

# On the Greatest Fixed Point of a Finitary Set Functor

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## Greatest fixed points

A **fixed point** of an endofunctor  $T: \mathcal{C} \rightarrow \mathcal{C}$  is a pair  $(A, f: A \rightarrow TA)$ , where  $A$  is an object of  $\mathcal{C}$  and  $f$  is an isomorphism. A **homomorphism** of fixed points  $(A, f)$  and  $(B, g)$  is an arrow  $h: A \rightarrow B$  such that

$$\begin{array}{ccc} A & \xrightarrow{f} & TA \\ h \downarrow & & \downarrow Th \\ B & \xrightarrow{g} & TB \end{array}$$

**Defn.** The **greatest fixed point** of  $T$  is the final object in the above category of fixed points.

## Examples

**Non-well-founded sets:** The greatest fixed point of the powerset functor  $\mathbb{P} : \text{Class} \rightarrow \text{Class}$  is a model of  $\text{ZF}^- + \text{anti-foundation axiom}$ .

**Reactive processes:** The class of  $A$ -labelled transition systems quotiented by **bisimilarity** is a greatest fixed point of  $\mathbb{P}(X \times A)$ .

**Domain equations:** Suppose  $\mathcal{C}$  is **algebraically compact**. Let  $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , and define  $\check{F} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$  by

$$\check{F}(A, B) = (F(B, A), F(A, B)).$$

The greatest fixed point of  $\check{F}$  has the form  $(D, D)$  where  $D \cong F(D, D)$ . Scott's  $D_\infty$  model of the  $\lambda$ -calculus can be seen in this way.

# Coalgebras

A **coalgebra** of an endofunctor generalises a **post-fixed point** of an monotone map.

A **coalgebra** of an endofunctor  $T: \mathcal{C} \rightarrow \mathcal{C}$  is a pair  $(A, f: A \rightarrow TA)$ , where  $A$ , the carrier of the coalgebra, is an object of  $\mathcal{C}$ , and  $f$ , the structure map, is an arrow of  $\mathcal{C}$ . A **homomorphism** of  $T$ -coalgebras  $(A, f)$  and  $(B, g)$  is an arrow  $h: A \rightarrow B$  such that

$$\begin{array}{ccc} A & \xrightarrow{f} & TA \\ h \downarrow & & \downarrow Th \\ B & \xrightarrow{g} & TB \end{array}$$

**Thm.** (Lambek) The final object in the category  $\mathcal{C}_T$  of  $T$ -coalgebras is a fixed point of  $T$ .

## Finitary functors

A cardinal  $\kappa$  is **regular** if it is not the sum of fewer than  $\kappa$  strictly smaller cardinals. For a regular cardinal  $\kappa$  we say that a partially ordered set  $I$  is  **$\kappa$ -directed** if each subset of  $I$  of size strictly less than  $\kappa$  has an upper bound in  $I$ .

A functor  $T : \mathcal{C} \rightarrow \mathcal{C}$  is  **$\kappa$ -accessible** if it preserves colimits of those diagrams indexed by  $\kappa$ -directed posets. An  $\omega$ -accessible functor is sometimes called **finitary**.

## GFP existence: a five-line proof

**Prop.** Let  $T : \text{Set} \rightarrow \text{Set}$  be an endofunctor. Then the forgetful functor  $U : \text{Set}_T \rightarrow \text{Set}$  creates (and preserves) colimits.

**Thm.** (Barr) If  $T : \text{Set} \rightarrow \text{Set}$  is accessible then there is a final  $T$ -coalgebra.

**Pf.** By the special adjoint functor theorem we must show that  $\text{Set}_T$  is cocomplete, well-copowered and has a set of generators. The cocompleteness and well-copoweredness follow from the same facts for  $\text{Set}$  by the above proposition. Makkai and Paré's weighted bilimit theorem implies that  $\text{Set}_T$  is accessible and thus has a set of generators.

