Semantics of Higher-order Probabilistic Programs

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Joint work with Chris Heunen, Ohad Kammar, Sam Staton, Frank Wood

What issues did we encounter in defining the denotational semantics of an idealised Anglican?

Continuous distribution and soft constraints

```
let x=sample(normal(0,1)) in
obs(normal(x,1),2); return(x)
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Which monad M should we pick?

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$p(x) \times p(y=2 \mid x)$

1. Lazy semantics (importance sampling):

$$M(T) = \mathrm{Prob}(R_{\geq 0} \times T)$$

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2. Eager semantics:

$$M(T) = R_{\geq 0} \times Prob(T)$$

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let x=sample(exponential(1)) in
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$\notin R_{\geq 0} \times Prob(R)$

Because p(y=0) = ∞ and p(x|y=0) undefined.

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Because $p(y=0) = \infty$ and $p(x|y=0)$ undefined.

- $p(x) = \exp(-x)$.
- $p(y=0 \mid x) = 1 / (c \times \exp(-x))$.
- $p(y=0) = \int p(y=0|x)p(x)dx = \int 1/c \, dx = \infty$.

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- $p(y=0) = \int p(y=0|x)p(x)dx = \int 1/c \, dx = \infty$.

We tried $(R_{\geq 0} \times Prob(R) + 1 + 1)$, but failed.

In the lazy semantics, conjugate-prior equations fail.

$$[\![\ \texttt{let x=sample(beta(1,1)) in obs(bern(x),true); return(x)}\ ]\!]$$

$$\neq$$

$$[\![\ \texttt{obs(bern(0.5),true); sample(beta(2,1))}\ ]\!]$$

In the lazy semantics, conjugate-prior equations fail.
But can be recovered with explicit normalisation (or nested query).

$$
\left[\!\left[\begin{array}{l}
\texttt{norm(} \\
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\end{array} \right]\!\right]
=
\left[\!\left[\begin{array}{l}
\texttt{norm(} \\
\texttt{obs(bern(0.5),true);} \\
\texttt{sample(beta(2,1)))}
\end{array} \right]\!\right]
$$

$[\![\mathrm{norm}]\!] : \mathrm{Prob}(R_{\geq 0} \times T) \to (R_{\geq 0} \times \mathrm{Prob}(T) + 1 + 1)$

Continuous densities

Density object

```
let x=sample(exponential(1)) in
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```

- Represents a probability density.
- Supports sample and obs methods.
- In Anglican, the densities of these objects are usually continuous functions.

$$[\![\text{Dens}[R]]\!] = \{\ f : R \rightarrow R_{\geq 0}\ \mid\ f \text{ is continuous and } \int f(x)dx = 1\}$$

$$[\![\text{sample}]\!]\ :\ [\![\text{Dens}[R]]\!] \rightarrow P(R_{\geq 0} \times R)$$

$$[\![\text{obs}]\!] :\ [\![\text{Dens}[R]]\!] \times [\![R]\!] \rightarrow P(R_{\geq 0} \times ())$$

$$[\![\mathrm{Dens}[R]]\!] = \{\, f : R \to R_{\geq 0} \;\mid\; f \text{ is continuous and } \int f(x)dx = 1 \,\}$$

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$$[\![\mathrm{obs}]\!] : [\![\mathrm{Dens}[R]]\!] \times [\![R]\!] \to P(R_{\geq 0} \times ())$$

[Q] Is $[\![\mathrm{obs}]\!]$ measurable?

Non-measurability of ev

$$ev : (R \rightarrow_m R) \times R \rightarrow R, \qquad ev(f,x) = f(x)$$

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But:

1. $ev_r = ev(-,r) : (R \to_m R) \to R$ is measurable.
2. $ev : (R \to_c R) \times R \to R$ is measurable.

$$[\![\mathrm{Dens}[R]]\!] = \{\, f : R \to R_{\geq 0} \;\mid\; f \text{ is continuous and } \int f(x)dx = 1 \,\}$$

$$[\![\mathrm{sample}]\!] \;:\; [\![\mathrm{Dens}[R]]\!] \to P(R_{\geq 0} \times R)$$

$$[\![\mathrm{obs}]\!] : [\![\mathrm{Dens}[R]]\!] \times [\![R]\!] \to P(R_{\geq 0} \times ())$$

[Q] Is $[\![\mathrm{obs}]\!]$ measurable? [A] Yes.

