On the Greatest Fixed Point of a Finitary Set Functor

James Worrell
Oxford University Computing Laboratory, UK

Greatest fixed points

A **fixed point** of an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ is a pair $(A, f : A \to TA)$, where A is an object of $\mathcal{C}$ and f is an isomorphism. A **homomorphism** of fixed points (A, f) and (B, g) is an arrow $h\colon A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & TA \arrow[d, "Th"] \\ B \arrow[r, "g"'] & TB \end{tikzcd}$$

Defn. The **greatest fixed point** of T is the final object in the above category of fixed points.

Examples

Non-well-founded sets: The greatest fixed point of the powerset functor $\mathbb{P} : \mathsf{Class} \to \mathsf{Class}$ is a model of ZF^- + **anti-foundation axiom**.

Reactive processes: The class of A-labelled transition systems quotiented by **bisimilarity** is a greatest fixed point of $\mathbb{P}(X \times A)$.

Domain equations: Suppose $\mathcal{C}$ is **algebraically compact**. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, and define $\breve{F} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ by

$$\breve{F}(A, B) = (F(B, A), F(A, B)) \, .$$

The greatest fixed point of $\breve{F}$ has the form (D, D) where $D \cong F(D, D)$. Scott's D_∞ model of the λ-calculus can be seen in this way.

Coalgebras

A **coalgebra** of an endofunctor generalises a **post-fixed point** of an monotone map.

A **coalgebra** of an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ is a pair $(A, f\colon A \to TA)$, where A, the carrier of the coalgebra, is an object of $\mathcal{C}$, and f, the structure map, is an arrow of $\mathcal{C}$. A **homomorphism** of T-coalgebras (A, f) and (B, g) is an arrow $h\colon A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & TA \arrow[d, "Th"] \\ B \arrow[r, "g"'] & TB \end{tikzcd}$$

Thm. (Lambek) The final object in the category $\mathcal{C}_T$ of T-coalgebras is a fixed point of T.

Finitary functors

A cardinal κ is **regular** if it is not the sum of fewer than κ strictly smaller cardinals. For a regular cardinal κ we say that a partially ordered set I is κ**-directed** if each subset of I of size strictly less than κ has an upper bound in I.

A functor $T : \mathcal{C} \to \mathcal{C}$ is κ**-accessible** if it preserves colimits of those diagrams indexed by κ-directed posets. An ω-accessible functor is sometimes called **finitary**.

GFP existence: a five-line proof

Prop. Let $T : \mathsf{Set} \to \mathsf{Set}$ be an endofunctor. Then the forgetful functor $U : \mathsf{Set}_T \to \mathsf{Set}$ creates (and preserves) colimits.

Thm. (Barr) If $T : \mathsf{Set} \to \mathsf{Set}$ is accessible then there is a final T-coalgebra.

Pf. By the special adjoint functor theorem we must show that Set_T is cocomplete, well-copowered and has a set of generators. The cocompleteness and well-copoweredness follow from the same facts for Set by the above proposition. Makkai and Paré's weighted bilimit theorem implies that Set_T is accessible and thus has a set of generators.

A more concrete approach

The **final sequence** of T is an ordinal-indexed sequence of objects $\langle T^{\alpha}1\rangle$, with maps $(f_{\gamma}^{\beta} : T^{\beta}1 \to T^{\gamma}1)_{\gamma \leqslant \beta}$, uniquely defined by the following conditions (where $\delta \leqslant \gamma \leqslant \beta$):

- FS-1 $T^{\beta+1}1 = T(T^{\beta}1)$;
- FS-2 $f_{\gamma+1}^{\beta+1} = T(f_{\gamma}^{\beta})$;
- FS-3 $f_{\delta}^{\beta} = f_{\delta}^{\gamma} \cdot f_{\gamma}^{\beta}$;
- FS-4 if β is a limit ordinal, then $(f_{\gamma}^{\beta} : T^{\beta}1 \to T^{\gamma}1)_{\gamma \leqslant \beta}$ is a limit cone.

Prop. If $f_{\kappa}^{\kappa+1}$ is an isomorphism, then $(T^{\kappa}1, (f_{\kappa}^{\kappa+1})^{-1})$ is a final coalgebra.

