Petri Net Semantics of Bunched Implications

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Abstract. Engberg and Winskel's Petri net semantics of linear logic is re-considered, from the point of view of the logic BI of bunched implications. We first show how BI can be used to overcome a number of difficulties pointed out by Engberg and Winskel, and we argue that it provides a more natural logic for the net semantics. We then briefly consider a more expressive logic based on an extension of BI with classical and modal features.

1 Introduction

A number of authors have explored the connection between Petri nets and linear logic [5, 8, 19, 9, 24, 25]. The basic idea is to encode markings as tensor products $a_1 \otimes \cdots \otimes a_n$ of atomic formulae, and transitions as judgements $M \vdash N$ with markings in antecedent and consequent positions. Given a judgement of this form as an axiom, for each transition in a net, one obtains that $M \vdash N$ is derivable from these axioms (for arbitrary markings M and N) iff N is reachable from M in the net. The reason for the noted connection with linear logic is that it is crucial for the rules of Weakening and Contraction to be absent for this encoding to work; otherwise we would always obtain that from a marked place a we could reach a marking $a \otimes a$ where a was marked twice, or we could always go from $a \otimes b$ to a.

However, just encoding reachability is a rather weak requirement: to achieve that, it would be enough to work with a "logic" with only one connective $\otimes$; implication is not even necessary. What one would like is to take the attractive connection between markings and tensor products, and include it in a richer surrounding logic. This is where the several works on linear logic and nets differ. Perhaps the most satisfying treatment has been given by Engberg and Winskel [16, 17] (and Brown's related [7]), to which we refer for discussion of the early works on linear logic and nets. Our purpose in this paper is to re-consider Engberg and Winskel's semantics, from the point of view of the logic BI of bunched implications [27].

The semantics of [16, 17] is organized by viewing a proposition as a set of markings closed under (backwards) reachability, with the idea that a proposition is a property consisting of the set of markings that can give rise to it. Engberg and Winskel define operations on these sets of markings that provide an interpretation of Intuitionistic Linear Logic (ILL). They also prove completeness results, and show by a number of examples how linear logic can be used to express properties of nets.

Despite these positive results, a number of problems show up in [16, 17], which point to both technical and conceptual discrepancies with linear logic. One is the need to consider an axiom of distribution

$$A\&(B \oplus C) \;\dashv\vdash\; (A\&B) \oplus (A\&C),$$

which holds in the net semantics, but not in linear logic. The logic that results from adding distribution is, however, very different in character to linear logic; indeed, distribution is often presented as a branch-point, which divides substructural logics into conceptually distinct families [6, 13].

Other problems pointed out by Engberg and Winskel include the need to consider an additional constant, in order to express negative properties (such as mutual exclusion), and a somewhat degenerate interpretation of !. Less technically, it is clear that the Petri net model does not conform to any of the prominent philosophical interpretations of linear logic connectives, such as the number-or-uses reading or the process reading [20, 21, 1]. Given these discrepancies, it seems reasonable to ask if there is better suited surrounding logic, which maintains the appealing connection between tensor products and markings while fitting with the semantics.

This is where BI comes in. The main difference between BI and ILL is that BI has a primitive (intuitionistic) additive implication $\rightarrow$ instead of linear logic's !. This implication exists naturally in the Petri net model; taking it as primitive allows the discord between logic and model observed by Engberg and Winskel to be largely avoided. The Petri net semantics also illustrates well the "resource interpretation" of BI advanced in [27, 26, 28].

The presentation in this paper is arranged by first giving the net semantics, and then exhibiting the properties of it which lead to BI. This essentially parallels, in abbreviated and concrete fashion, the derivation of BI described in [27]; the route from the net model is so direct that one might even argue that BI *could have* come from it.

Petri nets give rise to an intuitive semantics of BI, and we claim that BI provides a more natural surrounding logic for the model than previous attempts. But we also point out a number of further limitations it has as a logic of nets. One difficulty is that the intuitionistic additives in BI lead to a semantic restriction: a proposition denotes a set of markings that is closed under reachability (from the past). We make a start on addressing this problem by considering a variant of BI with classical additive connectives, and S4 modalities. This variant enables the closure restriction on propositions to be dropped, and also allows for simpler and more intuitive specifications. In the concluding section we point out other difficulties, that have not been addressed as of yet.

2 The Model

As in [17], we consider a basic notion of net, which can be infinite, and which does not have capacities. Formally, a net $\mathcal{N} = (P, T, pre, post)$ consists of sets P and T of places and transitions, and two functions $pre, post : T \to \mathcal{M}$ from transitions to markings, where a marking is a finite multiset of places and $\mathcal{M}$ denotes the set of all markings. A marking can be regarded as a function $M : P \to N$ from places to natural numbers that is zero on all but finitely many places. Addition of markings is given by $(M + N)p = Mp + Np$. We let $[]$ denote the empty marking.

The one-step reachability relation is given by

$$M \Rightarrow N \quad \text{iff} \quad \exists t. \exists M'. M = pre(t) + M' \text{ and } N = post(t) + M'$$

We can then define a preorder on markings by

$$M \sqsubseteq N \quad \text{iff} \quad \exists M_1, ..., M_n. M = M_1 \Rightarrow \cdots \Rightarrow M_n = N$$

$\sqsubseteq$ is the reflexive, transitive closure of $\Rightarrow$. The basic lemma is

Lemma 1. $(\mathcal{M}, [], +, \sqsubseteq)$ *is a preordered commutative monoid.*

That is, it is a commutative monoid for which $+$ is monotone. Note that $[]$ need not be a least element of the preorder.

The semantics of Engberg and Winskel is defined by considering the collection of down-closed subsets of markings as a quantale (an algebra model for ILL). Their semantics can equally be presented in a "forcing style," where a judgement $M \models A$ says that formula A is true at marking M. In this formulation, down-closure corresponds to the condition

Kripke Monotonicity: $N \models A$ and $M \sqsubseteq N$ implies $M \models A$.