Higher-order function

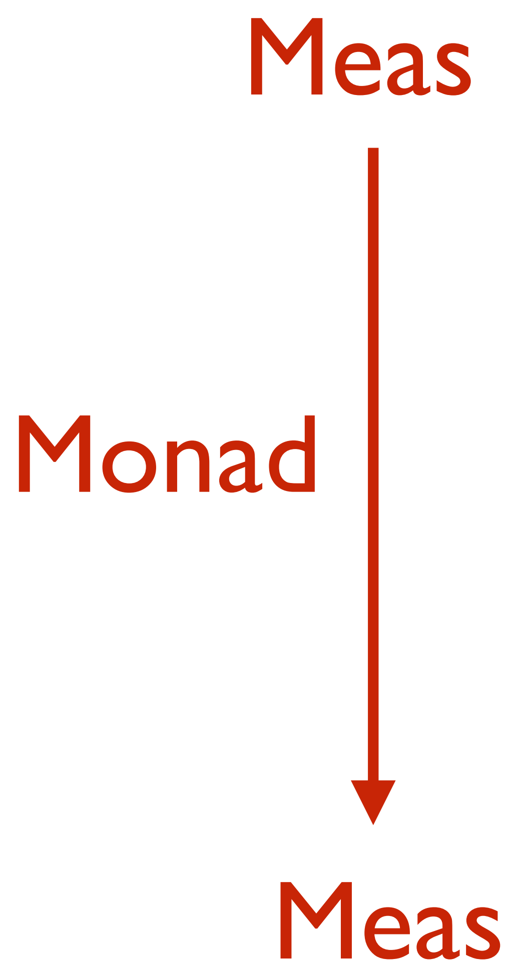
Difficulty

$$ev : (R \to_m R) \times R \to R, \qquad ev(f,x) = f(x)$$

[Aumann 61 & Halmos] ev is not measurable no matter which σ-algebra is used for $R \to_m R$.

[Lemma] The category of measurable spaces is not cartesian closed.

Use category theory to extend measure theory.



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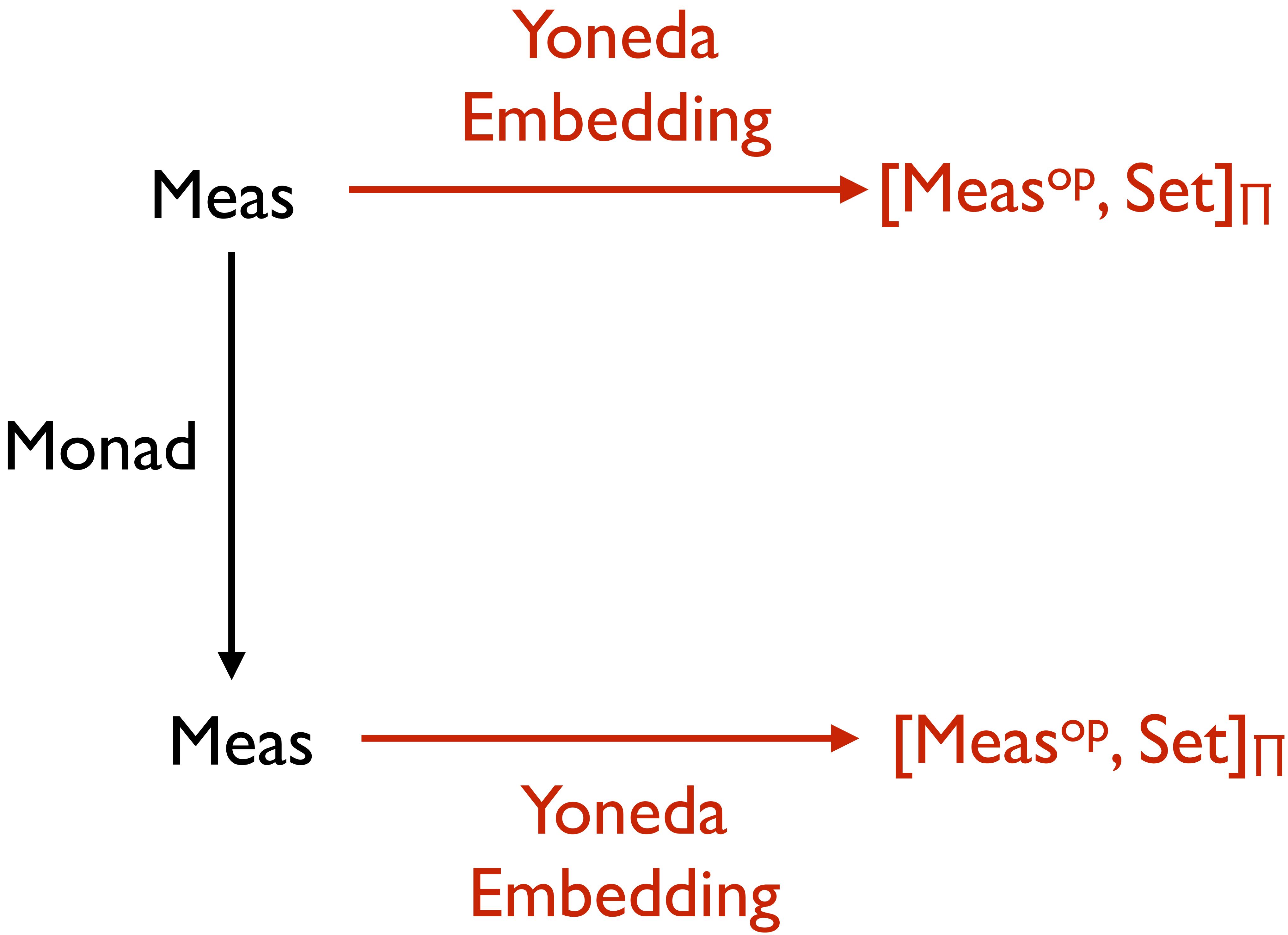