## A more concrete approach

The **final sequence** of  $T$  is an ordinal-indexed sequence of objects  $\langle T^\alpha 1 \rangle$ , with maps  $(f_\gamma^\beta : T^\beta 1 \rightarrow T^\gamma 1)_{\gamma \leq \beta}$ , uniquely defined by the following conditions (where  $\delta \leq \gamma \leq \beta$ ):

- FS-1  $T^{\beta+1} 1 = T(T^\beta 1)$ ;
- FS-2  $f_{\gamma+1}^{\beta+1} = T(f_\gamma^\beta)$ ;
- FS-3  $f_\delta^\beta = f_\delta^\gamma \cdot f_\gamma^\beta$ ;
- FS-4 if  $\beta$  is a limit ordinal, then  $(f_\gamma^\beta : T^\beta 1 \rightarrow T^\gamma 1)_{\gamma \leq \beta}$  is a limit cone.

**Prop.** If  $f_\kappa^{\kappa+1}$  is an isomorphism, then  $(T^\kappa 1, (f_\kappa^{\kappa+1})^{-1})$  is a final coalgebra.

## Motivating example

Our leading example of a finitary set functor is the finite powerset functor  $\mathbb{P}: \text{Set} \rightarrow \text{Set}$ . For a set  $X$ ,  $\mathbb{P}X$  is the collection of finite subsets of  $X$ .

For a function  $f: X \rightarrow Y$ ,  $\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$  is defined by

$$(\mathbb{P}f)(S) = f(S).$$

A  $\mathbb{P}$ -coalgebra  $(A, \alpha)$  can be seen as a graph or a transition system.

Next we investigate the final sequence on  $\mathbb{P}$ .

## Tree bisimulations

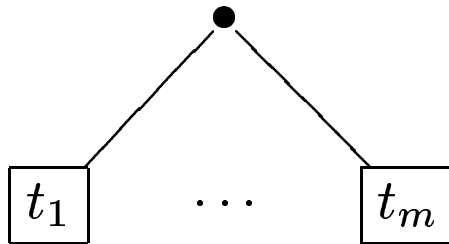
We consider trees as directed graphs with a distinguished root node from which every other node is reachable by a unique path.

A relation  $R$  on the set of nodes of a tree is a **tree bisimulation** if  $xRy$  implies the respective parents of  $x$  and  $y$  are related, each child of  $x$  is  $R$ -related to a child of  $y$ , and each child of  $y$  is  $R^{-1}$ -related to a child of  $x$ . We call a tree **strongly extensional** if no two distinct nodes are related by a bisimulation. For any tree  $t$  the union of all tree bisimulations is an equivalence, and the quotient of  $t$  by this equivalence is strongly extensional.

**Remark.** An infinite-depth tree can be extensional without being strongly extensional.

## Tree representation

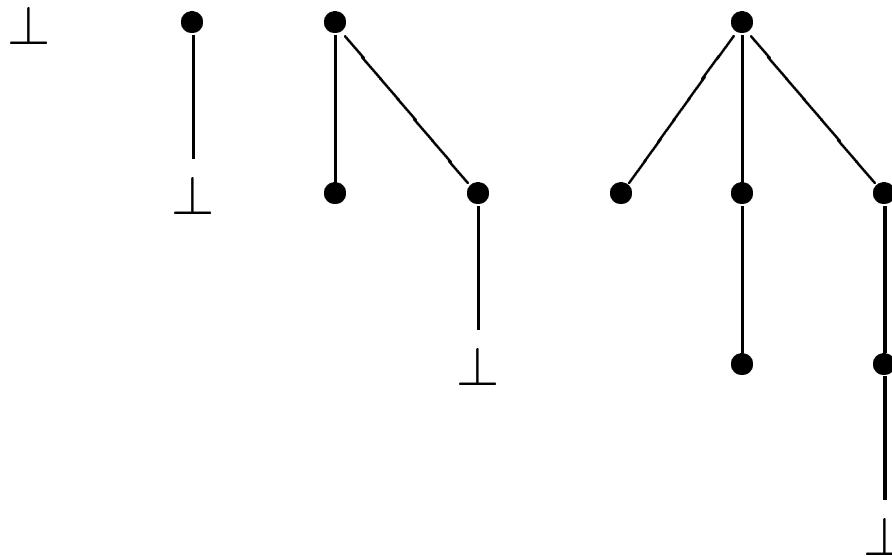
Write  $\{\perp\}$  for the final object in  $\text{Set}$ . For each  $n < \omega$  there is a bijection between  $\mathbb{P}^n 1$  and the set of finitely branching, strongly extensional trees of depth no greater than  $n$ , whose depth- $n$  nodes are labelled  $\perp$ . By induction: if  $x_1, \dots, x_m \in \mathbb{P}^n 1$  correspond to trees  $t_1, \dots, t_m$ , then  $\{x_1, \dots, x_m\} \in \mathbb{P}^{n+1} 1$  corresponds to