An interpretation i assigns a down-closed set of markings to each propositional letter p. An interpretation then determines the forcing relation as follows.

$$\begin{array}{lll}
M \models \mathrm{p} & \text{iff } M \in i(\mathrm{p}) \\
M \models I & \text{iff } M \sqsubseteq [] \\
M \models A * B & \text{iff } \exists N, N'. M \sqsubseteq N + N' \text{ and } N \models A \text{ and } N' \models B \\
M \models A \mathrel{-\!\!*} B & \text{iff } \forall N. (N \models A \text{ implies } N + M \models B) \\
M \models 1 & \quad \text{always} \\
M \models A \wedge B & \text{iff } M \models A \text{ and } M \models B \\
M \models 0 & \quad \text{never} \\
M \models A \vee B & \text{iff } M \models A \text{ or } M \models B
\end{array}$$

The connectives I, $*$ and $\mathrel{-\!\!*}$ are the multiplicative (or intensional) unit, conjunction and implication.

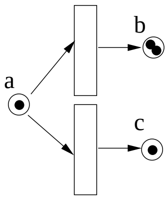
This semantics can be extended at once using Kripke's semantics of intuitionistic implication.

$$M \models A \to B \quad \text{iff} \quad \forall N \sqsubseteq M \,.\, (N \models A \;\text{ implies }\; N \models B)$$

With this addition to Engberg and Winskel's semantics, the additive (or extensional) connectives are those of intuitionistic propositional logic. The resulting forcing semantics is that of Day and Pym.

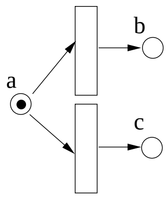
2.1 Examples

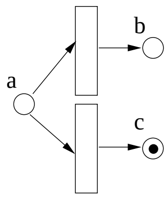
Suppose we are given a net. Then with any marking M we can associate a proposition $\overline{M}$ as follows. First, we require an injective map from places to propositional letters; we elide this and simply write a place as an atomic proposition. Second, we form a finite $*$-product whose components are places, with one copy of a place for each occurrence of the place in the marking. For example, a marking



is represented by the formula $\mathrm{a} * \mathrm{b} * \mathrm{b} * \mathrm{c}$. We can encode the transition system of a net using judgements of the form $M \models \overline{N}$.

It is important, however, to remember that for this we use an interpretation where each place b gets mapped to the set of markings that can reach the singleton multiset $\{\mathrm{b}\}$. So, we would have both $\{\mathrm{c}\} \models \mathrm{c}$ and $\{\mathrm{a}\} \models \mathrm{c}$ for the above net. That is, the formula c describes both of the following markings.





Note that $\{\mathrm{a}, \mathrm{a}\} \not\models \mathrm{c}$.

The markings described by $\mathrm{b} * \mathrm{c}$ for this net would be $\{\mathrm{b}, \mathrm{c}\}$, $\{\mathrm{a}, \mathrm{b}\}$, $\{\mathrm{a}, \mathrm{c}\}$ and $\{\mathrm{a}, \mathrm{a}\}$, because each of these could lead to the state where b and c are both marked. In contrast, $\mathrm{b} \wedge \mathrm{c}$ is only satisfied by $\{\mathrm{a}\}$. Note again that $\{\mathrm{a}, \mathrm{a}\} \not\models \mathrm{b} \wedge \mathrm{c}$; the reason is that $\{\mathrm{a}, \mathrm{a}\}$ can only lead to a marking where two places (which might be the same) are marked.

For examples involving implication, consider the Petri net shown in Fig. 1. This net describes processes either producing an item to a buffer or consuming an item from the buffer. Tokens in places r and t denote ready processes and terminated processes, respectively and b represents a buffer whose tokens are items produced. In this net, $M \models \mathrm{r} \mathrel{-\!\!*} \mathrm{t} * \mathrm{t}$ iff adding one ready process to M results in two terminated processes. From this we see the basic idea of $A \mathrel{-\!\!*} B$: it says that B will be obtained, if enough tokens are added to make A hold. On the other hand, $M \models \mathrm{r} \to \mathrm{t} * \mathrm{t}$ iff one ready process can't become M ($\{\mathrm{r}\} \not\sqsubseteq M$), because one ready process can't lead to two terminated processes ($\{\mathrm{r}\} \not\sqsubseteq \{\mathrm{t}, \mathrm{t}\}$). Therefore, when there is one item in the buffer and one terminated process ($\{\mathrm{t}, \mathrm{b}\}$), $\mathrm{r} \mathrel{-\!\!*} \mathrm{t} * \mathrm{t}$ holds but $\mathrm{r} \to \mathrm{t} * \mathrm{t}$ doesn't.

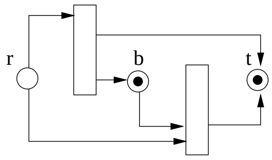


Fig. 1. example

The implication $\to$ allows the specification of negative properties using the negation $\neg A = A \to 0$. A basic example is mutual exclusion, where we say that two places cannot be marked at the same time. In the above net (which might be embedded in a larger net), we could say that a process is not both ready and terminated using $\neg(\mathrm{r} * \mathrm{t} * 1)$. Using $\mathrel{-\!\!*}$, we can further say that a process is not both ready and terminated in any marking reachable from a given marking M_0: $M_0 \mathrel{-\!\!*} \neg(\mathrm{r} * \mathrm{t} * 1)$. Note the role of 1 in $\mathrm{r} * \mathrm{t} * 1$. It enables the state, at a given time, to be partitioned into three parts where r is true in one, t in another, and where the third part is arbitrary. For example, $\mathrm{r} * \mathrm{t} * 1$ is true in $\{\mathrm{r}, \mathrm{t}, \mathrm{s}\}$, no matter what s is: the 1 component can be used to allow for additional state.