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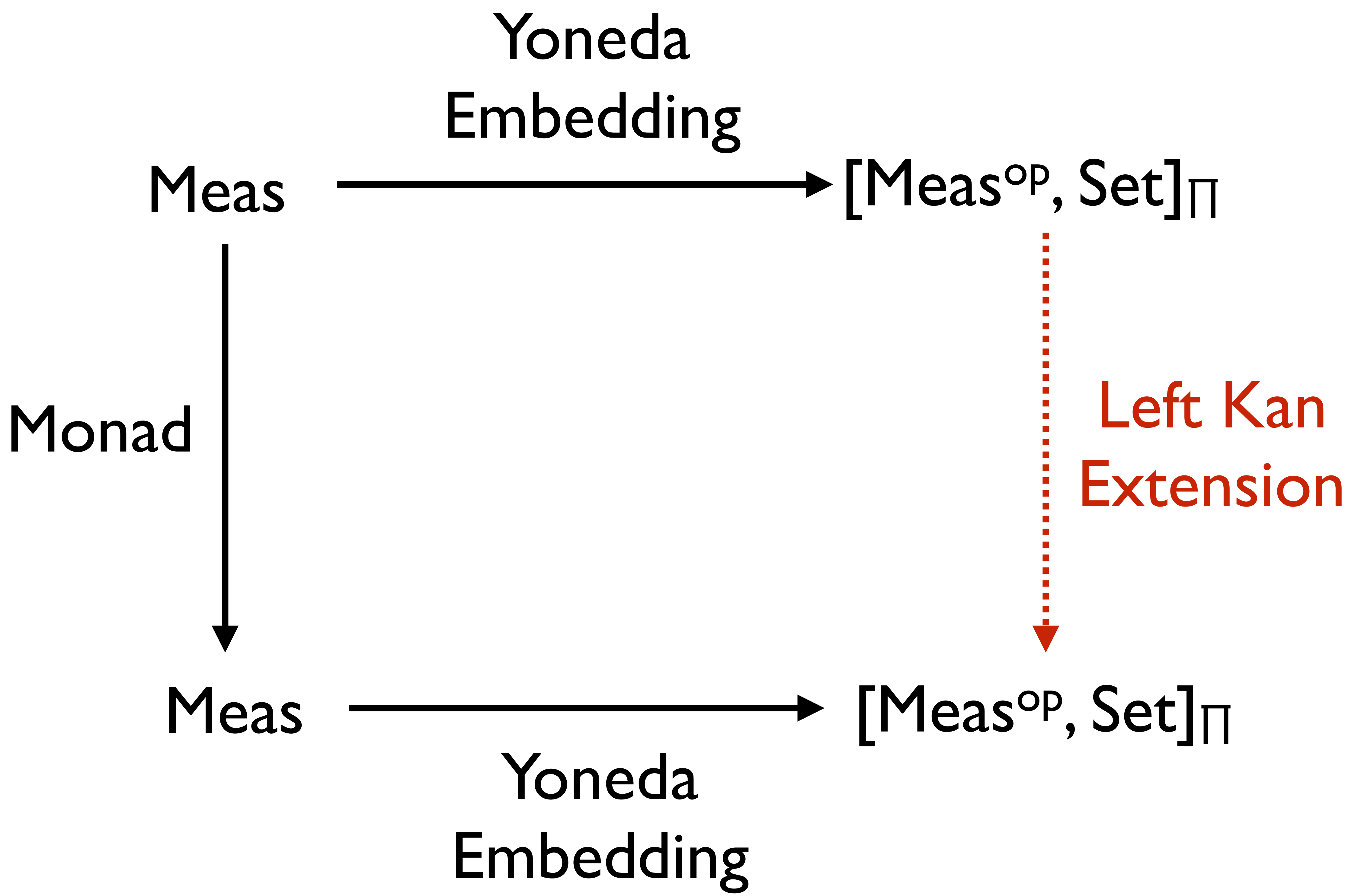