The projection map  $f_n^{n+1}$  takes a tree in  $\mathbb{P}^{n+1} 1$ , removes the depth- $(n + 1)$  nodes and computes the strongly extensional quotient.

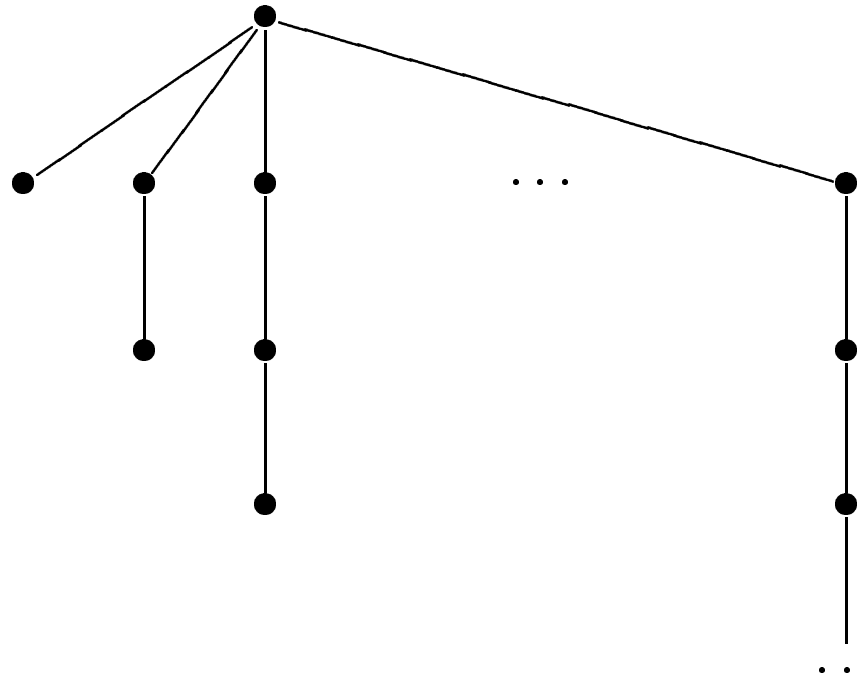
## Growing trees

Below we draw a row of trees taken respectively from  $\mathbb{P}^0 1$ ,  $\mathbb{P}^1 1$ ,  $\mathbb{P}^2 1$  and  $\mathbb{P}^3 1$ , with each tree projecting down to the tree to the left.



## In the limit

We can imagine the trees as projections of the following **compactly** branching infinite-depth tree in  $\mathbb{P}^\omega$ .



## A metric on trees

Given trees  $t, t'$ , write  $t \approx_n t'$  if the restrictions of  $t$  and  $t'$  to depth  $n$  have the same strongly extensional quotient, and define a pseudo-metric  $d_{\mathcal{T}}$  on the class of strongly extensional trees by

$$d_{\mathcal{T}}(t, t') = \inf\{2^{-n} \mid t \approx_n t'\}.$$

## To infinity and beyond!

$\mathbb{P}^\omega 1$  is the set of **compactly branching**, strongly extensional trees of possibly infinite depth.

$\mathbb{P}^{\omega+1} 1$  is the set of compactly branching strongly, strongly extensional trees **that are finitely branching at the root**.