It is instructive to unwind the formula $M_0 \mathrel{-\!\!*} \neg(\mathrm{r} * \mathrm{t} * 1)$. As a first approximation, what it describes is those markings M that, when added to M_0, do not give rise to a state where both r and t are marked. When we consider sequents $A \models B$ with formulae to the left, we will be able to pin M down more tightly (e.g. using I for A).

This first approximation is slightly inaccurate, in that it does not consider the quantification in the interpretation of $*$. That quantifies over future markings, those reachable in the forwards direction, so the formula really describes those states that, when M_0 is added, cannot reach a state from which both r and t can be marked in the future.

We considered this formula so closely in order to illustrate a point: Care is needed when translating informal statements into BI formulae, essentially because of the back-and-forth between future and past in formulae involving $*$ and $\to$. We will come back to this issue in Section 5.

2.2 Properties of the Semantics

We now define a notion $A \models B$ of entailment between formulae.

Definition 1. *1. Given a net $\mathcal{N}$ and interpretation i of the propositional letters, we write $A \models_{\mathcal{N},i} B$ if for all markings M, $M \models A$ under the interpretation implies that $M \models B$.*
2. We write $A \models_{\mathcal{N}} B$ if $A \models_{\mathcal{N},i} B$ for all interpretations i.
3. We write $A \models B$ if $A \models_{\mathcal{N}} B$ for all nets $\mathcal{N}$.

We have been using BI notation instead of that of linear logic. For reference, here are the correspondences between the connectives.

BI	1	$\wedge$	I	$*$	0	$\vee$	$\to$
ILL	$\top$	$\&$	I	$\otimes$	0	$\oplus$	$!$

These correspondences indicate similarity, but not an exact match-up, as $\to$ and ! impact the meanings of the other connectives in different ways [27]. In particular, the presence of $\to$ instead of ! in BI leads to fundamentally different behaviour, which is exhibited by the Petri net model. Most importantly, we have adjunction properties for both $*$ and $\wedge$:

$$\begin{array}{lll} A * B \models C \ \text{ iff } \ A \models B \mathrel{-\!\!*} C & & (* - adj) \\ A \wedge B \models C \ \text{ iff } \ A \models B \to C & & (\wedge - adj) \end{array}$$

The model also satisfies two forms of distribution:

$$\begin{array}{lll} A \wedge (B \vee C) \dashv\models (A \wedge B) \vee (A \wedge C) & & (\wedge\vee - dist) \\ A * (B \vee C) \dashv\models (A * B) \vee (A * C) & & (*\vee - dist) \end{array}$$

However, logically, we do not regard these forms of distribution as primitive; rather, they are consequences of the adjunction properties for $*$ and $\wedge$. That is, categorically, $\wedge - adj$ corresponds to $A \wedge (-)$ being a left adjoint, and since left adjoints preserve all colimits, it must preserve disjunction. This gives $(\wedge\vee - dist)$, and similarly $(* - adj)$ is responsible for $(* \vee - dist)$.

In contrast, in linear logic & does not have an implication adjoint, and linear logic does not satisfy

$$A\&(B \oplus C) \ \dashv\vdash \ (A\&B) \oplus (A\&C) \qquad (\& \oplus - dist)$$

Since $(\& \oplus - dist)$ is true in the Petri net model, Engberg and Winskel need to add it as an axiom. But, given that $\to$ exists in the model, it seems more natural to take it as primitive: doing so has the added advantage of relieving us of the need to include distribution as an additional, somewhat out of place, axiom.

These considerations lead to BI.

3 BI

The two adjunction properties $(* - adj)$ and $(\wedge - adj)$ are the starting point of BI [27]. To formulate introduction rules for $\mathrel{-\!\!*}$ and $\rightarrow$, we include context forming operations "," and ";" corresponding to $*$ and $\wedge$. The essential difference between these operations is that ";" admits Contraction and Weakening while "," does not.

CONNECTIVES

Additive connectives	$1 \ \wedge \ \rightarrow \ 0 \ \vee$
Multiplicative connectives	$I \ * \ \mathrel{-\!\!*}$

BUNCHES

$$
\begin{array}{rll}
\Gamma ::= & A & \text{propositional assumption} \\
& \mid \{\}_m & \text{multiplicative unit} \\
& \mid \Gamma, \Gamma & \text{multiplicative combination} \\
& \mid \{\}_a & \text{additive unit} \\
& \mid \Gamma; \Gamma & \text{additive combination}
\end{array}
$$

We write $\Gamma(\Delta)$ to indicate a bunch in which Δ appears as a subtree, and $\Gamma(\Delta')$ for the similar tree where Δ' replaces Δ. A notion of coherent equivalence factors out some representation details.

COHERENT EQUIVALENCE: $\Gamma \equiv \Gamma'$
$\equiv$ is the least equivalence relation on bunches satisfying

1 Commutative monoid equations for $\{\}_a$ and ;.
2 Commutative monoid equations for $\{\}_m$ and ,.
3 Congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.

Note that ";" and "," do not distribute over one another.

JUDGEMENTS

These are of the form

$$\Gamma \vdash A$$

where Γ is a bunch and A a formula. The rules for deriving judgements now follow.

