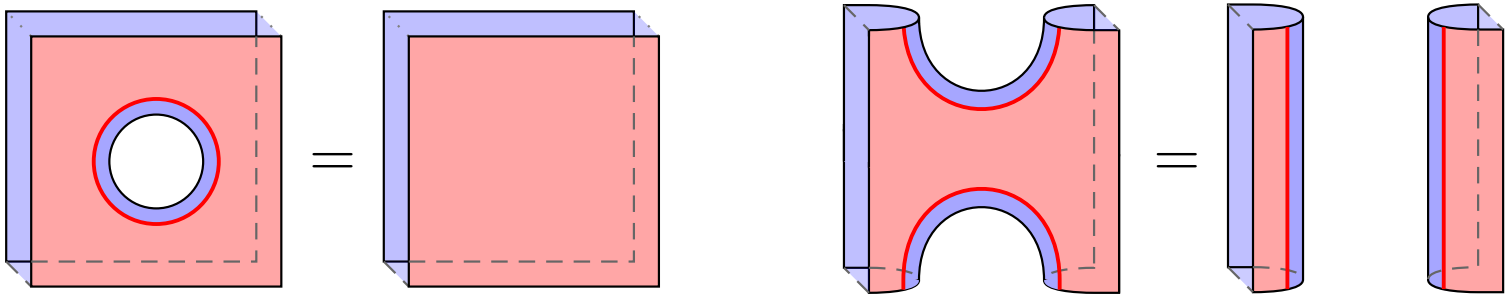
[2] [Horsman et al., *Surface code quantum computing by lattice surgery*, 2012]

The End

- Many open questions:
  - Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
  - Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
  - For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\widehat{\mathcal{C}}$. Is something similar true for Hopf algebras?
  - ...

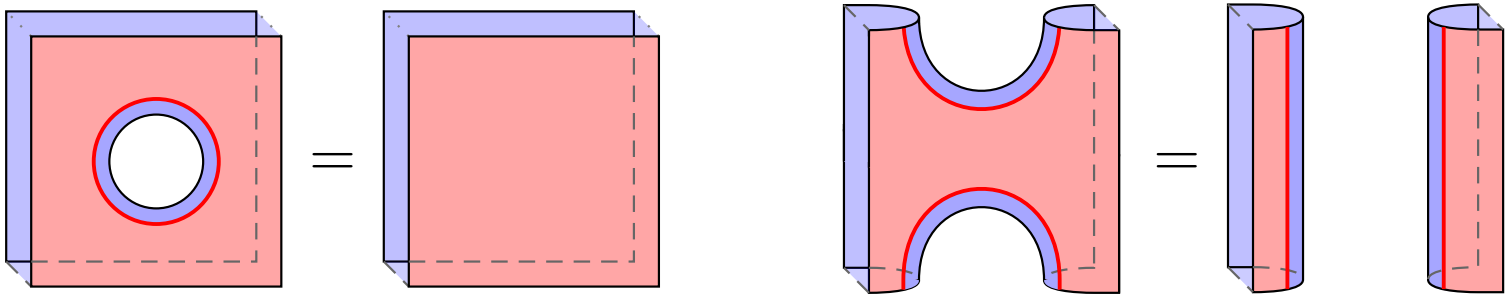
The End

- Many open questions:
  - Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
  - Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
  - For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\widehat{\mathcal{C}}$. Is something similar true for Hopf algebras?
  - ...
- Maybe most interestingly:
  For a defect TQFT, what is the physical meaning of the conditions:



The End

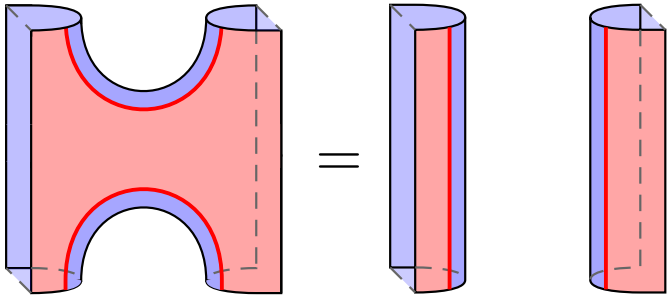
- Many open questions:
  - Can we drop dualizabilities in $\mathbb{H}$ to get more general Hopf algebras?
  - Can we make $\mathbb{H}$ into a symmetric monoidal 3-category with duals to talk about actual fully extended defect TQFTs?
  - For a Frobenius algebra in a monoidal category $\mathcal{C}$, there is a 2-category $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ such that the Frobenius algebra comes from a dualizable 1-morphism in $\widehat{\mathcal{C}}$. Is something similar true for Hopf algebras?
  - ...
- Maybe most interestingly:
  For a defect TQFT, what is the physical meaning of the conditions:



Thanks for listening!

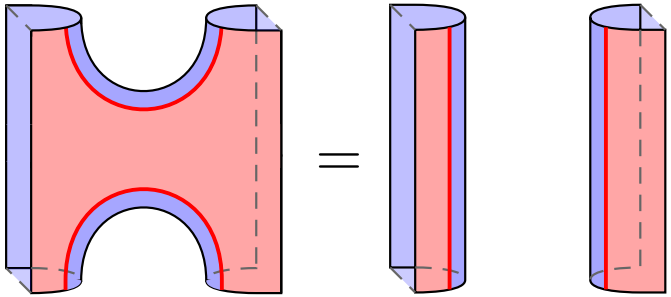
Weak Hopf algebras

If we *drop* the second condition

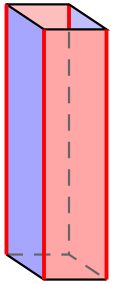


Weak Hopf algebras

If we *drop* the second condition



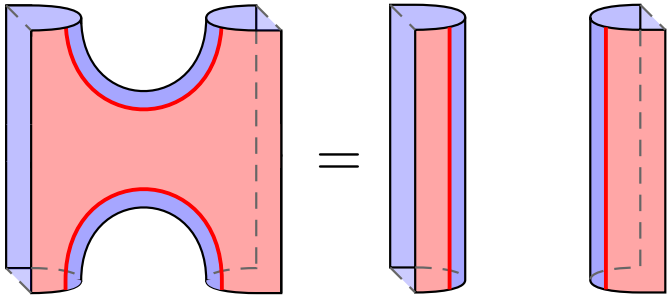
we only obtain a *weak* Hopf algebra on



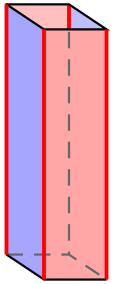
but have more functors $\mathbb{H} \to \mathbf{3Vect}$.

Weak Hopf algebras

If we *drop* the second condition



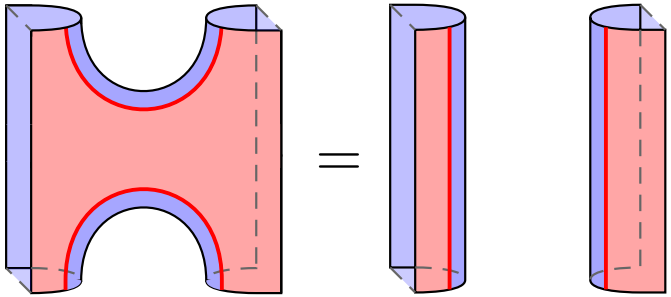
we only obtain a *weak* Hopf algebra on



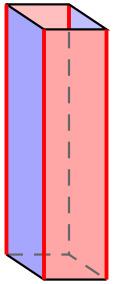
but have more functors $\mathbb{H} \to \mathbf{3Vect}$.
In fact, *every* fusion category induces such a functor.

Weak Hopf algebras

If we *drop* the second condition



we only obtain a *weak* Hopf algebra on



but have more functors $\mathbb{H} \to$ **3Vect**.
In fact, *every* fusion category induces such a functor.The corresponding Hopf algebra coincides with the Kitaev-Kong construction.