$\mathbb{P}^{\omega+n} 1$  is the set of compactly branching strongly, strongly extensional trees **that are finitely branching to depth  $n$** .

$\mathbb{P}^{\omega+\omega} 1$  is the set of finitely branching, strongly extensional trees.

Thus the final sequence of  $\mathbb{P}$  consists of  $\omega$  steps of **building up** higher and higher trees, followed by  $\omega$  steps of **pruning** level by level.

## The general case

We show that the final sequence  $\{T^\alpha 1, f_\beta^\alpha\}$  of a finitary functor  $T : \text{Set} \rightarrow \text{Set}$  stabilises in  $\omega + \omega = \omega 2$  steps.

Note that each  $T$ -coalgebra  $(E, e)$  extends to a **cone**  $(e_\alpha : E \rightarrow T^\alpha 1)$  over the final sequence, where  $e_{\alpha+1} = T(e_\alpha) \cdot e$ .

$$\begin{array}{ccccccc}
 E & \xrightarrow{e} & TE & \xrightarrow{Te} & T^2E & \xrightarrow{T^2e} & T^3E \xrightarrow{T^3e} \dots \\
 \downarrow ! & & \downarrow T! & & \downarrow T^2! & & \downarrow T^3! \\
 1 & \longleftarrow & T1 & \longleftarrow & T^21 & \longleftarrow & T^31 \longleftarrow \dots
 \end{array}$$

## The parsing map

**Prop.**  $f_\omega^{\omega+1}$  is injective (and hence  $f_\omega^\alpha$  is injective for all  $\alpha \geq \omega$ ).

Choose a **parsing map**  $p : T^\omega 1 \rightarrow T(T^\omega 1)$ , where  $p \cdot f_\omega^{\omega+1} = id$ , and extend  $p$  to a cone  $(p_\alpha : T^\omega 1 \rightarrow T^\alpha 1)$  over the final sequence of  $T$ .

**Prop.**  $p_\omega : T^\omega 1 \rightarrow T^\omega 1$  is an idempotent self map of the  $T$ -coalgebra  $(T^\omega 1, p)$ .

## A final coalgebra

Write  $p_\omega = s \cdot r$ , where the **retraction**  $r : T^\omega 1 \rightarrow G$  and the **section**  $s : G \rightarrow T^\omega 1$  satisfy  $r \cdot s = id$ . Notice that the pair  $Tr \cdot p$  and  $f_\omega^{\omega+1} \cdot Ts$  is also a splitting of  $p_\omega$ , thus, we have an isomorphism  $g$  making the diagram below commute.

$$\begin{array}{ccc}
 T^\omega 1 & \xrightarrow{p} & T^{\omega+1} 1 \\
 \downarrow r & & \downarrow Tr \\
 G & \xrightarrow{g} & TG \\
 \downarrow s & & \downarrow Ts \\
 T^\omega 1 & \xleftarrow{f_\omega^{\omega+1}} & T^{\omega+1} 1
 \end{array}
 \begin{array}{l}
 p_\omega \\
 \\
 \\
 \\
 Tp_\omega
 \end{array}$$

**Thm.**  $(G, g)$  is a final  $T$ -coalgebra.

## A squeezing argument

We show that the injection  $f_\omega^{\omega+\omega} : T^{\omega+\omega}1 \rightarrow T^\omega 1$  factors through  $s : G \rightarrow T^\omega 1$  and vice versa.