IDENTITY AND STRUCTURE

$$\frac{}{A \vdash A}\ Id \qquad \frac{\Gamma \vdash A}{\Delta \vdash A}\ \equiv\ (\text{where } \Delta \equiv \Gamma)$$

$$\frac{\Gamma(\Delta) \vdash A}{\Gamma(\Delta; \Delta') \vdash A}\ \text{Weakening} \qquad \frac{\Gamma(\Delta; \Delta) \vdash A}{\Gamma(\Delta) \vdash A}\ \text{Contraction}$$

Multiplicatives

$$\frac{}{\{\}_m \vdash I}\ I\, I \qquad \frac{\Gamma(\{\}_m) \vdash C \quad \Delta \vdash I}{\Gamma(\Delta) \vdash C}\ I\, E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \mathrel{-\!\!*} B}\ \mathrel{-\!\!*} I \qquad \frac{\Gamma \vdash A \mathrel{-\!\!*} B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ \mathrel{-\!\!*} E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A * B}\ *I \qquad \frac{\Gamma(A, B) \vdash C \quad \Delta \vdash A * B}{\Gamma(\Delta) \vdash C}\ *E$$

Additives

$$\frac{}{\{\}_a \vdash 1}\ 1\, I \qquad \frac{\Gamma(\{\}_a) \vdash C \quad \Delta \vdash 1}{\Gamma(\Delta) \vdash C}\ 1\, E$$

$$\frac{\Gamma; A \vdash B}{\Gamma \vdash A \to B}\ \to I \qquad \frac{\Gamma \vdash A \to B \quad \Delta \vdash A}{\Gamma; \Delta \vdash B}\ \to E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma; \Delta \vdash A \wedge B}\ \wedge I \qquad \frac{\Gamma(A; B) \vdash C \quad \Delta \vdash A \wedge B}{\Gamma(\Delta) \vdash C}\ \wedge E$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (i = 1, 2) \quad \vee I \qquad \frac{\Gamma \vdash A \vee B \quad \Delta(A) \vdash C \quad \Delta(B) \vdash C}{\Delta(\Gamma) \vdash C}\ \vee E$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash A}\ 0E$$

To see why $(\wedge \vee - dist)$ does not need to be included as an additional axiom, consider that intuitionistic logic is a subsystem of BI; we can simply use the usual intuitionistic proof.

The semantic consequence relation is extended to bunches by mapping "," to $*$, ";" to $\wedge$, $\{\}_m$ to I, and $\{\}_a$ to 1.

Lemma 2 (Soundness). *If $\Gamma \vdash A$ is derivable in BI, then $\Gamma \models A$.*

4 Technical Results

In this section we show some further properties of the semantics. We begin by considering completeness results, extending results of Engberg and Winskel. The proofs are straightforward: one is a variation on a Yoneda embedding argument, and another appeals to Kripke's completeness result for intuitionistic logic. We include them because they illustrate well the match between the model and logic. We also show an incompleteness result, indicating a mismatch. Then we move on to consider expressiveness and inexpressiveness issues.

4.1 Completeness and Incompleteness

Engberg and Winskel showed the completeness of the Petri net model for the fragment of ILL without $\oplus$, 0 or !. This result can easily be extended to cope with $\rightarrow$.

Proposition 1 (Completeness for $\vee, 0$-free fragment). *Suppose there are no occurrences of $\vee$ or 0 in Γ and A. Then*

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \models A$$

Proof. The proof essentially follows that of Engberg and Winskel in [17], which is in turn similar to the proof the Yoneda embedding preserves both monoidal closed and Cartesian closed structure [12, 32]. Consider the net whose places are formulae in the $\vee, 0$-free fragment, and where there is a transition from $\{A_1, \ldots, A_n\}$ to $\{B_1, \ldots, B_m\}$ just if $A_1 * \ldots * A_n \vdash B_1 * \ldots * B_m$ is provable in BI. Suppose interpretation i maps each propositional letter p to the set of markings which reach $\{\mathrm{p}\}$. In this net semantics, reachability is the same as provability: $\{A_1, \ldots, A_n\} \sqsubseteq \{B_1, \ldots, B_m\}$ iff $A_1 * \ldots * A_n \vdash B_1 * \ldots * B_m$ is provable in BI. With this fact, induction on the structure of A shows that $\{C_1, \ldots, C_n\} \models A$ iff $C_1 * \ldots * C_n \vdash A$. Since all of the cases except $\rightarrow$ are as in [17], we'll only show the case of $\rightarrow$.

Suppose $\{C_1, \ldots, C_n\} \models A_1 \rightarrow A_2$. Then, $(C_1 * \ldots * C_n) \wedge A_1 \vdash C_1 * \ldots * C_n$ and $(C_1 * \ldots * C_n) \wedge A_1 \vdash A_1$ are provable. Since reachability is equvalent to provability, a singleton marking $\{(C_1 * \ldots * C_n) \wedge A_1\}$ reaches $\{C_1, \ldots, C_n\}$. Also, by induction hypothesis, $\{(C_1 * \ldots * C_n) \wedge A_1\} \models A_1$. Therefore, $\{(C_1 * \ldots * C_n) \wedge A_1\} \models A_2$ because $\{C_1, \ldots, C_n\} \models A_1 \rightarrow A_2$. Again, by induction hypothesis, $(C_1 * \ldots * C_n) \wedge A_1 \vdash A_2$ is provable and so is $(C_1 * \ldots * C_n) \vdash A_1 \rightarrow A_2$.

For the other direction, suppose $C_1 * \ldots * C_n \vdash A_1 \rightarrow A_2$. Consider any marking $\{B_1, \ldots, B_m\}$ such that $\{B_1, \ldots, B_m\} \sqsubseteq \{C_1, \ldots, C_n\}$ and $\{B_1, \ldots, B_m\} \models A_1$. ¿From the equivalence between reachability and provability and induction hypothesis, $B_1 * \ldots * B_m \vdash C_1 * \ldots * C_n$ and $B_1 * \ldots * B_m \vdash A_1$ are provable. Therefore, $B_1 * \ldots * B_m \vdash A_2$ is also provable. Again, by induction hypothesis, $\{B_1, \ldots, B_m\} \models A_2$. *(End of Proof)*

We can also show completeness in the presence of disjunction, as long as there are no multiplicatives.