Motivating example

Our leading example of a finitary set functor is the finite powerset functor $\mathbb{P}\colon \mathsf{Set} \to \mathsf{Set}$. For a set X, $\mathbb{P}X$ is the collection of finite subsets of X. For a function $f : X \to Y$, $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is defined by $(\mathbb{P}f)(S) = f(S)$.

A $\mathbb{P}$-coalgebra (A, α) can be seen as a graph or a transition system.

Next we investigate the final sequence on $\mathbb{P}$.

Tree bisimulations

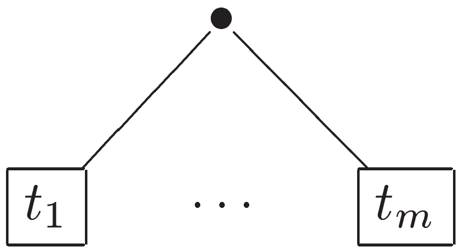
We consider trees as directed graphs with a distinguished root node from which every other node is reachable by a unique path.

A relation R on the set of nodes of a tree is a **tree bisimulation** if xRy implies the respective parents of x and y are related, each child of x is R-related to a child of y, and each child of y is R^{-1}-related to a child of x. We call a tree **strongly extensional** if no two distinct nodes are related by a bisimulation. For any tree t the union of all tree bisimulations is an equivalence, and the quotient of t by this equivalence is strongly extensional.

Remark. An infinite-depth tree can be extensional without being strongly extensional.

Tree representation

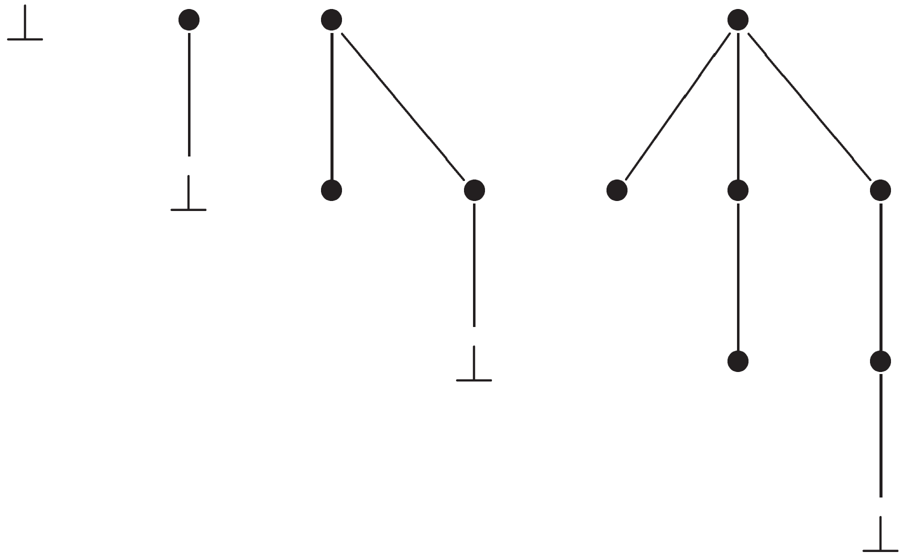
Write $\{\perp\}$ for the final object in Set. For each $n < \omega$ there is a bijection between $\mathbb{P}^n 1$ and the set of finitely branching, strongly extensional trees of depth no greater than n, whose depth-n nodes are labelled $\perp$. By induction: if $x_1, ..., x_m \in \mathbb{P}^n 1$ correspond to trees $t_1, ..., t_m$, then $\{x_1, \cdots, x_m\} \in \mathbb{P}^{n+1} 1$ corresponds to



The projection map f_n^{n+1} takes a tree in $\mathbb{P}^{n+1} 1$, removes the depth-$(n+1)$ nodes and computes the strongly extensional quotient.