**Prop.**  $p_\omega \cdot f_\omega^{\omega+\omega} = f_\omega^{\omega+\omega}$ .

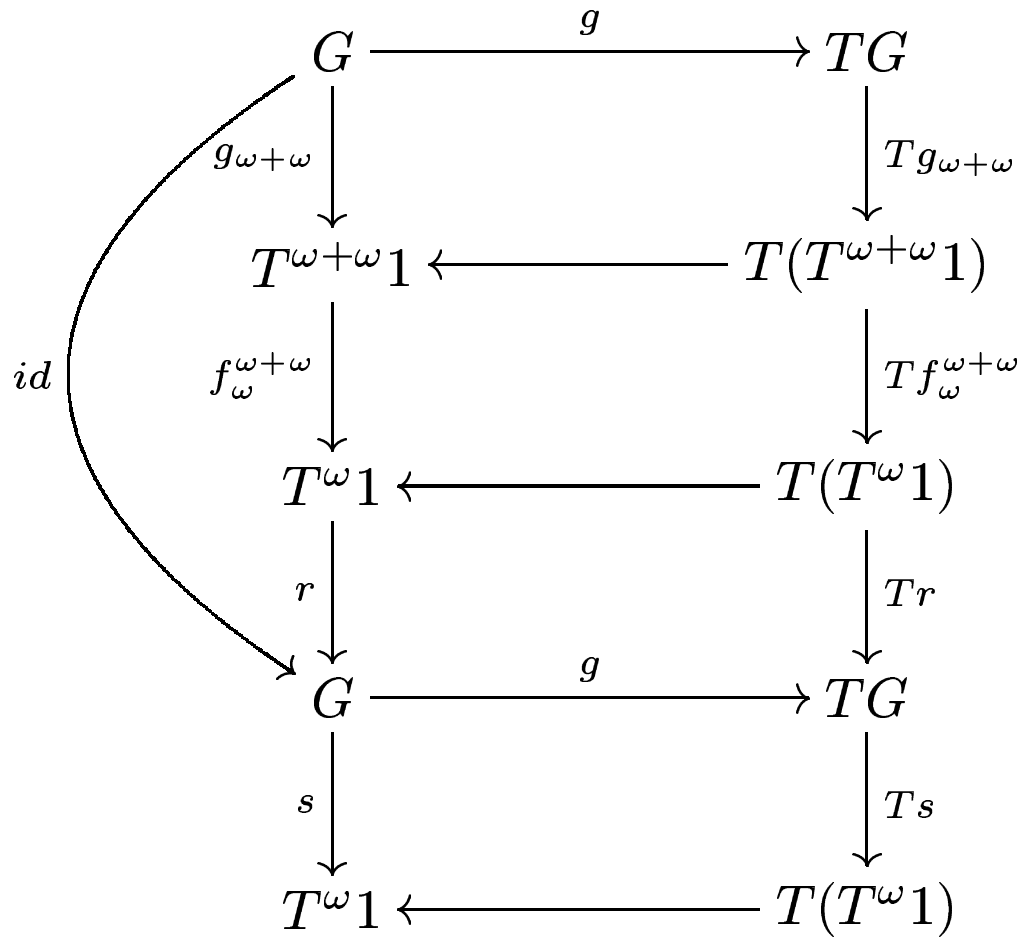
**Pf.** Note that  $p_n \neq f_n^\omega$  in general. We show that  $p_n \cdot f_\omega^{\omega+n} = f_n^{\omega+n}$  for each  $n < \omega$ .

$$\begin{aligned} p_{n+1} \cdot f_\omega^{\omega+n+1} &= T(p_n) \cdot p \cdot f_\omega^{\omega+n+1} \\ &= T(p_n) \cdot p \cdot f_\omega^{\omega+1} \cdot f_{\omega+1}^{\omega+n+1} \\ &= T(p_n) \cdot f_{\omega+1}^{\omega+n+1} \\ &= T(p_n \cdot f_\omega^{\omega+n}) \\ &= T(f_n^{\omega+n}) \\ &= f_{n+1}^{\omega+n+1}. \end{aligned}$$