Proposition 2 (Completeness for intuitionistic fragment). *Suppose there are no multiplicatives in Γ and A. Then*

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \models A$$

Proof. We know that intuitionistic logic is complete for Kripke's semantics wrt preorders of worlds [23]. Therefore, if $A \not\vdash B$ in intuitionistic logic, there is a preorder P, interpretation i, and element $p_0 \in P$ for which $p_0 \models A$ but $p_0 \not\models B$ in Kripke's semantics. We define a net whose places are the elements of P, where there is a transition from $\{p\}$ to $\{p'\}$ just if $p \leq p'$ in P. And let the interpretation i^* map an atomic proposition q to a set of markings $\{M \mid M \sqsubseteq \{i(\mathrm{q})\}\ \}$. In the corresponding net semantics, let p_0^* denote the marking where place p_0 is the only one marked. Then it is not difficult to verify that $p_0^* \models A$ while $p_0^* \not\models B$.

(*End of Proof*)

We do not get completeness for the whole of BI because of interactions between multiplicatives and the intuitionistic falsity 0. For instance, the formula $(A \mathrel{-\!\!*} 0) \to 0$ expresses (semantic) consistency in the following sense:

$$M \models (A \mathrel{-\!\!*} 0) \to 0 \text{ iff } \exists N.\, N \models A$$

Then the formula $(A * B \mathrel{-\!\!*} 0) \to 0$ expresses that A and B are consistent with one another, but we cannot infer that from the consistency of each of A and B on its own: consider the case when $A = (\mathrm{a} \mathrel{-\!\!*} 0)$ and $B = \mathrm{a}$. More formally,

Proposition 3 (Incompleteness). *The net semantics is not complete for the whole of BI. In particular, we have, for atomic propositions* p *and* q*,*

$$\begin{array}{ll} & (\mathrm{p} \mathrel{-\!\!*} 0) \to 0 \ \wedge\ (\mathrm{q} \mathrel{-\!\!*} 0) \to 0 \models (\mathrm{p} * \mathrm{q} \mathrel{-\!\!*} 0) \to 0 \\ \text{but} & (\mathrm{p} \mathrel{-\!\!*} 0) \to 0 \ \wedge\ (\mathrm{q} \mathrel{-\!\!*} 0) \to 0 \not\vdash (\mathrm{p} * \mathrm{q} \mathrel{-\!\!*} 0) \to 0 \end{array}$$

We do not know what extra axioms could be added to get an unrestricted completeness result.

4.2 Expressiveness

In Section 2.1 we used $\neg A = A \to 0$ to express negative properties. In order to express similar properties, Engberg and Winskel considered an additional constant, $\perp$, with forcing relation

$$M \models \perp \text{ iff } [\,] \not\sqsubseteq M$$

Note that the constant $\perp$ is not part of linear logic; in particular, it is not (multiplicative) falsity of classical linear logic, since we do not have $(A \mathrel{-\!\!*} \perp) \mathrel{-\!\!*} \perp \ = \ A$.

We can express $\perp$ using $\neg$, as long as the net in question is atomic. (A net N is atomic iff $M \sqsubseteq [\,]$ implies $[\,] \sqsubseteq M$.)

Lemma 3. *In any atomic net,* $\perp = \neg I$.

The fact that we do not have to go outside the logic to express negative properties underlines our contention of the natural fit between the net model and BI. Also, the use of intuitionistic negation to express $\perp$ lends further support to remarks in [16, 17], on the unsuitability of (classical) linear negation in the net case.

Two cousins of additive implication are considered in [16, 17]. One is $(A \mathrel{-\!\!*} \perp) \vee B$, and the other is $!A \mathrel{-\!\!*} B$. The forcing clause for $!A$ is

> $M \models !A$ iff there exists some set of markings X containing M such that for any M_1 in X, $M_1 \models I \wedge A$ and $M_1 \sqsubseteq M_2 + M_3$ for some M_2, M_3 in X

Neither $(A \mathrel{-\!\!*} \perp) \vee B$ nor $!A \mathrel{-\!\!*} B$ is the additive implication $A \to B$, and more generally we have:

Proposition 4. *There is a net $\mathcal{N}$ and interpretation i such that for some propositional letter* p, $\mathrm{p} * 1 \mathrel{-\!\!*} 0$ *can't be expressed by any formula A built from $\perp$, $!$ and the BI connectives minus $\to$. In other words, for any such A, there exists M such that either $M \models A$ but $M \not\models \mathrm{p} * 1 \mathrel{-\!\!*} 0$ or $M \not\models A$ but $M \models \mathrm{p} * 1 \mathrel{-\!\!*} 0$.*

Proof. Consider the Petri net $\mathcal{N}$ shown in Fig. 2 with the the following interpretation i:

$$\text{for all propositional letter q. } i(\mathrm{q}) = \{M \mid M \sqsubseteq \{\mathrm{b}\}\}$$

Then, for any propositional letter p, $M \models \mathrm{p} * 1 \to 0$ iff $M = n\mathrm{a}$ where $0\mathrm{a} = []$ and $(n+1)\mathrm{a} = n\mathrm{a} + \{\mathrm{a}\}$. Also, the following lemma holds for any formula A without $\to$:

For any marking M such that $M \models A$, if $M \neq []$, then $M + \{\mathrm{b}\} \models A$.

Since the formula $\mathrm{p} * 1 \to 0$ doesn't satisfy the lemma, it can't be expressed by any formula without $\to$. (*End of Proof*)
Notice that in an atomic net, $!A$ is equal to $I \wedge A$.