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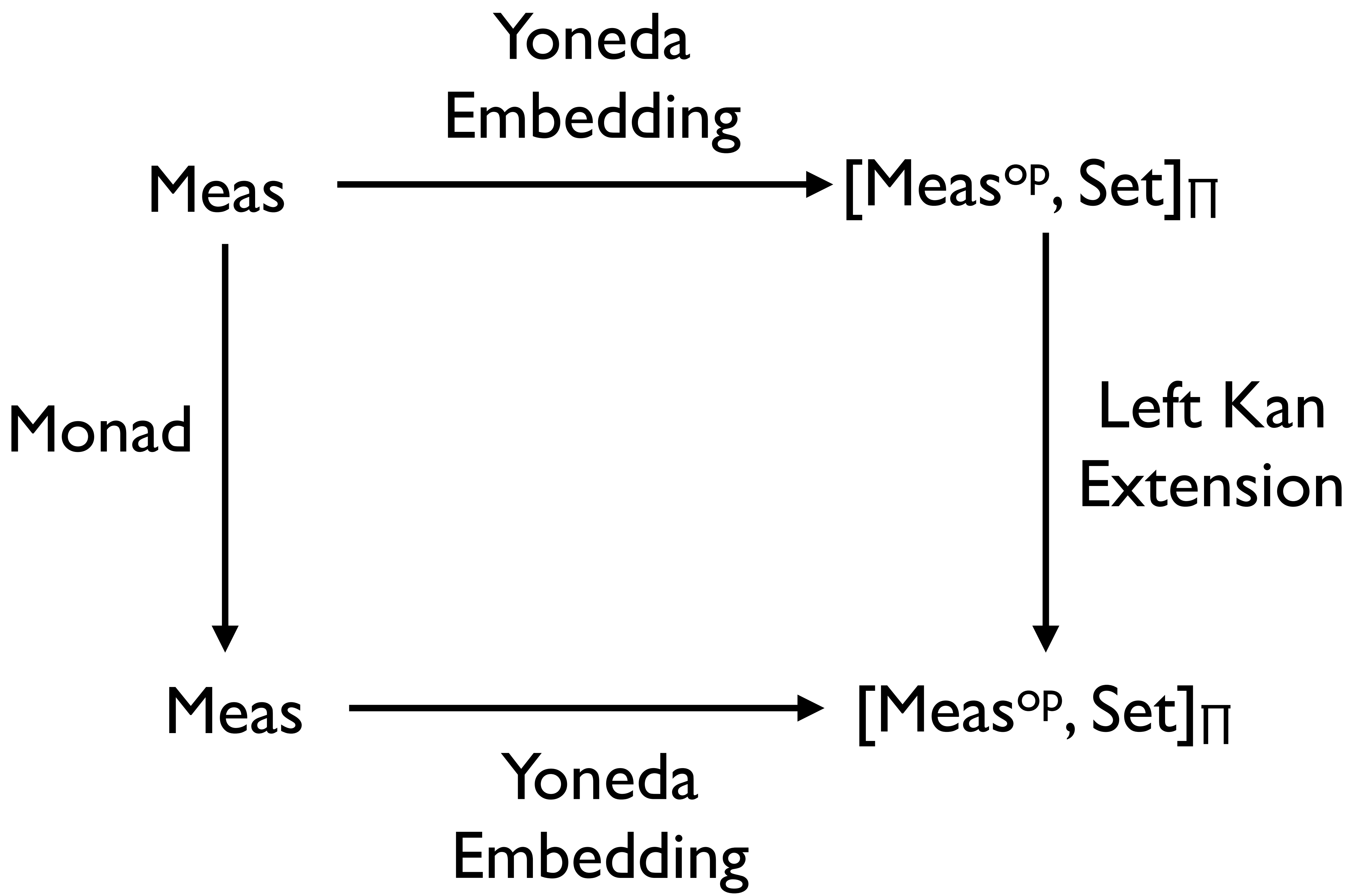

