Biunitary constructions in quantum information

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based on arXiv:1609.07775

The plan

- **Part 1.** Quantum structures
- **Part 2.** Higher algebra

Part 1
Quantum structures

Quantum structures

- A *Hadamard matrix* (Had) is a matrix $H \in \mathrm{Mat}_n(\mathbb{C})$ fulfilling

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- They provide the mathematical foundation for:

  mutually unbiased bases, quantum key distribution, quantum teleportation, dense coding, quantum error correction

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- How can we find them?

Part 2
Higher algebra

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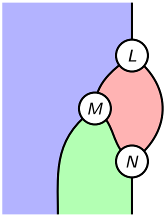
- *Higher algebra* lets us compose in higher dimensions:



Planar algebra = 2-category theory

The language describing algebra in the plane is *2-category theory*:

A	$A \xrightarrow{f} B$	$A \underset{f}{\overset{g}{\curvearrowright}} \Uparrow\eta \; B$
objects	1-morphism	2-morphism

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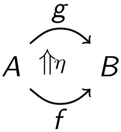
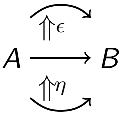
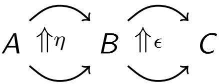


2-morphism

We can compose 2-morphisms like this:



vertical composition



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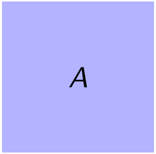

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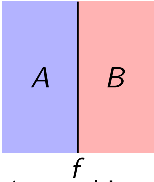

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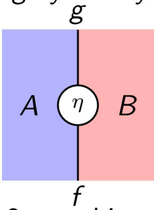

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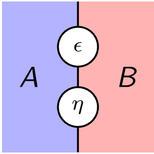

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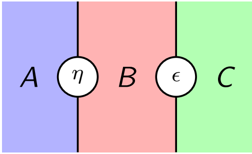

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The *dual* diagrams are the graphical calculus.

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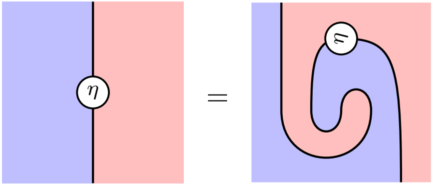

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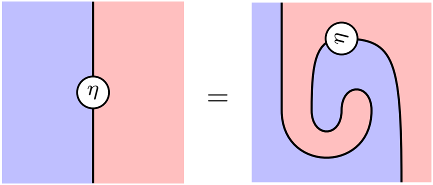

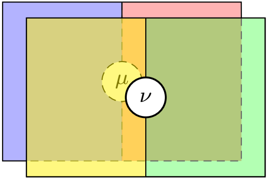

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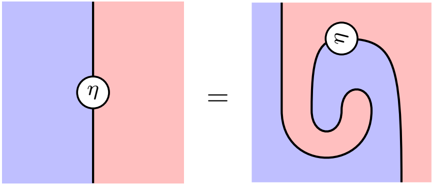

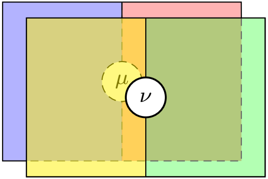

Summary. The graphical calculus of monoidal dagger pivotal 2-categories is given by surfaces, wires and vertices in three-dimensional space.

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Theorem. 2Hilb is a monoidal dagger pivotal 2-category.

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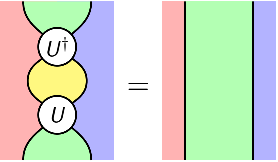

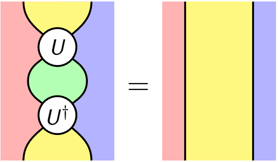

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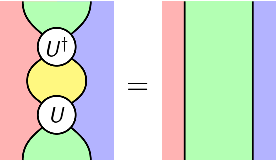

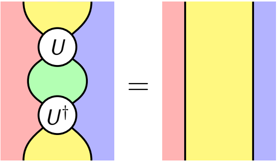

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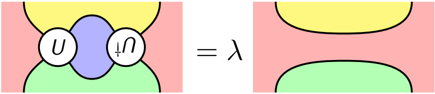

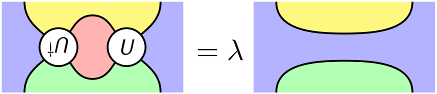

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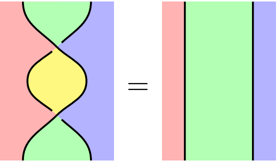

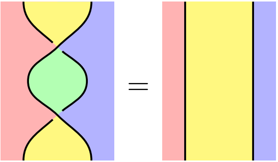

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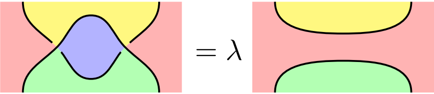

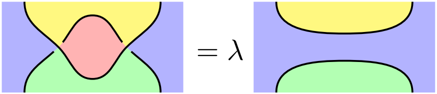

These look just like the *second Reidemeister move*.