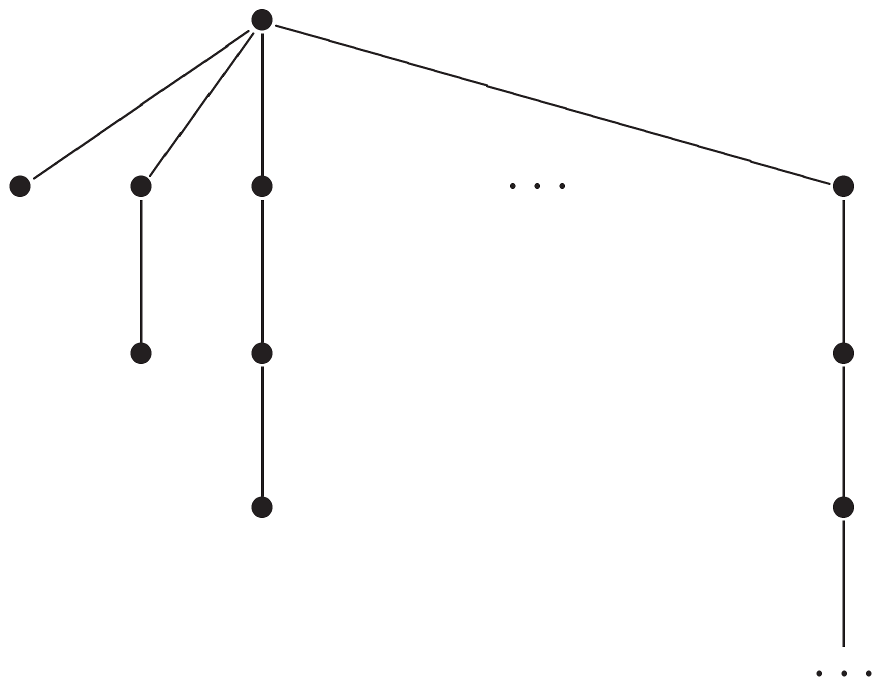
Growing trees

Below we draw a row of trees taken respectively from $\mathbb{P}^0 1$, $\mathbb{P}^1 1$, $\mathbb{P}^2 1$ and $\mathbb{P}^3 1$, with each tree projecting down to the tree to the left.



In the limit

We can imagine the trees as projections of the following **compactly** branching infinite-depth tree in $\mathbb{P}^\omega 1$.



A metric on trees

Given trees t, t', write $t \approx_n t'$ if the restrictions of t and t' to depth n have the same strongly extensional quotient, and define a pseudo-metric $d_{\mathcal{T}}$ on the class of strongly extensional trees by

$$d_{\mathcal{T}}(t, t') = \inf\{2^{-n} \mid t \approx_n t'\}.$$

To infinity and beyond!

$\mathbb{P}^{\omega}1$ is the set of **compactly branching**, strongly extensional trees of possibly infinite depth.

$\mathbb{P}^{\omega+1}1$ is the set of compactly branching strongly, strongly extensional trees **that are finitely branching at the root**.

$\mathbb{P}^{\omega+n}1$ is the set of compactly branching strongly, strongly extensional trees **that are finitely branching to depth** n.

$\mathbb{P}^{\omega+\omega}1$ is the set of finitely branching, strongly extensional trees.

Thus the final sequence of $\mathbb{P}$ consists of ω steps of **building up** higher and higher trees, followed by ω steps of **pruning** level by level.

The general case

We show that the final sequence $\{T^\alpha 1, f^\alpha_\beta\}$ of a finitary functor $T : \mathsf{Set} \to \mathsf{Set}$ stabilises in $\omega + \omega = \omega 2$ steps.

Note that each T-coalgebra (E, e) extends to a **cone** $(e_\alpha : E \to T^\alpha 1)$ over the final sequence, where $e_{\alpha+1} = T(e_\alpha) \cdot e$.

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "!"'] & TE \arrow[r, "Te"] \arrow[d, "T!"'] & T^2E \arrow[r, "T^2e"] \arrow[d, "T^2!"'] & T^3E \arrow[r, "T^3e"] \arrow[d, "T^3!"'] & \cdots \\
1 & T1 \arrow[l] & T^2 1 \arrow[l] & T^3 1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The parsing map

Prop. $f_\omega^{\omega+1}$ is injective (and hence f_ω^α is injective for all $\alpha \geqslant \omega$).

Choose a **parsing map** $p : T^\omega 1 \to T(T^\omega 1)$, where $p \cdot f_\omega^{\omega+1} = id$, and extend p to a cone $(p_\alpha : T^\omega 1 \to T^\alpha 1)$ over the final sequence of T.