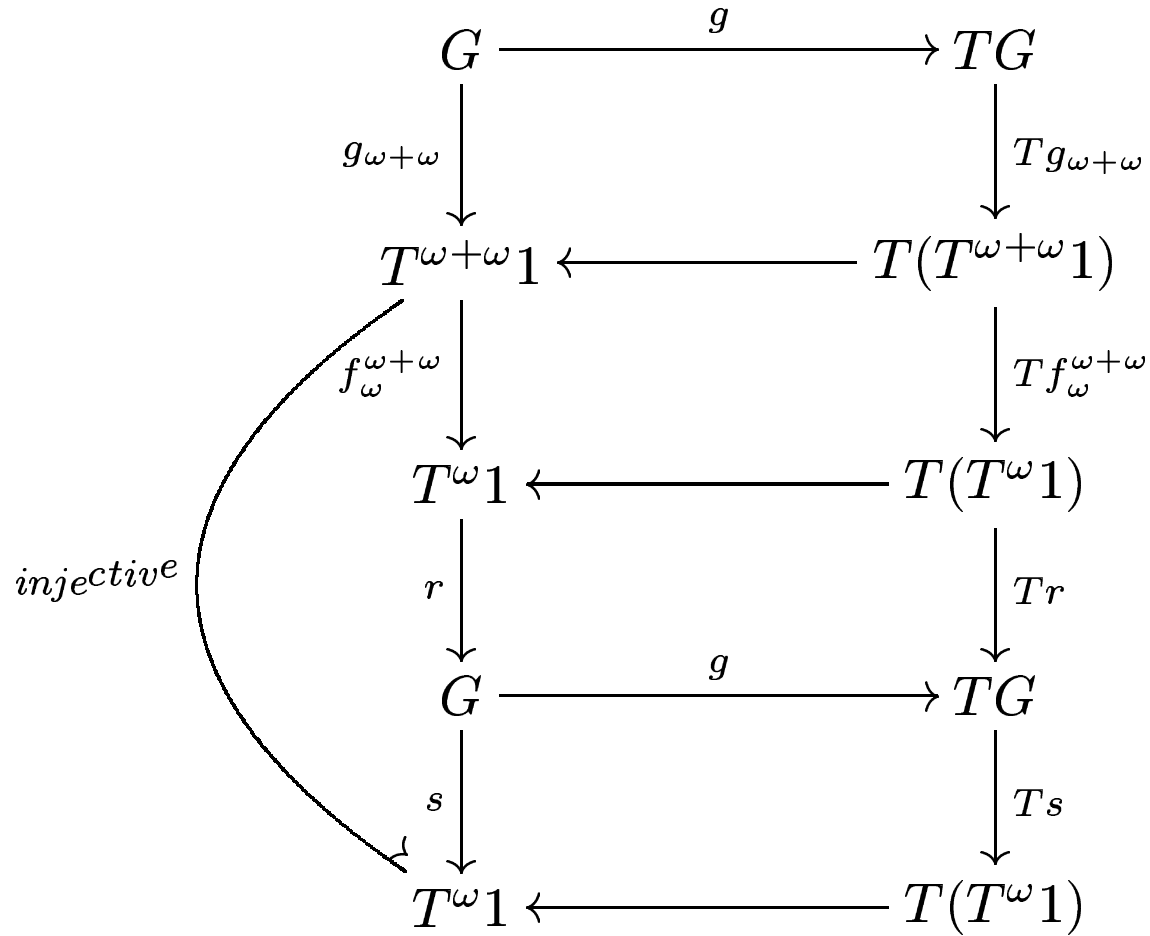
# A squeezing argument

$$\begin{array}{ccc}
 G & \xrightarrow{g} & TG \\
 \downarrow g_{\omega+\omega} & & \downarrow Tg_{\omega+\omega} \\
 T^{\omega+\omega}1 & \longleftarrow & T(T^{\omega+\omega}1) \\
 \downarrow f_{\omega}^{\omega+\omega} & & \downarrow Tf_{\omega}^{\omega+\omega} \\
 T^{\omega}1 & \longleftarrow & T(T^{\omega}1) \\
 \downarrow r & & \downarrow Tr \\
 G & \xrightarrow{g} & TG \\
 \downarrow s & & \downarrow Ts \\
 T^{\omega}1 & \longleftarrow & T(T^{\omega}1)
 \end{array}$$

# A squeezing argument



# A squeezing argument



## Main result

**Thm.** Let  $\kappa$  be a regular cardinal. The class of endofunctors on  $\mathbf{Set}$  whose final sequence converges in  $\kappa + \kappa$  steps is closed under

- 1  $\kappa$ -accessible functors;
- 2  $\kappa^{\text{op}}$ -continuous functors;
- 3 composition of functors;
- 4 arbitrary coproducts of functors;
- 5  $\mathcal{I}$ -indexed-limits of functors, for any small category  $\mathcal{I}$ .