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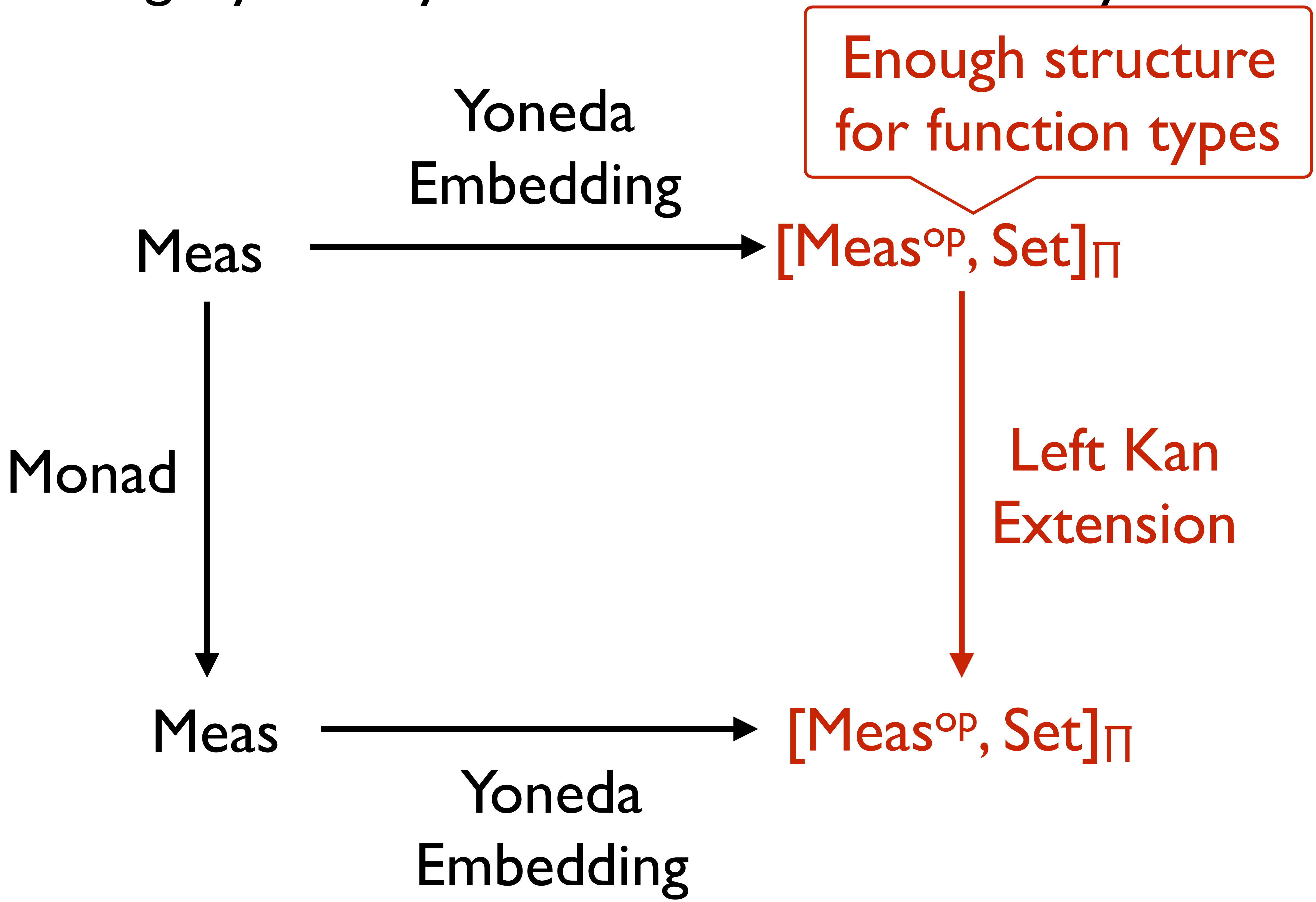