Prop. $p_\omega : T^\omega 1 \to T^\omega 1$ is an idempotent self map of the T-coalgebra $(T^\omega 1, p)$.

A final coalgebra

Write $p_\omega = s \cdot r$, where the **retraction** $r : T^\omega 1 \twoheadrightarrow G$ and the **section** $s : G \rightarrowtail T^\omega 1$ satisfy $r \cdot s = id$. Notice that the pair $Tr \cdot p$ and $f_\omega^{\omega+1} \cdot Ts$ is also a splitting of p_ω, thus, we have an isomorphism g making the diagram below commute.

$$\begin{tikzcd}
T^\omega 1 \arrow[r, "p"] \arrow[d, "r"] \arrow[dd, bend right=60, "p_\omega"'] & T^{\omega+1} 1 \arrow[d, "Tr"] \arrow[dd, bend left=60, "Tp_\omega"] \\
G \arrow[r, "g"] \arrow[d, "s"] & TG \arrow[d, "Ts"] \\
T^\omega 1 & T^{\omega+1} 1 \arrow[l, "f_\omega^{\omega+1}"]
\end{tikzcd}$$

Thm. (G, g) is a final T-coalgebra.

A squeezing argument

We show that the injection $f_\omega^{\omega+\omega} : T^{\omega+\omega}1 \to T^\omega 1$ factors through $s : G \rightarrowtail T^\omega 1$ and vice versa.

Prop. $p_\omega \cdot f_\omega^{\omega+\omega} = f_\omega^{\omega+\omega}$.

Pf. Note that $p_n \neq f_n^\omega$ in general. We show that $p_n \cdot f_\omega^{\omega+n} = f_n^{\omega+n}$ for each $n < \omega$.

$$\begin{aligned}
p_{n+1} \cdot f_\omega^{\omega+n+1} &= T(p_n) \cdot p \cdot f_\omega^{\omega+n+1} \\
&= T(p_n) \cdot p \cdot f_\omega^{\omega+1} \cdot f_{\omega+1}^{\omega+n+1} \\
&= T(p_n) \cdot f_{\omega+1}^{\omega+n+1} \\
&= T(p_n \cdot f_\omega^{\omega+n}) \\
&= T(f_n^{\omega+n}) \\
&= f_{n+1}^{\omega+n+1}.
\end{aligned}$$

A squeezing argument

$$
\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "g_{\omega+\omega}"'] & TG \arrow[d, "Tg_{\omega+\omega}"] \\
T^{\omega+\omega}1 \arrow[d, "f^{\omega+\omega}_{\omega}"'] & T(T^{\omega+\omega}1) \arrow[l] \arrow[d, "Tf^{\omega+\omega}_{\omega}"] \\
T^{\omega}1 \arrow[d, "r"'] & T(T^{\omega}1) \arrow[l] \arrow[d, "Tr"] \\
G \arrow[r, "g"] \arrow[d, "s"'] & TG \arrow[d, "Ts"] \\
T^{\omega}1 & T(T^{\omega}1) \arrow[l]
\end{tikzcd}
$$

A squeezing argument

$$
\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "g_{\omega+\omega}"'] \arrow[ddd, bend right=60, "id"'] & TG \arrow[d, "Tg_{\omega+\omega}"] \\
T^{\omega+\omega}1 \arrow[d, "f^{\omega+\omega}_{\omega}"'] & T(T^{\omega+\omega}1) \arrow[l] \arrow[d, "Tf^{\omega+\omega}_{\omega}"] \\
T^{\omega}1 \arrow[d, "r"'] & T(T^{\omega}1) \arrow[l] \arrow[d, "Tr"] \\
G \arrow[r, "g"] \arrow[d, "s"'] & TG \arrow[d, "Ts"] \\
T^{\omega}1 & T(T^{\omega}1) \arrow[l]
\end{tikzcd}
$$

A squeezing argument

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "g_{\omega+\omega}"'] & TG \arrow[d, "Tg_{\omega+\omega}"] \\
T^{\omega+\omega}1 \arrow[d, "f_{\omega}^{\omega+\omega}"'] \arrow[dddd, bend right=60, "injective"'] & T(T^{\omega+\omega}1) \arrow[l] \arrow[d, "Tf_{\omega}^{\omega+\omega}"] \\
T^{\omega}1 \arrow[d, "r"'] & T(T^{\omega}1) \arrow[l] \arrow[d, "Tr"] \\
G \arrow[r, "g"] \arrow[d, "s"'] & TG \arrow[d, "Ts"] \\
T^{\omega}1 & T(T^{\omega}1) \arrow[l]
\end{tikzcd}$$