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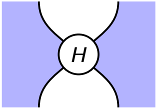

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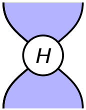

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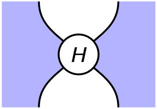

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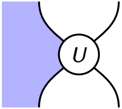

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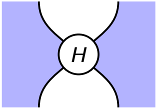

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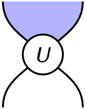

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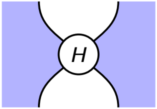

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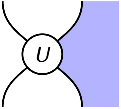

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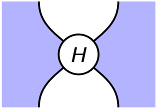

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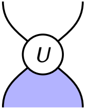

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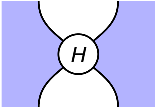

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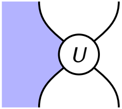

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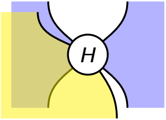

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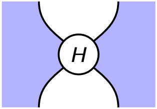

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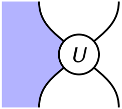

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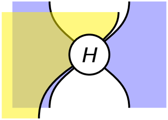
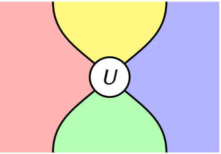
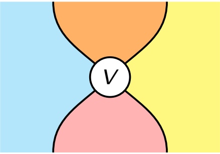


Composing biunitaries

Let U and V be biunitaries of the following types:





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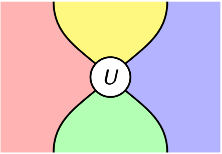
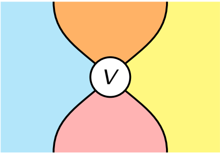
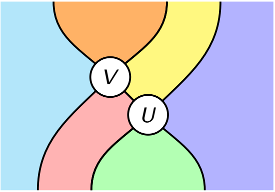




Then their diagonal composites are also biunitary:



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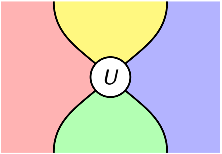

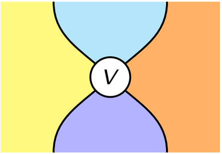

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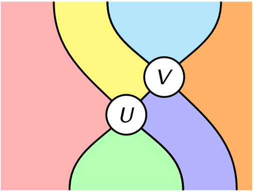

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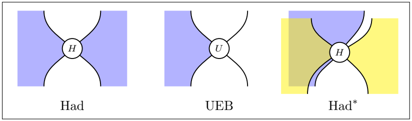

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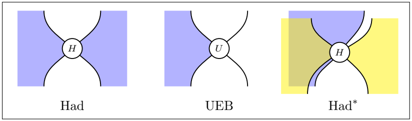

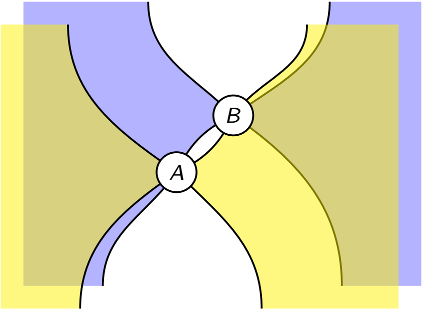

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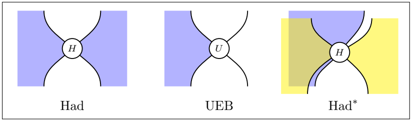
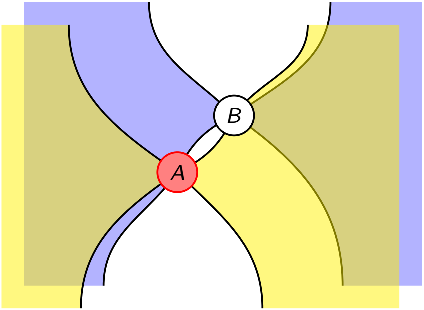