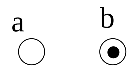


Fig. 2. counter example

Finally, one might wonder if there is a different interpretation of !, which one might use to decompose $\to$ in terms of $\mathrel{-\!\!*}$. In [27, 26], it has already been shown that there is no functor ! from a certain model of BI to itself such that $!E \mathrel{-\!\!*} F \cong E \to F$. A counter example can be made to show the same result for the net model.

Proposition 5. *There is a net $\mathcal{N}$ and interpretation i such that for some propositional letters* p *and* q, $\mathrm{p} \to \mathrm{q}$ *can't be expressed by* $!\mathrm{p} \mathrel{-\!\!*} \mathrm{q}$ *for any interpretation of* $!$*: there exists M such that $M \models \mathrm{p} \to \mathrm{q}$ but $M \not\models !\mathrm{p} \mathrel{-\!\!*} \mathrm{q}$ for any interpretation of* $!$.

Proof. When $i(\mathrm{p}) = \{M \mid M \sqsubseteq \{\mathrm{a}\}\}$ and $i(\mathrm{q}) = \{M \mid M \sqsubseteq \{\mathrm{b}\}\}$, the net in Fig. 3 and two formulae $\mathrm{p} \to \mathrm{q}$ and $!\mathrm{p} \mathrel{-\!\!*} \mathrm{q}$ show the proposition. $\{\mathrm{c}\} \models \mathrm{p} \to \mathrm{q}$ whereas for any interpretation of !, $\{\mathrm{c}\} \not\models !\mathrm{p} \mathrel{-\!\!*} \mathrm{q}$. (*End of Proof*)

5 Classical Additives and Modalities

One could go on and ask further expressiveness questions. However, perhaps more important are limitations that result from the use of down-closed sets, or from the Kripke monotonicity condition.

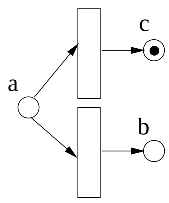


Fig. 3. counter example

For example, when we gave an example of the conjunction $\mathrm{b} * \mathrm{c}$ in Section 2.1, we found that it was satisfied by four markings, including the one $\{\mathrm{a}, \mathrm{a}\}$ where a was marked twice. Surely, it would be more immediate to consider the formula as simply denoting the marking $\{\mathrm{b}, \mathrm{c}\}$. Similarly, in describing negative properties we used a formula of the form $\neg(\mathrm{a} * \mathrm{b} * 1)$. Unrolling the definitions of $*$ and $\rightarrow$ results in a complex back-and-forth between past and future markings; this transit seems contrived and indirect, and more a hindrance than a help when one tries to use formulae to describe net properties.

Finally, many interesting properties of interest are not down-closed, such as "if place b is marked, the c will never become marked in the future." A look at a standard Petri net book [29] also reveals many other properties of interest that are not down- or up-closed, such as "transition t is not enabled in the future" (t-dead) and "from any future state, it is always possible to evolve the net in a way such that t is enabled" (t-live).

For these reasons, it is natural to consider a logic that talks about a "current" marking, and only mentions past or future markings when explicitly desired.

5.1 Semantics

We proceed by analogy with the relationship between Kripke's semantics of intuitionistic logic and the modal logic S4. In our case, we can use modalities to talk about order, the reachability relation for a Petri net, and formulate the semantics of additive and multiplicative connectives in a way that ignores the order. To complete the relationship, via a modal translation, we need both past and future modalities, the use of past modalities corresponding to the intuitionistic quantification in $\rightarrow$, and future modality corresponding to future quantification in $*$.

The syntax is obtained by adding two modalities, future ($\Box_+$) and past ($\Box_-$) necessities. Negation and two possibilities are defined as usual, and we make use of classical identities to restrict our additive (non-modal) connectives to $\rightarrow$ and 0:

$$\begin{array}{rcl@{\qquad\qquad}rcl} \neg A & \equiv & A \rightarrow 0 & 1 & \equiv & \neg 0 \\ \Diamond_+ A & \equiv & \neg\Box_+\neg A & A \vee B & \equiv & \neg A \rightarrow B \\ \Diamond_- A & \equiv & \neg\Box_-\neg A & A \wedge B & \equiv & \neg(\neg A \vee \neg B) \end{array}$$

An interpretation i now assigns a set of markings (not necessarily down-closed) to each propositional letter, and the forcing relation is

$$
\begin{array}{ll}
M \models \mathrm{p} & \text{iff } M \in i(\mathrm{p}) \\
M \models I & \text{iff } M = [] \\
M \models A * B & \text{iff } \exists N, N'.\,(M = N + N' \ \text{ and } \ N \models A \ \text{ and } \ N' \models B) \\
M \models A \mathrel{-\!\!*} B & \text{iff } \forall N.\,(N \models A \ \text{ implies } \ N + M \models B) \\
M \models 0 & \quad \text{never} \\
M \models A \to B & \text{iff } M \models A \ \text{ implies } \ M \models B \\
M \models \Box_{+} A & \text{iff } \forall N \sqsupseteq M.\, N \models B \\
M \models \Box_{-} A & \text{iff } \forall N \sqsubseteq M.\, N \models B
\end{array}
$$

This semantics is classical, in that $\to$ is interpreted pointwise; only the multiplicatives and modalities refer to markings other than the current one. As a result, any classical tautology, such as $\neg\neg A \to A$, is true at every marking.