Main result

Thm. Let κ be a regular cardinal. The class of endofunctors on Set whose final sequence converges in $\kappa + \kappa$ steps is closed under

1. κ-accessible functors;
2. κ^{op}-continuous functors;
3. composition of functors;
4. arbitrary coproducts of functors;
5. $\mathcal{I}$-indexed-limits of functors, for any small category $\mathcal{I}$.

A non-example

Consider the **subprobability-distribution functor** $\mathbb{D}\colon \mathsf{Set} \to \mathsf{Set}$. For a set X, $\mathbb{D}X$ is the set of functions $\mu\colon X \to [0,1]$ with countable support such that $\sum_{x\in X} \mu(x) \leqslant 1$. For $\mu \in \mathbb{D}X$ and $E \subseteq X$ define $\mu[E] = \sum_{x\in E} \mu(x)$. We can extend $\mathbb{D}$ to an endofunctor on Set by defining, for a function $f\colon X \to Y$,

$$(\mathbb{D}f)(\mu)(y) = \mu[f^{-1}(y)].$$

A non-example

Prop. In the final sequence $\{\mathbb{D}^\alpha 1, f^\alpha_\beta\}$, the map $f^{\omega+1}_\omega$ is injective.

Pf. Suppose $\mu, \rho \in \mathbb{D}(\mathbb{D}^\omega 1)$ and $f^{\omega+1}_\omega(\mu) = f^{\omega+1}_\omega(\rho)$. Then for each $\langle x_n \rangle \in \mathbb{D}^\omega 1$, since $\mu[-]$ (being a measure) preserves decreasing countable intersections,

$$\mu(\langle x_n \rangle) = \mu[\bigcap_{n<\omega} (f^\omega_n)^{-1}(x_n)] = \lim_{n\to\infty} \mu[(f^\omega_n)^{-1}(x_n)] = \lim_{n\to\infty} (\mathbb{D} f^\omega_n)(\mu)(x_n).$$

Similarly we get $\rho(\langle x_n \rangle) = \lim_{n\to\infty} (\mathbb{D} f^\omega_n)(\rho)(x_n)$. But

$$(\mathbb{D} f^\omega_n)(\mu) = f^{\omega+1}_{n+1}(\mu) = f^\omega_{n+1}(f^{\omega+1}_\omega(\mu)) = f^\omega_{n+1}(f^{\omega+1}_\omega(\rho)) = f^{\omega+1}_{n+1}(\rho) = (\mathbb{D} f^\omega_n)(\rho)$$

for all $n < \omega$. Thus $\mu = \rho$.

Other categories

Say that $C \in \mathcal{C}$ is a **perfect generator** if C is finitely presentable, projective and for each object K the canonical map

$$\varepsilon_K : \coprod_{\hom(X,K)} \longrightarrow K$$

is a **bimorphism**.

Examples. The one-element poset is a perfect generator in Pos, and the graph with one vertex and no edges is a perfect generator in Graph.

Thm. (Adámek) If $\mathcal{C}$ is locally finitely presentable with a perfect generator, and if $T : \mathcal{C} \to \mathcal{C}$ is a finitary functor preserving strong monos and bimorphisms, then the final sequence of T stabilises in $\omega + \omega$ steps.

Further work

The $\omega + \omega$ bound on the stabilisation of the final sequence of a set functor is tight. However, we do not have an example of an ω_1-accessible set functor whose final sequence takes fully $\omega_1 + \omega_1$ steps to stabilise. If the functor preserves limits of chains of monos, then one has stabilisation in $\omega_1 + \omega$ steps. This case seems to cover all of the ‘reasonable’ ω_1-accessible set functors one comes across, e.g., the countable powerset functor.

For functors on locally finitely presentable categories, are all (any) of Adámek’s side conditions necessary?

Thank you for your attention!

James Worrell. On the final sequence of a finitary set functor. *Theoretical Computer Science*, 338(1-3): 184–199, 2005.