Had*



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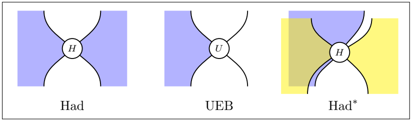
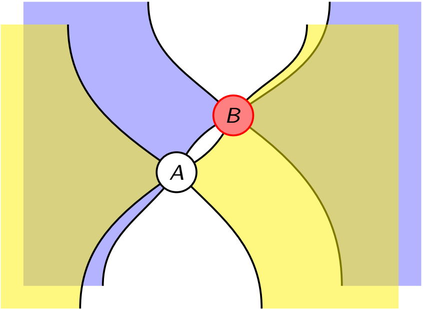


$$\text{Had}^* + \text{Had}^*$$



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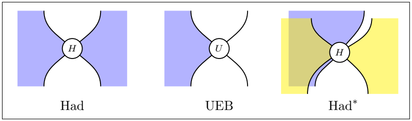
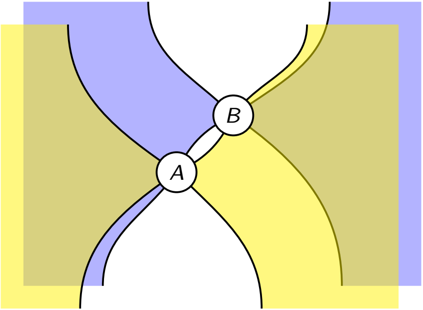


$$\text{Had}^* + \text{Had}^* \rightsquigarrow \text{Had}$$



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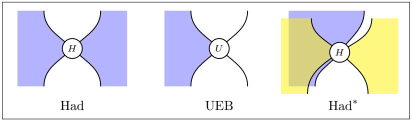

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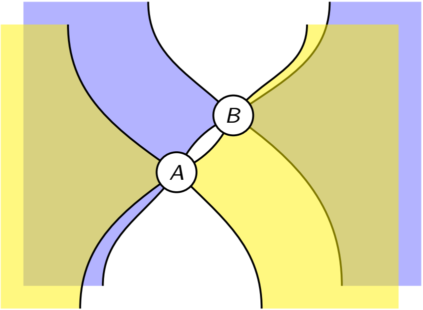

$$H_{ab,cd} = A^b_{a,c} B^c_{b,d}$$

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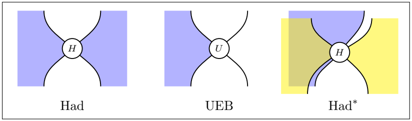

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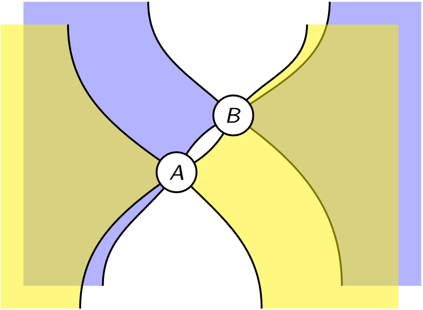

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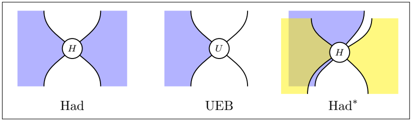

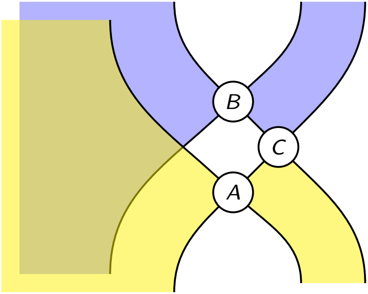

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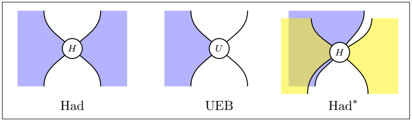

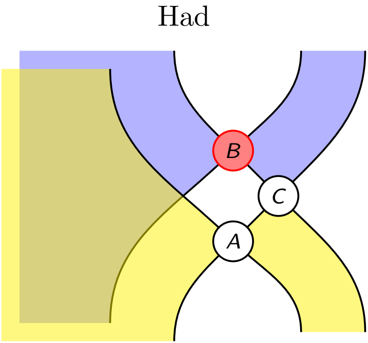

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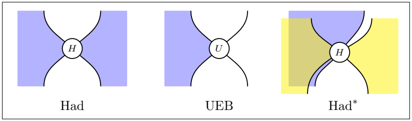
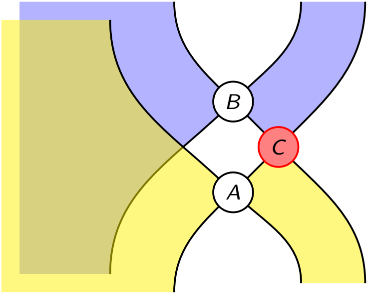


$\text{Had} + \text{Had}$



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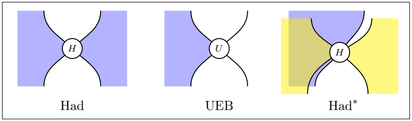