Lemma 4. *The semantics of this section validates all of the BI rules, along with*

Reduction Ad Absurdum

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

S4 Modalities

$$\frac{}{\Box A \vdash A} \qquad \frac{}{\Box(A \to B) \vdash \Box A \to \Box B}$$

$$\frac{1 \vdash A}{1 \vdash \Box A} \qquad \frac{}{\Box A \vdash \Box\Box A}$$

Future and Past Interaction

$$\frac{\Diamond_{+} A \vdash B}{A \vdash \Box_{-} B} \qquad \frac{\Diamond_{-} A \vdash B}{A \vdash \Box_{+} B}$$

Monotonicity

$$\frac{}{\Diamond(A * \Diamond B) \vdash \Diamond(A * B)}$$

Moreover, the above rules are complete as long as there are no multiplicatives. That is, for any Γ and A without multiplicative connectives,

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \models A$$

Proof. Since the soundness is immediate, we'll show the completeness part. The proof is similar to that of Proposition 2 because it uses an existing complete preordered model with respect to temporal logic with future and past modalities [18]. Consider the net whose place is a maximally consistent set of formulae in the multiplicative-free fragments (denoted by Π) and where there is a transition from Π to Π' only when for all $\Box_+ A \in \Pi$, $A \in \Pi'$. Let the interpretation be given by

$$\text{for any atomic proposition p. } i(\mathrm{p}) \quad = \quad \{\{\Pi\} \mid \Pi \in p\}$$

In this net semantics, we can show that a singleton marking $\{\Pi\}$ forces A iff $A \in \Pi$, using the following lemmas:

1. the following three are equivalent:
   - $\{\Pi\} \sqsubseteq \{\Pi'\}$.
   - for all $\Box_+ A \in \Pi$, $A \in \Pi'$.
   - for all $\Box_- A \in \Pi'$, $A \in \Pi$.
2. for any singleton marking $\{\Pi\}$,
   - if $\Diamond_+ A \in \Pi$, then for some singleton marking $\{\Pi'\}$, $A \in \Pi'$ and $\{\Pi\} \sqsubseteq \{\Pi'\}$.
   - if $\Diamond_- A \in \Pi$, then for some singleton marking $\{\Pi'\}$, $A \in \Pi'$ and $\{\Pi'\} \sqsubseteq \{\Pi\}$.

Now suppose $A \not\vdash B$. Then, $\{A, \neg B\}$ is consistent. Since any consistent set can be extended to maximally consistent set by the same process for classical logic, there exists a maximally consistent set Π including $\{A, \neg B\}$. Then, $\{\Pi\} \models A$ but $\{\Pi\} \not\models B$ since $\{\Pi\}$ forces $\neg B$.

(*End of Proof*)

We do not claim that the properties stated in this lemma are definitive; in particular, a proper proof-theoretic study is left to future work. (However, the properties are enough to secure the results of this section.)

The modal translation is from intuitionistic BI to modal classical BI. For any intuitionistic formula A, its translation A^{t} is defined as follows:

$$\begin{array}{llll} \mathrm{p}^{\mathrm{t}} \equiv \Box_- p & \qquad 1^{\mathrm{t}} \equiv 1 \\ I^{\mathrm{t}} \equiv \Diamond_+ I & \qquad (A \wedge B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \wedge B^{\mathrm{t}} \\ (A * B)^{\mathrm{t}} \equiv \Diamond_+(A^{\mathrm{t}} * B^{\mathrm{t}}) & \qquad 0^{\mathrm{t}} \equiv 0 \\ (A \mathrel{-\!\!*} B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \mathrel{-\!\!*} B^{\mathrm{t}} & \qquad (A \vee B)^{\mathrm{t}} \equiv A^{\mathrm{t}} \vee B^{\mathrm{t}} \\ & \qquad (A \to B)^{\mathrm{t}} \equiv \Box_-(A^{\mathrm{t}} \to B^{\mathrm{t}}) \end{array}$$

A bunch Γ is translated by two steps; replacing ; and , by $\wedge$ and $*$ and then applying the translation to the resulting formula. Given a net $\mathcal{N}$ and an "down-closed" interpretation i, it is immediate that semantics of a formula remains the same via this translation: for any formula A in intuitionistic BI, $M \models A$ in intuitionistic BI iff $M \models A^{\mathrm{t}}$ in modal classical BI.

Proposition 6. *If $\Gamma \vdash A$ is derivable in intuitionistic BI, then $\Gamma^{\mathrm{t}} \vdash A^{\mathrm{t}}$ is provable in BI with the addition of the rules in Lemma 4.*

Proof. The proof proceeds by induction on the depth of the derivation of $\Gamma \vdash A$. The following two lemmas play the crucial role in dealing with additive implication (first lemma) and elimination rules (second lemma):

1. for any intuitionistic BI formula A, $A^{\mathrm{t}} \vdash \Box_{-} A^{\mathrm{t}}$ is provable in BI with additional rules in Lemma 4.
2. let the context C be given by the grammar $C \equiv [\,] \mid \Diamond_{+}(C * A) \mid C \wedge A$. Then, the following rules are admissible with rules of BI plus those in Lemma 4:

$$\frac{A \vdash B}{C[A] \vdash C[B]} \qquad \frac{}{C[A \vee B] \vdash C[A] \vee C[B]}$$

All additional rules in Lemma 4 are essential in the proofs of the main proposition as well as two lemmas. As an illustration, we'll show a part of the proof of the first lemma, which is just one case of the induction on the structure of A.

$$\cfrac{\cfrac{\Diamond_{+}\Diamond_{+} I \vdash \Diamond_{+}\Diamond_{+} I}{\Diamond_{+} I \vdash \Box_{-}\Diamond_{+}\Diamond_{+} I}\ \mathit{Interaction}}{\Diamond_{+} I \vdash \Box_{-}\Diamond_{+} I}$$

where the last step is a derived rule mainly by $S4$. (*End of Proof*)

5.2 Expressing Net Properties

Given a marking M, we define the tensor formula $\overline{M}$ as before. But now we have a choice: Do we represent a place b by a down-closed set of markings, or simply by the single marking $\{\mathrm{b}\}$? We choose the latter, as it is more direct.