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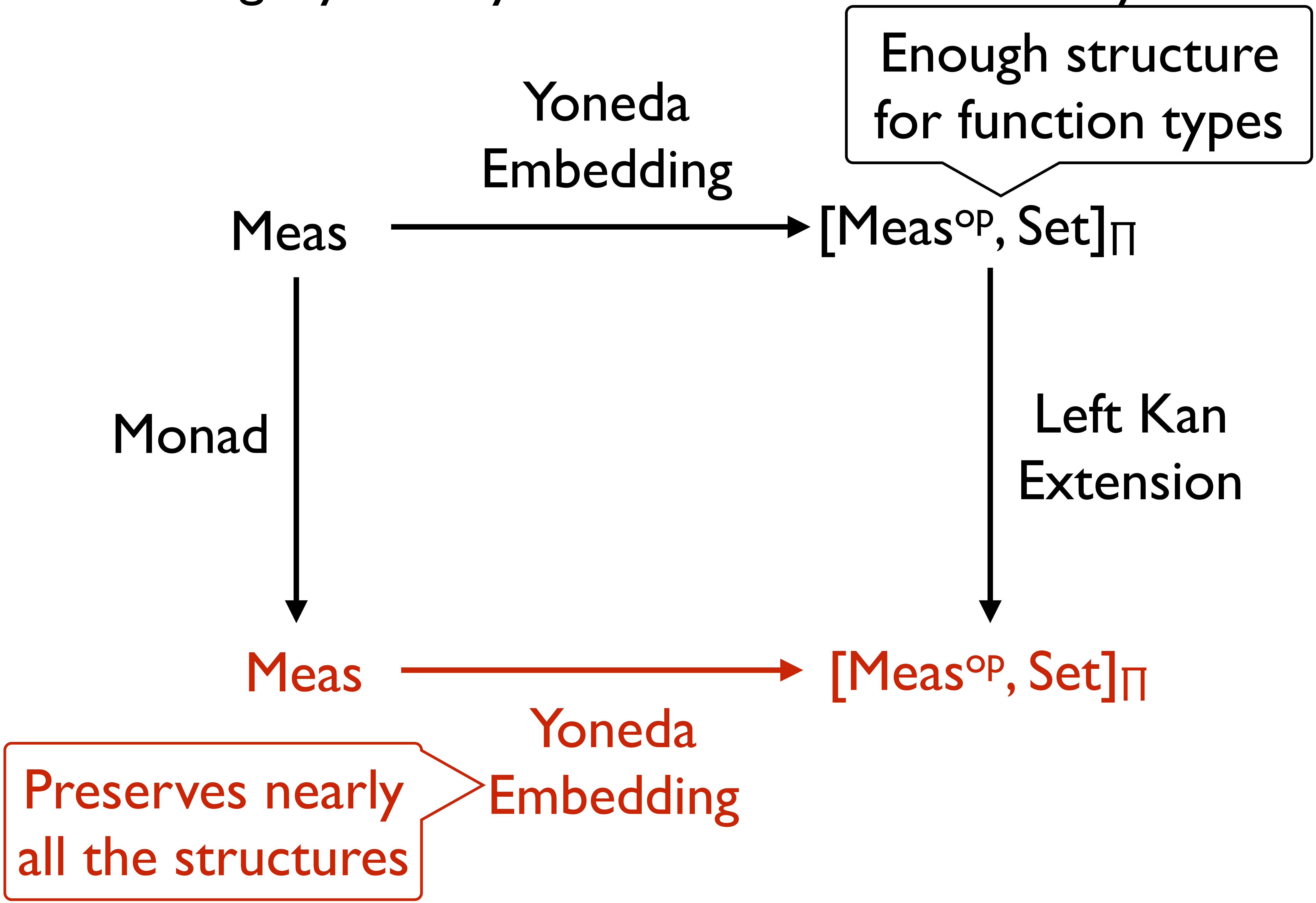

[Question] Are all definable functions from R to R in a high-order probabilistic PL measurable?

Our semantics says that the answer is yes for a core call-by-value language.

The monad $\underline{M}([\![R \to R]\!])$ at $[\![R \to R]\!]$ consists of:

equivalence classes of measurable functions $f : \Omega \times R \to R$ for probability spaces Ω.

The function f is what probabilists call a measurable stochastic process.

The extended monad $\underline{M}$ describes computations with dynamically allocated read-only variables.

$$\underline{M}(F)(w) =$$
$$\{\ [(a, f)]_{\sim}\ \mid\ \exists v.\ a \in F(v)\ \wedge\ f : w \to_m \mathrm{Prob}(R_{\geq 0} \times v)\ \}$$

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F is the type of a value.
w represents a space of all random vars so far.
v extends w with new random variables according to f.

Further details

- Can be found in our archive paper:

  http://arxiv.org/pdf/1601.04943.pdf