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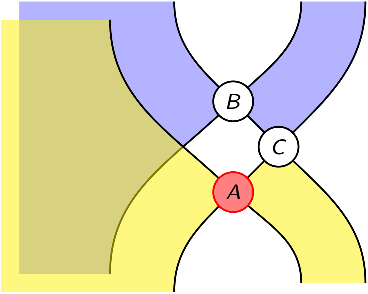

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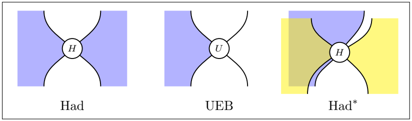
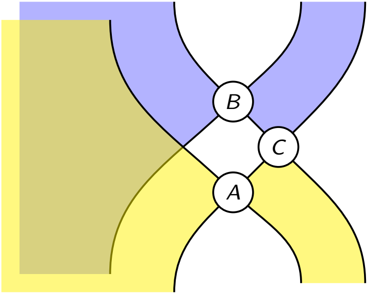


$$\mathrm{Had} + \mathrm{Had} + \mathrm{Had} \rightsquigarrow \mathrm{UEB}$$



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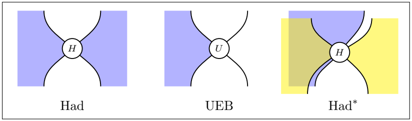

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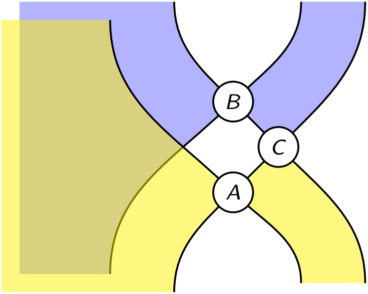

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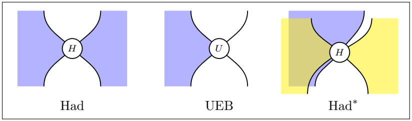

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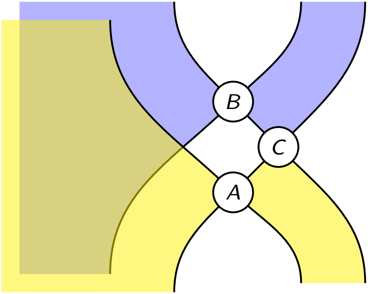

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✓

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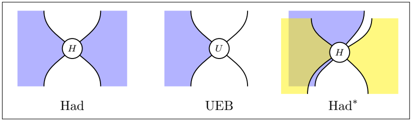
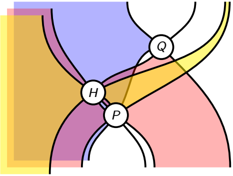
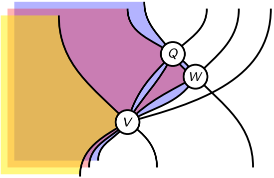




$U_{abc,de,fg} = H^{b,c}_{a,eg} P^{c,g}_{e,b,f} \; Q_{c,g,d}$



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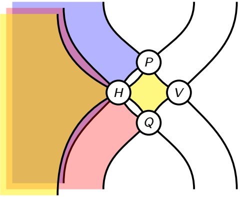

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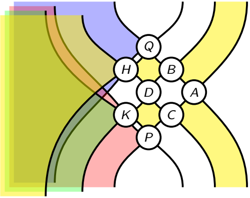

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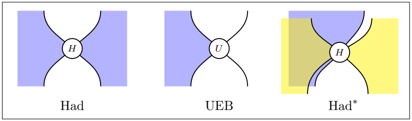

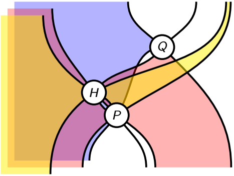

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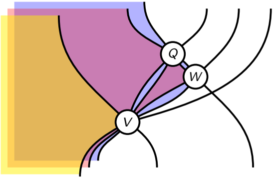

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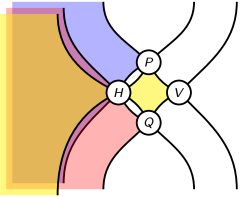

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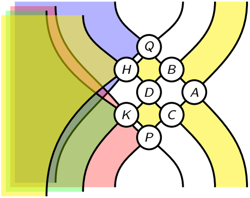

$U_{abcd,ef,gh} = \frac{1}{n} \sum_{r,s} A_{f,h} B_{s,f} C_{r,h} D_{s,r} H^{d}_{a,s} K^{c}_{b,r} Q_{d,s,e} P_{r,c,g}$ ✓

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Thanks for listening!