But now, we must represent reachability differently than in intuitionistic BI. There, a judgement $\overline{M} \models \overline{N}$ indicated that marking N was reachable from M; down-closure was crucial for this. In the present interpretation, $\overline{M} \models \overline{N}$ simply means that markings M and N are equal.

We can represent reachability using judgements of the form $\overline{M} \vdash \Diamond_{+}\overline{N}$. But we would like to go further than this, and "compile" any net into a formula, which represents its reachability capabilities. So, given a net $\mathcal{N}$, we define

$$F(\mathcal{N}) \;=\; \Box_{+}(\bigwedge_{t} \mathit{sub}(\mathit{pre}(t) \rightarrow \Diamond_{+}\mathit{post}(t)))$$

where $\mathit{sub}(A)$ says that A is true for all submarkings: $\mathit{sub}(A) \equiv \neg(\neg A * 1)$. (Notice that the past modalities are not used, as is typical in the temporal logic approach to concurrency.)

Proposition 7. *For a given net $\mathcal{N}$ and interpretation i which maps places to singleton markings as above, the following are equivalent.*

1. *Marking N is reachable from M.*
2. *$M \models \Diamond_{+}\overline{N}$ holds in the model.*
3. *$F(\mathcal{N}) \vdash \overline{M} \rightarrow \Diamond_{+}\overline{N}$ is derivable, using the rules of BI plus those of Lemma 4.*

Proof. The only interesting equivalence is between the first part and the third. Since $F(\mathcal{N})$ is true for all markings in $\mathcal{N}$, the third part implies the first by

Lemma 4. To get the other direction, we use induction on the steps of transitions with the following derived rule:

$$\frac{}{sub(A \to \Diamond_+ B) \vdash A * C \to (\Diamond_+ B) * C}$$

¿From the above rule and Monotonicity in Lemma 4,

$$\frac{\overline{\Box_+ sub(A \to \Diamond_+ B) \vdash \Diamond_+(A * C) \to \Diamond_+((\Diamond_+ B) * C)}}{\Box_+ sub(A \to \Diamond_+ B) \vdash \Diamond_+(A * C) \to \Diamond_+(B * C)}$$

is derivable, from which we can easily construct induction on the steps of transitions. (*End of Proof*)

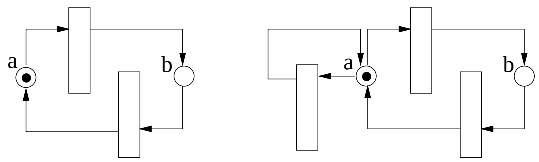
A source of incompleteness in the compilation formula $F(\mathcal{N})$ is that it does not include frame axioms, which would say that the only transitions are those in the conjunction. As a result, it is impossible to infer that a given marking is *not* reachable, reasoning syntactically from $F(\mathcal{N})$ alone.

Finally, we show by a number of examples how a number of other typical net properties can be expressed.

1. If a token is added to b, then a state with c marked becomes reachable: $\mathrm{b} \mathrel{-\!\!*} \Diamond_+(\mathrm{c} * 1)$.
2. b and c will never become marked at the same time: $\neg\Diamond_+(\mathrm{b} * \mathrm{c} * 1)$.
3. Either b or c is always marked, but never both, and never anything else: $\Box_+(\mathrm{b} \vee \mathrm{c})$.
4. Either b or c is always marked, along with possibly more: $\Box_+((\mathrm{b} \vee \mathrm{c}) * 1)$.
5. From any future state, it is always possible to evolve the net in a way such that b becomes marked and thereafter c: $\Box_+(\Diamond_+((\mathrm{b} * 1) \wedge \Diamond_+(\mathrm{c} * 1)))$.
6. Transition t is currently live: $pre(t) * 1$.
7. The net (with initial marking M_0) is live: $M_0 \to \bigwedge_t \Box_+\Diamond_+(pre(t) * 1)$.
8. t_0 and t_1 are detatched, so can that they can fire concurrently: $pre(t_0) * pre(t_1) * 1$.
9. t_0 and t_1 are in conflict: $(pre(t_0) * 1) \wedge (pre(t_1) * 1) \wedge \neg(pre(t_0) * pre(t_1) * 1)$.

6 Discussion

In this paper we have mainly looked at the semantics of BI we obtain from Engberg and Winskel's (and also Brown's) Petri net model. However, when we ask whether BI is a good logic of nets, the emphasis on reachability must be questioned. That is, the basic step of setting up a preorder of markings, determined by reachability, limits the net properties that can be expressed. For example, a typical liveness property might be: "for any way of evolving the net (from a given marking), the place b will eventually become marked." The problem is that it is easy to find two nets with the same sets of places, and the same derived reachability relation, but where one of the nets satisfies the liveness property while the other does not.



This suggests to take a different structure than a reachability preorder as primitive, as in branching time temporal logics, Hennessy-Milner logic or temporal logic based on partial order semantics [15, 33, 4, 30, 31], but in a way compatible with multiplicative conjunction.

There have, to date, been a number of suggestive uses of substructural logics in concurrency (e.g. [11, 3, 2, 22, 8, 19, 9, 24, 25]). While there has not yet been a definitive or completely convincing account, the idea of using multiplicative conjunction to describe spatial distribution of resources or processes remains appealing.

In this connection it is worth stressing, in conclusion, that there appears to be no compelling reason to limit oneself to *constructive* substructural logics, such as linear logic or (intuitionistic) BI. A particularly vivid illustration of this point is provided by the recent logic for mobile ambients of Cardelli and Gordon [10]. Their logic is similar to BI in the way it mixes multiplicatives and additives, but it also includes a novel modality (the ambient match), which is used to talk about location of processes. Their model supports classical additives, which seems entirely natural when one considers process specifications. Similarly, we found the classical version of the net semantics to be much more intuitive and direct, as well as being more expressive, than in the intuitionistic case.

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