## A non-example

Consider the **subprobability-distribution functor**  $\mathbb{D}: \mathbf{Set} \rightarrow \mathbf{Set}$ . For a set  $X$ ,  $\mathbb{D}X$  is the set of functions  $\mu: X \rightarrow [0, 1]$  with countable support such that  $\sum_{x \in X} \mu(x) \leq 1$ . For  $\mu \in \mathbb{D}X$  and  $E \subseteq X$  define  $\mu[E] = \sum_{x \in E} \mu(x)$ . We can extend  $\mathbb{D}$  to an endofunctor on  $\mathbf{Set}$  by defining, for a function  $f: X \rightarrow Y$ ,

$$(\mathbb{D}f)(\mu)(y) = \mu[f^{-1}(y)].$$

## A non-example

**Prop.** In the final sequence  $\{\mathbb{D}^\alpha 1, f_\beta^\alpha\}$ , the map  $f_\omega^{\omega+1}$  is injective.

**Pf.** Suppose  $\mu, \rho \in \mathbb{D}(\mathbb{D}^\omega 1)$  and  $f_\omega^{\omega+1}(\mu) = f_\omega^{\omega+1}(\rho)$ . Then for each  $\langle x_n \rangle \in \mathbb{D}^\omega 1$ , since  $\mu[-]$  (being a measure) preserves decreasing countable intersections,

$$\mu(\langle x_n \rangle) = \mu\left[\bigcap_{n < \omega} (f_n^\omega)^{-1}(x_n)\right] = \lim_{n \rightarrow \infty} \mu[(f_n^\omega)^{-1}(x_n)] = \lim_{n \rightarrow \infty} (\mathbb{D}f_n^\omega)(\mu)(x_n).$$

Similarly we get  $\rho(\langle x_n \rangle) = \lim_{n \rightarrow \infty} (\mathbb{D}f_n^\omega)(\rho)(x_n)$ . But

$$(\mathbb{D}f_n^\omega)(\mu) = f_{n+1}^{\omega+1}(\mu) = f_{n+1}^\omega(f_\omega^{\omega+1}(\mu)) = f_{n+1}^\omega(f_\omega^{\omega+1}(\rho)) = f_{n+1}^{\omega+1}(\rho) = (\mathbb{D}f_n^\omega)(\rho)$$

for all  $n < \omega$ . Thus  $\mu = \rho$ .

## Other categories

Say that  $C \in \mathcal{C}$  is a **perfect generator** if  $C$  is finitely presentable, projective and for each object  $K$  the canonical map

$$\varepsilon_K : \coprod_{\text{hom}(X, K)} \longrightarrow K$$

is a **bimorphism**.

**Examples.** The one-element poset is a perfect generator in  $\text{PoS}$ , and the graph with one vertex and no edges is a perfect generator in  $\text{Graph}$ .

**Thm.** (Adámek) If  $\mathcal{C}$  is locally finitely presentable with a perfect generator, and if  $T : \mathcal{C} \rightarrow \mathcal{C}$  is a finitary functor preserving strong monos and bimorphisms, then the final sequence of  $T$  stabilises in  $\omega + \omega$  steps.

## Further work

The  $\omega + \omega$  bound on the stabilisation of the final sequence of a set functor is tight. However, we do not have an example of an  $\omega_1$ -accessible set functor whose final sequence takes fully  $\omega_1 + \omega_1$  steps to stabilise. If the functor preserves limits of chains of monos, then one has stabilisation in  $\omega_1 + \omega$  steps. This case seems to cover all of the ‘reasonable’  $\omega_1$ -accessible set functors one comes across, e.g., the countable powerset functor.

For functors on locally finitely presentable categories, are all (any) of Adámek’s side conditions necessary?

**Thank you for your attention!**

James Worrell. On the final sequence of a finitary set functor. *Theoretical Computer Science*, 338(1-3): 184–199, 2005.